The Price of Anarchy in the Markovian Single Server Queue

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Abstract

The Price of Anarchy (PoA) is a measure for the loss of optimality due to decentralized behavior. It has been studied in many settings. Surprisingly, it has not been studied in the most fundamental queueing system involving customers' decisions, namely, the single server Markovian queue. While much study has been devoted to reducing the inefficiency of customers' selfishness in this model, this paper is the first to investigate the significance of this inefficiency. We find that the loss of efficiency is bounded by 50% in most practical cases, and that it has an odd behavior in two aspects: First, it sharply increases as the arrival rate comes close to the service rate; Second, it becomes unbounded exactly when the arrival rate is greater than the service rate, which is odd because the system is always stable. Knowing these bounds is important for the queue controller, for example when considering an investment in added service capacity or better control of the arrival process.

I. Introduction

Non-optimality of equilibrium behavior is an intrinsic feature of observable queues: customers' joining behavior, which is based on the queue length, does not maximize the social welfare. Naor [21] was the first to demonstrate this phenomena assuming a simple M/M/1 observable queue with linear waiting costs and a fixed service value. The arrival process can be controlled by a central entity (the *manager*) to optimize social welfare. Control can be achieved directly by admitting or rejecting customers , for example [7], [15], [20], or by appropriate pricing of the service, for example [16], [17], [21], [27]. Implementing control mechanisms may be costly. Therefore, it is important to ask in what cases it is worthy to invest in the regulation of the customers, and in what cases it is not.

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The inefficiency of selfish behavior is often measured by the *Price of Anarchy (PoA)* [14], [19]. PoA bounds the ratio of the social welfare under optimum to the social welfare under equilibrium. Thus, PoA measures the extent to which non-cooperation approximates cooperation. The PoA has been studies in various settings: congestion games [14], routing [9], [23], [25], toll competition in a parallel network [26], network-creation game [4], [8], supply chains [5], [6], system resource allocation [13], [24], greedy auctions [18], multiple-items auctions [25], network resource allocation games [12], spectrum-sharing games [9], network-pricing games [1] and more.

Surprisingly, there has been little research on quantifying the inefficiency of queuing systems. Haviv and Roughgarden [11] consider a multi-server queueing system, in which the arrivals are routed to the servers, and the routing decisions are not based on the queue lengths. The PoA in such system is bounded from above by the number of servers. Anselmi and Gaujal [2] considers a system of parallel unobservable queues, in which the router has a memory of previous dispatching choices and the demands grow with the network size. We explore the PoA of an observable $M/M/1$, which is the most fundamental queueing system that involves customers' decision. This model has been studied by Noar [21].

Naor's M/M/1 model assumes a First-Come First-Served observable queue (the length can be observed by the decision maker) with a single server, Poisson arrivals, exponential service, linear waiting costs and fixed rewards from obtaining service. Balking is associated with zero reward. The equilibrium solution in this model is very simple since there exists a dominant *pure threshold strategy*. Namely, for some integer n, an arriving customer joins the queue if and only if the observed queue length upon arrival is shorter than n, and this strategy maximizes the individual's expected welfare no matter what strategies are adopted by the others. The socially optimal behavior is also characterized by a threshold strategy. Naor observes that the threshold of the optimal strategy is in general smaller than that of the Nash equilibrium strategy. This result also holds for more general queueing models (see §2 in [10] for a survey of strategic behavior in observable queueing systems).

In this paper, we explore the behavior of the PoA and bound its value as a function of the model's parameters. After some preliminary derivations in Section II, we investigate in Section III the PoA as a function of the normalized service values. Our main results are obtained in Section IV were we investigate the behavior of the PoA as a function of the system's utilization, and we conclude with some comments in Section V.

II. PoA GENERAL BEHAVIOR

Following Naor's notation, λ denotes the arrival rate, and μ denotes the service rate. The customer obtains a reward of value R upon completing service, and a cost of C per unit of time spent waiting or in service. The model's parameters can be normalized so that there are only two relevant parameters: $\rho = \frac{\lambda}{\mu}$, which is the arrival rate normalized in service capacity units, and $\nu_s = \frac{R\mu}{C}$, which is the value of service in terms of expected waiting cost during a service duration.[1]

The Nash equilibrium and optimal thresholds, n_e and n^* respectively, and their associated social welfare, have been studied by Naor [21]. It is straightforward that $n_e = \left\lfloor \frac{R\mu}{C} \right\rfloor = \lfloor \nu_s \rfloor$. Define $g(\nu) = \frac{\nu(1-\rho)-\rho(1-\rho^\nu)}{(1-\rho)^2}$, then $n^* = \lfloor \nu^* \rfloor$, where ν^* is the unique solution to

$$g(\nu) = \nu_s. \tag{1}$$

Naor showed that $n^* \leq n_e$. Moreover, $n^* = n_e$ if and only if $n_e = 1$.

The social welfare associated with a threshold n is

$$S_n = R\lambda \frac{1-\rho^n}{1-\rho^{n+1}} - C\left[\frac{\rho}{1-\rho} - \frac{(n+1)\rho^{n+1}}{1-\rho^{n+1}}\right].$$

PoA is defined as the ratio of the expected optimal net gain per time unit S_{n^*}, and the expected equilibrium one S_{n_e}:

$$\mathrm{PoA}(\rho,\nu_s) = \frac{\frac{1-\rho^{n^*}}{1-\rho^{n^*+1}} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(n^*+1)\rho^{n^*}}{1-\rho^{n^*+1}}\right]}{\frac{1-\rho^{\lfloor\nu_s\rfloor}}{1-\rho^{\lfloor\nu_s\rfloor+1}} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(\lfloor\nu_s\rfloor+1)\rho^{\lfloor\nu_s\rfloor}}{1-\rho^{\lfloor\nu_s\rfloor+1}}\right]} \tag{2}$$

$$= \frac{p_{\text{join}}(n^*) - \frac{1}{\nu_s}q(n^*)}{p_{\text{join}}(n_e) - \frac{1}{\nu_s}q(n_e)}, \tag{3}$$

where $q(n)$ is the expected queue length, and $p_{\text{join}}(n)$ is the probability that an arriving customer joins the queue. under a threshold n. $\mathrm{PoA}(\rho,\nu_s)$ is shown in Figures 1.

[1]To avoid triviality, $\nu_s \geq 1$. Otherwise, an arriving customers would balk even if the system is empty.

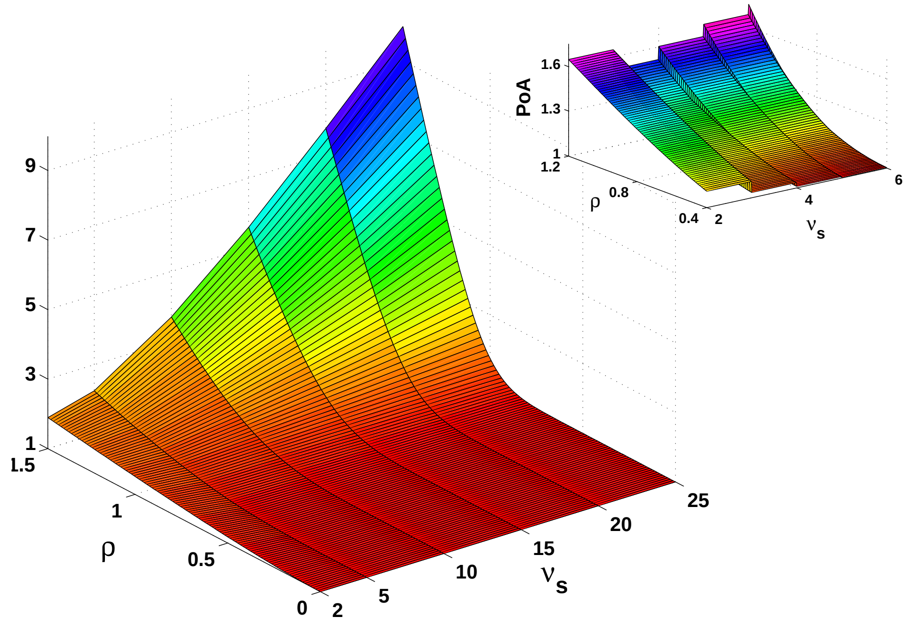

Fig. 1. $\mathrm{PoA}(\rho, \nu_s)$. The inset is a zoom-in of the results, where the PoA is non monotonous.

The definition of the PoA in M/M/1 is not an explicit function of ρ and ν_s. Instead, it is also a function of the optimal strategy n^*. This strategy has to be computed according to a numeric procedure, designed by Naor [21]. Accordingly, the investigation of the PoA is not a straightforward analysis of a 2-variable function. It is a combination of analytical and numerical approaches. In the following subsections, we isolate the influence of ν_s and ρ on the PoA.

III. PoA as a function of ν_s

Figure 2 demonstrates the behavior of PoA(ν_s) for various values of ρ. We pay a special interest to the behavior of the PoA when $\nu_s \to \infty$, because an infinite asymptotic limit indicates that the PoA is unbounded. The three following lemmas describe the three possible limits in association with the value of ρ:

Lemma III.1. $\lim_{\nu_s \to \infty} \mathrm{PoA}(1, \nu_s) = 2.$

Proof: Consider the function $h(x) = \frac{x}{x+1} - \frac{x}{2\nu_s}$. For $\rho = 1$, the optimal social welfare S_{n^*} equals $h(n^*)$, because the queue length distribution is uniform on $\{0, 1, \dots, n\}$, and in particular $q(n) = n/2$ and $p_{\text{join}} = n/(n+1)$.

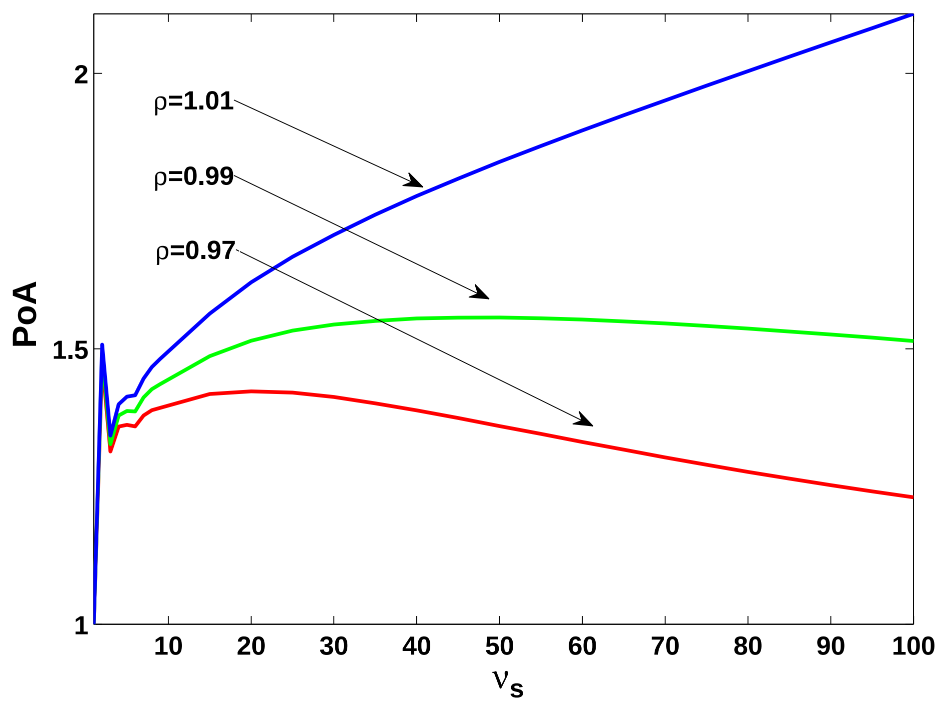

Fig. 2. PoA as a function of ν_s for various values of ρ

A continuous (with respect to x) analysis of $h(x)$ yields that it is maximized by $x = \sqrt{2\nu_s} - 1$. Thus, $S_{n*} = h(n*) \leq h(\sqrt{2\nu_s} - 1)$. By Naor's results, n_e is the floor of ν_s; $n* < n_e$; and the social welfare is an unimodal function which is decreasing for $x > n* + 1$. Thus, $S_{n_e} = h(\lfloor \nu_s \rfloor) \geq h(\nu_s)$. We conclude that:

$$\mathrm{PoA}(1, \nu_s) \leq \frac{\frac{\sqrt{2\nu_s}-1}{\sqrt{2\nu_s}} - \frac{\sqrt{2\nu_s}-1}{2\nu_s}}{\frac{\nu_s}{\nu_s+1} - \frac{1}{2}} \tag{4}$$

Specifically, we explore the case when $\nu_s \to \infty$. According to Naor's results: n^* is either the floor or the ceil of $\sqrt{2\nu_s} - 1$; n_e is the floor of ν_s; both n_e and n^* goes to infinity when ν_s goes to infinity. Thus, when $\nu_s \to \infty$, $S_{n*} = h(n*)$ goes to $h(\sqrt{2\nu_s} - 1)$ and $S_{n_e} = h(\lfloor \nu_s \rfloor)$ goes to $h(\nu_s)$. Therefore,

$$\lim_{\nu_s \to \infty} \mathrm{PoA}(1, \nu_s) = \lim_{\nu_s \to \infty} \frac{\frac{\sqrt{2\nu_s}-1}{\sqrt{2\nu_s}} - \frac{\sqrt{2\nu_s}-1}{2\nu_s}}{\frac{\nu_s}{\nu_s+1} - \frac{1}{2}} = 2 \tag{5}$$

■

Lemma III.2. $\forall \rho > 1, \lim_{\nu_s \to \infty} \text{PoA}(\rho, \nu_s) \to \infty$.

Proof: When $\rho > 1$ and when $\nu_s \to \infty$, by (1)

$$n^* = \log_\rho \nu_s + o(\log_\rho \nu_s) \tag{6}$$

By (2)

$$\text{PoA}(\rho, \nu_s) \;\sim\; \frac{\frac{1-\rho^{\log_\rho \nu_s}}{1-\rho^{\log_\rho \nu_s+1}} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(\log_\rho \nu_s+1)\rho^{\log_\rho \nu_s}}{1-\rho^{\log_\rho \nu_s+1}}\right]}{\frac{1-\rho^{\lfloor \nu_s \rfloor}}{1-\rho^{\lfloor \nu_s \rfloor+1}} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(\lfloor \nu_s \rfloor+1)\rho^{\lfloor \nu_s \rfloor}}{1-\rho^{\lfloor \nu_s \rfloor+1}}\right]}.$$

Since

$$\frac{1-\rho^{\log_\rho \nu_s}}{1-\rho^{\log_\rho \nu_s+1}} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(\log_\rho \nu_s+1)\rho^{\log_\rho \nu_s}}{1-\rho^{\log_\rho \nu_s+1}}\right] = \frac{1-\nu_s}{1-\rho\nu_s} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(\log_\rho \nu_s+1)\nu_s}{1-\rho\nu_s}\right] \sim$$
$$\frac{1}{\rho} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} + \frac{\log_\rho \nu_s}{\rho}\right] = \frac{\nu_s(1-\rho)-\rho-(1-\rho)\log_\rho \nu_s}{\nu_s(1-\rho)\rho} \sim \frac{1}{\rho}$$

and

$$\frac{1-\rho^{\lfloor \nu_s \rfloor}}{1-\rho^{\lfloor \nu_s \rfloor+1}} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(\lfloor \nu_s \rfloor+1)\rho^{\lfloor \nu_s \rfloor}}{1-\rho^{\lfloor \nu_s \rfloor+1}}\right] \sim \frac{1}{\rho} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} + \frac{\lfloor \nu_s \rfloor}{\rho}\right]$$
$$= \frac{\nu_s(1-\rho)-\rho-(1-\rho)\lfloor \nu_s \rfloor}{\nu_s(1-\rho)\rho} = \frac{(\nu_s-\lfloor \nu_s \rfloor)(1-\rho)-\rho}{\nu_s(1-\rho)\rho} < \frac{1-2\rho}{\nu_s(1-\rho)\rho} \xrightarrow[\nu_s \to \infty]{} 0,$$

we conclude that $\text{PoA}(\rho, \nu_s) \xrightarrow[\nu_s \to \infty]{} \infty$. ■

In contrast, $\forall \rho < 1$, the PoA is decreasing to 1 when $\nu_s \to \infty$:

Lemma III.3. $\forall \rho < 1$, $\lim_{\nu_s \to \infty} \text{PoA}(\rho, \nu_s) = 1$.

Proof: When $\rho \leq 1$, the solution of Equation (1) satisfies

$$\lim_{\nu_s \to \infty} \frac{n^*}{\nu_s} = 1 - \rho. \tag{7}$$

Substituting this relation into (2) gives:

$$\lim_{\nu_s \to \infty} \text{PoA}(\rho, \nu_s) =$$
$$\frac{\frac{1-\rho^{\nu_s(1-\rho)}}{1-\rho^{\nu_s(1-\rho)+1}} - \frac{1}{\rho}\left[\frac{1}{1-\rho} - \frac{(\nu_s(1-\rho)+1)\rho^{\nu_s(1-\rho)}}{1-\rho^{\nu_s(1-\rho)+1}}\right]}{\frac{1-\rho^{\lfloor \nu_s \rfloor}}{1-\rho^{\lfloor \nu_s \rfloor+1}} - \frac{1}{\nu_s}\left[\frac{1}{1-\rho} - \frac{(\lfloor \nu_s \rfloor+1)\rho^{\lfloor \nu_s \rfloor}}{1-\rho^{\lfloor \nu_s \rfloor+1}}\right]} \xrightarrow[\nu_s \to \infty]{} 1$$

■

By the three lemmas above, we conclude that the PoA is not bounded when $\rho > 1$. In the next section we will prove that the PoA is bounded when $\rho \leq 1$ and we will find the bound.

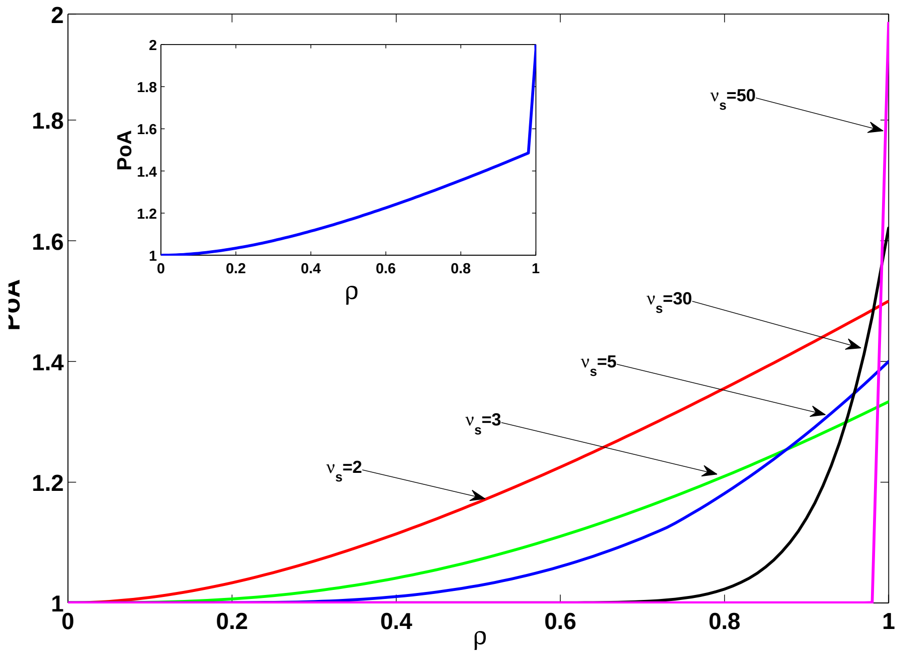

Fig. 3. PoA as a function of ρ for different values of ν_S. The inset is the upper envelope which is defined by 8

IV. POA AS A FUNCTION OF ρ

Figure 3 demonstrates the behavior of PoA(ρ) for various values of ν_s.

We consider the upper envelope of the function $\mathrm{PoA}(\rho, \nu_s)$ for $\nu_s \geq 1$, which is demonstrated at the inset of Figure 3:

$$\mathrm{PoA}(\rho) = \sup_{\nu_s \geq 1} \mathrm{PoA}(\rho, \nu_s) \tag{8}$$

We describe the characteristics of the upper envelope $\mathrm{PoA}(\rho)$ in association with the value of ρ by Lemma IV.1,Lemma IV.2 and Theorem IV.5:

Lemma IV.1. *For any $\rho > 1$, $\mathrm{PoA}(\rho)$ is unbounded.*

Proof: By Lemma III.2. ∎

Remark: Under the limit case, when $\rho \to \infty$, given any threshold n, the number of customers in the system is always n. This is because at any departure instant, an arrival occurs, due to the infinite arrival rate. The social welfare per unit of time is then $R\mu - nC$. Thus, $n^* = 1$. An explanation to the latter is that in general, under optimization, customers are allowed to wait to prevent idleness of the server in the future. When the arrival rate is infinite, idleness never occurs. On the other hand, for fixed μ, C and R, n_e does not depend on λ and n_e remains the same. The PoA is

$$\frac{R\mu - C}{R\mu - Cn_e} = \frac{1 - \frac{1}{\nu_s}}{1 - \frac{\lfloor \nu_s \rfloor}{\nu_s}} = \frac{\nu_s - 1}{\nu_s - \lfloor \nu_s \rfloor} \xrightarrow{\nu_s \to \infty} \infty$$

Lemma IV.2. $\mathrm{PoA}(1) = 2$.

Proof: By (4)

$$PoA(1, \nu_s) \leq \frac{\frac{\sqrt{2\nu_s}-1}{\sqrt{2\nu_s}} - \frac{\sqrt{2\nu_s}-1}{2\nu_s}}{\frac{\nu_s}{\nu_s+1} - \frac{1}{2}} = \frac{(\sqrt{2\nu_s} - 1)^2(\nu_s + 1)}{\nu_s(\nu_s - 1)}, \tag{9}$$

For $\nu_s = 2, 3$, the right-hand side is at most 1.5, and for $\nu_s > 2 + \sqrt{3}$ its derivative is positive. Therefore, $PoA(1, \nu_s)$ is bounded by the limit when ν_s goes to infinity, which is 2 by III.1. ■

The following lemmas will be used in the proof of Theorem IV.5.

Lemma IV.3. *For any $\rho < 1$, if $\mathrm{PoA}(\rho, x)$ is maximized at $x = \nu_s$ then ν_s is an integer.*

Proof: We show that PoA is monotone decreasing with ν_s in the range where n_e is fixed, i.e. $\nu_s \in [n, n + 1)$. This range is divided into a finite number of intervals, such that in each interval n^* is also fixed. The PoA is continuous where n^* changes because at these values $S_{n*} = S_{n*+1}$. Therefore it is sufficient to show that PoA is monotone decreasing with ν_s in each of the intervals, where both n^* and n_e are fixed.

Consider two values $\nu_s^1 < \nu_s^2$ in such interval. The Derivative of (3) with respect to ν_s is

$$\frac{d}{d\nu_s} PoA(\rho, \nu_s) = \frac{q(n^*)p_{\text{join}}(n_e) - q(n_e)p_{\text{join}}(n^*)}{\nu_s^2 \left[p_{\text{join}}(n_e) - \frac{1}{\nu_s}q(n_e)\right]^2}$$

which is negative when $q(n^*)p_{\text{join}}(n_e) < q(n_e)p_{\text{join}}(n^*)$:

$$\left(\frac{1}{1-\rho} - \frac{(n^*+1)\rho^{n^*}}{1-\rho^{n^*+1}}\right)\left(\frac{1-\rho^{n_e}}{1-\rho^{n_e+1}}\right) < \left(\frac{1}{1-\rho} - \frac{(n_e+1)\rho^{n_e}}{1-\rho^{n_e+1}}\right)\left(\frac{1-\rho^{n^*}}{1-\rho^{n^*+1}}\right).$$

This inequality holds when

$$(1 - \rho^{n_e})(1 - \rho^{n^*+1}) - (n^* + 1)\rho^{n^*}(1 - \rho^{n_e})(1 - \rho) -$$

$$(1 - \rho^{n^*})(1 - \rho^{n_e+1}) + (n_e + 1)\rho_e^n(1 - \rho^{n^*})(1 - \rho) < 0$$

Simplifying the last expression, it is left to show that $n_e\rho^{n_e}(1-\rho^{n^*}) - n^*\rho^{n^*}(1-\rho_e^n) < 0$ or $\frac{\rho^{-n^*}(1-\rho^{n^*})}{n^*} - \frac{\rho^{-n_e}(1-\rho^{n_e})}{n_e} < 0$, which is true since $n^* < n_e$ and $\frac{\rho^{-n}(1-\rho^n)}{n}$ is increasing with n when $\rho < 1$.

We conclude that $\mathrm{PoA}(\rho, \nu_s)$ is decreasing with $\nu_s \in [n, n+1]$. In particular this means that the maximum value of PoA in this range is obtained at $\nu_s = n$. ■

Lemma IV.4. *For $\rho < 1$, $\frac{1+(n+1)\rho^n - \rho^{n+1}(n+2)}{1-\rho^{n+1}}$ is a monotone decreasing function of n.*

Proof: We have

$$\frac{1+(n+1)\rho^n - \rho^{n+1}(n+2)}{1-\rho^{n+1}} = 1 + \frac{(n+1)(1-\rho)\rho^n}{1-\rho^{n+1}} = 1 + \frac{n+1}{\sum_{i=0}^{n}\rho^{i-n}}.$$

Thus, we want to show that $\frac{n+1}{\sum_{i=0}^{n}\rho^{i-n}}$ is decreasing, or alternatively $\frac{\sum_{i=0}^{n}\rho^{-i}}{n+1}$ is increasing. As the sequence ρ^{-n} is increasing (because $\rho < 1$), so is the sequence of averages. ■

Theorem IV.5. *Suppose that $\rho < 1$, then $\mathrm{PoA}(\rho) < 2$.*

Proof: By (2) , we need to prove that $\forall\ \rho < 1$ and $\nu_s \geq 1$:

$$\frac{1-\rho^{n^*}}{1-\rho^{n^*+1}} - \frac{1}{\nu_s}\left(\frac{1}{1-\rho} - (n^*+1)\frac{\rho^{n^*}}{1-\rho^{n^*+1}}\right) \leq 2\left[\frac{1-\rho^{\lfloor\nu_s\rfloor}}{1-\rho^{\lfloor\nu_s\rfloor+1}} - \frac{1}{\nu_s}\left(\frac{1}{1-\rho} - (\lfloor\nu_s\rfloor+1)\frac{\rho^{\lfloor\nu_s\rfloor}}{1-\rho^{\lfloor\nu_s\rfloor+1}}\right)\right].$$

By Lemma IV.3 it is sufficient to consider $\nu_s = \lfloor\nu_s\rfloor$. Since ν_s is an integer and $n^* < \nu_s$, it is sufficient to prove that for any two integers $n < m$:

$$\frac{1-\rho^{n}}{1-\rho^{n+1}} - \frac{1}{m}\left(\frac{1}{1-\rho} - (n+1)\frac{\rho^{n}}{1-\rho^{n+1}}\right) \leq 2\left[\frac{1-\rho^{m}}{1-\rho^{m+1}} - \frac{1}{m}\left(\frac{1}{1-\rho} - (m+1)\frac{\rho^{m}}{1-\rho^{m+1}}\right)\right].$$

Since $\frac{1-\rho^n}{1-\rho^{n+1}}$ is monotone increasing in n, it is sufficient to prove

$$\frac{1}{m}\frac{1}{1-\rho} + \frac{1}{m}(n+1)\frac{\rho^n}{1-\rho^{n+1}} \leq \frac{1-\rho^m}{1-\rho^{m+1}} + \frac{2}{m}(m+1)\frac{\rho^m}{1-\rho^{m+1}}.$$

Multiplying by $m(1-\rho)$, it is sufficient to prove

$$\frac{1+(n+1)\rho^n - \rho^{n+1}(n+2)}{1-\rho^{n+1}} \leq \frac{m+\rho^m(m+2)}{1-\rho^{m+1}}(1-\rho).$$

By Lemma IV.4, the left-hand side is monotone decreasing with n. Therefore it is sufficient to show that the inequality holds for $n = 1$:

$$\frac{1+2\rho-3\rho^2}{1-\rho^2} < \frac{m+\rho^m(m+2)(1-\rho)}{1-\rho^{m+1}},$$

or

$$\begin{aligned} 0 &< (m-1) - 2\rho - (m-3)\rho^2 + (m+2)\rho^m - (m+1)\rho^{m+1} - m\rho^{m+2} + (m-1)\rho^{m+3} \\ &= \left[0 < 2 + (m-3)(1+\rho) + (m+1)\rho^m + \rho^m(1+\rho) - (m-1)\rho^{m+2}\right](1-\rho). \end{aligned} \tag{10}$$

Since $\rho < 1$, it is left to show that

$$0 < 2 + (m-3)(1+\rho) + (m+1)\rho^m + \rho^m(1+\rho) - (m-1)\rho^{m+2}$$

which is true by $m > n \geq 2$ and $(m+1)\rho^m > (m-1)\rho^{m+2}$. ■

Following the analytic results, we would like to add some numerical observations from Figure 3. The inset demonstrates that except from a small range near $\rho = 1$, the boundary is much smaller than the boundary proved in Theorem IV.5. Specifically, when $\rho < 0.98175$, $\mathrm{PoA}(\rho) = \mathrm{PoA}(\rho, 2)$, which leads to the following observation:

Observation IV.6. *If* $\rho \in [0, 0.98175]$, $\mathrm{PoA}(\rho, \nu_s) \leq \frac{1+\rho+\rho^2}{1+\rho} < 1.48635$

Proof: We observe that $\mathrm{PoA}(\rho) = \mathrm{PoA}(\rho, 2)$ when $\rho < 0.98175$. Substituting $\nu_s = 2$ in (2) we have

$\mathrm{PoA}(\rho, \nu_s) \leq \frac{1+\rho+\rho^2}{1+\rho}$. We also have that PoA is uniformly bounded by the maximum of $\frac{1+\rho+\rho^2}{1+\rho}$ over $[0, 0.98175]$ which is 1.48635. ■

In contrast, when $\rho \in [0.98175, 1]$, $\mathrm{PoA}(\rho)$ is not equal to a single function $\mathrm{PoA}(\rho, \nu_s)$. Instead, there is an infinite number of functions which define the upper envelope of the PoA in this range. In this small range, when ρ becomes close to 1, the boundary sharply increases to 2.

V. Summary and conclusions

This study explores the PoA in the observable M/M/1 model. The first result is that the PoA in M/M/1 is bounded if $\rho \leq 1$. We emphasize that the model doesn't need to assume $\rho \leq 1$ for stability, and in fact this number is of no significance in Naor's results. It comes therefore as a surprise that the PoA is bounded if and only if $\rho \leq 1$. We assume that this finding is related to the fact that $\frac{n^*}{\nu_s}$ is bounded if and only if $\rho \leq 1$ (see (7) and (6)).

Another interesting result is that for most real situations the PoA is small in comparison with other models discussed in the literature. In particular, when ρ is in $[0, 0.98175]$, the bound is $\frac{1+\rho+\rho^2}{1+\rho} < 1.5$. In most real situations ρ falls into this range. When ρ is in the small range between 0.98175 and 1, the PoA is not bounded by a single function. We prove that the tight bound for this range is 2.

A further study could assess the PoA in other queueing systems, in which the self-optimization by individual customers does not optimize public good. For example, it would be interesting to explore the PoA in a GI/M/1 queue, where the arrival process is a general one, and in a GI/M/s system, where there are parallel servers. Like the M/M/1, these systems also have a pure threshold strategy for the Nash equilibrium solution, according to Yechialy [27], [28].

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