Equilibrium in a two dimensional queueing game: When inspecting the queue is costly

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1 Introduction

This research is concerned with congestion-prone environments like queues, roads, computer systems, and data networks. In such systems, a decision made by a user affects the utility of other users. Users wish to maximize their own utility, unwilling (or unable) to coordinate their actions. They choose among alternative actions, while taking into consideration various parameters such as waiting time and cost, inspection cost, service quality, etc. An appropriate model for analyzing such situations is that of non-cooperative games, where the solution concept commonly used is the *Nash equilibrium*.

In most models of congestion one tries to avoid others, and hence responds inversely to their actions. For example, if more individuals join a queue then it tends to discourage the individual from joining. This behavior is called *avoid-the-crowd* (ATC). The opposite behavior of *follow-the-crowd* (FTC) is also common. For example, the more customers buy priority in a queueing system, the more is an individual inclined to follow them and buy priority for himself. FTC behavior typically results in multiple equilibria, whereas ATC provides a unique equilibrium (Hassin and Haviv [2003], p. 6-7).

This discussion is limited to cases where the customer's decision is of one dimension. Adding another action as a possibility for the deciding customer increases the model's complexity, and requires a new definition of the ATC and FTC concepts. This research is an important step in this direction.

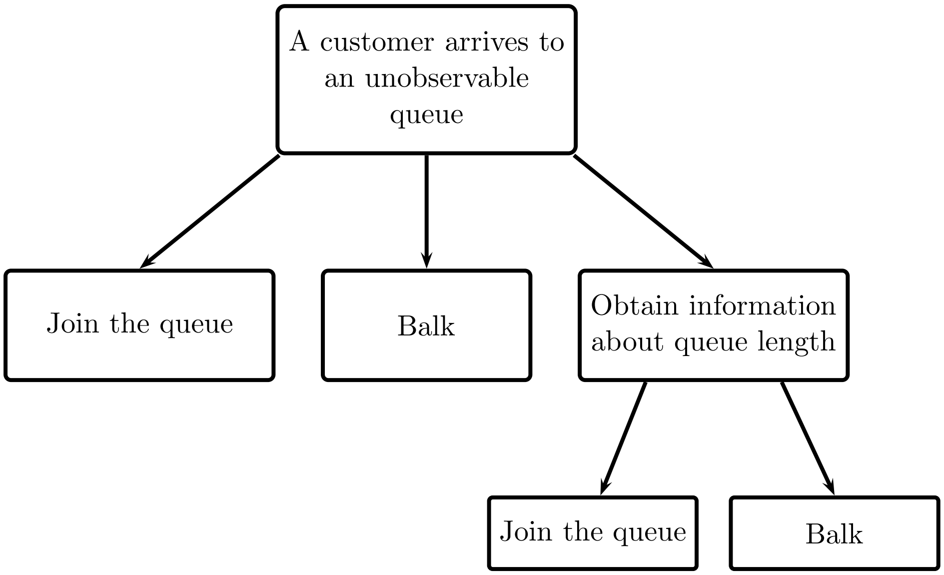
Related definitions of FTC and ATC behavior are established in supermodular and submodular games (Topicks (1979), Yao (1995)). Supermodular (submodular) games are strategic games, with a utility function that exhibits increasing (decreasing) differences in the other customers' strategy. These games require that the set of actions, S, is defined on a sublattice, which means that if x, y are vectors in S, then $x \vee y = (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_n, y_n))$ are also vectors in S. But that is impossible when the set of actions is a set of probability distribution vectors (for example: let $x = (0, 1) \in S$ and $y = (1, 0) \in S$, then $x \vee y = (1, 1)$ should also be in S, but it is not a probability distribution vector and therefore it is not in S). According to that, the existence theorems which have been proved for supermodular games does not apply here.

In this paper we focus on systems where the customer's decision is made out of three actions. This results in a complicated structure of the equilibrium solution, compared with the threshold solution concept which is common in queueing systems (Hassin and Haviv (2003), p.7-9).

The model that we present here combines two of the fundamental perceptions in strategic queueing theory: the observable queue (Naor (1969)), and the unobservable queue (Edelson and Hildebrand (1975)). Naor considered an observable queue, where identical customers make a decision whether or not to join it. The decision is based on the length of the queue, and the solution is a threshold strategy: customers join the queue only if it is shorter than a threshold. For the same decision making problem, Edelson and Hildebrand assumed that the queue length is unobservable to the customers. Therefore, their decision is based on the expected utility from each action, and the solution is the probability of joining in equilibrium.

Large and Norman (2010) presented another example of a model that bridges between the observable and unobservable cases. In their model, a single server produces to stock when there is no demand, and satisfies the demand when the queue is not empty. Customers have heterogeneous valuations of the product. A fraction of them see the queue length and the others are uninformed. The latter group joins according to a threshold (only those with valuation higher than a critical value join), which in equilibrium stabilizes the system so that the demand rate equals the production/service rate. The informed customers join as in Naor's model.

In our model, customers arrive to an unobservable single server queue, and choose among three options: join the queue, balk, or inspect the queue length. The solution consists of the probability of each action in equilibrium, based on the expected utilities, as in Edelson and Hildebrand's unobservable queue model. If customers decide to inspect the queue, it becomes observable to them, and the decision making problem becomes similar to Naor's observable queue model. The following flowchart shows customers' decision problem in our model:



Since inspecting the queue and waiting has a positive cost, and service gives a positive benefit to the customer, the equilibrium is not straightforward. For example, if more customers tend to inspect the queue, the individual customer might be motivated to join it without inspection, since the probability for a long queue is decreasing. But if more customers tend to join the unobserved queue, then it is not clear if the best response strategy of the individual customer is to inspect the queue or to balk, or a mixture of the two.

This model has applications in real life. For example, hospitals in the USA publish their emergency rooms (ER) average wait time on their web sites. Some even offer a service of acquiring this information as a text message to the patient mobile phone. By using this service, the lines in the ER become observable to the patient, and that affects his decision on attending the ER (for examples, see JFK medical center at http://jfkmc.com/our-services/er-wait-time.dot, Reston hospital center at http://www.restonhospital.com/, Brandon regional hospital at http://brandonhospital.com/our-services/er-wait-time.dot).

An interesting property that exists in this model is the *pairwise ATC* property: if we keep one of the three actions fixed, customers would choose to act according to the ATC policy regarding the two remaining actions. For example, if a constant fraction of customers balk, and the rest can only choose among inspecting and joining, then if more customers inspect the queue, the individual customer is less motivated to inspect it. This ATC policy is adopted by customers since it reduces the probability of long queues while saving the inspecting fee. In the same way, if a constant fraction of customers inspect the queue, and more customers tend to join the queue, the individual customer is less motivated to join it, since the probability of a long queue increases. For the same reason, if a constant fraction of customers join the queue, and more customers tend to inspect it, the individual customer is more motivated to balk.

Notice that we refer to this model as a symmetric game, where all customers have the same set of actions and the same utility function. Therefore, we seek a symmetric Nash equilibrium, which is a strategy profile that is identical to all customers, and is a best response against itself. Such an equilibrium is assured in a symmetric finite game, which has a finite set of players and a finite set of pure strategies (Nash, 1951). In our game, the set of pure strategies is finite, but the set of customers is infinite. Peleg (1969) proved existence of equilibrium in a game with infinite many players N, but his proof does not apply here either, since it requires that a finite set of players $A \subseteq N$ will determine the equilibrium. This condition is not fulfilled in our model. Therefore, the existence of the equilibrium requires a new proof, which we represent in this paper.

In Section 2, we present the mathematical model of this game. In Section 3, we prove the existence and uniqueness of a symmetric equilibrium. In Section 4 we demonstrate how to calculate the equilibrium point in various ways. In Section 5, we analyze numerically the sensitivity of the model towards its parameters, and show that there exists an equilibrium in which customers are indifferent among all three actions. In Section 6 we summerize our results and discuss future work.

2 The Model

Consider an unobservable single server queue. Arriving customers choose among three options: join the queue, balk, or inspect the queue length and then decide whether or not to join. The cost of waiting a unit of time is $C_W > 0$, and the cost of inspecting the queue length is $C_I \geq 0$.

We assume Poisson arrivals with parameter λ, and exponential service time with parameter μ. Upon service completion the customer gains a reward, $R \geq 0$. Following Naor (1969), we denote by $\nu = \frac{R\mu}{C_W}$ the normalized value of service measured in units of expected waiting cost for a single service completion. To resolve ties, we assume without loss of generality, that ν is not an integer.

Once a customer knows the queue length then as in Naor's observable queue model, he joins if the length is strictly smaller than the threshold n_e,

$$n_e = \left\lfloor \frac{R\mu}{C_W} \right\rfloor = \lfloor \nu \rfloor. \tag{1}$$

Assume that each customer inspects the queue with probability P_I, balks without inspecting the queue with probability P_B, and joins the queue without inspecting it with probability $P_J = 1 - P_I - P_B$. Then, the effective arrival rate is

$$\lambda_e = \begin{cases} (1 - P_B)\lambda & i < n_e, \\ P_J \lambda & i \geq n_e. \end{cases} \tag{2}$$

We now calculate the stationary probability vector. Denote by π_i the stationary probability that the queue length is i. If $\frac{R\mu}{C_W} < 1$, then $n_e = 0$, and $P_B = 1$ is a dominant strategy for all customers. Therefore, we assume that $n_e \geq 1$. The balance equations for $0 \leq i < n_e$ are:

$$(1 - P_B)\lambda\pi_i = \mu\pi_{i+1},$$

and for $i \geq n_e$:

$$P_J\lambda\pi_i = \mu\pi_{i+1}.$$

Define $\rho = \frac{\lambda}{\mu}$, $\xi = (1 - P_B)\rho$ and $\eta = 1 - P_J\rho$. Note that $0 \leq \xi \leq \rho$, and $1 - \rho \leq \eta \leq 1$. In equilibrium, $P_J\lambda < \mu$, meaning $P_J\rho < 1$, and $\eta > 0$. But since balking is an option, ρ can be greater than 1, and therefore ξ can be greater than 1.

From the balance equations we calculate the stationary distribution:

$$\pi_i = \begin{cases} \xi^i \pi_0 & i < n_e, \\ \xi^{n_e}(1 - \eta)^{i - n_e}\pi_0 & i \geq n_e, \end{cases} \tag{3}$$

and

$$\pi_0 = \left[\sum_{i=0}^{n_e - 1} \xi^i + \xi^{n_e} \sum_{i=n_e}^{\infty} (1 - \eta)^{i - n_e} \right]^{-1} = \left[\sum_{i=0}^{n_e - 1} \xi^i + \xi^{n_e} \cdot \frac{1}{\eta} \right]^{-1} \tag{4}$$

$$= \left[\frac{1 - \xi^{n_e}}{1 - \xi} + \frac{\xi^{n_e}}{\eta} \right]^{-1}. \tag{5}$$

Notice, that $\xi = 1$ is not a singularity point of π_0, since from equation (4) we get that for $\xi = 1$, π_0 is defined and is equal to

$$\frac{\eta}{n_e\eta + 1}. \tag{6}$$

When a new customer arrives, he calculates his expected utility from each possible action. We denote these expected utilities by U_B for balking without inspecting the queue, U_J for joining without inspecting, and U_I for inspecting. The expected utility from balking without inspecting is

$$U_B = 0. \tag{7}$$

The expected utility from inspecting the queue and joining it only if it is shorter than n_e is

$$\begin{aligned} U_I &= \sum_{i=0}^{n_e-1} \pi_i \left(R - \frac{C_W(i+1)}{\mu}\right) - C_I = \sum_{i=0}^{n_e-1} \xi^i \pi_0 \left(R - \frac{C_W(i+1)}{\mu}\right) - C_I \\ &= \pi_0 \left[R \sum_{i=0}^{n_e-1} \xi^i - \frac{C_W}{\mu} \sum_{i=0}^{n_e-1} \xi^i (i+1)\right] - C_I \qquad (8) \\ &\overset{(*)}{=} \pi_0 \left[R \cdot \frac{1-\xi^{n_e}}{1-\xi} - \frac{C_W}{\mu} \cdot \frac{1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1}}{(1-\xi)^2}\right] - C_I, \qquad (9) \end{aligned}$$

where $(*)$ is calculated by the sums:

$$\sum_{i=0}^{n_e-1} \xi^i = \frac{1-\xi^{n_e}}{1-\xi},$$

and

$$\sum_{i=0}^{n_e-1} \xi^i (i+1) = \left(\sum_{i=0}^{n_e-1} \xi^{i+1}\right)' = \left(\frac{\xi(1-\xi^{n_e})}{1-\xi}\right)' = \frac{1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1}}{(1-\xi)^2}. \tag{10}$$

Notice that for $\xi = 1$, U_I is defined, and from equations (6) and (8) we get that it equals to

$$\frac{n_e\eta[2\mu R - C_W(n_e+1)]}{2\mu(n_e\eta+1)}. \tag{11}$$

The expected utility from joining the queue without inspecting its length is

$$\begin{aligned} U_J &= \sum_{i=0}^{\infty} \pi_i \left(R - \frac{C_W(i+1)}{\mu}\right) = R - \frac{C_W}{\mu} \sum_{i=0}^{\infty} \pi_i (i+1) \\ &= R - \frac{C_W}{\mu} \pi_0 \left[\sum_{i=0}^{n_e-1} \xi^i (i+1) + \sum_{i=n_e}^{\infty} \xi^{n_e} (1-\eta)^{i-n_e} (i+1)\right] \qquad (12) \\ &\overset{(**)}{=} R - \frac{C_W}{\mu} \pi_0 \left[\frac{1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1}}{(1-\xi)^2} + \frac{\xi^{n_e}(n_e\eta+1)}{\eta^2}\right], \qquad (13) \end{aligned}$$

where $(**)$ is calculated by (10) and by

$$\begin{aligned}\sum_{i=n_e}^{\infty}(1-\eta)^{i-n_e}(i+1) &= \sum_{j=0}^{\infty}(1-\eta)^j(j+n_e+1) = n_e\sum_{j=0}^{\infty}(1-\eta)^j + \sum_{j=0}^{\infty}(1-\eta)^j(j+1) \\ &= n_e\frac{1}{\eta} - \left(\sum_{j=0}^{\infty}(1-\eta)^{j+1}\right)' = n_e\frac{1}{\eta} - \left(\frac{1-\eta}{\eta}\right)' = \frac{n_e\eta+1}{\eta^2}.\end{aligned}$$

Notice that for $\xi = 1$, U_J is defined, and from equations (6) and (12) we get that it equals to

$$R - \frac{C_W}{\mu}\left[\frac{n_e(n_e+1)\eta}{2(n_e\eta+1)} + \frac{1}{\eta}\right]. \tag{14}$$

3 Equilibrium

Customers wish to maximize their utility, and therefore choose **best response strategies**. Since in this model the customers are homogeneous, we are interested in a **symmetric equilibrium**. A strategy profile (P_I, P_B, P_J) is a **symmetric equilibrium profile** if it is a best response against itself. The best response strategies (P_I, P_B, P_J) satisfy $P_I \geq 0$, $P_B \geq 0$, $P_J \geq 0$, $P_I + P_B + P_J = 1$, and in addition

$$\begin{cases} U_I > \max(U_J, 0) \Rightarrow P_I = 1 \\ 0 > \max(U_I, U_J) \Rightarrow P_B = 1 \\ U_J > \max(U_I, 0) \Rightarrow P_J = 1 \\ U_J = 0 > U_I \Rightarrow P_I = 0 \\ U_J = U_I > 0 \Rightarrow P_B = 0 \\ U_I = 0 > U_J \Rightarrow P_J = 0 \\ U_I = U_J = 0 \Rightarrow 0 \leq P_I, P_J, P_B \leq 1 \end{cases} \tag{15}$$

Notice, that when $C_I = 0$, all customers inspect the queue, and we get Naor's observable queue model. Therefore, we assume from now on that $C_I > 0$. When C_I is very high, inspecting the queue becomes a dominated strategy, and we get Edelson and Hildebrand's unobservable queue model.

A special case of an equilibrium occurs when $R < \frac{C_W}{\mu}$, meaning $n_e = 0$. In this case, $U_J \leq 0$. And since $C_I > 0$ we also get that $U_I < 0$. Therefore, the equilibrium strategy is $(P_I, P_B, P_J)^e = (0, 1, 0)$, and balking becomes a dominant strategy. From now on we assume that $R > \frac{C_W}{\mu}$, and as a result $n_e \geq 1$.

We wish to prove that there exists a unique equilibrium strategy in this model. The ATC property does not apply here directly, since customers have three optional actions. We do find a **pairwise ATC** property: For a constant P_B, if P_I of all others increases, then the queue is less congested and therefore the best response would be to avoid the crowd and increase P_J. For a constant P_I, if P_J of all others increases, the best response would be to avoid the crowd and balk. For a constant P_J, if P_B of all others increases, the best response would be to increase P_I.

In order to prove existence and uniqueness of the equilibrium, we define the **expected utility set (EUS)**, as the set of the achievable expected utility values. For a given set of parameters: $\lambda, \mu, R, C_W, C_I$ and for every possible strategy (P_I, P_B) that is adopted by all customers, we calculate (U_I, U_J), and therefore:

$$\text{EUS} = \{(x, y)|\text{ there exists }(P_I, P_B)\text{ such that }U_I = x, U_J = y\}. \tag{16}$$

For example consider $\lambda = 0.5, \mu = 1, R = 10, C_W = 7, C_I = 1$. Figure 1(a) shows U_I and U_J for $P_B = 0.05, 0.1, \ldots, 1$ and $0 \leq P_I < 1 - P_B$. In each curve, P_B is constant and P_I is changing between 0 and $1 - P_B$. Similarly, with the same parameters, we create the EUS by keeping P_I constant in each curve in Figure 1(b) and P_J constant in each curve in Figure 1(c). For the same parameters, we refine the values of P_B and P_I, as shown in Figure (2).

Our proof is based on the topological properties of the set of strategies in this game. We lean on the fact that the strategy set is a compact set. By proving that the mapping from the strategy set to the expected utility set (EUS) is continuous, we conclude that the EUS is also a compact set. We also prove that the mapping has a continuous inverse, and therefore it is a homeomorphism. As a result, the topological properties of the set of strategies are preserved in the EUS. Later, we prove monotonicity of the EUS boundaries. As a result, we could determine that the EUS has a unique shape, and use its properties in order to prove that the equilibrium exists and is unique.

The properties of the EUS are proven in the following lemmas.

Lemma 1 *The EUS is a compact set.*

Proof: The EUS is the image of the mapping $(P_I, P_B) \rightarrow (U_I, U_J)$. We start from a triangle:

$$T = \{(P_I, P_B) : P_I, P_B \geq 0, P_I + P_B \leq 1\}. \tag{17}$$

This triangle is a compact set. To prove that the EUS is also a compact set, we use the following lemma (Kelley (1955), p.141):

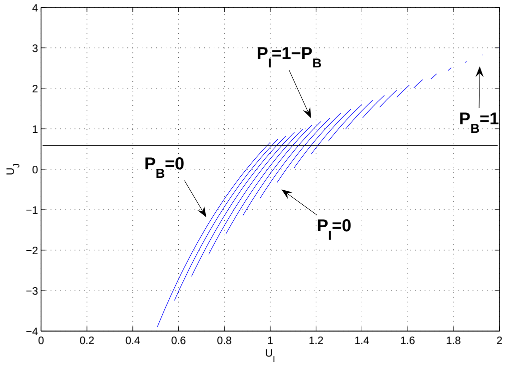
Lemma 2 *Let f be a continuous function carrying the compact topological space X onto the topological space Y. Then Y is compact.*

Therefore, we need to show that the mapping is continuous. From equations (8) and (12) it follows that both U_I and U_J are continuous for any fixed set of parameters, and therefore the mapping $(P_I, P_B) \rightarrow (U_I, U_J)$ is continuous, and following Lemma 2 the EUS is a compact set. ■

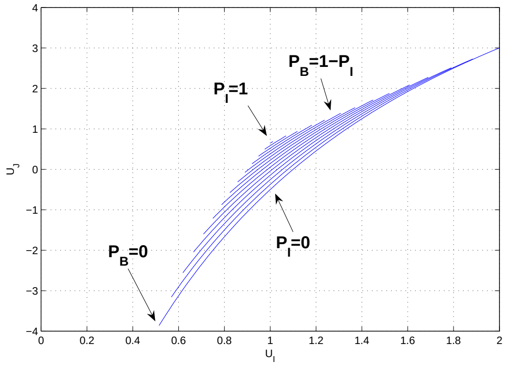
Lemma 3 *The interior of the EUS is a simply connected domain.*

Proof: In order to prove this claim, we use the following definition:

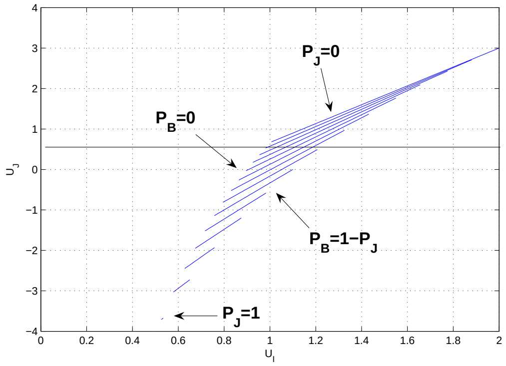
Definition 1 *Homeomorphism is a continuous one-to-one map of a topological space X onto a topological space Y such that f^{-1} is also continuous (Kelley (1955), p.87).*



(a) Curves for constant values of P_B



(b) Curves for constant values of P_I



(c) Curves for constant values of P_J

Figure 1: U_I vs. U_J curves

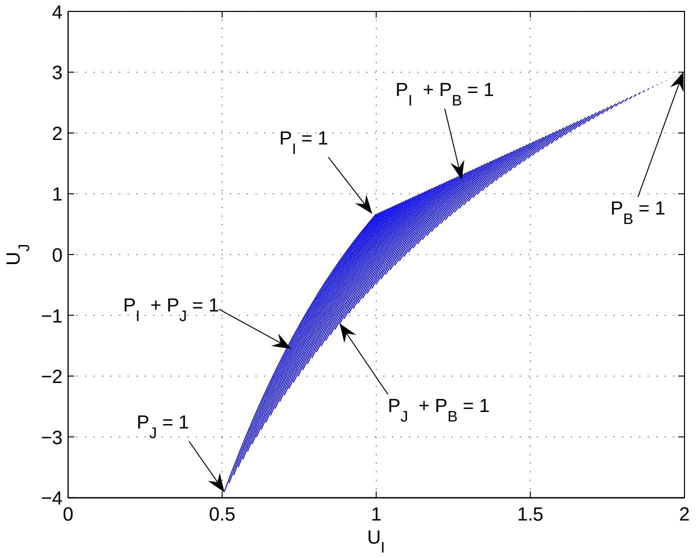


Figure 2: Expected utility set (EUS)

The property of being a simply connected domain is a topological property, and a homeomorphism preserves topological properties (Reid and Szendroi, 2005, pp.113-118).

We start again from the triangle T, that was defined in (17). The interior of this triangle is a simply connected domain. By proving that the mapping $(P_I, P_B) \to (U_I, U_J)$ is a homeomorphism, we conclude that the EUS is also a simply connected domain. Figure 3 show the homeomorphism between the probability set to the EUS.

Lemma 4 *For any set of parameters* $R, C_W, C_I, \lambda, \mu$ *such that* $R > \frac{C_W}{\mu}$*, the mapping* $(P_B, P_I) \to (U_I, U_J)$ *is a homeomorphism.*

Proof: We use the following lemma (from Ma, 2002, p. 39), to prove that the mapping $(P_I, P_B) \to (U_I, U_J)$ is a homeomorphism:

Lemma 5 *Let* $f : X \to Y$ *be a continuous bijection. If* X *is compact, then* f *is a homeomorphism.*

We already showed in the proof of Lemma 6 that the mapping is continuous. We now wish to prove that it is bijective. In order to do so, we first show that the mapping: $(P_B, P_I) \to (\xi, \eta)$ is bijective. Then, we show that the mapping: $(\xi, \eta) \to (U_I, U_J)$ is bijective. Therefore, the mapping $(P_B, P_I) \to (U_I, U_J)$ is bijective.

Since $\xi = (1 - P_B)\rho$ and $\eta = 1 - (1 - P_B - P_I)\rho$, the Jacobian of the transformation $(P_B, P_I) \to (\xi, \eta)$ is:

$$J(\xi, \eta) = \begin{vmatrix} \xi'_{P_B} & \xi'_{P_I} \\ \eta'_{P_B} & \eta'_{P_I} \end{vmatrix} = \xi'_{P_B} \cdot \eta'_{P_I} - \eta'_{P_B} \cdot \xi'_{P_I} = -\rho^2, \tag{18}$$

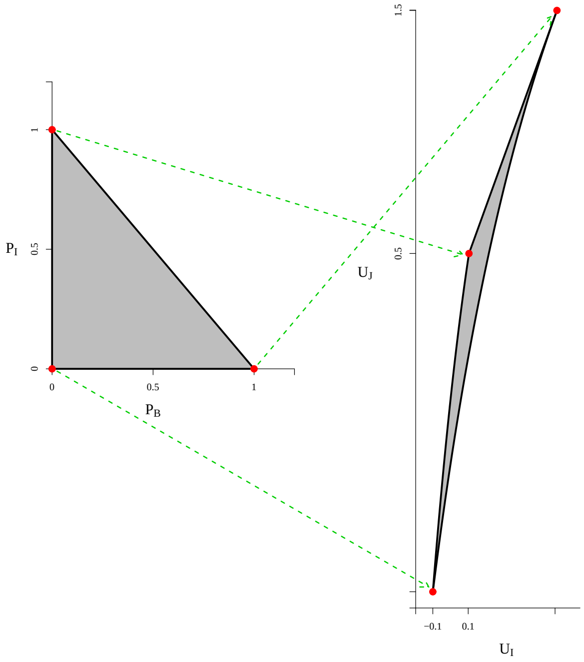


Figure 3: Homeomorphism between the probability set and the EUS

and since $\rho > 0$, the Jacobian is non-zero and by the inverse function theorem we get that the mapping is bijective. We now want to use the same argument in order to prove that the mapping $(\xi, \eta) \to (U_I, U_J)$ is bijective. For simplicity, we use the presentation of U_I, U_J in (8) and (12). We divide each equation by $\frac{C_W}{\mu}$, and use the definition: $\nu = \frac{R\mu}{C_W}$. Then (8) and (12) transform into:

$$U_I = R \cdot \left[\frac{(1-\xi^{n_e})\eta}{(1-\xi^{n_e})\eta + \xi^{n_e}(1-\xi)} - \frac{1}{\nu} \cdot \frac{(1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1})\eta}{(1-\xi)((1-\xi^{n_e})\eta + \xi^{n_e}(1-\xi))}\right] - C_I, \tag{19}$$

$$U_J = R \cdot \left[1 - \frac{1}{\nu}\left(\frac{(1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1})\eta}{(1-\xi)((1-\xi^{n_e})\eta + \xi^{n_e}(1-\xi))} + \frac{\xi^{n_e}(n_e\eta+1)(1-\xi)}{\eta((1-\xi^{n_e})\eta + \xi^{n_e}(1-\xi))}\right)\right]. \tag{20}$$

We look at the Jacobian of this transformation:

$$\begin{aligned} J(U_I, U_J) &= \begin{vmatrix} {U_I}'_\xi & {U_I}'_\eta \\ {U_J}'_\xi & {U_J}'_\eta \end{vmatrix} = {U_I}'_\xi \cdot {U_J}'_\eta - {U_I}'_\eta \cdot {U_J}'_\xi \\ &= \frac{R^2\xi^{n_e-1}}{\eta\nu^2(1-\xi)(\eta - \xi^{n_e}(\xi-1+\eta))^3} \cdot \Big({n_e}^3(1-\xi)^2\xi^{n_e}\eta^2 - {n_e}^2(1-\xi)^2\xi^{n_e}\eta(\eta\nu-2) \\ &+ n_e\left(-\xi\eta^2 + \xi^{2n_e+1}[(1-\xi)^2 - \eta^2] + \xi^{n_e}[(1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3\nu]\right) \\ &+ \xi(1-\xi^{n_e})[\xi^{n_e}(\xi-1+\eta)[2 + (\xi-1-\eta)\nu] + \eta(\eta\nu-2)]\Big) \end{aligned} \tag{21}$$

Lemma 6 *For all* $R \geq 0$, $\nu > 1$, $\xi > 0$ *and* $0 < \eta < 1$, $J(U_I, U_J) < 0$.

Proof: In Appendix A.1. ■

By proving Lemma 6, we complete the proof of Lemma 4. ■

By proving Lemma 4, we complete the proof of Lemma 3. ■

The following lemma is a result of the **pairwise ATC** property.

Lemma 7 *: The EUS curves* $U_J(U_I)$ *for constant* P_B *are monotonically increasing with slope* > 1. *Similarly, for constant* P_I *or* P_J.

Proof: First, look at the constant P_B curves in Figure 1(a). On each curve, when traversed from bottom to top, P_I is increasing from 0 to 1, and P_J is decreasing from 1 to 0. When P_I increases and P_J decreases, customers tend to inspect the queue rather then joining it without inspection. Then for all given n, the probability that the queue length is no longer than n, increases. More formally, consider π_x, $x \in [0, 1]$ to be the steady state distribution of the queue state when P_B is constant and $P_I = x$. Then for $x \geq y$, π_x stochastically dominates π_y. As a result, the value of both U_I and U_J increases. U_J is more sensitive to the queue length distribution, because customer who choose to join without inspecting cannot balk when the queue is long. Therefore, since the probability for a short queue increases, we get $\Delta U_J > \Delta U_I$. Therefore, for a curve of constant P_B, U_J is strictly monotonically increasing in U_I with slope > 1.

Next, look at the constant P_I curves in Figure 1(b). On each curve, when traversed from bottom to top, P_B is increasing from 0 to 1, and P_J is decreasing from 1 to 0. When P_B increases and P_J decreases, customers tend to balk from the queue rather then joining it. Then for all given n, the probability that the queue length is no longer than n, increases, and the same argument as above yields the conclusion that for a curve of constant P_I, U_J is strictly monotonically increasing in U_I with slope > 1.

Last, look at the constant P_J curves in Figure 1(c). On each curve, when traversed from bottom to top, P_B is increasing from 0 to 1, and P_I is decreasing from 1 to 0. When P_B increases and P_I decreases, customers tend to balk from the queue rather then inspecting it, and again for a curve of constant P_J, U_J is also strictly monotonically increasing in U_I with slope > 1. ■

Corollary 8 U_J *is strictly monotonically increasing in* U_I *on the EUS boundaries.*

Proof: The corollary immediately follows from Lemma 7. ■

In Appendix A.2, we give an explicit expression of the boundary curves.

We now prove the existence and uniqueness of the equilibrium based on the EUS properties. We start by dividing the expected utility space into seven different regions:

1. $U_J > \max(U_I, 0)$ (region 1).
2. $U_I > \max(U_J, 0)$ (region 2).
3. $\max(U_I, U_J) < 0$ (region 3).
4. $U_I = U_J > 0$ (region/line 4).
5. $U_I = 0 > U_J$ (region/line 5).
6. $U_J = 0 > U_I$ (region/line 6).
7. $U_I = U_J = 0$ (region 7).

These regions are shown in Figure (4):

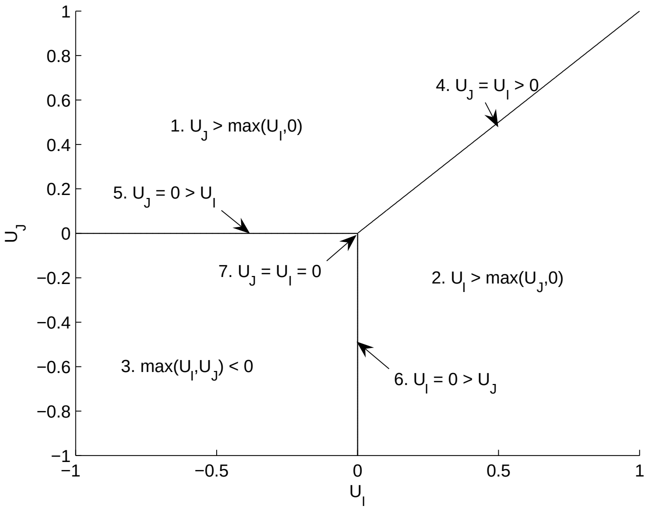


Figure 4: Expected utility space

Recall that a symmetric equilibrium is a strategy which satisfies condition (15). Therefore, the equilibrium is obtained in the intersection point between the EUS and the relevant region. For example, if the point $P_I = 1$ of the EUS is in region 2, then $P_I = 1$ is a best response against itself and therefore it is a symmetric equilibrium strategy. Another example: if the left boundary of the EUS, in which $P_I + P_J = 1$, intersects with region (line) 4, then the intersection point defines a symmetric equilibrium values P_I and P_J such that $P_I + P_J = 1$ and $P_B = 0$.

There is only one scenario where no such intersection exists, but in this case the origin is included in the EUS, and the symmetric equilibrium is obtained in the origin (region 7), where $0 \leq P_I, P_J, P_B \leq 1$.

Corresponding to the seven cases of condition (15), we distinguish between different types of equilibrium in this game:

(a) If the point where $P_B = 1$ is in region 3, then it defines a pure equilibrium. But since we assumed that $R - \frac{C_W}{\mu} > 0$, we get that for $P_B = 1$, $U_J = R - \frac{C_W}{\mu} > 0$. We also get that $U_I = R - \frac{C_W}{\mu} - C_I$ and therefore $U_J > U_I$. As a result, the point $P_B = 1$ is always obtained in region 1, and therefore this pure equilibrium does not occur in our model.

(b) If the point where $P_J = 1$ is in region 1, then it defines a pure equilibrium (Figure 5(a)).

(c) If the point where $P_I = 1$ is in region 2, then it defines a pure equilibrium (Figure 5(b)).

(d) If the left boundary of the EUS, where $P_I + P_J = 1$, intersects with line 4, then the intersection point is a mixed equilibrium (Figure 5(c)).

(e) If the lower boundary of the EUS, where $P_B + P_J = 1$, intersects with line 5, then the intersection point is a mixed equilibrium. (Figure 5(d)).

(f) If the upper boundary of the EUS, where $P_I + P_B = 1$, intersects with line 6, then the intersection point is a mixed equilibrium (Figure 5(e)).

(g) If the origin is in the EUS, then the origin is a mixed equilibrium (Figure 5(f)).

Notice that the point $P_B = 1$ is obtained on the EUS for $U_I = R - \frac{C_W}{\mu} - C_I$ and $U_J = R - \frac{C_W}{\mu}$. Since we assumed that $C_I > 0$, we get that for $P_B = 1$, $U_J > U_I$. According to the assumption that $R - \frac{C_W}{\mu} > 0$, we also get that for $P_B = 1$, $U_J > U_I > 0$. Therefore, in our model, $P_B = 1$ is always obtained in region 1. According to that, there exist six types of equilibrium:

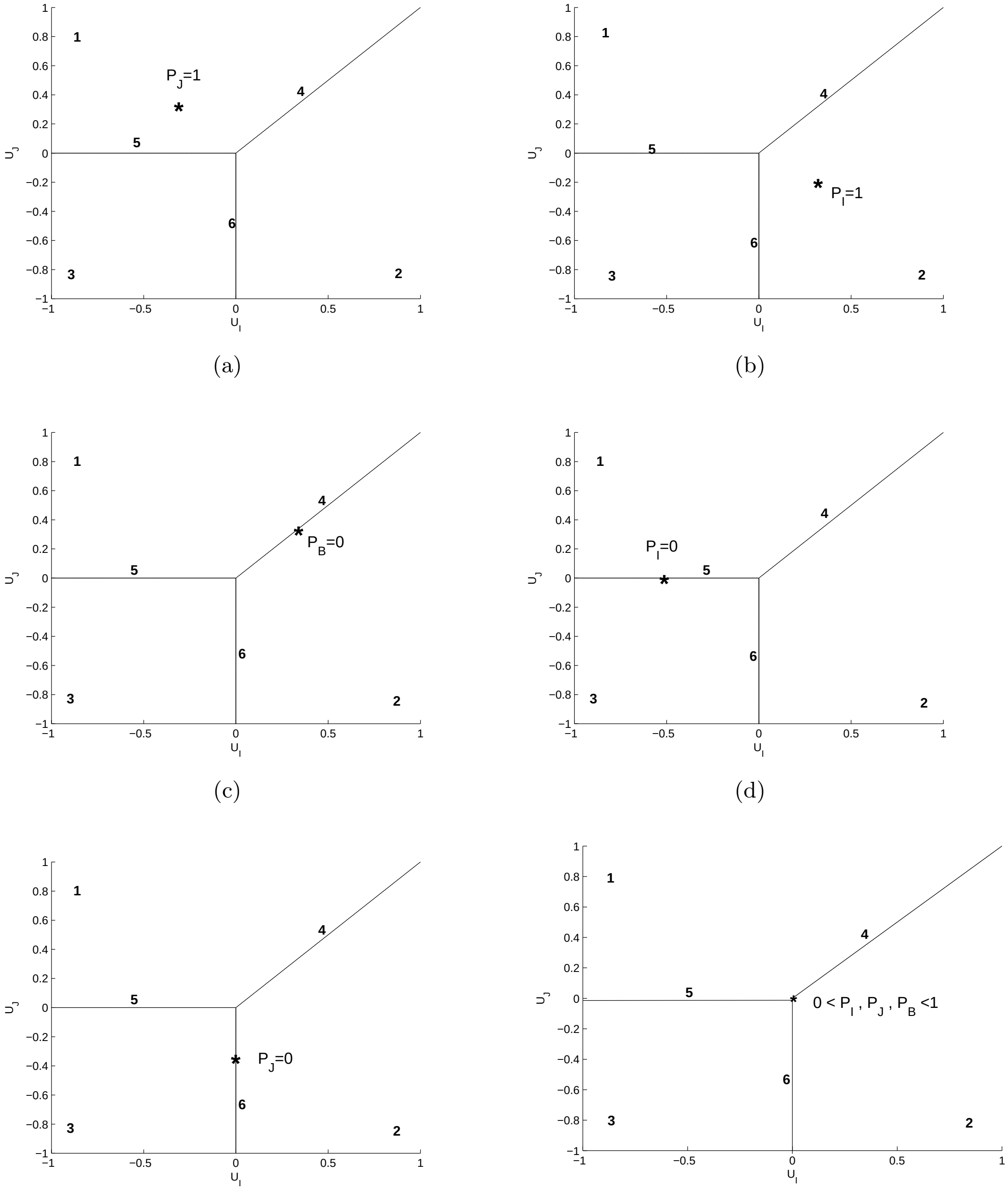
Theorem 1 *For each set of parameters $\lambda, \mu, R, C_W, C_I$, there exist a symmetric Nash equilibrium strategy in this game.*

Proof: For all possible chosen of parameters, the EUS appears on the expected utility space that was defined above (see Figure 4). We go through all possibilities, and prove that there exist an equilibrium in each one of them: Assume that the point where $P_J = 1$ is in region 1, then $P_J = 1$ is a pure equilibrium. For example see Figure (6). Otherwise, $P_J = 1$ has to be in region 2 or 3. First, assume that $P_J = 1$ is in region 2. If $P_I = 1$ is also in region 2, then $P_I = 1$ is a pure equilibrium (see Figure 7(a)). Otherwise, based on the EUS properties, $P_I = 1$ has to be in region 1. In that case, the boundary $P_I + P_J = 1$ must intersects with line (region) 4, and $P_I + P_J = 1$ is a mixed equilibrium (see Figure 8).

Next, assume that the point where $P_J = 1$ is in region 3. If $P_I = 1$ is in region 2, then $P_I = 1$ is a pure equilibrium (see Figure 7(b)). Otherwise, $P_I = 1$ has to be in region 1 or 3. If $P_I = 1$ is in region 1, one of the following three cases must occur:

(1) $P_I + P_J = 1$ intersects with line 4, and then $P_I + P_J = 1$ is a mixed equilibrium (see Figure 9(a)).

(2) $P_B + P_J = 1$ intersects with line 5, and then $P_B + P_J = 1$ is a mixed equilibrium (see Figure 9(b)).



(a) (b) (c) (d) (e) (f)

Figure 5: All types of equilibrium

(3) The origin is contained in the EUS, and then by Lemma 3, $P_I + P_J + P_B = 1$ is a mixed equilibrium (see Figure 9(c)).

Otherwise, $P_I = 1$ has to be in region 3. Then one of the following three cases must occur:

(1) $P_I + P_B = 1$ intersects with line 6, and then $P_I + P_B = 1$ is a mixed equilibrium (see Figure 10(a)).

(2) $P_B + P_J = 1$ intersects with line 5, and then $P_B + P_J = 1$ is a mixed equilibrium (see Figure 10(b)).

(3) The origin is contained in the EUS, and then $P_I + P_J + P_B = 1$ is a mixed equilibrium (see Figure 10(c)).

■

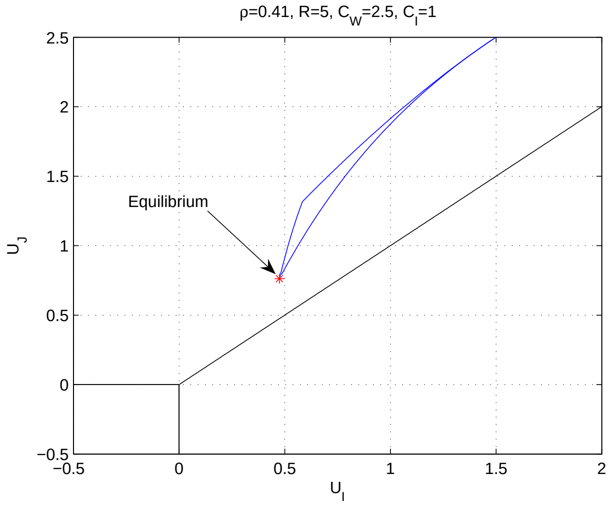
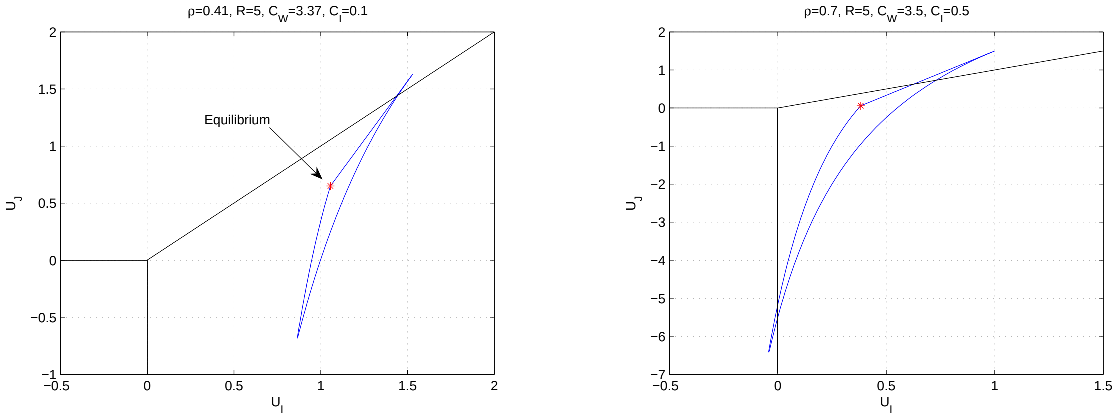


Figure 6: A pure equilibrium at $P_J = 1$, when $P_J = 1$ is in region 1



(a) Equilibrium when $P_J = 1$ is in region 2

(b) Equilibrium when $P_J = 1$ is in region 3

Figure 7: Pure equilibria at $P_I = 1$

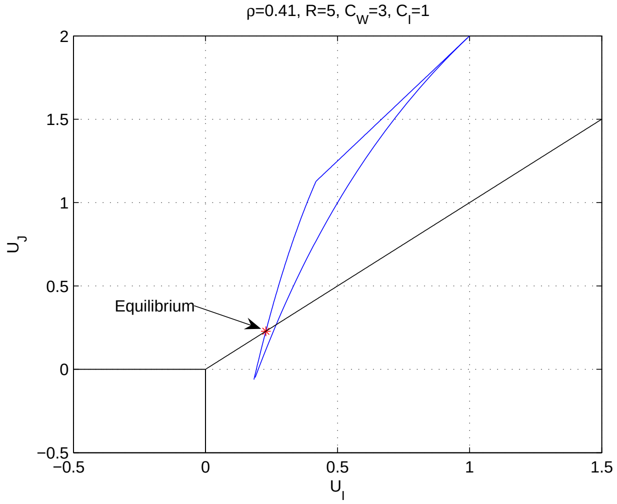
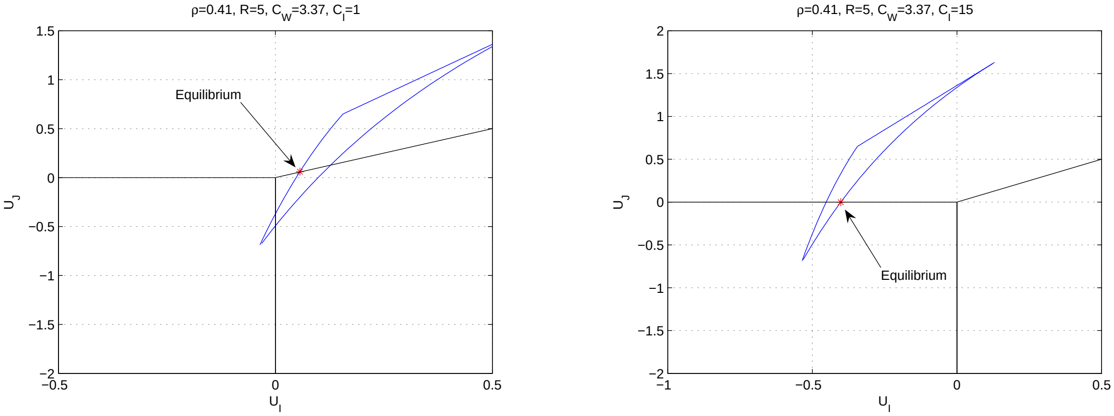
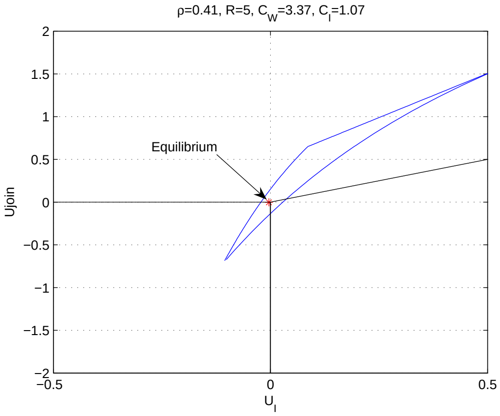


Figure 8: A mixed equilibrium at $P_I + P_J = 1$, when $P_J = 1$ is in region 2 and $P_I = 1$ is in region 1



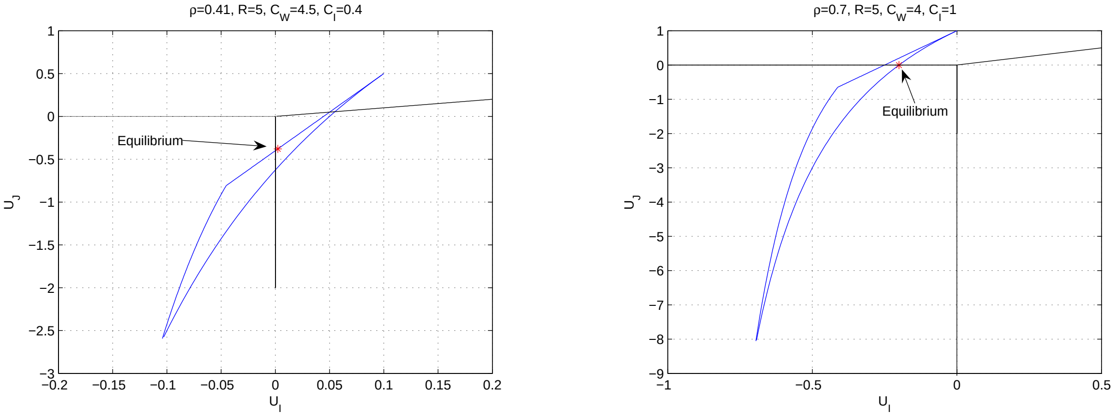
(a) Equilibrium at $P_I + P_J = 1$

(b) Equilibrium at $P_B + P_J = 1$



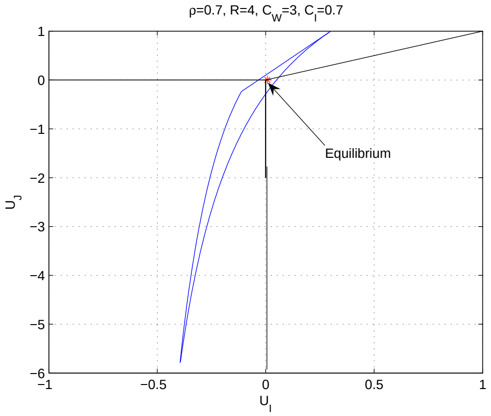
(c) Equilibrium at $P_I + P_J + P_B = 1$

Figure 9: Mixed equilibria when $P_I = 1$ is in region 1 and $P_J = 1$ is in region 3



(a) Equilibrium at $P_I + P_B = 1$

(b) Equilibrium at $P_B + P_J = 1$



(c) Equilibrium at $P_I + P_J + P_B = 1$

Figure 10: Mixed equilibria when $P_I = 1$ and $P_J = 1$ are in region 3

Theorem 2 *For each set of parameters $\lambda, \mu, R, C_W, C_I$, the symmetric Nash equilibrium strategy is unique.*

Proof: The proof is based on the unique shape of the expected utility set. By Lemma 7, the EUS boundaries are all increasing monotonic functions, with slope > 1. By Lemma 3, the EUS is compact and simply connected, meaning that it does not include any "holes". Using these two results, we show that it is not possible that more than one of the six scenarios that were introduced in the existence proof can occur under the assumption $R > \frac{C_W}{\mu}$.

Assume that the point where $P_J = 1$ is in region 1, then $P_J = 1$ is a pure equilibrium. From Lemma 7 we get that the entire EUS is in region 1, and therefore the equilibrium is unique. For example see Figure 6. Otherwise, $P_J = 1$ has to be in region 2 or 3. First, assume that $P_J = 1$ is in region 2. Then $P_I = 1$ can be in region 1 or 2:

(I) If $P_I = 1$ is in region 2, then $P_I = 1$ is a pure equilibrium. The EUS boundaries $P_I + P_B = 1$ and $P_J + P_B = 1$ intersect with line 4, but none of these intersections define another equilibrium (see Figure 7(a)).

(II) If $P_I = 1$ is in region 1, then the boundary $P_I + P_J = 1$ must intersect with line 4, and $P_I + P_J = 1$ is a mixed equilibrium. The EUS boundary $P_J + P_B = 1$ also must intersect with line 4, but this intersection does not define another equilibrium (see Figure 8).

Next, assume that the point where $P_J = 1$ is in region 3. Then $P_I = 1$ can be in region 1, 2 or 3:

(I) If $P_I = 1$ is in region 2, then $P_I = 1$ is a pure equilibrium. In this case, the EUS boundaries $P_I + P_B = 1$ and $P_J + P_B = 1$ intersect with line 4, and the boundaries $P_I + P_J = 1$ and $P_J + P_B = 1$ intersect with line 6, but none of these intersections define another equilibrium, and therefore this equilibrium is unique (see Figure 7(b)).

(II) If $P_I = 1$ is in region 1, one of the following three cases must occur:

  (1) $P_I + P_J = 1$ intersects with line 4, and then $P_I + P_J = 1$ is a mixed equilibrium. Notice that no other equilibrium exists, since the intersections $P_I + P_J = 1$ and $P_J + P_B = 1$ with line 6 and $P_B + P_J = 1$ with line 4 do not define another equilibrium, and therefore this equilibrium is unique (see Figure 9(a)).

  (2) $P_B + P_J = 1$ intersects with line 5, and then $P_B + P_J = 1$ is a mixed equilibrium. The intersection of the boundary $P_I + P_J = 1$ with line 5 does not yield another equilibrium, and therefore this equilibrium is unique (see Figure 9(b)).

  (3) The origin is contained in the EUS, and then $P_I + P_J + P_B = 1$ is a mixed equilibrium. Notice that no other equilibrium exists, since the intersections $P_I + P_J = 1$ with line 5, $P_J + P_B = 1$ with line 6 do not define another equilibrium, and therefore this equilibrium is unique (see Figure 9(c)).

(III) If $P_I = 1$ is in region 3, one of the following three cases must occur:

  (1) $P_I + P_B = 1$ intersects with line 6, and then $P_I + P_B = 1$ is a mixed equilibrium. The intersection of $P_J + P_B = 1$ with line 6 does not define another equilibrium, and therefore this equilibrium is unique (see Figure 10(a)).

(2) $P_B + P_J = 1$ intersects with line 5, and then $P_B + P_J = 1$ is a mixed equilibrium. The intersection of $P_J + P_B = 1$ with line 5 does not define another equilibrium, and therefore this equilibrium is unique (see Figure 10(b)).

(3) The origin is contained in the EUS, and then $P_I + P_J + P_B = 1$ is a mixed equilibrium. The intersections of $P_J + P_B = 1$ with lines 4 and 6, and the intersection of $P_I + P_J = 1$ with line 5 do not define another equilibrium, and therefore this equilibrium is unique (see Figure 10(c)).

■

4 Numerical Methods

We describe three different ways to find the equilibrium point in this queueing game. One of them was already described in section 3: we look for the intersection point of the EUS with one of the lines: $U_J = 0$ where $U_I < 0$, $U_I = 0$ where $U_J < 0$, or $U_J = U_I > 0$.

Another way to find the equilibrium is by fixing one of the probabilities and moving to a single-dimension problem. For example, let $\lambda = 0.41, \mu = 1, R = 5, C_W = 3.37, C_I = 1.07$. We fix $P_B = 0.1$. That means that customers balk with probability 0.1, and the decision problem reduces to choosing P_I (or P_J), and therefore it becomes a single dimensional problem. For fixed P_B, customers would tend to inspect the queue when others tend to join it, and vice versa, which is an ATC policy. In that case, the best response strategy function P_I^* is decreasing step function with a single step and there exists a unique equilibrium, as shown in Figure 11:

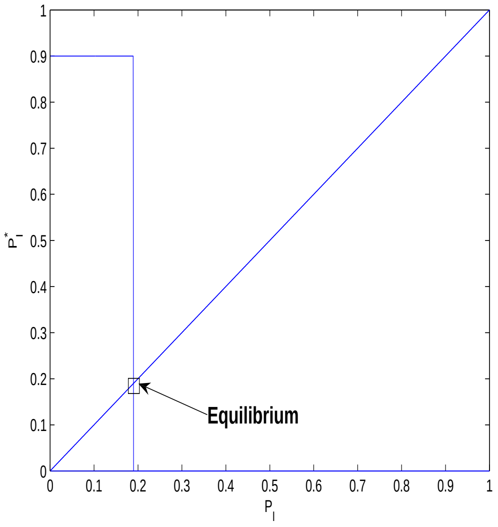
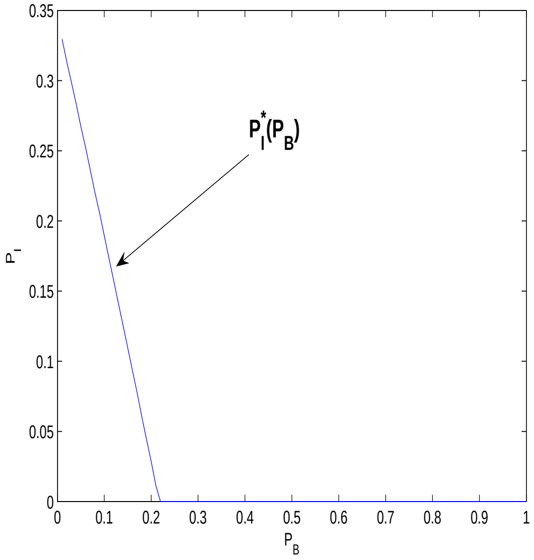
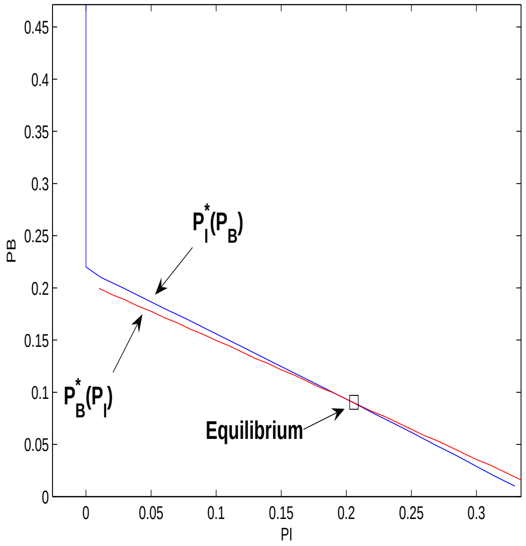


Figure 11: Equilibrium for P_B=0.1

We repeat this process for each fixed value: $P_B = 0, \dots, 1$. We now create the function $P_I^*(P_B)$, that for each P_B returns the P_I equilibrium strategy, as shown in Figure 12(a). Next, we repeat the same process, but for fixed P_I. For each P_I, we calculate the P_B equilibrium strategy, and create the function $P_B^*(P_I)$. Then, we find the intersection of $P_I^*(P_B)$ and $P_B^*(P_I)$. This point is the equilibrium of the model, as is shown in Figure 12(b). Notice, that both $P_B^*(P_I)$ and $P_I^*(P_B)$ are not linear functions, though it might seem like that in the graphs.



(a) P_I equilibrium function over fixed P_B



(b) Intersection point is the equilibrium

Figure 12: Equilibrium is reached where $P_I^*(P_B) = P_B^*(P_I)$

Another way to find the equilibrium is by the following algorithm, which returns a best response strategy (P_I^*, P_B^*, P_J^*).

1. Choose an arbitrary strategy (P_I, P_B, P_J), and define tolerance parameters ϵ, δ.
2. Compute U_I, U_J using (P_I, P_B, P_J).
3. If $U_I > \max\{U_J, 0\} + \epsilon$, then set $(P_I^*, P_B^*, P_J^*) = (1, 0, 0)$.
4. If $U_B > \max\{U_J, U_I\} + \epsilon$, then set $(P_I^*, P_B^*, P_J^*) = (0, 1, 0)$.
5. If $U_J > \max\{U_I, 0\} + \epsilon$, then set $(P_I^*, P_B^*, P_J^*) = (0, 0, 1)$.
6. If $|U_I| < \epsilon$, $U_J < 0$ and $\min\{U_I, 0\} > U_J + \epsilon$, then set $(P_I^*, P_B^*, P_J^*) = (\frac{P_I}{P_I+P_B}, \frac{P_B}{P_I+P_B}, 0)$.
7. If $|U_J - U_I| < \epsilon$, $U_J > 0, U_I > 0$ and $\min\{U_J, U_I\} > \epsilon$, then set $(P_I^*, P_B^*, P_J^*) = (P_I, 0, 1 - P_I)$.
8. If $|U_J| < \epsilon$, $U_I < 0$ and $\min\{U_J, 0\} > U_I + \epsilon$, then set $(P_I^*, P_B^*, P_J^*) = (0, P_B, 1 - P_B)$.
9. If at least two of the following conditions are fulfilled: $|U_I| < \epsilon$, $|U_J| < \epsilon$ and $|U_J - U_I| < \epsilon$, then set $(P_I^*, P_B^*, P_J^*) = (P_I, P_B, P_J)$.
10. If $|P_I^* - P_I| + |P_B^* - P_B| < \delta$, stop, and declare the equilibrium strategy as $(P_I^e, P_B^e, P_J^e) = (P_I^*, P_B^*, P_J^*)$.
11. If $|P_I^* - P_I| + |P_B^* - P_B| \geq \delta$, define a "new" strategy $(P_I^{\text{new}}, P_B^{\text{new}}, P_J^{\text{new}})$ as a convex combination of the "old" strategy and its best response, using a random number $\gamma \in (0, 1)$ as a weight. Go back to second step and repeat the process, until you reach the stopping criteria.

When $\epsilon, \delta \to 0$, the algorithm converges into the symmetric equilibrium point. We applied this algorithm with $\epsilon = \delta = 0.005$ in our numerical examples.

Using the algorithm to find the equilibrium of the example that we discussed before: $\lambda = 0.41, \mu = 1, R = 5, C_W = 3.37, C_I = 1.07$, we get the equilibrium $(P_I^*, P_B^*, P_J^*) = (0.2215, 0.08, 0.6985)$, the same result that we obtained by the other two methods.

5 Sensitivity Analysis

We now wish to check the influence of each parameter on the equilibrium strategy, while the others are kept fixed. When R increases, both U_I and U_J increase. For high enough values of R, $U_I, U_J > 0$, and we look at the *value of the information*, which we define as the expected loss from joining the queue without obtaining the information:

$$V = \sum_{i=n_e}^{\infty} \pi_i \left(\frac{C_W(i+1)}{\mu} - R \right). \tag{22}$$

As long as $V < C_I$, all customers join the queue without inspection it, meaning $P_J = 1$. When $V = C_I$, all customers mix among inspecting and joining, meaning $P_J + P_I = 1$. When $V > C_I$, they all inspect before joining, meaning $P_I = 1$. When R increases, n_e increases, and therefore V increases and the equilibrium point changes according to that. An example is shown in Figure 13.

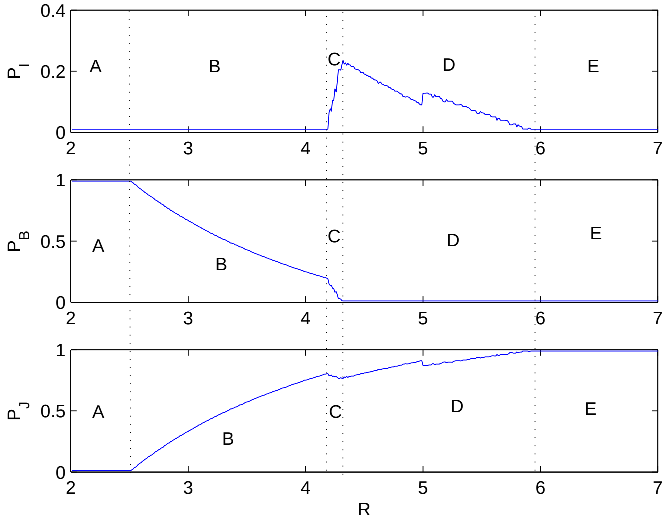


Figure 13: Change of equilibrium when R increases

In this graph, $\lambda = 0.5, \mu = 1, C_W = 2.5, C_I = 1$. In the x-axis, R changes from 2 to 7, and in the y-axis we get the equilibrium strategy P_I, P_B, P_J (the same structure appears in all the following graphs). In region A, the observable queue threshold is $n_e = 0$, and therefore $U_I, U_J < 0$ and $P_B = 1$. When R increases, n_e increases and U_I, U_J increase as well. In region B, $U_J = 0 > U_I$, and therefore customers are indifferent in equilibrium between joining and balking, and we get a mixed strategy. As R increases, customers tend to join over balk. In region C, $U_I = U_J = 0$, and therefore we get a mixed strategy where $P_I, P_B, P_J > 0$. As R continue to increase, in region D, $U_I = U_J > 0$, meaning the value of information is equal to its price, and customers mix among joining and obtaining the information. When R is large, as in region E, V decreases and joining with probability 1 becomes the equilibrium strategy.

Notice that a discontinuity occurs when $R = 5$, where $U_I = U_J > 0$. In this point, the value of n_e changes from 1 to 2. As a result, the values of U_I and U_J jump down (since U_I, U_J are not continuous in n_e). The new value $U_I = U_J$ is similar to the value that was reached for lower value of R, and therefore P_I jumps down while P_J jumps up. When n_e increases, the probability for a short queue decreases, and customers tend to join the queue with higher probability, at the expense of of obtaining the information.

Another example of discontinuities is shown in Figure 14, where $\lambda = 0.7, \mu = 1, C_W = 1, C_I = 0.2$, and R changes from 0 to 10:

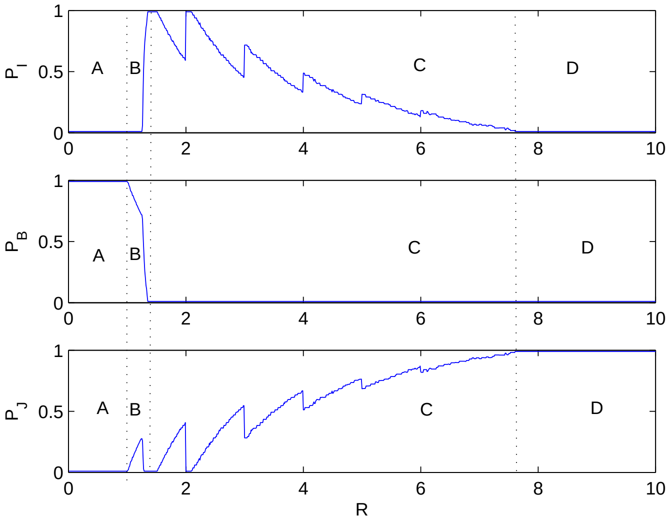


Figure 14: Change of equilibrium while R increases

The change of equilibrium is about the same as in Figure 13, and the main difference is at region C, where $U_I = U_J > 0$. We get a discontinuity in every integer value of R, since $C_W = \mu = 1$ and therefore n_e changes there. Unlike the previous graph, some jumps leads to $P_I = 1$. Figure 15 show the jump in U_I and U_J values near $n_e = 2$ and $n_e = 3$.

We can conclude that when R is small enough, such that $n_e = 0$, we get $P_B = 1$. As R increases, P_B decreases to 0. When R is large enough, such that $V < C_I$, we get $P_J = 1$. In between, we get $V \geq C_I$, which leads to $P_I > 0$.

We now wish to determine the change of equilibrium when increasing C_I while all other parameters are kept fixed. Naturally, U_I decreases, and P_I decreases accordingly. As long as $U_I > 0$, if $U_J > 0$, then increasing C_I would results in increasing P_J at the expense of of P_I. Otherwise, if $U_J < 0$, it would results in increasing P_B on behalf of P_I. An example is shown in Figure 16, where $\lambda = 0.5, \mu = 1, R = 4, C_W = 2$, and C_I changes from 0 to 3.

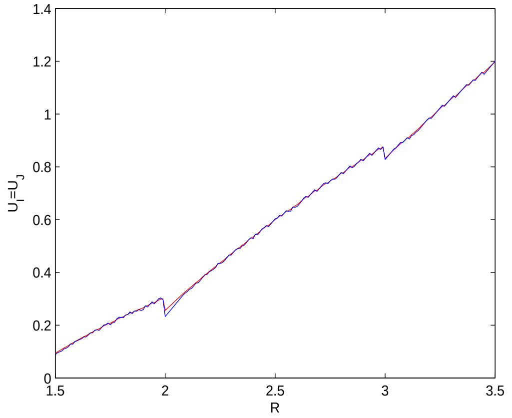


Figure 15: Discontinuities in the value of $U_I = U_J$ when R increases

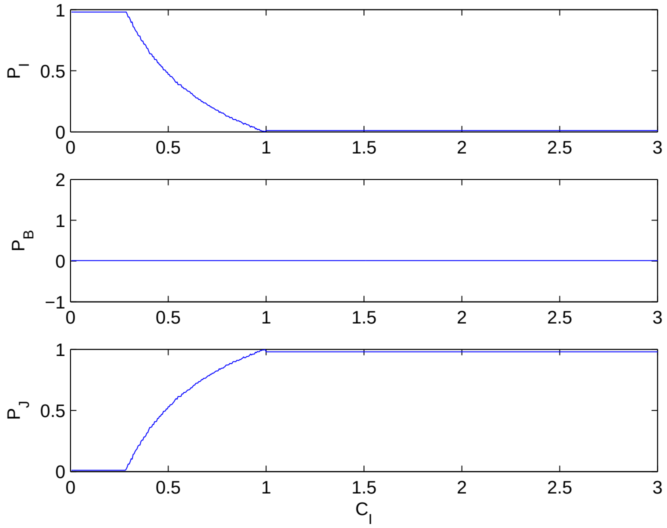


Figure 16: P_I decreases in C_I

In Figure 16, $U_I, U_J > 0$ and therefore $P_B = 0$. For small values of C_I, $P_I = 1$, but as C_I increases, P_I decreases while P_J increases. For large values of C_I, $P_I = 0$. A more complicated structure of equilibrium is demonstrated in Figure 17, where $\lambda = 0.8, \mu = 1, R = 7, C_W = 3$, and C_I changes from 0 to 4.

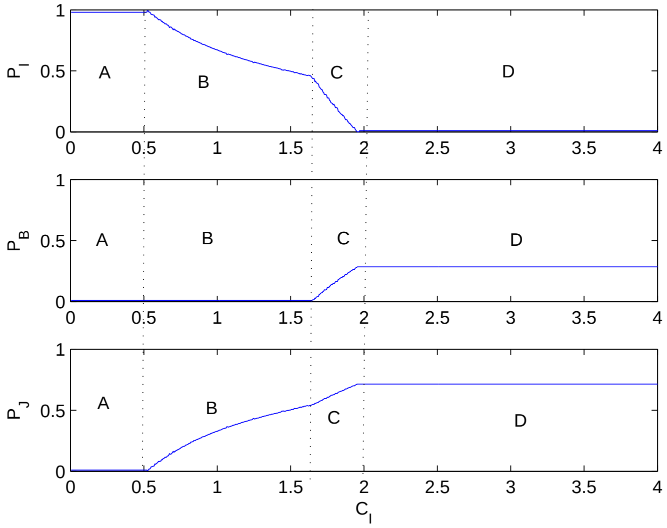


Figure 17: P_I decreases in C_I

Figure 17 is an example of how increasing C_I reduces not only the expected utility from inspecting the queue, but also the expected utility form joining it without inspection. In region A, $U_I > \max\{U_J, 0\}$, and therefore $P_I = 1$, which is Naor's model. While C_I increases, as in region B, the value of information becomes equal to its price, meaning $U_I = U_J > 0$. In region C, we get a decrease both in U_I and U_J values. In this region, $U_I = U_J = 0$, and customers are indifferent among all actions. In region D, C_I is large, and as a result inspection becomes a dominated strategy. And since $U_J = 0$, the equilibrium stabilizes on a mixed strategy in which $P_J, P_B > 0$ for all $C_I > 2$. This is an example of Edelson and Hildebrand's (1975) equilibrium.

The expected utilities from joining and from obtaining the information are also functions of $\frac{C_W}{\mu}$, which is the average waiting cost for a complete service. These are the normalized parameters of the model. In equilibrium, when the ratio $\frac{C_W}{\mu}$ is low, P_I increases. But it comes to a point where P_I starts to decrease, until $P_I = 0$ when $\frac{C_W}{\mu}$ approaches the reward value, R. For example, Figure 18 shows the equilibrium strategy (P_I, P_B, P_J) as a function of $\frac{C_W}{\mu}$, for $\lambda = 0.5, \mu = 1, R = 5$ and $C_I = 0.5$.

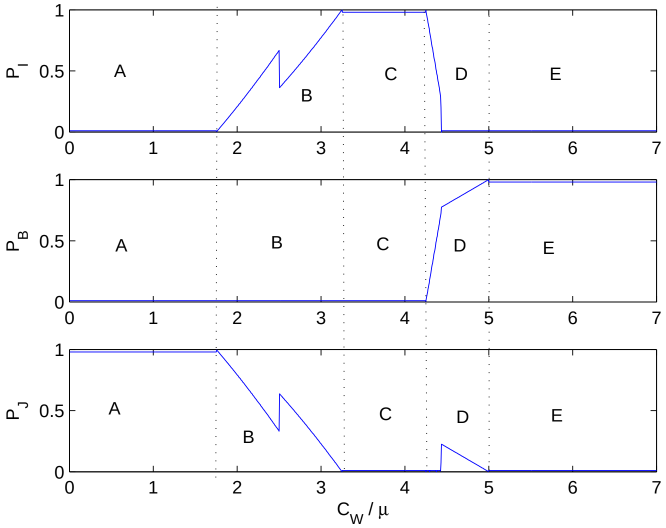


Figure 18: P_I is non-monotonic in $\frac{C_W}{\mu}$

In region A, $U_J > \max\{U_I, 0\}$, and therefore $P_J = 1$. In region B, $U_J = U_I > 0$, and therefore $P_B = 0$ and customers mix among joining and obtaining the information. In region C, all customers inspect the queue, meaning $P_I = 1$, and then P_I drops to 0. In region D, customers are indifferent among balking and inspecting the queue, and therefore $P_B > 0$ and $P_J = 1 - P_B$. In region E, all customers balk from the queue, meaning $P_B = 1$.

Since $R = 5$, we get a discontinuity at $\frac{C_W}{\mu} = 2.5$, where n_e jumps down from 2 to 1. At this point, $U_I = U_J > 0$, and both U_I and U_J increase when n_e decreases (since the probability for a short queue increases as well). The new value of U_I, U_J is similar to a previous one that was reached for a lower value of $\frac{C_W}{\mu}$, and therefore P_I jumps down while P_J jumps up.

It is also possible to find that, in equilibrium, customers are indifferent among all three options. Figure 19 shows the equilibrium strategy for $\rho = 0.5, R = 5$ and $C_I = 1$. In region C, we get a mixed strategy among all actions.

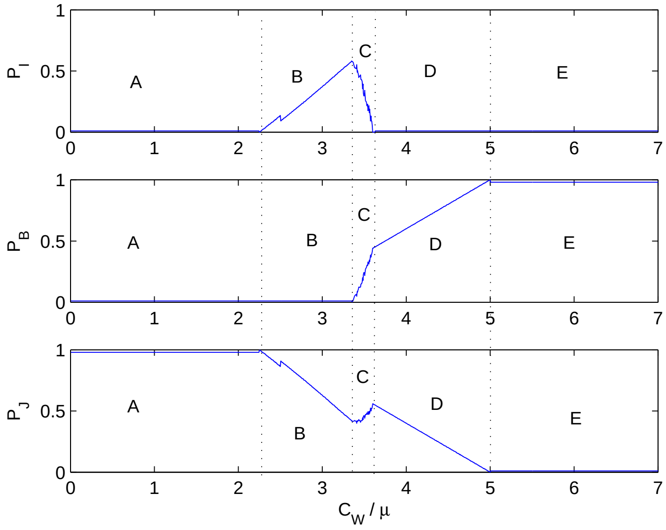


Figure 19: In region C, customers are indifferent among all options

In region A, $P_J = 1$, meaning that all customers join the queue without inspecting it. In region B, customers are indifferent among joining and inspecting the queue, and therefore $0 < P_I < 1$ and $P_J = 1 - P_I$. In region C, all customers inspect the queue, meaning $P_I = 1$, and then P_I drops to 0. In region D, customers are indifferent among balking and inspecting the queue, and therefore $P_B > 0$ and $P_J = 1 - P_B$. In region E, customers balk from the queue, and $P_B = 1$.

Changing the ratio $\rho = \frac{\lambda}{\mu}$, also has an impact on the equilibrium point. When ρ is small, the probability for short queues is high, since arriving rate (λ) is low comparing to service rate (μ). Therefore, in equilibrium, P_J is high while P_I is low. But as ρ increases, the probability for short queue decreases, and therefore information becomes significant and P_I increases at the expense of P_J. As an example, see Figure 20, where $R = 5, C_W = 2$ and $C_I = 0.5$.

Figure 21, for which $R = 5, C_W = 3.37$ and $C_I = 1$, is an example of mixed strategy equilibrium when ρ is large enough. In region A, B the evaluation of the equilibrium is the same as in figure 20, but in region C, when $\rho > 0.5$, the expected utility from all possible actions is 0, and therefore customers mix among them.

In Appendix A.3, we prove the following proposition:

Proposition 9 *For fixed values of R, μ, C_I, when $U_J = U_B = 0 > U_I$, the graph of P_B is linear in C_W, and*

$$P_B = \frac{1}{\rho R} \cdot \frac{C_W}{\mu} - \frac{1-\rho}{\rho}. \tag{23}$$

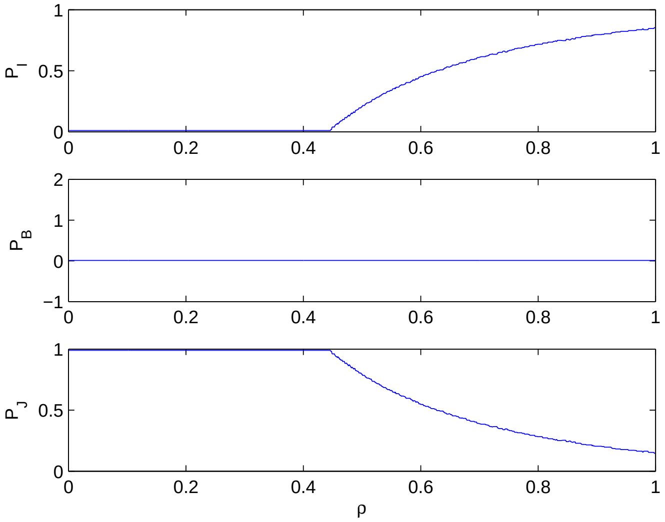


Figure 20: When ρ increases, information becomes significant

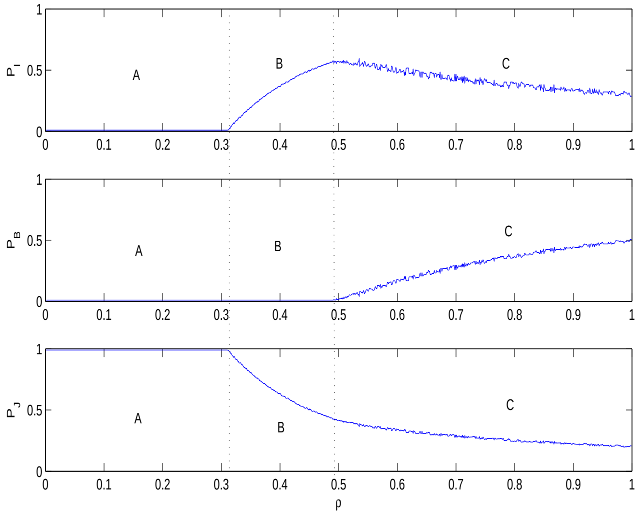


Figure 21: In region C, customers indifferent among all options

6 Conclusion

Our main goal in this paper was to prove the existence and uniqueness of equilibrium in a two dimensional strategic game. Our proof is based on the topological properties of the set of strategies in this game. We prove that the mapping from the compact strategy set to the expected utility set (EUS) is continuous, and therefore the EUS is also a compact set. By proving that the mapping is also a homeomorphism, we show that the topological properties of the set of strategies, specifically that its interior is simply connected, is also preserved in the EUS. We also proved monotonicity of the EUS boundaries, based on the *pairwise ATC* property. As a result, we could determine that the EUS has a unique shape, and use its properties in order to prove that the equilibrium exists and is unique.

We were also interested in the question of what kind of equilibrium will arise in this game, according to its complicated structure. We showed that for different set of parameters, this model provides all kinds of equilibrium: pure, mixed among two actions, and mixed among all actions. Sensitivity analysis showed that strategies can change in non-monotonic way as a reaction to change in one of the parameters. In addition, we showed that the discontinuities in Naor's threshold strategy, $n_e = \left\lfloor \frac{R\mu}{C_W} \right\rfloor = \lfloor \nu \rfloor$, result in discontinuities in the equilibrium.

We showed that when R increases, the value of information decreases, and customers tend to join without inspecting. When the cost of information, C_I, increases, customers tend to join at the expense of inspecting the queue. When ρ increases, the probability of high congestion queue increases, and therefore the value of information increases, and customers tend to inspect the queue. Changing the ratio $\frac{C_W}{\mu}$ leads to non-monotonic behavior: when the ratio is low, customers tend to join without inspecting; when the ratio is high, customers tend to balk; in between, customers tend to inspect the queue.

Future research may be able to derive similar proofs for proving existence and uniqueness of equilibrium in other multi-dimensional queueing games. It may also give an answer to another question, which is finding the social optimum behavior of the customers in this game. In Section 5, we saw the influence of changes in the parameters on customers' equilibrium strategy. It is especially interesting here, since this model concerns both positive and negative externalities. Externality is the impact of customers' decision on others: for example, the decision to join the queue results in longer queue, and that affects other customers' welfare. Indeed, in most cases there are *negative externalities* associated with an increased congestion when a customer joins a queue. However, in other cases there are *positive externalities.* In our model, when a customer inspects the queue, the probability of long queues is reduced, and therefore inspection has a positive impact on others. The direction and strength of the externalities has an impact on the deviation of the equilibrium solution from the socially desired behavior, and on the means used to reduce the gap.

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A Appendix: Proofs

A.1 Proof of Lemma 6

For simplicity, we use in the proof n instead of n_e. We start by proving that the denominator is always positive. The denominator can be represented as a product of the following terms: $\eta > 0$, $\nu > 1$. The two other terms: $(1-\xi)(\eta - \xi^n(\xi - 1 + \eta))^3 = (1-\xi)(\eta(1-\xi^n) + (1-\xi)\xi^n)^3$. For $\xi < 1$, both terms are positive, and for $\xi > 1$, both terms are negative. Therefore, the product is positive and the denominator is positive. For $\xi = 1$, the denominator is zero, but it is possible to calculate the limit of the Jacobian, as we will show later.

We now wish to determine the sign of the numerator. Denote the numerator as M:

$$\begin{aligned} M &= R^2\xi^{n-1}\cdot\Big(n^3(1-\xi)^2\xi^n\eta^2 - n^2(1-\xi)^2\xi^n\eta(\eta\nu-2) \\ &+ n[-\xi\eta^2 + \xi^{2n+1}[(1-\xi)^2-\eta^2] + \xi^n((1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3\nu)] \\ &+ \xi(1-\xi^n)[\xi^n(\xi-1+\eta)[2+(\xi-1-\eta)\nu] + \eta(\eta\nu-2)]\Big). \end{aligned} \tag{24}$$

Since $R^2\xi^{n-1} > 0$, it is enough to find the sign of the rest of the numerator. For fixed n, notice that M is a linear function of ν where $n \le \nu < n+1$. Therefore, it is enough to prove that M has the same sign when $\nu = n$ and when $\nu \to n+1$. First, we consider the case where $\nu = n$. Denote:

$$\begin{aligned}
\widetilde{M} &= n^3(1-\xi)^2\xi^n\eta^2 - n^2(1-\xi)^2\xi^n\eta(\eta n - 2) \\
&+ n[-\xi\eta^2 + \xi^{2n+1}[(1-\xi)^2 - \eta^2] + \xi^n((1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3 n)] \\
&+ \xi(1-\xi^n)[\xi^n(\xi - 1 + \eta)(2 + (\xi - 1 - \eta)n) + \eta(\eta n - 2)].
\end{aligned}$$

We look at the first two terms:

$$n^3(1-\xi)^2\xi^n\eta^2 - n^2(1-\xi)^2\xi^n\eta(\eta n-2) = n^3(1-\xi)^2\xi^n\eta^2 - n^3(1-\xi)^2\xi^n\eta^2 + 2n^2(1-\xi)^2\xi^n\eta = 2n^2(1-\xi)^2\xi^n\eta.$$

Then:

$$\begin{aligned}
\widetilde{M} &= 2n^2(1-\xi)^2\xi^n\eta - n\xi\eta^2 + n\xi^{2n+1}((1-\xi)^2 - \eta^2) + n\xi^n((1-\xi)^2 + 2n\xi\eta^2 - (1-\xi)^3 n^2) \\
&+ 2\xi^{n+1}(1-\xi^n)(\xi - 1 + \eta) + n\xi^{n+1}(1-\xi^n)((1-\xi)^2 - \eta^2) \\
&+ \xi n(1-\xi^n)\eta^2 - 2\eta\xi(1-\xi^n).
\end{aligned}$$

The third term is also balanced with 8th term:

$$n\xi^{2n+1}((1-\xi)^2 - \eta^2) + n\xi^{n+1}(1-\xi^n)((1-\xi)^2 - \eta^2) = n\xi^{n+1}((1-\xi)^2 - \eta^2),$$

and we get:

$$\begin{aligned}
\widetilde{M} &= 2n^2(1-\xi)^2\xi^n\eta - n\xi\eta^2 + n\xi^n((1-\xi)^2 + 2n\xi\eta^2 - (1-\xi)^3 n^2) \\
&+ 2\xi^{n+1}(1-\xi^n)(\xi - 1 + \eta) + n\xi^{n+1}((1-\xi)^2 - \eta^2) + \xi n(1-\xi^n)\eta^2 - 2\eta\xi(1-\xi^n).
\end{aligned}$$

By gathering all the elements that contain η^2, we get:

$$-n\xi\eta^2 + 2n\xi^{n+1}\eta^2 - n\xi^{n+1}\eta^2 + \xi n(1-\xi^n)\eta^2 = 0.$$

Then $\widetilde{M}$ becomes:

$$\begin{aligned}
\widetilde{M} &= 2n^2(1-\xi)^2\xi^n\eta + n\xi^n(1-\xi)^2 - n^2\xi^n(1-\xi)^3 \\
&+ 2\xi^{n+1}(1-\xi^n)(\xi - 1 + \eta) + n\xi^{n+1}(1-\xi)^2 - 2\xi(1-\xi^n)\eta.
\end{aligned}$$

By gathering all the elements that contain η, we get:

$$2n^2(1-\xi)^2\xi^n\eta + 2\xi^{n+1}(1-\xi^n)\eta - 2\xi(1-\xi^n)\eta = 2n^2(1-\xi)^2\xi^n\eta - 2\xi(1-\xi^n)^2\eta.$$

By gathering all the rest of the elements, we get:

$$n\xi^n(1-\xi)^2 - n^2\xi^n(1-\xi)^3 + 2\xi^{n+1}(1-\xi^n)(\xi-1) + n\xi^{n+1}(1-\xi)^2 = n\xi^n(1-\xi)^2(1-n(1-\xi)+\xi) - 2\xi^{n+1}(1-\xi^n)(1-\xi).$$

Then $\widetilde{M}$ becomes:

$$\widetilde{M} = 2n^2(1-\xi)^2\xi^n\eta - 2\xi(1-\xi^n)^2\eta + n\xi^n(1-\xi)^2(1 - n(1-\xi) + \xi) - 2\xi^{n+1}(1-\xi^n)(1-\xi). \quad (25)$$

For $n = 1$, $\widetilde{M} = 0$. We now show that $\widetilde{M} < 0$ for all $n \geq 2$. We first prove that

$$2n^2(1-\xi)^2\xi^n\eta - 2\xi(1-\xi^n)^2\eta < 0 \tag{26}$$

$$2n^2(1-\xi)^2\xi^n\eta - 2\xi(1-\xi^n)^2\eta = 2\xi(1-\xi)^2\eta \cdot [n^2\xi^{n-1} - (1+\xi+\xi^2+\ldots+\xi^{n-1})^2]$$

Since $2\xi(1-\xi)^2\eta > 0$, it is enough to show that $n^2\xi^{n-1} < (1+\xi+\xi^2+\cdots+\xi^{n-1})^2$. Both terms are positive, and therefore we can look at the square root: $n\xi^{\frac{n-1}{2}} < 1+\xi+\xi^2+\cdots+\xi^{n-1}$ For all ξ and $n > 1$, the function: $f(n) = \xi^{n-1}$ is positive, monotonic decreasing and convex. If n is even, then from convexity we get a system of $\frac{n}{2}$ inequalities:

$$\begin{aligned}
\frac{1}{2}(1+\xi^{n-1}) &> \xi^{\frac{n-1}{2}} \\
\frac{1}{2}(\xi+\xi^{n-2}) &> \xi^{\frac{n-1}{2}} \\
&\vdots \\
\frac{1}{2}(\xi^{n/2-1}+\xi^{n/2}) &> \xi^{\frac{n-1}{2}}
\end{aligned}$$

If n is odd we get a system of $\frac{n-1}{2}$ inequalities and one equality:

$$\begin{aligned}
\frac{1}{2}(1+\xi^{n-1}) &> \xi^{\frac{n-1}{2}} \\
\frac{1}{2}(\xi+\xi^{n-2}) &> \xi^{\frac{n-1}{2}} \\
&\vdots \\
\frac{1}{2}(\xi^{\frac{n-3}{2}}+\xi^{\frac{n+1}{2}}) &> \xi^{\frac{n-1}{2}} \\
\xi^{\frac{n-1}{2}} &= \xi^{\frac{n-1}{2}}
\end{aligned}$$

In both cases, when summing all the inequalities in each system, we get

$$n\xi^{\frac{n-1}{2}} < 1+\xi+\xi^2+\cdots+\xi^{n-1}.$$

Next, we prove that the remaining elements are negative as well:

$$\begin{aligned}
& n\xi^n(1-\xi)^2(1-n(1-\xi)+\xi) - 2\xi^{n+1}(1-\xi^n)(1-\xi) \\
= \; & \xi^n(1-\xi)^2[n(1-n(1-\xi)+\xi) - 2\xi(1+\xi+\xi^2+\cdots+\xi^{n-1})]
\end{aligned}$$

Since $\xi^n(1-\xi)^2 > 0$, it is enough to show that $n[1-n(1-\xi)+\xi] - 2\xi(1+\xi+\xi^2+\cdots+\xi^{n-1}) < 0$. We prove it by induction. For $n = 2$, we get:

$$2[1-2(1-\xi)+\xi] - 2\xi(1+\xi) = 2(-1+2\xi-\xi^2) = -2(1-\xi)^2 < 0.$$

Next, we assume that the statement is true for n, and use it to prove that it is also true for $n+1$.

$$\begin{aligned} & (n+1)(1-(n+1)(1-\xi)+\xi) - 2\xi(1+\xi+\xi^2+\cdots+\xi^n) \\ = \ & n[1-n(1-\xi)+\xi] - 2\xi(1+\xi+\xi^2+\cdots+\xi^{n-1}) - n(1-\xi) + 1 - (\xi+1)(1-\xi) + \xi - 2\xi^{n+1} \\ < \ & -n(1-\xi) + 1 - (n+1)(1-\xi) + \xi - 2\xi^{n+1} = -2n(1-\xi) + 2\xi(1-\xi^n) \\ = \ & -2(1-\xi)[n - \xi(1+\xi+\xi^2+\cdots+\xi^{n-1})] < -2(1-\xi)(n-\xi n) = -2n(1-\xi)^2 < 0 \end{aligned}$$

which complete the proof of equation (26).

We now show that M (Equation (24)) is strictly negative when $\nu \to n+1$:

$$\begin{aligned} \lim_{\nu \to n+1} M &= R^2\xi^{n-1} \cdot \Big(n^3(1-\xi)^2\xi^n\eta^2 - n^2(1-\xi)^2\xi^n\eta(\eta(n+1)-2) \\ &+ n\left[-\xi\eta^2 + \xi^{2n+1}\left((1-\xi)^2-\eta^2\right) + \xi^n\left((1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3(n+1)\right)\right] \\ &+ \xi(1-\xi^n)\left(\xi^n(\xi-1+\eta)\left[2+(\xi-1-\eta)(n+1)\right] + \eta(\eta(n+1)-2)\right)\Big). \end{aligned} \tag{27}$$

Following the same process, we reduce the problem into determining the sign of $\widehat{M}$, where

$$\widehat{M} = \widetilde{M} - n^2(1-\xi)^2\xi^n\eta^2 - n\xi^n(1-\xi)^3 + \xi^{n+1}(1-\xi^n)[(1-\xi)^2-\eta^2] + \xi(1-\xi^n)\eta^2 \tag{28}$$

We first gather all terms that contain η^2:

$$\begin{aligned} & -n^2(1-\xi)^2\xi^n\eta^2 - \xi^{n+1}(1-\xi^n)\eta^2 + \xi(1-\xi^n)\eta^2 \\ = \ & -\eta^2[n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2] \end{aligned}$$

We already proved that $n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2 < 0$, and therefore this is a positive frame. But by adding it to the first two terms of $\widetilde{M}$ (Equation (26)), we get:

$$(2\eta-\eta^2)[n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2] = \eta(2-\eta)[n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2] < 0.$$

Last, we show that $-n\xi^n(1-\xi)^3 + \xi^{n+1}(1-\xi^n)(1-\xi)^2 < 0$:

$$\begin{aligned} & -n\xi^n(1-\xi)^3 + \xi^{n+1}(1-\xi^n)(1-\xi)^2 \\ = \ & \xi^n(1-\xi)^3[-n + \xi(1+\xi+\xi^2+\cdots+\xi^{n-1})] < \xi^n(1-\xi)^3[-n+n\xi] \\ = \ & -n\xi^n(1-\xi)^4 < 0 \end{aligned}$$

To complete the proof, we need to show that $M < 0$ for all $1 < \nu < 2$. Notice, that when $\nu = 1$, $\widetilde{M} = 0$. But when $\nu = 1$, $R = \frac{C_W}{\mu}$, and the expected utility from inspecting the queue, regardless of other customers' strategy, becomes negative. Therefore, this action becomes dominated by the other two actions, and can be omitted. As a result, the game becomes a two actions game, which has a unique solution due to ATC. And since $\widehat{M} = -\xi(1-\xi)^4 < 0$ for $\nu \to 2$, we get that $M < 0$ for all $1 < \nu < 2$.

We also need to show what happens when $\xi = 1$. As we saw above, this is not really a singularity point of the mapping, and if we calculate the Jacobian that point at directly form equations (8) and (12), we get that it is equal to the limit of the representation of the Jacobian that appears in equation (21) when $\xi \to 1$:

$$\lim_{\xi \to 1} J(U_I, U_J) = -\frac{n(n+1)(-4+6\nu-2n+n(n-1)t(2+nt-t\nu))}{12t(1+nt)^3\nu^2}.$$

The denominator is a product of strictly positive terms, and therefore it is strictly positive. In the numerator, $n(n+1) > 0$. We want to show that the other term in the product is strictly positive. Since $\nu > n \geq 1$, we get that $-4+4\nu > 0$ and $2\nu - 2n > 0$. $2+nt-t\nu = 2-t(\nu-n) > 1$, since both $t < 1$ and $\nu - n < 1$. Combining all this terms we get that the numerator is strictly positive, and so is the entire fraction. From the minus sign we conclude that the limit exists and it is strictly negative.

By proving that $\widetilde{M}$ and $\widehat{M}$ are strictly negative for all $\nu > 1$, we conclude that the Jacobian of the mapping is non-zero and therefore the continuous mapping is a homeomorphism.

A.2 Corollary 8: Explicit expression of the boundaries curves

The lower boundary of the EUS is where $P_B + P_J = 1$ and $P_I = 0$. Substituting this into equations (3) and (4), we get:

$$\pi_i = \xi^i(1-\xi) \qquad i = 0, 1, 2, \ldots$$

Therefore:

$$\begin{aligned} U_I &= \sum_{i=0}^{n_e-1} \xi^i \pi_0 \left(R - \frac{C_W(i+1)}{\mu}\right) - C_I \\ &= R(1-\xi^{n_e}) - \frac{C_W}{\mu}\left(\frac{1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1}}{1-\xi}\right) - C_I, \end{aligned}$$

and

$$U_J = R - \frac{C_W}{\mu}\left(\frac{1-(n_e+1)\xi^{n_e}(1-\xi) - \xi^{n_e+2}}{1-\xi}\right).$$

Notice that for $\xi \neq 1$ the lower boundary is continuous. When $\xi \to 1$, we get that $U_I \to -C_I$ and $U_J \to R - \frac{C_W}{\mu}$.

The left boundary of the EUS is the curve where $P_B = 0$. Therefore, on this curve $P_I + P_J = 1$. Substituting this into equations (3) and (4), we get:

$$\pi_i = \begin{cases} \rho^i \pi_0 & i < n_e, \\ \rho^{n_e}(1-\eta)^{i-n_e}\pi_0 & i \geq n_e, \end{cases}$$

where

$$\pi_0 = \left[\frac{1-\rho^{n_e}}{1-\rho} + \frac{\rho^{n_e}}{\eta}\right]^{-1}.$$

Therefore:

$$\begin{aligned} U_I &= \sum_{i=0}^{n_e-1} \rho^i \pi_0 \left(R - \frac{C_W(i+1)}{\mu} \right) - C_I \\ &= \pi_0 \left[R \cdot \frac{1-\rho^{n_e}}{1-\rho} - \frac{C_W}{\mu} \cdot \frac{1-(n_e+1)\rho^{n_e} + n_e \rho^{n_e+1}}{(1-\rho)^2} \right] - C_I, \end{aligned}$$

and

$$\begin{aligned} U_J &= U_I + C_I + \sum_{i=n_e}^{\infty} \rho^{n_e} (P_J \rho)^{i-n_e} \pi_0 \left(R - \frac{C_W(i+1)}{\mu} \right) \\ &= R - \frac{C_W}{\mu} \pi_0 \left[\frac{1-(n_e+1)\rho^{n_e} + n_e \rho^{n_e+1}}{(1-\rho)^2} + \frac{\rho^{n_e}(n_e \eta + 1)}{\eta^2} \right]. \end{aligned}$$

For $\rho \neq 1$ and $\eta > 0$ the left boundary is continuous. Since we assumed that $\eta > 0$, it is enough to show what happens when $\rho \to 1$:

$$\begin{aligned} \lim_{\rho \to 1} U_I &= R \cdot \frac{n\eta}{n\eta + 1} - \frac{C_W}{\mu} \cdot \frac{n(n+1)\eta}{2n\eta + 2} - C_I \\ \lim_{\rho \to 1} U_J &= R - \frac{C_W}{\mu} \cdot \left[\frac{n(n+1)\eta}{2n\eta + 2} + \frac{1}{\eta} \right] \end{aligned}$$

The upper boundary of the EUS is the curve where $P_B + P_I = 1$. Therefore, on this curve $P_J = 0$. Substituting this into equations (3) and (4), we get:

$$\pi_i = \begin{cases} \xi^i \pi_0 & i \leq n_e, \\ 0 & i > n_e, \end{cases}$$

where

$$\pi_0 = \frac{1-\xi}{1-\xi^{n_e+1}}.$$

Therefore:

$$\begin{aligned} U_I &= \sum_{i=0}^{n_e-1} \xi^i \pi_0 \left(R - \frac{C_W(i+1)}{\mu} \right) - C_I \\ &= R \frac{1-\xi^{n_e}}{1-\xi^{n_e+1}} - \frac{C_W}{\mu} \left(\frac{1-(n_e+1)\xi^{n_e} + n_e \xi^{n_e+1}}{(1-\xi^{n_e+1})(1-\xi)} \right) - C_I, \end{aligned}$$

and

$$\begin{aligned} U_J &= U_I + C_I + \frac{\xi^{n_e}(1-\xi)}{1-\xi^{n_e+1}} \left(R - \frac{C_W(n_e+1)}{\mu} \right) \\ &= R - \frac{C_W}{\mu} \frac{\xi^{n_e}(1-\xi)(n_e+1)}{1-\xi^{n_e+1}}. \end{aligned}$$

Notice that for $\xi \neq 1$ the upper boundary is continuous. When $\xi \to 1$ we get:

$$\begin{aligned}
\lim_{\xi \to 1} U_I &= R \cdot \frac{n}{n+1} - \frac{C_W}{\mu} \cdot \frac{n}{2} - C_I \\
\lim_{\xi \to 1} U_J &= R - \frac{C_W}{\mu}
\end{aligned}$$

A.3 Proof of Proposition 9

By assumption, $U_J = 0$ and $P_I = 0$. Then:

$$\sum_{i=0}^{\infty} \pi_i \left(R - \frac{C_W(i+1)}{\mu} \right) = 0.$$

The stationary probabilities for $i = 1, 2, \ldots$ become:

$$\pi_i = ((1-P_B)\rho)^i \pi_0,$$

and therefore,

$$\pi_0 = 1 - (1-P_B)\rho.$$

Substituting into U_J, we get

$$\begin{aligned}
0 &= \sum_{i=0}^{\infty} ((1-P_B)\rho)^i (1-(1-P_B)\rho) \left(R - \frac{C_W(i+1)}{\mu} \right) \\
&= \frac{C_W}{\mu} \sum_{i=0}^{\infty} ((1-P_B)\rho)^i (i+1) - R \sum_{i=0}^{\infty} ((1-P_B)\rho)^i \\
&= \frac{C_W}{\mu} \left(\sum_{i=0}^{\infty} ((1-P_B)\rho)^{i+1} \right)' - \frac{R}{1-(1-P_B)\rho} \\
&= \frac{C_W}{\mu} \left(\frac{(1-P_B)\rho}{1-(1-P_B)\rho} \right)' - \frac{R}{1-(1-P_B)\rho} \\
&= \frac{C_W}{\mu} \left(\frac{1}{(1-(1-P_B)\rho)^2} \right) - \frac{R}{1-(1-P_B)\rho}
\end{aligned}$$

$$\Rightarrow \frac{C_W}{\mu} = [1-(1-P_B)\rho]R,$$

and therefore

$$P_B = \frac{1}{\rho R} \cdot \frac{C_W}{\mu} - \frac{1-\rho}{\rho}.$$