

Optimal Consumption with a Stochastic Income Stream and Two Interest Rates

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I. Introduction

An individual's choice between consumption and investment is very complex and simplifications must be introduced in every model building. A common simplification is the elimination of uncertainty from the model by replacing random variables by their expected values, since in general the deterministic case is much simpler than its probabilistic counterpart. Examples of such random variables are future labor incomes and investment returns. Another type of simplification is the elimination of variables or opportunities from the model. For example, borrowing may be excluded from the individual's financial opportunities or it may be assumed that there exists only one interest rate, common to borrowers and lenders.

A combination of the above attitudes has led to an important result obtained by Mirman [8] and Leland [4] in two period models, and Miller [6] and Mendelson and Amihud [5] in a multiperiod model. They have proved that in the case of one interest rate and additive utility functions, concavity and positive third derivative of the utility functions imply an increase in consumption when going from a random labor income case to its deterministic counterpart. In [5] and [7] this result is extended, and it is shown that consumption also increases when a random sequence of labor income is replaced by a less risky sequence.

The importance of the above results is twofold; they supply qualitative knowledge on the impact of increased uncertainty in future labor income on the consumer's decisions, and they give an upper bound on the immediate consumption in probabilistic models. These results can be explained as follows: The assumptions imply convexity of the marginal utility functions which tend therefore to increase when randomness is introduced or increased. Therefore the expected marginal utility of future consumption (related to present savings) increases. This leads to an increased preference for future consumption at the expense of immediate consumption.

This paper analyzes a multiperiod nonstationary model with a random non-capital income stream and two deterministic interest rates. Its major purpose is to demonstrate that the results of Miller and of Mendelson and Amihud do not generally hold when the interest rate on debt is greater than the interest rate on assets.

The problem of optimal consumption under a differential in the borrowing and lending rate was introduced by Fisher [2]. In a two period model with deterministic income stream, Fisher showed the existence of a wealth interval in which optimal consumption is exactly equal to wealth. An individual whose initial wealth is in this interval wishes to lend when confronted with the borrowing rate and to borrow when confronted with the lending rate. For levels of wealth below this interval optimal consumption is greater than wealth and above the interval optimal consumption is less than wealth. These results were extended to the multiperiod case by Watkins [12] who also describes a solution procedure to determine an optimal lifetime consumption program.

Optimal consumer behavior has been widely analyzed in the economic literature. We discussed here only those papers directly relevant to this paper. A list of related papers would include [1; 3; 9; 10; 11; 13; 14]. The various components of the problem are described and put into a formal model in the next section. The transition equations governing the process are derived in Section III. The character of the optimal consumption program is examined in Section IV. In Section V we investigate the effect of uncertainty in future labor income on consumption. This section is concluded with a numerical example in which the introduction of uncertainty increases the optimal level of consumption.

II. The Model

The individual's consumption opportunities occur at discrete, equally spaced points in time. These points divide the consumer's lifetime into T periods. The state of the system at the beginning of each period $j = 1, 2, \dots, T$ is described by the variable w_j , the amount of capital then on hand. This amount (which may be negative) already includes the labor income Y_{j-1} received at the end of period $j - 1$. At this time the individual chooses to consume some nonnegative amount c_j . If the consumed amount is less than the capital, the difference is left as savings to grow at a rate defined by the interest rate for lending $r_{L,j}$. If it is greater, the difference is borrowed at a rate $r_{B,j}$.

The individual's objective is postulated to be maximization of his expected lifetime utility. His resources consist of an initial capital position and the income stream. We assume that the individual's utility consists of the utility of consumption accumulated during his lifetime plus his utility of bequest. Therefore we assume that in addition to the utility functions of consumption u_1, \dots, u_T a utility function of bequest exists with similar properties of concavity and positive third derivative. This function is, however, defined also for negative values. The case in which the individual is allowed to borrow only against certain future income can be obtained if we assume that the utility bequest is $-\infty$ for negative values.

For the reader's convenience we list below the notation employed throughout this paper. We also define the underlying assumptions.

- T length of the planning horizon, assumed to be finite and deterministic.
 c_j consumption in period j , required to be nonnegative.
 w_j wealth on hand at the beginning of period j ; this amount is allowed to be negative.
 $c_j(w_j)$ optimal consumption in period j .
 w_{T+1} amount of wealth left as bequest at the end of period T ; this amount is allowed to be negative.
 $F_{T+1}(w_{T+1})$ the bequest utility function: we assume that this function is concave and that its first derivative is convex and equal to zero for $w_{T+1} = \infty$.

$$U(c_1, \dots, c_T, w_{T+1}) = \sum_{j=1}^T u_j(c_j) + F_{T+1}(w_{T+1})$$

the lifetime utility function defined over all possible consumption programs (c_1, c_2, \dots, c_T) , where $u_j(c_j)$ $j = 1, \dots, T$ are assumed to possess the same properties as F_{T+1} .

- Y_j random income received at the end of period j ; we assume that Y_j $j = 1, \dots, T$ are independent, but not necessarily identically distributed.
 $r_{L,j}$ and $r_{B,j}$ lending and borrowing interest rates respectively in period j ; we assume $r_{L,j} \leq r_{B,j}$.
 $h_j(w_j - c_j)$ wealth at the end of period j before receiving Y_j .

$$h_j(w_j - c_j) = \begin{cases} (1 + r_{L,j})(w_j - c_j) & \text{if } w_j - c_j \geq 0 \\ (1 + r_{B,j})(w_j - c_j) & \text{if } w_j - c_j \leq 0. \end{cases}$$

In the following, we simplify the exposition by omitting the subscript j from u_j , $r_{L,j}$ and $r_{B,j}$.

III. Derivation of the Functional Equations

According to our assumptions the consumption decisions are made at the beginning of each period and c_j is consumed immediately, so that the level of savings (or debt) is $w_j - c_j$. Thus the sequence of events in each period includes consumption and deposit of savings at the beginning of the period and receipt of the random income at the end of it. This process is expressed by the following set of equations:

$$w_{j+1} = h_j(w_j - c_j) + Y_j \quad j = 1, 2, \dots, T. \quad (1)$$

The consumer's objective is to maximize his expected lifetime utility $EU(c_1, c_2, \dots, c_T, w_{T+1})$ given his initial wealth w_1 and subject to (1) and to the nonnegativity of c_1, \dots, c_T .

We define $F_j(w_j)$ as the maximum expected value of $U(c_j, \dots, c_T, w_{T+1})$ given the initial wealth w_j in period j and subject to $c_i \geq 0$ $i = j, \dots, T$ and (1). This function may be interpreted as a utility function of wealth at the beginning of period j . The problem reduces now to a sequence of T problems, each involving a single decision variable, the decision which must be taken at the current moment. This approach leads to the following functional equation:

$$F_j(w_j) = \max_{c \geq 0} F_j(w_j, c) \text{ where} \\ F_j(w_j, c) = u(c) + EF_{j+1}[h_j(w_j - c) + Y_j] \quad j=1, \dots, T. \quad (2)$$

The expectation in equation (2) is with respect to the random income Y_j . Letting $c_j(w_j)$ denote the optimal consumption level at period j as a function of the capital w_j , we obtain the following relation:

$$F_j(w_j) = u(c_j(w_j)) + EF_{j+1}[h_j(w_j - c_j(w_j)) + Y_j]. \quad (3)$$

The results of the paper hold for general distributions of the incomes Y_j . To simplify the exposition, we assume continuous probability distributions and thus continuous expectations taken with respect to Y , specifically EF' . If this assumption is relaxed, then EF' may have a finite number of discontinuities and equalities involving EF' must be replaced by inequalities involving the right and left derivatives at these points.

Differentiating (2) with respect to c we obtain for $c < w_j$

$$dF_j(w_j, c)/dc = u'(c) - (1+r_L)EF'_{j+1}[(1+r_L)(w_j - c) + Y_j] \quad (4)$$

and for $c > w_j$

$$dF_j(w_j, c)/dc = u'(c) - (1+r_B)EF'_{j+1}[(1+r_B)(w_j - c) + Y_j]. \quad (5)$$

IV. Characterization of the Optimal Consumption Policy

In period j the consumer allocates his wealth w_j between instant consumption c_j and savings $w_j - c_j$. The utility of instant consumption is given by $u(c_j)$ while the utility of savings is given by the expected value of the derived utility of the next period $EF_{j+1}(h_j(w_j - c_j) + Y_j)$. The marginal utility of instant consumption $u'(c_j)$ is decreasing, as is $F'_{T+1}(w_{T+1})$, and we assume (as will be inductively shown in the following) that F'_j is decreasing for $j = 1, \dots, T$. The consumer maximizes his total utility by equating his marginal utility of instant consumption and his marginal utility of savings. This is illustrated in Figure 1. The marginal utility function of instant consumption is a differentiable convex function (Figure 1a). Since the derivative of EF_{j+1} with respect to savings is $(1+r_B)EF'_{j+1}$ for negative savings, and $(1+r_L)EF'_{j+1}$ for positive savings, then when $r_L < r_B$ the marginal utility of savings is discontinuous when savings equal zero (Figure 1b). Figure 1c shows the marginal utility of wealth, F'_j , where wealth is the sum of instant consumption and savings. This function is the horizontal sum of the functions in Figures 1a and 1b. We see that F'_j is continuous and decreasing but nondifferentiable at the critical points w^1 , w^2 , and w^3 . More important, we see that the function is not convex.

To formulate the above results we now define for each period three critical levels of wealth w_j^1 , w_j^2 , and w_j^3 . Since $r_L \leq r_B$, $u'(0)$ satisfies exactly one of the following the relations:

$$u'(0) > (1+r_B)EF'_{j+1}(Y_j) \quad (6)$$

$$(1+r_L)EF'_{j+1}(Y_j) \leq u'(0) \leq (1+r_B)EF'_{j+1}(Y_j) \quad (7)$$

$$u'(0) < (1+r_L)EF'_{j+1}(Y_j). \quad (8)$$

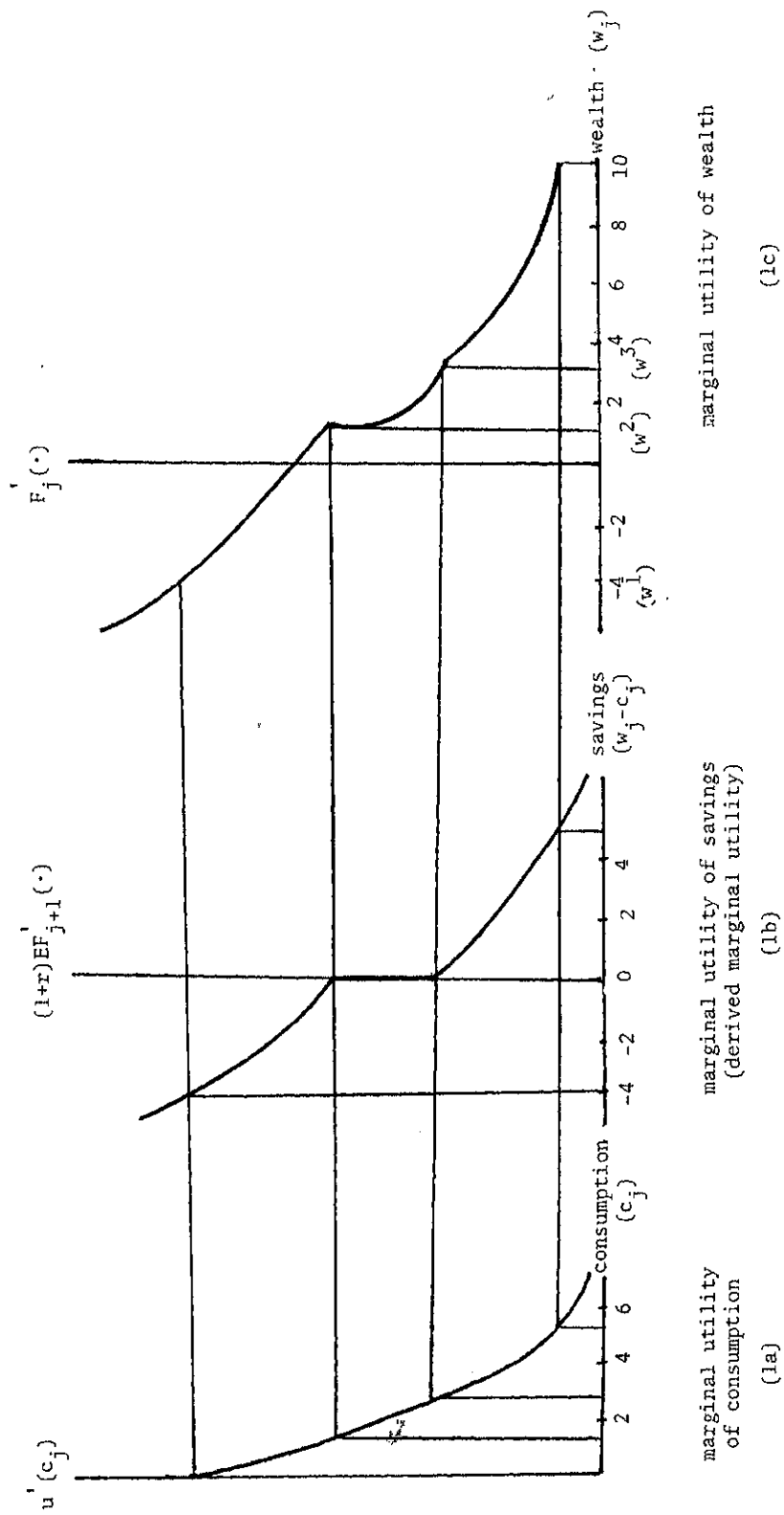


Figure 1.

Definition of w_j^1 :

If (6) holds, then w_j^1 is the (negative) level of wealth satisfying

$$u'(0) = (1 + r_B)EF'_{j+1}(w_j^1(1 + r_B) + Y_j). \quad (9)$$

If (6) holds and (9) has no finite solution (e.g., when $u'(0) = \infty$), then w_j^1 is defined to be $-\infty$. If (7) holds, then $w_j^1 = 0$.

If (8) holds, then w_j^1 is the (positive) level of wealth satisfying

$$u'(0) = (1 + r_L)EF'_{j+1}(w_j^1(1 + r_L) + Y_j).$$

Definition of w_j^2 :

If (6) holds, then w_j^2 is the (positive) level of wealth satisfying

$$u'(w_j^2) = (1 + r_B)EF'_{j+1}(Y_j), \quad (10)$$

else we define $w_j^2 = w_j^1$.

Definition of w_j^3 :

If (8) does not hold, then w_j^3 is the (nonnegative) level of wealth satisfying

$$u'(w_j^3) = (1 + r_L)EF'_{j+1}(Y_j). \quad (11)$$

If (8) holds, we define $w_j^3 = w_j^2$.

The following lemma is directly implied by the definitions.

LEMMA 1. Assuming that F' is monotone decreasing, then one of the following holds:

- (a) If (6) holds, then $w_j^1 < 0 < w_j^2 < w_j^3$.
- (b) If (7) holds, then $w_j^1 = w_j^2 = 0 < w_j^3$.
- (c) If (8) holds, then $w_j^1 = w_j^2 = w_j^3 > 0$.

Theorem 1 below results directly from the above discussion and from Figure 1. Its first part formally characterizes the optimal consumption level as a function of the individual's capital. The results are very similar to those obtained by Fisher [2] and Watkins [12]. When the interest rates are unequal there exists a wealth interval (w_j^2, w_j^3) in which the marginal propensity to consume is one, i.e., an additional increment of capital now will be immediately consumed. Outside this interval the marginal propensity to consume is smaller than one so that the expenditure of additional income will be spread out over an extended period.

The second part of the theorem shows that the derived utility function of wealth inherits the basic properties of the utility function of consumption. This result has already been obtained by other authors for the case of one interest rate (e.g. Miller's Lemma 12 in [7]). However, there is one difference when the interest rates are unequal; the marginal utility of wealth is not necessarily convex. This property will be shown in Section 4 to be crucial in the analysis of the impact of uncertainty on the optimal level of consumption.

THEOREM 1.

I. $c_j(w_j)$ is a monotone increasing, continuous function, differentiable everywhere but at w_j^1 , w_j^2 , and w_j^3 , satisfying:

- a. For $w_j \in (-\infty, w_j^1)$: $c_j(w_j) = c_j'(w_j) = 0$.
- b. For $w_j \in (w_j^1, w_j^2)$: $c_j(w_j) > \max(0, w_j)$ and $0 < c_j'(w_j) < 1$.

$c_j(w_j)$ is solved by equating (5) to zero.

- c. For $w_j \in (w_j^2, w_j^3)$: $c_j(w_j) = w_j$, and $c_j'(w_j) = 1$.

- d. For $w_j \in (w_j^3, \infty)$: $c_j(w_j) < w_j$ and $0 < c_j'(w_j) < 1$.

$c_j(w_j)$ is solved by equating (4) to zero.

II. The function $F_j''(w_j)$ is monotone increasing everywhere except at w_j^2 .

The results of part I of Theorem 1 are summarized in Figure 2, which describes the three cases detailed in Lemma 1. Notice that as a result of the existence of the interval (w_j^2, w_j^3) , consumption is not a concave function of wealth even when $c_j > 0$.

Recall that r_B must satisfy the following inequalities: $r_L \leq r_B \leq \infty$.

The two extreme cases will later be of interest to us:

- i. $r_B = \infty$. In this case (6) does not hold, and by Lemma 1, $w_j^1 = w_j^2$.
- ii. $r_B = r_L$. Then by definition $w_j^2 = w_j^3$.

The following corollary to Theorem 1 holds in both cases:

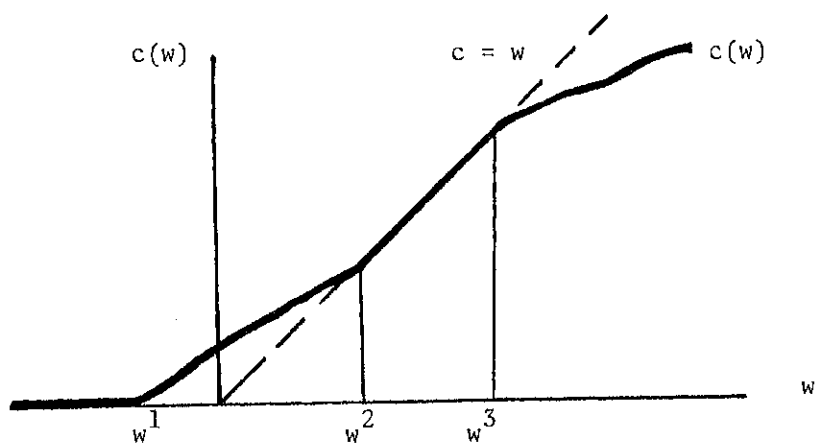
COROLLARY 1. In the two extreme cases $r_B = \infty$ and $r_B = r_L$, $F_j(w_j)$ is a convex function.

Proof: The corollary clearly holds when $r_L = r_B$, since in this case F' is the horizontal sum of two nonincreasing convex functions. If $r_B = \infty$ then Figure 1c is identical to Figure 1a for $w < w^2$ and equal the horizontal sum of Figure 1a and 1b for $w > w^2$. Therefore F'' increases also in w^2 and F' is convex.

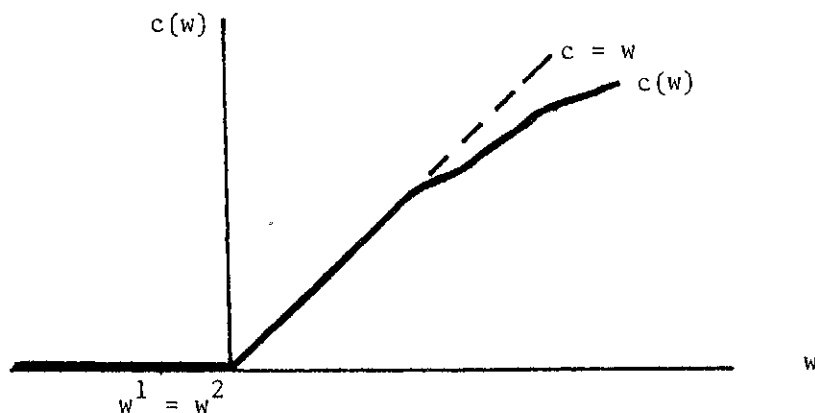
V. Comparing the Random Case with the Deterministic Case

Miller [6] shows that when lending and borrowing interest rates are equal, the individual will tend to reduce his present consumption when his labor income is exposed to risk. In other words; the optimal consumption decreases when going from the deterministic labor income case to a random labor income case in which the expected values of the incomes are preserved. Therefore optimal consumption in the deterministic case is an upper bound for optimal consumption in the random case. We show that this result does not generally hold when the borrowing rate is greater than the lending rate, i.e., $r_B > r_L$. However, the upper bound always holds for the extreme cases where $r_B = \infty$ (equivalent to the case in which borrowing is forbidden), and $r_B = r_L$.

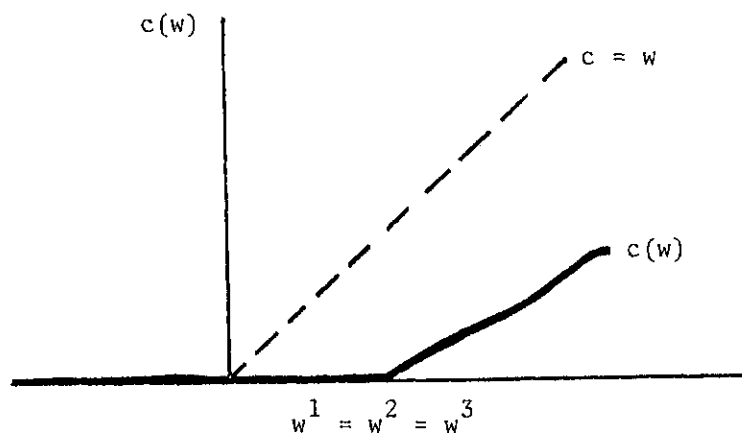
The following theorem gives conditions under which Miller's upper bound holds for two interest rates.



case a; equation (6) holds



case b: equation (7) holds



case c: equation (8) holds

Figure 2.

THEOREM 2.

Suppose F'_{j+1} is convex. Optimal period j consumption in the deterministic case is an upper bound for optimal period j consumption in the random case.

Proof: We prove the theorem by induction. For simplicity we write $u' = dEF/dc$ when it is meant that u' is between the right and left derivatives of EF . We also denote by $G_j(W)$ the utility of wealth in the deterministic case.

Suppose that

$$G'_{j+1}(h + EY_j) \leq EF'_{j+1}(h + Y_j) \quad (12)$$

for every value of h . This assumption holds for $j=T$ since $G'_{T+1}(h+EY_T) = F'_{T+1}(h + EY_T)$ and by Jensen's Inequality $F'_{T+1}(h+EY_T) \leq EF'_{T+1}(h + Y_T)$.

Let c_r and c_d denote the optimal consumption levels in period j in the random and deterministic cases, respectively. Then

$$u'(c_r) = dEF_{j+1}[h_j(w_j - c_r) + Y_j]/dc \quad (13)$$

and

$$u'(c_d) = dG_{j+1}[h_j(w_j - c_d) + EY_j]/dc. \quad (14)$$

By (12) and (13) and since u' and F' are nonincreasing, c_r must be decreased to satisfy (14). Hence $c_d \geq c_r$. Moreover, denote by c_d and c_r the optimal consumption levels when $w_j = h + EY_{j-1}$, then

$$G'_j(w_j) = u'(c_d) \leq u'(c_r) = F'_j(w_j).$$

Substituting $h + EY_{j-1}$ for w_j and using Jensen's Inequality we obtain

$$G'_j(h + EY_{j-1}) \leq F'_j(h + EY_{j-1}) \leq EF'_j(h + Y_j),$$

so that (12) holds also when $j+1$ is replaced by j .

The proof is illustrated in Figure 3 where again the graph of marginal utility of wealth $F'_j(w_j)$ is the horizontal sum of the graphs of marginal utilities from consumption and saving, and the optimal consumption level $c_j(w_j)$ is the value equating the marginal utilities of consumption and saving.

COROLLARY 2. If $r_L = r_B$, or if $r_B = \infty$, then optimal consumption under uncertainty is less than or equal to optimal consumption when incomes are replaced by their expected values.

The proof is immediate from Corollary 1 and Theorem 2.

When F' is not convex, for certain values of wealth the elimination of uncertainty may increase the expected marginal utility of wealth, thus causing $c_1(w_1)$ to decrease. This possibility is illustrated by the following example.

Example. Consider a two period problem, i.e. $T = 2$, with the following data:

$w_1 = 0$, $F_3(w) = 1 - e^{-w}$; $u(c) = 1 - e^{-10c}$; $r_L = 0.5$, $r_B = 1$;
 Y_1 is assigned the values 0.42 and 0.26, each with probability 0.5, and
 $Y_2 = 0$ with certainty.

We first solve for w_2^1 , w_2^2 , and w_2^3 . Since $u'(0) > (1+r_B)EF_3(Y_2)$, (notice that $EF_3(Y_2) = F_3(0)$)

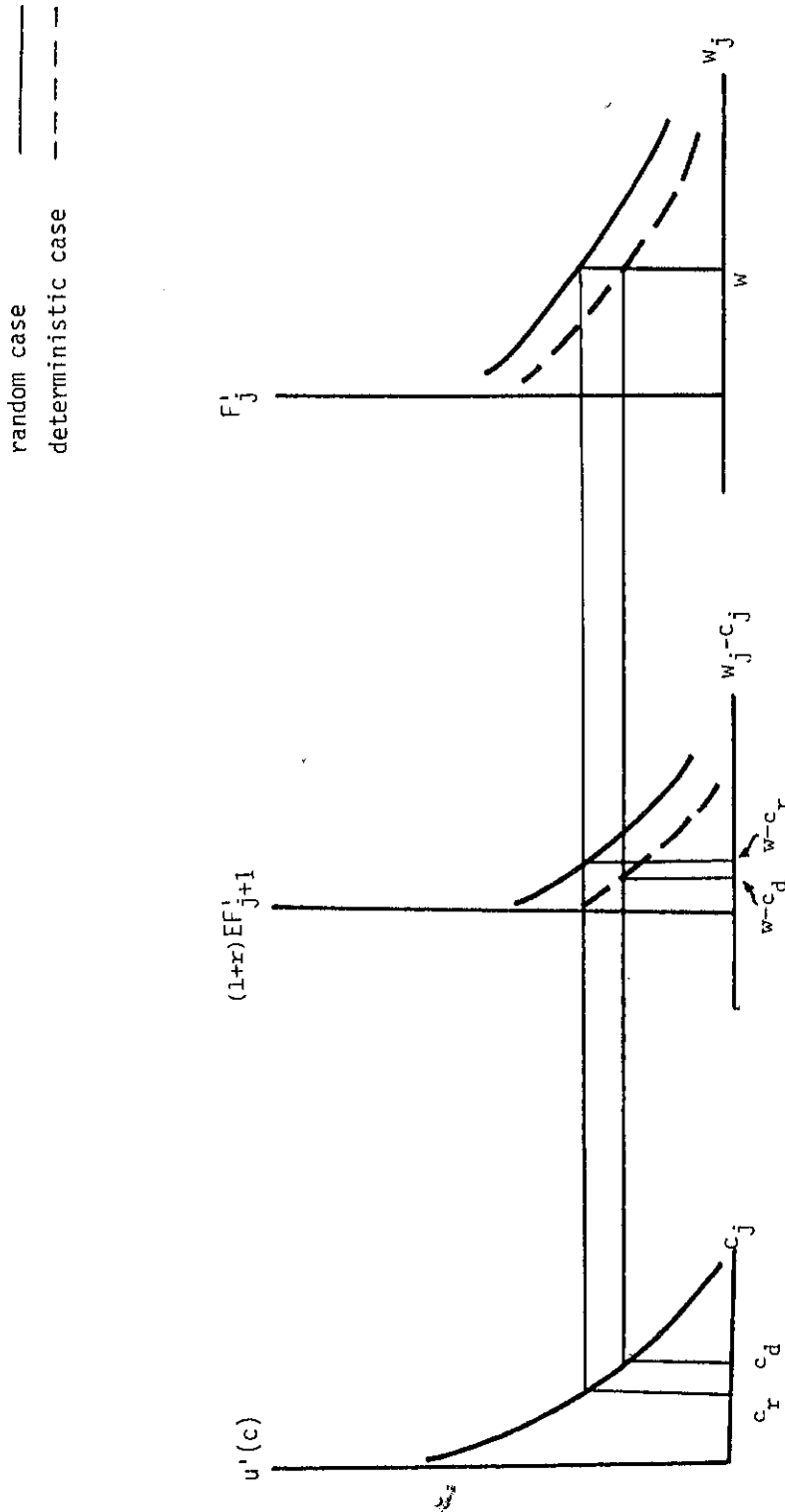


Figure 3.

$= 1$), then (6) holds. Thus we use (9), (10) and (11) to get: $w_2^1 = 0.8047$; $w_2^2 = 0.1609$; $w_2^3 = 0.1897$.

Using part I of Theorem 1 we solve for $c_2(w_2)$. Using Part II, we then solve for $F_2'(w_j)$, obtaining:

$$c_2(w_2) = \begin{cases} 0 & w_2 \in (-\infty, -0.8047) \\ 0.1341 + w_2/6 & w_2 \in (-0.8047, 0.1609) \\ w_2 & w_2 \in (0.1609, 0.1897) \\ 0.1650 + 0.1304w_2 & w_2 \in (0.1897, \infty) \end{cases}$$

$$F_2'(w_2) = \begin{cases} 2e^{-2w_2} & w_2 \in (-\infty, 0.8047) \\ 10e^{-1.341-1.66w_2} & w_2 \in (-0.8047, 0.1609) \\ 10e^{-10w_2} & w_2 \in (0.1609, 0.1897) \\ 10e^{1.650-1.304w_2} & w_2 \in (0.1897, \infty) \end{cases}$$

Similarly we can solve for $j = 1$. From F_2' we note that (6) is satisfied. Thus by Lemma 1, $w_1^1 \leq 0$, $w_1^2 \geq 0$. Since initial wealth is given as $w_1 = 0$ then $w_1 \in (w_1^1, w_1^2)$, implying by Theorem 1 that $c_1(w_1) > w_1 = 0$. Thus the relevant interest rate for period 1 is r_B and equation (5) holds. Equating (5) to zero with $j = 1$ and $w_1 = 0$ we obtain $c_1 = 0.098$.

We now solve the same problem, replacing Y_1 by its expected value 0.34. $F_2'(w_2)$ is the same as above, and equations (6) and (5) still hold. The solution is now $c_1 = 0.091$. Thus with uncertainty eliminated, consumption decreased from 0.098 to 0.091.

VI. Concluding Remarks

This paper serves to shed lights on the effect of income uncertainty on consumption, extending previous works on this issue and presenting a new result. The prevailing knowledge is that for individuals whose marginal utility is convex, i.e., for those with non-decreasing absolute risk aversion, increased income uncertainty decreases consumption. This may imply that a policy-maker may induce an increase in saving by making income more uncertain, or may boost demand by stabilizing income.

However, in the more realistic scenario, where the interest rates for borrowers are greater than for lenders, this result may no longer hold. We have shown that in this case, and for convex marginal utility, income uncertainty does not necessarily lead to a reduction in the individual's level of consumption. It may well be, that income uncertainty will have the opposite effect, i.e., it may lead to an increase in consumption. This result is extremely important, due to its implication on income stabilization policies: Now, stable income may be preferred if the policy maker desires to stimulate savings. Of course, our results pertain only to the individual behavior, and it is worth studying the aggregate consumption behavior.

It is worth noting that our results encompass a wide range of behavior. Firstly, our consumer's utility function is of a more general form than that in other works on multiperiod consumption. In particular, we do not assume anything about the inter-period relationship between the utility functions. Ours represent non-stationary preferences, while the others assumed identical utility functions in each period with a constant discount factor. Secondly, we allow for different interest rates for borrowers and lenders. In the case where these two rates are identical, our results on the effect of income uncertainty are similar to those derived by others. When the two rates differ, the effect of income uncertainty on consumption no longer correspond with the prevailing results. Surprisingly, in the other extreme case, where borrowing is not feasible, i.e. (when the rate on borrowing is extremely high), our results again agree with those in the literature. It may be of interest to explore the effect of income uncertainty under a more general — and perhaps more realistic — scenario, where the consumer is faced with a multiple of interest rates, which depend on the amount borrowed or lent.

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