Reconstructing edge-disjoint paths

M. Conforti\(^a\), H. Hassin\(^b\), \(^{+,\ast}\), R. Ravi\(^c\), \(^{\ast,2}\)

\(^{a}\)Department of Mathematics for Computer Science, University of Padova, Piazza Buonarroti 3, 35131 Padova, Italy
\(^b\)Department of Statistics and Operations Research, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel
\(^{+,\ast}\)GEIA, Carnegie Mellon University, Pittsburgh, PA 15213, USA
\(^{\ast,2}\)

Received 18 September 2002; received in revised form 21 January 2003; accepted 27 January 2003

Abstract

For an undirected graph \(G = (V, E)\), the edge connectivity values between every pair of nodes of \(G\) can be succinctly recorded in a flow-equivalent tree that contains the edge connectivity value for a linear number of pairs of nodes. We generalize this result to show how we can efficiently recover a maximum set of disjoint paths between any pair of nodes of \(G\) by storing such sets for a linear number of pairs of nodes. At the heart of our result is an observation that combining two flow solutions of the same value, one between nodes \(s\) and \(r\) and the second between nodes \(s\) and \(t\), into a feasible flow solution of value \(|V|\) between nodes \(s\) and \(r\), is equivalent to solving a stable matching problem on a bipartite multigraph.

Our observation, combined with an observation of Chazelle, leads to a data structure, which takes \(O(n^3)\) time to generate, that can construct the maximum number \(\lambda_{\ast}(s, t)\) of edge-disjoint paths between any pair \((s, t)\) of nodes in time \(O(n^2 \log(n^2))\). time.

© 2003 Elsevier Science B.V. All rights reserved.

Keywords: flow equivalent trees; stable matching.

1. Introduction

Given an undirected graph \(G = (V, E)\) with \(|V| = n\), let \(\lambda(s, t)\) be the \(s\)-\(t\) edge connectivity of \(G\), i.e., the maximum number of edge-disjoint \(s\)-\(t\) paths. Gomory and Hu [4] showed that the edge connectivity function \(\lambda = \{\lambda(s, t) : s, t \in V\}\) has a compact tree representation, i.e., there exists a weighted spanning tree on \(V\) such that for every pair of nodes \(s, t \in V\), \(\lambda(s, t)\) is the minimum weight of an edge on the (unique) \(s\)-\(t\) path in this tree. This tree is known as a flow-equivalent tree of \(G\).

Suppose that a set of \(\lambda(s, t)\) edge-disjoint \(s\)-\(t\) paths are given for every edge \((s, t)\) of the \(|V| - 1\) edges of the flow equivalent tree: Can we efficiently construct \(\lambda(s, t)\) edge-disjoint \(s\)-\(t\) paths for an arbitrary pair \(u, v \in V\)? Such a question may potentially arise in applications that need to compute the maximum flow, or alternatively the maximum number of edge-disjoint paths, between arbitrary pairs of vertices at several points in the course of its execution.

5167-6377/03/$ see front matter © 2003 Elsevier Science B.V. All rights reserved.
doi:10.1016/S0166-6737(03)00022-1
In this paper we describe a compact representation of the sets of the \( \mathcal{A}(u,v) \) edge-disjoint paths for every pair \( u,v \in P \). This representation consists of a graph with node set \( P \) and \( O(n) \) edges, where \( n = |P| \). Each edge \( (s,t) \) in this graph is associated with \( \mathcal{A}(s,t) \) edge-disjoint sr-paths. This data structure can be computed in a preprocessing step that takes time \( O(n^2) \) and \( O(n^2) \) space. We then show how to construct \( \mathcal{A}(u,v) \) edge-disjoint ur-paths for an arbitrary pair \( u,v \in P \) in \( O(n^2) \) time, where \( \mathcal{A}(u,v) \) is the inverse Ackermann function.

2. Stable matchings

Let \( G = (P,Q,E) \), \( |P| = |Q| \), be a bipartite multigraph which is complete, i.e. every pair of nodes in \( P \) and \( Q \) are adjacent. Assume further that every node \( p \in P \) ranks the edges having \( p \) as end node according to its preference and every node \( q \in Q \) also ranks the edges having \( q \) as end node, so that every edge is ranked twice, at both end nodes. A perfect matching \( M \) of \( G \) is stable if for every edge \( e \in E \setminus M \) with end nodes \( p \) and \( q \), either in the p-ranking, \( e \) is less desirable than the edge \( e_s \in M \) that saturates \( p \), or in the q-ranking, \( e \) is less desirable than the edge \( e_t \in M \) that saturates \( q \). Gale and Shapley in their seminal paper [3] (see also [5]) show that every complete bipartite simple graph has a stable matching. Their proof is algorithmic and we give below a straightforward adaptation to the multigraph case.

**StableMatch(bipartite multigraph)**

1. Start with \( M = \emptyset \). Initially, all nodes in \( P \) are exposed and all edges are unexplored.

2. While a node \( p \in P \) is exposed, explore the unexplored edge \( e \) that has highest p-ranking.

   a. Let \( q \) be the other end node of \( e \).

   b. If \( q \) is exposed, then set \( M = M \cup \{e\} \).

   c. Else if the edge \( e_s \in M \) that saturates \( q \) is less preferable than \( e \) in the q-ranking, set \( M = M \cup \{e\} \setminus \{e_s\} \).

The matching \( M \) is stable upon termination of the algorithm. At the end, \( M \) is a perfect matching. For, assume not: then all the edges incident to \( p \) are explored, for some exposed node \( p \). Note that when an edge is explored, its end node in \( Q \) is saturated and remains saturated throughout the algorithm. So when the least desirable edge in the p-ranking is explored and its end node in \( Q \) is saturated, all nodes in \( Q \) are saturated. This is impossible since \( |P| = |Q| \) and \( p \) is exposed.

Finally, an edge is explored at most once in the algorithm, so its complexity is \( O(n^2) \).

We remark that the above problem can be interpreted as a multi-ethnic marriage problem, in which \( P \) represents the set of suitors, \( Q \) the set of brides, and the edges with end nodes \( p \) and \( q \) represent the set of possible marriage ceremonies that can unite \( p \) and \( q \). A perfect matching that is stable corresponds to a set of ceremonies \( C \) that unites all the suitors to all the brides so that no suitor \( p \) and bride \( q \) would both prefer a ceremony not in \( Q \) (possibly with other partners).

3. Composing flow solutions

Let \( P = \{p_1, \ldots, p_k\} \) be a set of \( f \) edge-disjoint sr-paths and \( Q = \{q_1, \ldots, q_l\} \) be a set of \( f \) edge-disjoint rt-paths. Since each flow path has at most \( n \) edges, it is straightforward to find a set of \( f \) edge-disjoint sr-paths in the graph formed by the union of the sr- and rt-paths having \( O(n^2) \) edges. Using a classical flow-augmenting algorithm to find such a decomposition takes \( O(n^2) \) time [1]. Using a method of Karger and Levine [6], this can be accomplished in time \( O(f n^2) \).

**Theorem 3.1.** Let \( P = \{p_1, \ldots, p_k\} \) be a set of \( f \) edge-disjoint sr-paths and \( Q = \{q_1, \ldots, q_l\} \), a set of \( f \) edge-disjoint rt-paths where each flow path has at most \( n \) edges. Then, there exists a set of \( f \) edge-disjoint sr-paths such that each path in this set is the concatenation of a "prefix" of a path in \( P \) and a "suffix" of a path in \( Q \). Moreover, this set can be computed in \( O(n^2) \) time.

**Proof.** Construct the following complete bipartite multigraph \( B = (P,Q,E) \): The node sets \( P \) and
Q represent the paths in P = \{p_1, \ldots, p_l\} and Q = \{q_1, \ldots, q_l\}. For every edge e, that is common to paths p_i and q_j, B contains an edge e with end nodes p_i and q_j. If, after adding all these edges the resulting bipartite multigraph is not complete, add a “dummy” edge between each pair of nonadjacent nodes in P and Q, to make it complete. The priority (from most desirable to least desirable) of the edges of B having p_i as end node is given by the order in which the edges are encountered when traversing path p_i from s to r. The “dummy” edges receive the lowest possible priority (the ranking among them is immaterial). The priority of the edges of B having q_j as end node is given by the order in which the edges are encountered when traversing path q_j from r to t. Again, the “dummy” edges receive the lowest possible priority.

From a stable perfect matching M of B one can construct the desired x-paths as follows: For every edge e in M with end nodes p_i and q_j which is not a dummy edge, traverse path p_i starting from s until e is met and then continue on q_j to t. (Edge e may or may not belong to the path thus constructed.) For every edge e in M with end nodes p_i and q_j which is a dummy edge, traverse path p_i starting from s to r and then traverse q_j from r to t.

The fact that the matching M is stable on B insures that the f x-paths thus constructed are edge disjoint. Indeed, suppose for a contradiction that an edge e is used in two of these concatenated paths, which are represented by two edges in the stable matching, say (p', q') and (p'', q''). These edges are witnessed by the fact that there are edges q'' and q' common to p'' and q'' and q' common to p'' and q'. Since p_i and p_j are disjoint, the edge e must occur in only one of them, so assume that e occurs in p'' and q'. Since e is in the concatenated path from p'' and q'', it must be the case that e occurs before g_i in p'' going from s to r. This means that an edge between p'' and q' in the auxiliary bipartite multigraph has higher priority than the edge (p', q') witnessed by e in the P-ranking. Similarly, since the edge e occurs in the concatenated path from p'' and q', it must be the case that e occurs before g_j in q' going from t to r. This means that the edge between p'' and q' in the auxiliary bipartite multigraph has higher priority than (p', q') witnessed by e in the Q-ranking. Thus, this unmatched edge (p'', q') violates the definition of stability of the matching found, a contradiction.

Note however, that the concatenated paths constructed as above, while being edge disjoint, may not be simple, in which case we can delete cycles without destroying the ‘prefix-suffix’ property. This clean-up step takes time proportional to the size of the paths.}

We finally remark that every stable matching problem in a complete bipartite multigraph can be converted into a path-transitive problem of the above type between f edge-disjoint x-paths and f edge-disjoint r-paths.

4. Augmenting flow-equivalent trees

Given the above method for composing a pair of edge-disjoint path solutions, we now show how we can maintain the maximum edge-disjoint paths solution for O(n) pairs of nodes in an n-node undirected graph, so that the maximum edge-disjoint paths solution for any arbitrary pair of nodes s and r can be recovered by applying the stable matching procedure O(n log n) times. We exploit the natural connection that the maximum number of edge-disjoint paths in unit capacity undirected graph between a pair of nodes is equal to the value of the maximum flow between them [1], and use the flow-equivalent tree as our starting point. Consider a pair of nodes s and t separated by k edges in a given flow-equivalent tree. To compute the maximum flow between them using the above procedure, we must use k applications of the procedure. The key to speeding this up is to add O(n) additional flow solutions in such a way that for any pair of nodes, there always exists a small number of pairs of nodes connecting them from which we can compose the required flow. Note that:for any pair of nodes, the flow decomposition of a maximum flow (say f) solution can be computed as mentioned above in time O(f^{3/2} log n) = O(n^{3/2} log n). This will lead to a total time complexity of O(n^{3/3} log n) for this preprocessing step since we need to do this for \(\binom{n}{2}\) pairs. In the unit capacity case, every pair of nodes can have O(1) paths in their max-flow decomposition leading to a space requirement of O(n^2) for this data structure. Next, we describe how to specify these pairs.

To do this, we use a method due to Chazelle [2]:

Given an n-node edge-weighted tree, he provides an
algorithm to choose $O(n)$ shortcut edges with weights on them such that for any path in the given tree, it is possible to compute the partial sum of the weights in the path using $O(x, n)$ summations involving the original and added edges (see Theorem 2 in [2]). More formally, Chazelle proved the following under the RAM model of computation.

Theorem 4.1 (Chazelle [2]). Let $T$ be a free tree with $n$ weighted edges. There exists a constant $c > 1$ such that, for any integer $m > cn$, it is possible to sum up weights along an arbitrary query path of $T$ in time $O(a(m, n))$. The data structure is of size at most $n$ and can be constructed in time $O(m)$.

Chazelle's result is frauded in a more general setting where the weight function maps the edges to a semigroup, and the partial sum in the above theorem can be replaced with the semigroup operation. We use this generalization and observe that $(2^c, \min)$ is a semigroup, and hence Chazelle's construction applies to deriving the minimum-weight edge along a tree path (rather than the sum) using shortcut edges. In fact, this is accomplished by weighting every shortcut edge with the minimum weight of an edge along the tree-path between its endpoints. We also maintain a maximum flow decomposition between pairs of nodes connected by shortcut edges. This enables us to reconstruct the maximum flow for any pair of nodes using $O(a(n, m))$ flow compositions. Each flow composition was argued earlier to take time $O(f(n))$ for a flow of value $f$ giving the claimed time of $O(a(n, m)a(n, f))$ time to reconstruct the maximum number of flow paths $\lambda(u, v)$ between any pair of nodes $(u, v)$. We thus have our main theorem.

Theorem 4.2. Given an undirected unit capacity graph on $n$ nodes, in time $O(n^3)$, a data structure using space $O(n^2)$ can be constructed that, given any pair of nodes, can compute the maximum number $f$ of edge-disjoint paths between them in time $O(a(n, m)f(n))$ where $a(n, m)$ is the inverse Ackermann function.

References