Correction to Proposition 4 in [1]

Section 3 of [1] deals with the case $s > i$. The results are correct. But Proposition 4 is inaccurately stated and its proof contains an error. The proof supposed that because the second derivative of π is positive wherever it exists, the function has no interior maximum. But the interior maximum might be at a point where the derivative does not exist. (For example, consider the minimum of two linear functions, one increasing and the other decreasing). This supplement proves the claimed result: the profit-maximizing policy either has continuous sales with $p = w$, or else has sales only at the ordering points $0, T, 2T, \ldots$.

Recall that in any no-sales interval (t_1, t_2), a proportion $\beta = \frac{s}{i+s}$ of the arrivals in this interval prefer to buy the good at t_1, and a proportion $1 - \beta$ prefer to buy at t_2. By assumption, $s > i$ and therefore $\beta > \frac{1}{2}$.

Observation 1 *Consider a policy of discrete sales, with sales at τ_0 and at $\tau > \tau_0$, and exactly one other sales point $x \in (\tau_0, \tau)$.*
1. If $\tau < T$ then the costs associated with selling at $\tau_0 + x \in (\tau_0, \tau)$ or at $\tau - x$ are identical.
2. If $\tau = T$ then selling at $\tau_0 + x < \tau - x$ yields higher profits than selling at $\tau - x$.
Proof: Without loss of generality, assume that $\tau_0 = 0$, since the difference is a constant independent of the decision which we consider.

1. Consider selling at x. The quantity sold at time x is $\lambda[(1-\beta)x + \beta(\tau - x)]$; the quantity sold at time τ is $\lambda(\tau - x)(1-\beta)$. The inventory holding cost is

$$\begin{aligned} C(x) &= \lambda h\{x[(1-\beta)x + \beta(\tau - x)] + [\tau(\tau - x)(1-\beta)]\} \\ &= \lambda h[(x^2 + (\tau - x)^2)(1-\beta) + x(\tau - x)]. \end{aligned}$$

This expression is symmetric in x and $\tau - x$.

2. The cost associated with selling at x is $\lambda h\{x[(1-\beta)x + \beta(\tau - x)]$. The cost associated with selling at $\tau - x$ is higher, namely $\lambda h\{(\tau - x)[(1-\beta)(\tau - x) + \beta x]$. ■

Observation 2 *Suppose the longest no-sales interval has length Δ, and suppose there are consecutive sales at $\tau_0 < x < \tau$ such that both $x - \tau_0 < \Delta$ and $\tau - x < \Delta$. Then the firm can increase its profits by increasing the longer of these no-sales intervals.*
Proof: By assumption, increasing the longer of these no-sales intervals does not affect the profits, since p is determined by the longest no-sales interval

Δ. Again assume, w.l.o.g., that $\tau_0 = 0$. Moreover, by Observation 1 we can assume w.l.o.g. that $x > \frac{\tau}{2}$. The inventory cost associated with selling at x and τ is $C(x) = \lambda h\{x[(1-\beta)x + \beta(\tau - x)] + \tau(\tau - x)(1-\beta)\}$, where the second term does not exist if $\tau = T$. Thus,

$$\begin{aligned} C'(x) &= 2(1-2\beta)x + [2\beta\tau - \tau] \\ &> 2(1-2\beta)\frac{\tau}{2} + [(2\beta - 1)\tau] \geq 0. \end{aligned}$$

The strict inequality follows since $\beta > \frac{1}{2}$ and $x < \frac{\tau}{2}$. The second inequality is strict if $\tau = T$, since in this case the term in square brackets does not exist. ∎

Corollary *Suppose T and the maximum length of a no-sales interval Δ are fixed, and $T = (k+\alpha)\Delta$ where $k \in \{1, 2, \ldots\}$ and $0 \leq \alpha < 1$. Then, the firm maximizes profits by selling at $0, \Delta, 2\Delta, \ldots, (k-1)\Delta$, and $(k-1+\alpha)\Delta$.*[1]
Proof: By the first observation, the order of no-sales intervals, except for the last one, has no effect on the firm's costs. If there exist two no-sales intervals of lengths shorter than Δ, then by Observation 1 w.l.o.g. they are consecutive, and this contradicts Observation 2. Thus, at most one such interval exists. Using again the first observation, we assume that this is the interval before the last. ∎

From this point we analyze policies that satisfy the properties given in the corollary.

At the end of each of the first $k-2$ no-sale intervals the firm sells a quantity $\lambda\Delta$. Of this quantity, $\lambda(1-\beta)\Delta$ is sold to customers who desired the good at an earlier time, and $\lambda\beta\Delta$ to customers who buy it earlier than their most desired time of purchase. Similarly, at $t = (k-1)\Delta$ the firm sells $\lambda\Delta(1 - \beta + \alpha\beta)$, and at $t = (k-1+\alpha)\Delta$ the firm sells $\lambda\Delta(\alpha(1-\beta) + \beta)$. The total inventory holding cost is therefore

$$\begin{aligned} C_I(\Delta) &= \lambda h\Delta^2[(1 + 2 + \cdots + k - 2) + (k-1)(1 - \beta + \alpha\beta) + (k - 1 + \alpha)(\alpha - \alpha\beta + \beta)] \\ &= \lambda h\Delta^2[(1 + 2 + \cdots + k - 1) + (k-1)\alpha + \alpha^2 + \alpha\beta(1-\alpha)] \\ &= \lambda h\Delta^2\left[\frac{k(k-1)}{2} + (k-1)\alpha + \alpha^2 + \alpha\beta(1-\alpha)\right] \\ &\geq \frac{1}{2}\lambda h\Delta^2(k + \alpha - 1)(k + \alpha), \end{aligned}$$

[1] W.l.o.g. $k \geq 1$. If $k = 0$, i.e., $T < \Delta$, we choose $p = w_1$ and sell continuously.

where the inequality follows from $\beta \geq \frac{1}{2}$.[2]

Since $\alpha < 1$ and the price is determined by the largest no-sales interval, $p = w - \Delta\frac{is}{i+s}$, and the profit rate satisfies

$$\pi \leq \lambda\left(w - \Delta\frac{is}{i+s}\right) - \frac{\frac{1}{2}\lambda h\Delta^2(k+\alpha-1)(k+\alpha)+A}{(k+\alpha)\Delta} \equiv \lambda w - C(\Delta).$$

Keeping α and k at constant levels while optimizing Δ yields

$$C(\Delta) \geq 2\sqrt{\left(\frac{is}{i+s} + \frac{h(k+\alpha-1)}{2}\right)\frac{\lambda A}{k+\alpha}}.$$

Lemma *If* $h\left(\frac{1}{i}+\frac{1}{s}\right) \leq 2$ *then* $C(\Delta) \geq C_{EOQ}$.

Proof: Recall from (12) in [1] that $C_{EOQ} = \sqrt{2\lambda hA}$. Hence, it suffices to prove that $2\left(\frac{is}{i+s}+\frac{h(k+\alpha-1)}{2}\right)\frac{1}{k+\alpha} \geq h$, or equivalently, that $\frac{2}{k+\alpha}\frac{is}{i+s} \geq h\left(1-\frac{k+\alpha-1}{k+\alpha}\right)$. The last inequality follows from the assumption of the lemma. ■

Lemma *If* $h\left(\frac{1}{i}+\frac{1}{s}\right) \geq 2$ *then* $C(\Delta) \geq C_{0,T,\ldots}$*, where* $C_{0,T,\ldots}$ *is the minimum cost of a policy that sells only at the beginning of the cycle.*

Proof: Recall from (10) that $C_{0,T,\ldots} = 2\sqrt{\frac{\lambda A is}{i+s}}$. Hence, it suffices to prove that $\left(\frac{is}{i+s}+\frac{h(k+\alpha-1)}{2}\right)\frac{1}{k+\alpha} \geq \frac{is}{i+s}$, or equivalently, $\frac{h(k+\alpha-1)}{2(k+\alpha)} \geq \frac{is}{i+s}\left(1-\frac{1}{k+\alpha}\right)$. The last inequality follows from the assumption of the lemma. ■

References

[1] A. Glazer and R. Hassin, "A deterministic single-item inventory model with seller holding cost and buyer holding and shortage costs," *Operations Research* **34** (1986) 613-618.

[2] $C_I(\Delta)$ is nondecreasing in β. Therefore, it is minimized when =0.5 . In this case, however, the firm gains nothing from a no-sale interval that does not end at T, and the cost is identical to that obtained with a single no-sales interval at the end of the cycle.