# The impact of inspection cost on equilibrium, revenue, and social-welfare in a single server queue

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**Abstract**

Classical models of customer decision making in unobservable queues assume acquiring queue-length information is too costly. However, due to recent advancements in communication technology, various services now make this kind of information accessible to customers at a reasonable cost. In our model, which reflects this new opportunity, customers choose among three options: join the queue, balk, or inspect the queue length before deciding whether to join. Inspection is associated with a cost. We compute the equilibrium in this model and prove its existence and uniqueness. Based on two normalized parameters - congestion and service valuation - we map all possible input parameter sets into three scenarios. Each scenario is characterized by a different impact of inspection cost on both equilibrium and revenue-maximization queue-disclosure policy: fully observable (when inspection cost is very low), fully unobservable (when inspection cost is too high), or observable by demand (when inspection cost is at an intermediate level). We show that when maximizing social welfare, the optimal disclosure policy is zero inspection cost. We show the structure remains the same when a fraction of the customers are considered urgent, that is, they always join, whereas the others are non-urgent and therefore join according to their equilibrium strategy.

**Keywords:** Queueing theory, Game theory, M/M/1 Markovian queue, Strategic customers.

# 1 Introduction

The classical model of strategic customers' decisions in an unobservable queue assumes acquiring queue-length information is too costly. However, today's communication technology makes online information accessible at a reasonable cost, and customers can make use of this information when deciding whether to join a queue. In many cases, this information is available at no charge, but still it is not costless to the customers who spend time and effort to obtain it, whether they pre-register, download relevant application, or even acquire relevant terminal interface equipment. Several papers discuss another type of cost associated with obtaining information, especially online through the service provider's site. Customers are often required to reveal personal information that they might not otherwise be willing to provide (Miyazaki and Fernandez 2001, Sheehan 2002, Hann et al. 2002, and Huang and Van Mieghem 2014). In this sense, the inspection cost is real and not a transfer payment from customer to service provider.

Online queue-length information is a relatively new option. Most of the services providing such information are health-care services, and hospitals all over the United States and Canada have recently started publishing their emergency rooms (ERs) expected waiting times on their web sites. For example, more than 20 distinct hospitals and dozens of support facilities are under the management of Florida Hospital, which provides current ER waiting times for every location on its web site, at `www.floridahospital.com`. Patients are encouraged to check this information prior to arrival, and decide whether to arrive to the ER or to turn to an alternative facility.

The Ontario government also provides such information to the public. Its web site `http://www.health.gov.on.ca/en/public/programs/waittimes/edrs/default.aspx` publishes current waiting times at 126 Ontario ERs and urgent care centers. According to the site, the Ontario government considers this reporting "an important part of our commitment to being open and accountable about how well we are doing in achieving our two top health-care priorities: reducing ER wait times and improving access to family health care." Patients in Ontario can also observe queue lengths for specific medical procedures such as surgery, MRIs, or CT scans. For more ER waiting-time examples, see JFK medical center at `jfkmc.com/our-services/er-wait-time.dot` and Reston hospital center at `www.restonhospital.com`.

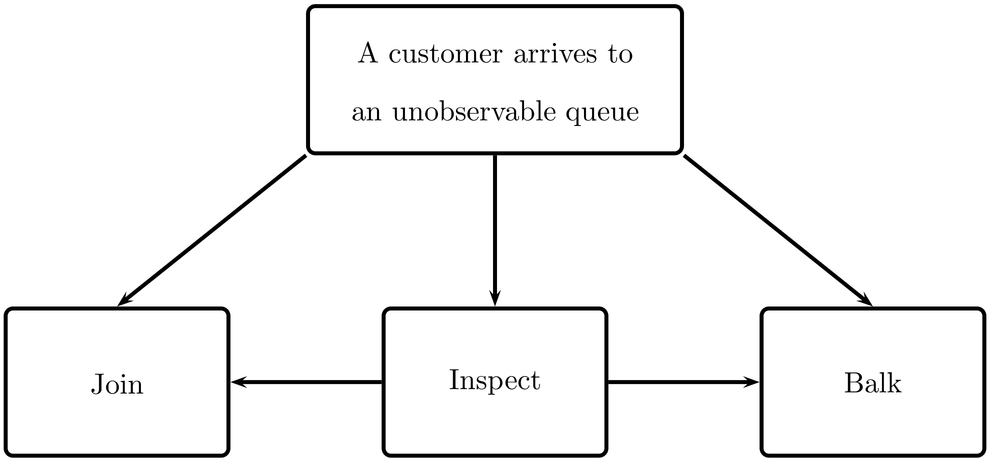



Figure 1: The decision process

Other services also publish online queue information such as waiting times for voting locations (see `www.votepinellas.com`), security check-in times at international airports (see Atlanta International Airport `www.atlanta-airport.com/passenger/waittimes`), border-crossing times (see the Canada Border Services `www.cbsa.gc.ca/bwt-taf`), and amusement-park waits (see Disney `touringplans.com/disney-world-app`).

We consider customers arriving to a single-server queue. To the traditional question of join-or-balk, we add a third option: customers may choose to acquire information about the queue length prior to making their decision.

Our model's solution consists of the probability of each action in equilibrium. If a customer inspects the queue, the next step is similar to Naor's observable-queue model.

When the inspection cost is very high, customers do not inspect the queue, and the decision is as in the unobservable-queue model. On the other hand, when the inspection cost is negligible, all customers inspect the queue and join according to the observable-queue model. The flowchart in Figure 1 shows the customers' decision problem.

We approach the model from two main directions: (a) we characterize the (Nash) equilibrium strategy and prove its existence and uniqueness; (b) we analyze the impact of inspection cost on equilibrium, revenue, and social welfare.

### 1.1 Existence and uniqueness of the equilibrium

A major challenge of this research is to prove the existence and uniqueness of the equilibrium strategy in our symmetric game of homogenous customers. In a typical model of congestion, participants try to avoid others and hence respond inversely to their actions. For example, an increase in the propensity of others to join a queue tends to discourage the individual from joining. This behavior is called *avoid-the-crowd* (ATC). The opposite behavior of *follow-the-crowd* (FTC) is also common. For example, the more customers buy priority in a queueing system, the more an individual is inclined to follow and buy priority for himself. FTC behavior typically results in multiple equilibria, whereas ATC provides a unique equilibrium (Hassin and Haviv 2003, pp. 6-7).

However, this discussion is limited to models in which the decision is one dimensional. Adding another option to the set of actions, as we suggest here, increases the model's complexity and requires a new examination of the ATC and FTC concepts. This research is a pioneering step in that direction.

We prove the existence and uniqueness of the equilibrium by analyzing geometric properties of the customers' expected utility set. We used this proof in another paper for a model of sequential inspection of queueing systems (Hassin and Roet-Green, 2013).

### 1.2 The impact of inspection cost

Consider a profit-maximizing service provider wishing to maximize system revenue by maximizing the throughput (effective joining rate). This assumption coincides with settings such as ERs, where entrance fees are exogenously given. Can the provider use inspection cost to maximize throughput?

The answer depends on the system parameters. Unsurprisingly, the throughput associated with an observable queue is higher in some cases, and in others, it is lower than that of an unobservable queue (see Hassin 1986, and Chen and Frank 2004). For example, when the (potential) arrival rate is small, the equilibrium throughput is larger in the unobservable case, where all join in, whereas in the observable case, some may still observe a long queue and balk. By contrast, when the arrival rate is very high and the service valuation is not much larger than the waiting cost incurred during service, providing the queue information increases the throughput.

Our contribution to the literature lies in mapping all possible input parameter sets into three scenarios based on the changing of the equilibrium as a function of inspection cost. For each scenario, we provide the revenue-maximizing solution, which points to a disclosure policy: observable queue (low inspection cost), unobservable queue (high inspection cost), or inspect-by-demand queue (at an intermediate-level inspection cost). Our analysis confirms that for many cases, the revenue-maximization solution is achieved with an intermediate level of inspection cost. Controlling the accessibility of the queue information can affect the inspection cost. For example,

- Publishing the queue length on the service provider's web site. The inspection cost increases as the search becomes harder: Does the information pop up on the web site's home page, or does it require an additional search? Does it involve a registration process? How intrusive is the registration process?

- Smart-phone application: How easy is it to download?

- Offering the information by phone call or text messaging.

We also show that for all scenarios, as the cost of inspection increases, social welfare decreases. Maximum welfare is achieved when the cost of inspection is zero, which is the observable model. Therefore, the social planner should encourage service providers to reduce the inspection cost.

We discuss the provider's decision process with respect to health-care applications: ERs in private and public hospitals, walk-in clinics, and also hospital service of private rooms upon request. Health-care analysis raises another interesting question: How would the analysis change when considering both urgent and non-urgent customers arriving to the same facility? Research has shown that at least 40 percent of patients attending ERs in the United States are non-urgent patients (see, e.g., Gill 1994, Padgett and Brodsky 1992, Petersen et al. 1998). By assuming urgent customers join the queue without inspection, we focus on the behavior of the non-urgent customers at equilibrium. We show the general structure of the analysis is maintained, and the range of parameter values representing congestion and service valuation for which inspection at a positive cost maximizes revenue increases with the fraction of non-urgent customers.

### 1.3 Related literature

Our model bridges two of the fundamental perceptions in strategic queueing theory: the observable queue (Naor 1969), and the unobservable queue (Edelson and Hildebrand 1975). Several other papers also bridge the two models. In the work of Marianov et al. (2005), customers also face a two-phase decision problem. First, customers decide whether to travel to the unobservable queue center. After arriving, the queue becomes observable and the customers' next decision is whether to join. Customers are heterogeneous in their travel and waiting costs. In equilibrium, a fraction of the customers balk, whereas the rest arrive to the queue, observe its length, and join only if it is under a certain threshold. Due to different waiting costs, each customer has his own threshold. The main difference between this model and ours is that it introduces a one-dimensional decision to customers at the first phase, whereas in our model, customers face a two-dimensional decision in the first phase. Our idea of an inspection cost is different from a travel cost, because customers in our model can avoid the inspection and costlessly join the unobservable queue, but they cannot avoid travel costs.

Other papers assume some of the customers are informed but customer types are predetermined and not a decision variable as in our model. Large and Norman (2010) assume a make-to-stock producer selling to customers with heterogeneous product valuations. A fraction of the customers see the queue length, whereas the others are uninformed and only those with valuations higher than a critical value join.

Hu, Li, and Wang (2014) consider a single queue with two customer types, informed customers who observe the queue length and uninformed customers. The fraction of informed customers is exogenous. These authors characterize the equilibrium and, similar to our finding, conclude the throughput function is unimodal in the fraction of informed customers.

Xu and Hajek (2013) present a multi-server model with unobservable queues. Inspection of the queues prior to joining is possible, but has a cost. Customers decide prior to arrival on the number

$k$ of queues to inspect. After inspecting these queues, they join the shortest one. Also, this model bridges the unobservable case with $k = 0$ and the observable case with $k = N$. The authors solve the model asymptotically when the number of servers grows to infinity, and show the existence of a symmetric equilibrium strategy in this game. For a comprehensive survey of queueing models with strategic customers, see Hassin (2016).

Only a few other papers analyze customer decisions of whether to buy queue-length information. Hassin and Haviv (1994) consider a system with two identical parallel servers. An arriving customer can acquire information about which queue is shorter by paying a fixed amount, and then join the shorter queue. A customer who does not purchase the information chooses one of the queues randomly. After joining, customers jockey costlessly from one queue to another, when the difference between them reaches a given threshold of $N$. The authors compute the value of information and the equilibrium threshold strategies. Intuitively, we expect the value of information to be a decreasing function of the proportion of informed customers $p$, inducing an ATC type of behavior. However, the authors show that under certain parameters, this value increases with $p$, inducing an FTC type of behavior.

Hassin and Roet-Green (2013) assume customers arrive to a system of parallel queues where inspection of the queue is required prior to joining. The queues vary in their service rates and inspection costs. The customer chooses which queue to inspect first, and based on that information, decides whether to continue inspecting another queue. After each inspection, the customer can decide whether to join one of the inspected queues. In many cases, the equilibrium strategy contains cascades: customers choose one action (join or inspect) when they observe $i$ and $i + 2$ customers in their first observed queue, and the other action when they observe $i + 1$ customers in their first observed queue.

Sundar and Ravikumar (2014) consider a model with two service providers that dynamically set their prices in a market with two customer types. Some customers randomly select a server and either join or balk, whereas the others first observe both queues and then decide to join one or balk. The main difference from our model is that a customer's type is predetermined and not a decision variable.

Another aspect we consider in our paper is social welfare maximization. We show the best result is achieved when the inspection cost is zero. Although this result may look intuitive, it is not that simple, because customers behave selfishly and do not always use the search information in a socially desirable way. For example, when the proportion of informed customers is exogenous, for some parameters, social welfare decreases as the fraction of informed customers increases (see Hu et al.2014). By contrast, our result implies that when the provider controls only the inspection cost, and customers maximize their own utility, the observable queue is optimal. Our different result is driven by our model assumption that the proportion of informed customers is endogenous, that is, determined in equilibrium.

The remainder of this paper is structured as follows: In section 2, we present the mathematical model. In section 3, we prove the existence and uniqueness of a symmetric equilibrium. In section 4, we analyze the impact of inspection cost on equilibrium, revenue, and social welfare. In section 5, we provide managerial insights for service providers, given their service parameters. In section 6, we analyze the impact urgent customers have on the equilibrium. We conclude our main results in section 7.

## 2 The Model

We consider an unobservable M/M/1 first-come first-served queue. Arriving customers face three options: join the queue, balk, or inspect queue length first and then decide whether to join. We use the following notation:

- $C_W$ is waiting cost per unit time. We assume $C_W > 0$.
- $C_I$ is the cost of inspecting the queue. We assume $C_I \geq 0$.
- $\lambda$ is the arrival rate.
- $\mu$ is the service rate.
- $R$ is the service valuation.
- A customer inspects the queue with probability $P_I$.

- A customer joins the queue without inspecting it with probability $P_J$.
- A customer balks with probability $P_B$.

Following Naor (1969), we denote

$$\nu = \frac{R\mu}{C_W} \tag{1}$$

as the *normalized value of service* measured in units of expected waiting cost for a single service completion. In the same sense, and as derived from the numerical analysis of this paper, we denote the *normalized cost* parameter

$$\kappa = \frac{C_I\mu}{C_W} \tag{2}$$

and the *normalized congested* parameter (i.e., the system utilization factor)

$$\rho = \frac{\lambda}{\mu}. \tag{3}$$

As in Naor's model, a customer observing a queue of length $i$, joins if $i \leq n_e - 1$, where

$$n_e = \left\lfloor \frac{R\mu}{C_W} \right\rfloor = \lfloor \nu \rfloor. \tag{4}$$

Note that when $n_e$ is an integer, the customer is indifferent between joining and balking when $i = n_e - 1$. We adopt Naor's tie-breaking rule and assume that in this case, the customer joins.

A consequence of this strategy is that the effective arrival rate is

$$\lambda_e = \begin{cases} (1 - P_B)\lambda & i \leq n_e - 1, \\ P_J\lambda & i \geq n_e. \end{cases} \tag{5}$$

Denote by $\pi_i$ the stationary probability of queue length $i$. If $\nu < 1$, then $n_e = 0$ and $P_B = 1$ is a dominant strategy. To avoid trivialities, we assume $\nu \geq 1$. The balance equation for $0 \leq i \leq n_e - 1$ is

$$(1 - P_B)\lambda\pi_i = \mu\pi_{i+1},$$

and for $i \geq n_e$, the balance equation is

$$P_J\lambda\pi_i = \mu\pi_{i+1}.$$

To simplify the presentation, we define

$$\begin{aligned} \xi &= (1-P_B)\rho, \\ \eta &= 1-P_J\rho. \end{aligned} \tag{6}$$

Note that in equilibrium, $P_J\rho < 1$, and as a result, $0 < \eta < 1$. The stationary distribution for $i > 0$ is

$$\pi_i = \begin{cases} \xi^i \pi_0 & i \le n_e - 1, \\ \xi^{n_e}(1-\eta)^{i-n_e}\pi_0 & i \ge n_e, \end{cases}$$

and for $i = 0$, if $\xi \neq 1$,

$$\pi_0 = \left[\sum_{i=0}^{n_e-1} \xi^i + \xi^{n_e} \sum_{i=n_e}^{\infty} (1-\eta)^{i-n_e}\right]^{-1} = \left[\frac{1-\xi^{n_e}}{1-\xi} + \frac{\xi^{n_e}}{\eta}\right]^{-1},$$

and if $\xi = 1$,

$$\pi_0 = \frac{\eta}{n_e\eta + 1}.$$

The expected utility from balking without inspecting is

$$U_B = 0.$$

The expected utility from inspecting the queue when $\xi \neq 1$ is

$$U_I = \sum_{i=0}^{n_e-1} \pi_i \left(R - C_W \frac{i+1}{\mu}\right) - C_I = \pi_0 \left[R \cdot \frac{1-\xi^{n_e}}{1-\xi} - \frac{C_W}{\mu} \cdot \frac{1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1}}{(1-\xi)^2}\right] - C_I, \tag{7}$$

and for $\xi = 1$, it is

$$U_I = \frac{n_e\eta[2\mu R - C_W(n_e+1)]}{2\mu(n_e\eta+1)}. \tag{8}$$

The expected utility from joining the queue without inspecting it if $\xi \neq 1$ is

$$U_J = \sum_{i=0}^{\infty} \pi_i \left(R - C_W \frac{i+1}{\mu}\right) = R - \frac{C_W}{\mu}\pi_0 \left[\frac{1-(n_e+1)\xi^{n_e} + n_e\xi^{n_e+1}}{(1-\xi)^2} + \frac{\xi^{n_e}(n_e\eta+1)}{\eta^2}\right], \tag{9}$$

and if $\xi = 1$, it is

$$U_J = R - \frac{C_W}{\mu}\left[\frac{n_e(n_e+1)\eta}{2(n_e\eta+1)} + \frac{1}{\eta}\right]. \tag{10}$$

We assume customers are strategic in the sense that they maximize their own expected utility and therefore choose their best-response strategies. Because customers in this model are homogeneous, we are interested in a symmetric equilibrium. A strategy profile $(P_I, P_B, P_J)$ is a symmetric equilibrium profile if it is a best response against itself. The best-response strategies $(P_I, P_B, P_J)$ satisfy $P_I \geq 0$, $P_B \geq 0$, $P_J \geq 0$, $P_I + P_J + P_B = 1$, and in addition,

$$\begin{cases} U_I > \max\{U_J, 0\} \Rightarrow P_I = 1 \\ 0 > \max\{U_I, U_J\} \Rightarrow P_B = 1 \\ U_J > \max\{U_I, 0\} \Rightarrow P_J = 1 \\ U_J = 0 > U_I \Rightarrow P_I = 0 \\ U_J = U_I > 0 \Rightarrow P_B = 0 \\ U_I = 0 > U_J \Rightarrow P_J = 0 \\ U_I = U_J = 0 \Rightarrow 0 \leq P_I, P_J, P_B \leq 1. \end{cases} \tag{11}$$

## 3 Existence and Uniqueness of the Equilibrium Strategy

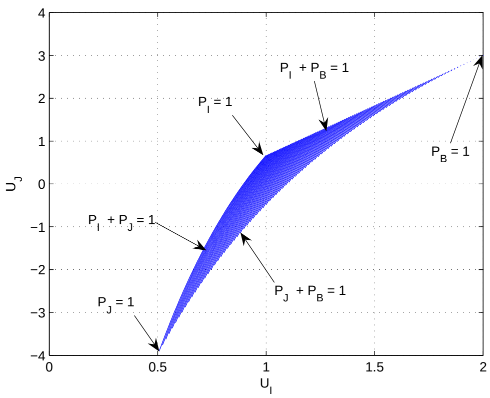
As discussed in the introduction, when customers choose among three actions, the solution is two dimensional. Therefore, the standard one-dimensional ATC property does not apply here and we are required to prove the existence and uniqueness of the equilibrium by using two-dimensional topological analysis. Our main result in this section is summarized in Theorem 1.

**Theorem 1.** *For each set of normalized parameters $\rho, \nu, \kappa$, a unique symmetric Nash equilibrium strategy exists.*

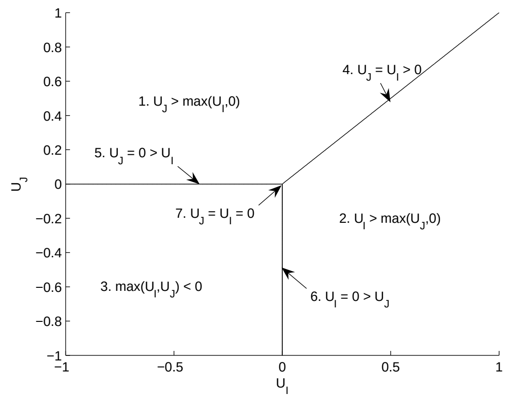
The full proof of Theorem 1 is given in Appendix A, and we only provide the outline of the proof here. The proof is based on the topological and geometrical properties of the **expected utility set (EUS)**, which represents the expected utilities from every possible symmetric strategy. Given the set of normalized parameters, $\rho, \nu, \kappa$, we define the EUS as follows:

$$\text{EUS} = \{(x, y) | \text{ there exists } (P_I, P_B) \text{ such that } U_I = x, U_J = y\}. \tag{12}$$

An example of the EUS for $\rho = 0.5, \nu = 1.43, \kappa = 0.14$ is shown in Figure 2(a); the boundaries and vertices of the EUS are marked with the corresponding strategy type.




(a) Expected utility set (EUS) for $\rho = 0.5$, $\nu = 1.43$, and $\kappa = 0.14$.




(b) Expected utility space

Figure 2: The EUS is defined on the expected utility space

The EUS is defined on the expected utility space. This space is divided into seven regions shown in Figure 2(b):

1. $U_J > \max\{U_I, 0\}$ (region 1).

2. $U_I > \max\{U_J, 0\}$ (region 2).

3. $\max\{U_I, U_J\} < 0$ (region 3).

4. $U_I = U_J > 0$ (region 4).

5. $U_I = 0 > U_J$ (region 5).

6. $U_J = 0 > U_I$ (region 6).

7. $U_I = U_J = 0$ (region 7).

Recall that a symmetric equilibrium strategy satisfies the equilibrium conditions (11). Therefore, the equilibrium is obtained at an intersection point between the EUS and the region that represents the corresponding relation among the utilities. For example, if the EUS vertex that represents $P_I = 1$ is in region 2, where $U_I > \max\{U_J, 0\}$, then $P_I = 1$ is a best response against itself and therefore defines a symmetric equilibrium strategy. The following is another example. If the left boundary of the EUS, representing all the strategies that satisfy $\{P_I > 0, P_J > 0, P_B = 0\}$, intersects with region (line) 4, where $U_I = U_J > 0$, the intersection point defines a symmetric equilibrium.

In only one scenario does no such intersection exist, namely, when the origin is included in the EUS. Any inner point of the EUS represents a mixed strategy where $0 < P_I, P_J, P_B < 1$. At the origin, $U_I = U_J = 0$, and therefore if the origin appears as an inner point of the EUS it defines a symmetric equilibrium strategy.

To show existence, we need to show such an intersection is always defined. To do so, we prove two topological properties of the EUS, summarized in Lemma 1.

**Lemma 1.** *The EUS is a non-empty compact set and its interior is a simply connected domain.*

The non-emptiness property of the EUS ensures that given the set of parameters, the EUS appears on the expected utility space. The compactness of the EUS ensures its boundaries are well defined, and as a result, an equilibrium can be defined on the boundaries. By proving the EUS is simply connected, we ensure that if the origin is included in the EUS, which implies the equilibrium is defined as an inner point, this point is also included in the EUS.

To prove these properties, we rely on the definition of the EUS as the image of the mapping $f\colon (P_I, P_B) \to (U_I, U_J)$. The strategy set is a triangle, defined as follows:

$$T = \{(P_I, P_B) : P_I, P_B \geq 0, P_I + P_B \leq 1\}. \tag{13}$$

This triangle is a non-empty compact set, and its interior is a simply connected domain. To prove these properties are preserved in the EUS, we prove the mapping $f$ is a ***homeomorphism***: a continuous one-to-one map of a topological space onto another topological space such that the inverse mapping is also continuous. By proving the mapping is a homeomorphism, we prove the topological properties of the strategy set are preserved in the EUS. An illustration of the mapping is given in Figure 3.

The proof of homeomorphism includes two steps. First, we prove $f$ is continuous. Second, we

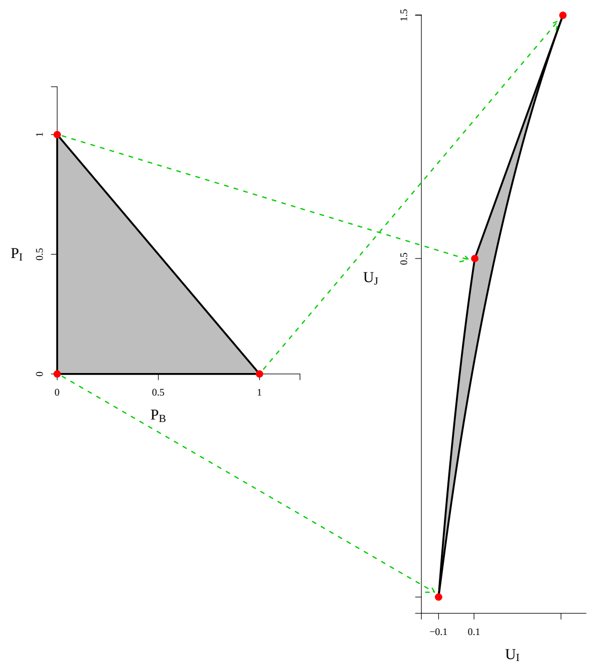



Figure 3: Homeomorphism between the probability set and the EUS

prove it has a continuous inverse by showing the Jacobian of $f$ is non-zero. Then, by the inverse function theorem, the mapping $f$ is bijective and therefore a homeomorphism.

By proving Lemma 1, we prove the existence of a symmetric equilibrium strategy in this game. Moreover, we distinguish between seven different types of equilibrium, which correspond to the seven cases of condition (11). A symmetric equilibrium is defined when

(a) The projection of $P_B = 1$ on the EUS appears in region 3, where $0 > \max\{U_I, U_J\}$.

(b) The projection of $P_J = 1$ on the EUS appears in region 1, where $U_J > \max\{U_I, 0\}$.

(c) The projection of $P_I = 1$ on the EUS appears in region 2, where $U_I > \max\{U_J, 0\}$.

(d) The left boundary of the EUS, which is the projection of $\{P_I > 0, P_J > 0, P_B = 0\}$, intersects with line 4, where $U_I = U_J > 0$.

(e) The lower boundary of the EUS, which is the projection of $\{P_I = 0, P_J > 0, P_B > 0\}$, intersects with line 5, where $U_J = 0 > U_I$.

(f) The upper boundary of the EUS, which is the projection of $\{P_I > 0, P_J = 0, P_B > 0\}$, intersects with line 6, where $U_I = 0 > U_J$.

(g) The origin, which is the projection of $P_I + P_B + P_J = 0$, is included within the EUS, where $U_I = U_J = 0$.

Recall that by assuming $\nu \geq 1$ or equivalently $R - \frac{C_W}{\mu} > 0$, we exclude $P_B = 1$ from our analysis. All six other types of equilibrium are shown in Figure 4. What is left to prove is that no matter where the EUS appears on the expected utility space, at least one intersection that defines an equilibrium occurs. To prove that assertion, we locate the projection of $P_J = 1$ on the EUS in every possible region and go through all possible appearances of the EUS on the expected utility space given the position of that vertex. We prove that for every possibility, an equilibrium must exist. We repeat the process with the projection of $P_I = 1$.

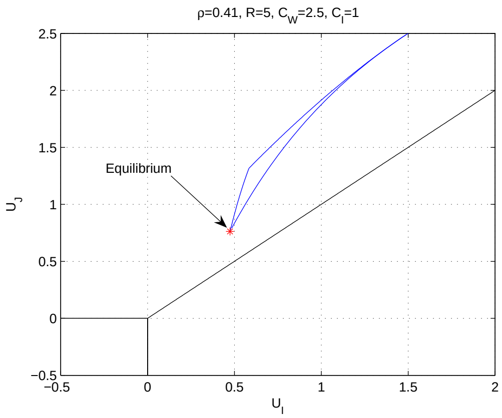
Next, we prove the equilibrium is unique by showing that for every possible EUS no more than one intersection exists with the space regions such that this intersection defines an equilibrium. To do so, we prove Lemma 2.

**Lemma 2.** *: The EUS boundaries are strictly increasing with a slope $> 1$.*

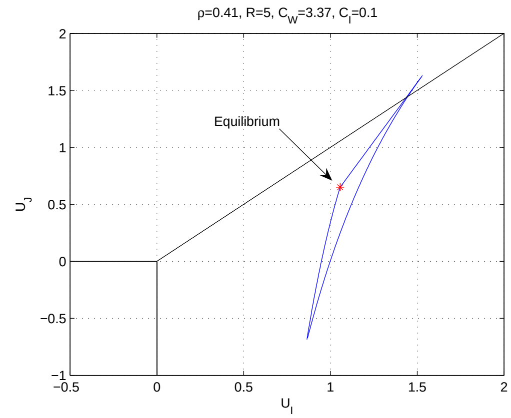
We use the geometrical property of the EUS as described in Lemma 2 to show that for any possible location of the EUS, only one intersection between the EUS and the different regions of the expected utility space defines a symmetric equilibrium. To do so, we fix the vertex that projects the strategy $P_J = 1$ in all possible regions, and check every possible intersection, eliminating those intersections that cannot exist given the boundaries' geometrical properties. We repeat the process for the vertex that represents $P_I = 1$. The detailed proof is given in Appendix A.

# 4 The impact of inspection cost

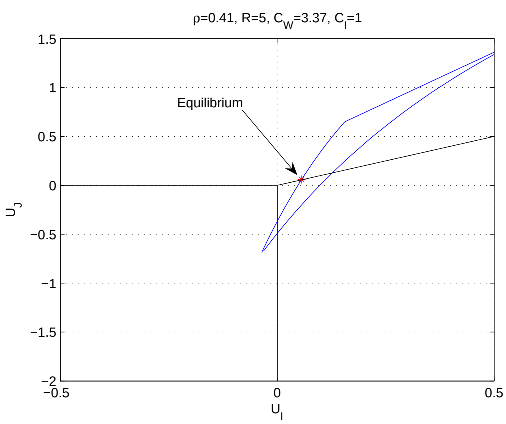
In this section, we analyze the impact of inspection cost on equilibrium strategy, revenue, and social welfare.



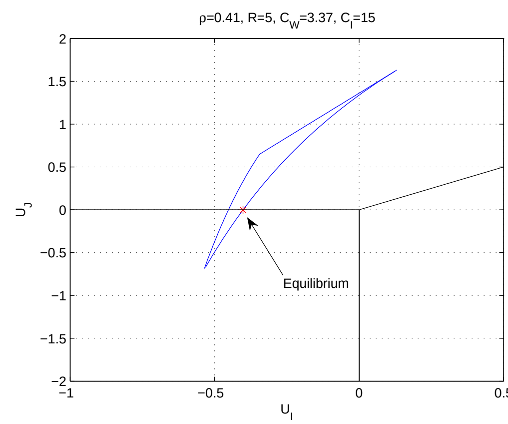

(a) Pure equilibrium at $P_J = 1$



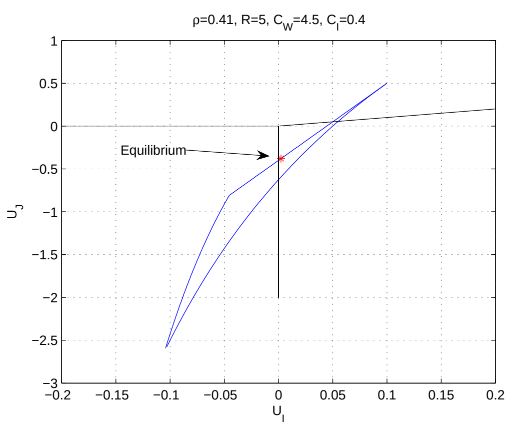

(b) Pure equilibrium at $P_I = 1$



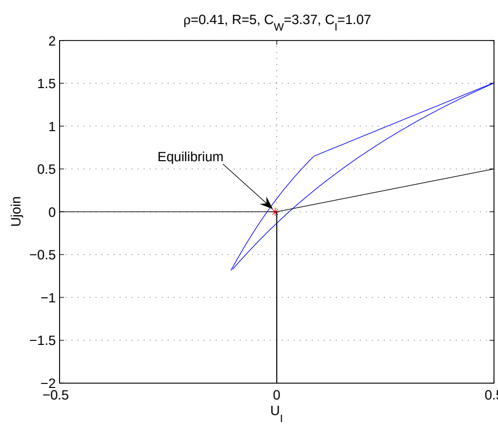

(c) Mixed equilibrium at $\{P_I > 0, P_J > 0, P_B = 0\}$




(d) Equilibrium at $\{P_I = 0, P_J > 0, P_B > 0\}$




(e) Equilibrium at $\{P_I > 0, P_J = 0, P_B > 0\}$




(f) Equilibrium at $\{P_I > 0, P_J > 0, P_B > 0\}$

Figure 4: Defining pure and mixed equilibrium points via the EUS

### 4.1 Equilibrium as a function of inspection cost

The equilibrium strategy is a function of inspection cost. When inspection cost is very low, all customers inspect the queue. In this case, the queue becomes observable.

When inspection cost is very high, customers do not inspect the queue and it remains unobservable. The equilibrium strategy then would be the same as in the unobservable-queue model. Given our assumption that $R - \frac{C_W}{\mu} > 0$, the probability of joining the unobservable queue is strictly positive ($0 < P_J \geq 1$). What would be the equilibrium when inspection cost is at an intermediate level?

We now look into the evolution of the equilibrium strategy as the inspection cost $\kappa$ increases from 0 to infinity. We use the following parameters:

$$
\begin{aligned}
K_1 &= \min\{S_1, S_2\}, \\
&\quad \text{where} \quad S_1 = \frac{(1-\rho)\rho^{n_e}}{1-\rho^{n_e+1}}(n_e + 1 - \nu), \\
&\quad \text{and} \quad S_2 = \nu \frac{1-\rho^{n_e}}{1-\rho^{n_e+1}} - \frac{1-(n_e+1)\rho^{n_e} + n_e\rho^{n_e+1}}{(1-\rho)(1-\rho^{n_e+1})}, \\
K_2 &= \frac{x(1-\rho)\rho^{n_e}h(\rho)}{(1-\rho^{n_e})x + \rho^{n_e}(1-\rho)}, \\
&\quad \text{where } x \text{ solves} \quad x = \frac{n_e - \nu + \sqrt{(n_e-\nu)^2 + 4h(\rho)}}{2h(\rho)} \\
&\quad \text{and} \quad h(\rho) = \frac{1}{\rho^{n_e}}\left[\nu\frac{1-\rho^{n_e}}{1-\rho} - \frac{1-(n_e+1)\rho^{n_e} + n_e\rho^{n_e+1}}{(1-\rho)^2}\right], \\
K_3 &= \frac{\xi^{n_e}(1-\xi)}{1-\xi^{n_e+1}}(n_e + 1 - \nu), \\
&\quad \text{where } \xi \text{ solves} \quad \nu = \frac{1 + \xi + \xi^2 + \ldots + \xi^{n_e} - (n_e+1)\xi^{n_e+1}}{1-\xi^{n_e+1}}, \\
K_4 &= n_e\left(1 - \frac{1}{\nu}\right)^{n_e}, \\
K_5 &= \rho^{n_e}\left(n_e - \nu + \frac{1}{1-\rho}\right). \qquad (14)
\end{aligned}
$$

We can provide the following interpretation for the constants $K_1, K_2, K_3, K_4,$ and $K_5$. We already argued $P_I = 1$ for $\kappa = 0$. We define $K_1$ as the maximum inspection-cost level for which the equilibrium strategy is $P_I = 1$. This level is achieved in one of two scenarios: either $U_I = U_J > 0$

or $U_I = 0 > U_J$ . Substituting into the definitions of $U_I$ and $U_J$ and solving the corresponding equalities, we get $K_1 = S_1$ for $U_I = U_J > 0$, and $K_1 = S_2$ for $U_I = 0 > U_J$. Moreover, for $\nu > \frac{\sum_{i=0}^{n_e} \rho^i - (n_e+1)\rho^{n_e+1}}{1-\rho^{n_e+1}}$, $S_1 > S_2$; otherwise, $S_1 \leq S_2$.

Next we define the minimum cost for which the EUS contains the origin. Denote $K_2$ for the case where $K_1 = S_1$, and $K_3$ for the case where $K_1 = S_2$.

We also define the minimum cost for which the equilibrium strategy implies all customers do not inspect the queue length, that is, $P_I = 0$. Denote $K_4$ for the case where $P_J < 1$, and $K_5$ for the case where $P_J = 1$.

We distinguish among three scenarios:

**Scenario 1** In equilibrium, $\{P_I > 0, P_J > 0, P_B = 0\}$ for $K_1 < \kappa \leq K_5$, and $P_J = 1$ for $\kappa > K_5$.

**Scenario 2** In equilibrium, $\{P_I > 0, P_J > 0, P_B = 0\}$ for $K_1 < \kappa \leq K_2$, $\{P_I > 0, P_J > 0, P_B > 0\}$ for $K_2 < \kappa \leq K_4$, and $\{P_I = 0, P_J > 0, P_B > 0\}$ for $\kappa > K_4$.

**Scenario 3** In equilibrium, $\{P_I > 0, P_J = 0, P_B > 0\}$ for $K_1 < \kappa \leq K_3$, $\{P_I > 0, P_J > 0, P_B > 0\}$ for $K_3 < \kappa \leq K_4$, and $\{P_I = 0, P_J > 0, P_B > 0\}$ for $\kappa > K_4$.

Proposition 3 shows for every set of parameters, if we increase the inspection cost from 0 to $\infty$, only one of those three scenarios exists. Later, we show that when maximizing revenue, each scenario implies a different queue-disclosure policy.

**Proposition 3.** *For $\kappa \leq K_1$, $P_I = 1$ in equilibrium.*

*Suppose $\kappa > K_1$. If $\rho < 1$, then*

1. *if $\nu > \frac{1}{1-\rho}$, Scenario 1 holds.*
2. *if $1 + \frac{\rho}{1+\rho} < \nu \leq \frac{1}{1-\rho}$, Scenario 2 holds.*
3. *if $\nu \leq 1 + \frac{\rho}{1+\rho}$, Scenario 3 holds.*

*Otherwise, if $\rho \geq 1$, then for every $\nu$ and its corresponding $n_e$,*

1. *if $\nu > \frac{\sum_{i=0}^{n_e} \rho^i - (n_e+1)\rho^{n_e+1}}{1-\rho^{n_e+1}}$, Scenario 2 holds.*

*2. if* $\nu \leq \frac{\sum_{i=0}^{n_e} \rho^i - (n_e+1)\rho^{n_e+1}}{1-\rho^{n_e+1}}$*, Scenario 3 holds.*

The proof of Proposition 3 is given in Appendix B. Figure 5 maps every pair $(\rho, \nu)$ to an equilibrium scenario 1, 2, or 3. Note the discontinuity of equilibrium scenario 3 follows from the discontinuity of the observable-queue threshold, $n_e$.

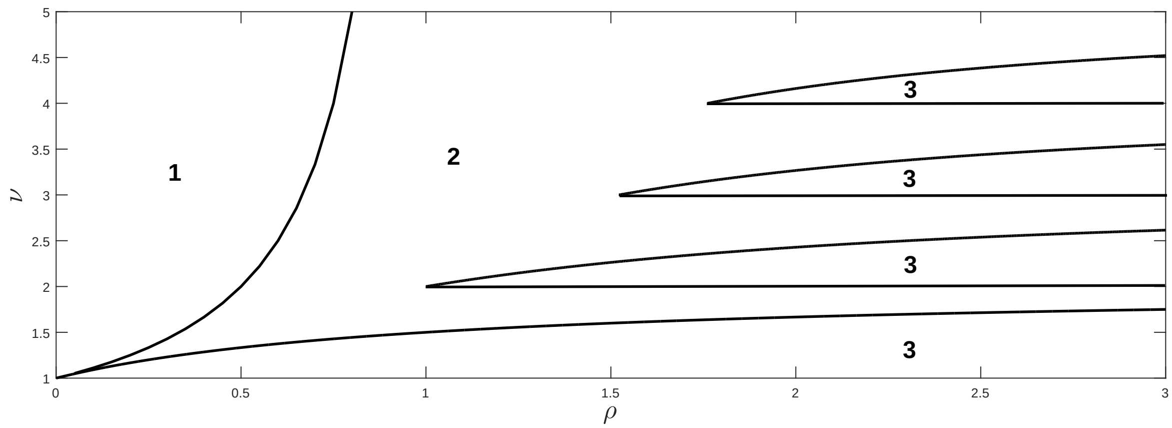



Figure 5: Mapping of equilibrium scenarios

**Corollary 4.** *As* $\kappa$ *increases, the probability* $P_I$ *of the equilibrium strategy decreases while* $P_J$ *and* $P_B$ *increase.*

Corollary 4 follows immediately from Proposition 3. Figure 6 is an example of how inspection cost transforms the queue from observable to unobservable, according to Scenario 2. The fixed parameters are $\rho = 0.8, \nu = 2.33$. The top graph shows $P_I$; the middle graph shows $P_B$; the bottom graph shows $P_J$.

## 4.2 Revenue as a function of inspection cost

This section focuses on the optimization of inspection cost when the service provider can monitor this cost. We define expected revenue as a monotone increasing function of $P_E$, the expected throughput in the system, that is, the probability that an arriving customer is served. Hence, the

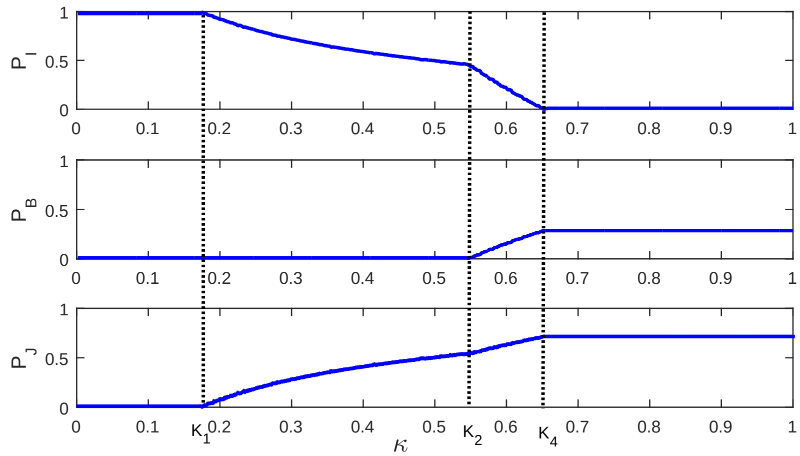



Figure 6: For $\rho = 0.8$ and $\nu = 2.33$: As $\kappa$ increases, the observable queue becomes unobservable

provider's problem is to select the value of $\kappa$ that maximizes $P_E$, where

$$\begin{aligned} P_E &= P(\text{Enter}|\text{Inspect}) \cdot P(\text{Inspect}) + P(\text{Join}) \\ &= \sum_{i=0}^{n_e-1} \pi_i \cdot P_I + P_J = \frac{1-\xi^{n_e}}{1-\xi} \cdot \pi_0 \cdot P_I + P_J. \end{aligned} \tag{15}$$

Denote by $P_E^{\text{obs}}$ (and $P_E^{\text{unobs}}$) the probability of entering the queue when $\kappa$ is small (large) such that all customers (do not) inspect the queue. Then

$$\begin{aligned} P_E^{\text{obs}} &= \frac{1-\rho^{n_e}}{1-\rho^{n_e+1}} < 1, \\ P_E^{\text{unobs}} &= \min\left\{\frac{\nu-1}{\nu\rho}, 1\right\}. \end{aligned} \tag{16}$$

Note that when $\rho \geq 1$, $P_E^{\text{unobs}} = \frac{\nu-1}{\nu\rho} < 1$.

To characterize the change in $P_E$ as $\kappa$ increases, we use the following lemma and observation:

**Lemma 5.**

1. *While $U_I = U_J > 0$, $P_E$ increases with $\kappa$.*

2. *While $U_I = 0 > U_J$, $P_E$ decreases with $\kappa$.*

**Observation 1.** *While $U_I = U_J = 0$, $P_E$ decreases with $\kappa$.*

**Proposition 6.** *For any pair $(\rho, \nu)$,*

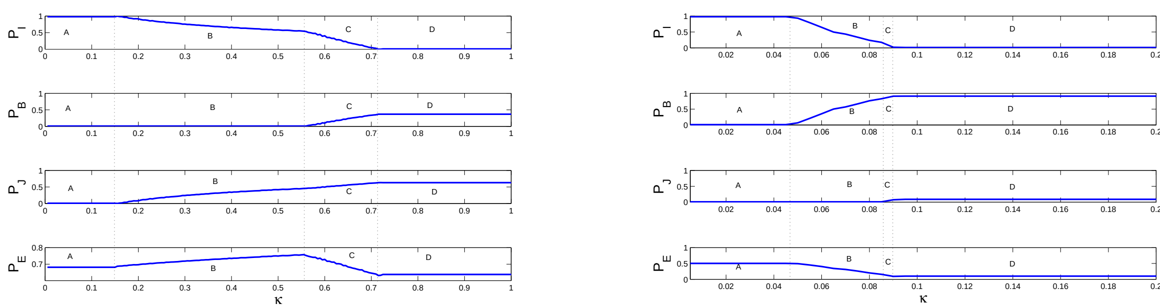
1. *if Scenario 1 holds,* $P_E$ *increases with* $\kappa$ *and reaches the maximum at every* $\kappa \geq K_5$*.*

2. *if Scenario 2 holds,* $P_E$ *is a unimodal function of* $\kappa$ *and reaches the maximum at* $K_2$*.*

3. *if Scenario 3 holds,* $P_E$ *decreases with* $\kappa$ *and reaches the maximum at every* $\kappa \leq K_1$*.*

The proofs of Lemma 5, Proposition 6, and the numerical analysis that implied Observation 1 are given in Appendix B. The following proposition determines the optimal inspection cost at each scenario.

Figure 7 shows two examples of how, in different scenarios, the equilibrium strategy ($P_I, P_B$, and $P_J$ are shown in the three top graphs) and $P_E$ (bottom graph) change as $\kappa$ increases (on the x-axis).



(a) When $\rho = 0.95$ and $\nu = 2.5$, Scenario 2 holds, and $P_E$ is a unimodal function of $\kappa$. The maximum is obtained when $\kappa = 0.5595 = K_2$.

(b) When $\rho = 0.99$ and $\nu = 1.1$, Scenario 3 holds, and $P_E$ is monotonically decreasing with $\kappa$.

Figure 7: $P_E$ as a function of $\kappa$ under different scenarios

## 4.3 Social welfare as a function of inspection cost

We now examine the effect of the inspection cost on social welfare. Recall that inspection cost is not considered a transfer payment to the service provider, but a real cost of the effort customers incur to obtain queue-length information. Therefore, the inspection cost is included in social welfare, which we define as

$$SW = \lambda(P_J U_J + P_I U_I).$$

**Proposition 7.** *For any pair $(\rho, \nu)$, $SW$ is monotonically decreasing with $\kappa$ and obtains its maximum when $\kappa = 0$.*

The proof of Proposition 7 is given in Appendix B.

# 5 Discussion

We discuss our results while referring to various health-care service settings, namely, private ERs, public ERs, walk-in clinics, and private hospital rooms upon request. We represent these settings by (illustrative) level of congestion ($\rho$) and customers' service valuation ($\nu$), as shown in Table 1.

| Service | $\rho$ | $\nu$ | Corresponds to scenario no. |
|---|---|---|---|
| Private ER | 0.3 (low) | 5 (high) | 1 |
| Public ER | 0.9 (high) | 5 (high) | 2 |
| Walk-in clinic | 0.6 (moderate) | 3 (moderate) | 2 |
| Private hospital rooms upon request | 0.5 (moderate) | 1.3 (low) | 3 |

Table 1: Three different service settings corresponding to three possible scenarios

Our analysis elicits managerial insights for each service setting. For private ERs, where congestion is low, the equilibrium follows Scenario 1: customers join the unobservable queue with probability 1. In this case, keeping the queue unobservable is in the best interests of the private health-care provider. Revealing information might discourage potential customers from joining.

For public ERs where congestion is high, the case is different and equilibrium follows Scenario 2. Our results as described in Proposition 6 suggest that for revenue maximization, the provider should control the cost of inspection, thereby raising throughput and revenue. But a public health-care provider is more likely to prefer to maximize social welfare over revenue and therefore make the queue observable, which would reduce congestion as compared to the revenue-maximization solution. This policy also applies for many public services characterized by high congestion, such as Department of Motor Vehicles (DMV) and Social Security administration offices.

However, for walk-in clinics, Proposition 6 implies that providers should make queue length observable to some extent. Because walk-in clinics are managed by private health-care providers, one can assume revenue management is in their best interests. The inspection cost can be, for example, a requirement for registration at the provider's web site, or it can be an exposure to advertisements for other commercial goods alongside the waiting-time publication. This solution applies to many other private services such as banks, popular restaurants, and department stores.

Last, we consider the example of requesting a private room at admission to a hospital. The valuation of the service is much lower when compared to the other cases, such that it follows Scenario 3, where an observable queue maximizes both revenue and social welfare.

# 6 Heterogeneous service valuations

We consider two classes of customers. Class 1 customers value the service so highly that their dominant strategy is to join the queue without inspection. Following the motivating example of ERs, we refer to them as *urgent customers*. Class 2 customers have a smaller service valuation. We refer to them as *non-urgent* customers. For simplicity, we assume all customers have the same waiting cost.

Let $q \in (0,1)$ be the proportion of urgent customers in the population. Then

$$\lambda_i = \begin{cases} [q + (1-q)(1-P_B)]\lambda & i < n_e, \\ [q + (1-q)P_J]\lambda & i \geq n_e. \end{cases} \tag{17}$$

Denote:

$$\begin{aligned} \bar{\xi} &= [q + (1-q)(1-P_B)]\rho, \\ \bar{\eta} &= 1 - [q + (1-q)P_J]\rho. \end{aligned} \tag{18}$$

The analysis of sections 2 and 3 of this paper holds with $\bar{\xi}$ instead of $\xi$ and $\bar{\eta}$ instead of $\eta$, including Theorem 1. In the same way, the analysis of section 4 regarding the impact of inspection cost also holds when substituting $\bar{\xi}$ and $\bar{\eta}$. Let $\bar{K}_1, \bar{K}_2, \bar{K}_3, \bar{K}_4$, and $\bar{K}_5$ be the corresponding thresholds to $K_1, K_2, K_3, K_4$, and $K_5$. We use the corresponding thresholds for defining Scenarios 1-3 for the heterogeneous case, in corresponding to Proposition 3.

**Proposition 8.** *For $\kappa \leq \bar{K}_1$, the equilibrium strategy is $P_I = 1$.*

*Suppose $\kappa > \bar{K}_1$. If $\rho > 1$, then, using $\bar{K}_2, \bar{K}_3, \bar{K}_4$, and $\bar{K}_5$, respectively,*

1. *if $\nu > \frac{1}{1-\rho}$, Scenario 1 holds;*

2. *if $1 + \frac{1}{1-q\rho} - \frac{1}{1+(1-q)r} < \nu \leq \frac{1}{1-\rho}$, Scenario 2 holds;*

3. *if $\nu \leq 1 + \frac{1}{1-q\rho} - \frac{1}{1+(1-q)r}$, Scenario 3 holds.*

*Otherwise, if $\rho \geq 1$, then for every $\nu$ and its corresponding $n_e$,*

1. *if $\nu > \left(\frac{1-\rho^{n_e}}{1-\rho} + \frac{\rho^{n_e}}{1-q\rho}\right)^{-1} \left[\frac{1-(n_e+1)\rho^{n_e}+n_e\rho^{n_e+1}}{(1-\rho)^2} + \frac{\rho^{n_e}(n_e(1-q\rho)+1)}{(1-q\rho)^2}\right]$, Scenario 2 holds;*

2. *if $\nu \leq \left(\frac{1-\rho^{n_e}}{1-\rho} + \frac{\rho^{n_e}}{1-q\rho}\right)^{-1} \left[\frac{1-(n_e+1)\rho^{n_e}+n_e\rho^{n_e+1}}{(1-\rho)^2} + \frac{\rho^{n_e}(n_e(1-q\rho)+1)}{(1-q\rho)^2}\right]$, Scenario 3 holds.*

Following Proposition 8, the region of Scenario 1 on the $\rho - \nu$ space does not change with $q$. However, as $q$ decreases, the region of Scenario 2 grows larger at the expense of the region of Scenario 3. Corollary 9 summarizes this result.

**Corollary 9.** *The region in the $\rho - \nu$ space for which inspection at a positive cost maximizes revenue increases with the fraction of non-urgent customers.*

# 7 Concluding remarks

Our contribution to the literature lies in two aspects. From the analytical point of view, we provided a proof of the existence and uniqueness of the equilibrium using the topological properties of the expected utility set.

From the operations point of view, our main result is that enabling inspection of queue length at a cost may increase throughput. This result may appear counter-intuitive because more customers join the queue when the information has a cost as opposed to when it is free.

By adding this third action to the join-or-balk classical discussion, we expand the provider's ability to control the customers' behavior, allowing the provider to maximize revenue by increasing throughput. We provide a concrete characterization of the model parameters for which adding inspection at a cost leads to the desirable result.

In relation to social welfare, we show the social planner should aspire to decrease the inspection cost to the minimum because maximum welfare is achieved when the queue is observable. We argue the provider can influence the inspection cost in several ways. Future research could analyze the extent to which a provider should invest in reducing the inspection cost. Assuming such an investment is monotonically decreasing with $C_I$, and given that social welfare is also decreasing with $C_I$, a socially optimal cost investment exists for the provider to reduce inspection cost, and the corresponding inspection cost is greater than zero.

We focus on analyzing the classical M/M/1 queue model. The results remain very similar under the general arrival assumption. Yechiali (1971) showed that in the case of a G/M/1 observable-queue model, an equilibrium threshold strategy exists. Because the calculation of the threshold in the G/M/1 model is different than in the M/M/1 model, it would require changes in the equations, and the numerical results would be different. Yet we expect the nature of the results to stay the same as in the M/M/1 model.

The general service-rate case, however, is different. Kerner (2011) showed that customer strategy in the M/G/1 model is not always characterized by a threshold. In addition, the equilibrium in the M/G/1 model is not always unique. Therefore, whether our results would hold for that model is unclear, and pursuing such a different model is left for future research.

We focused on analyzing a single-queue model as a primary and necessary step in understanding the influence of inspection costs on the unobservable-queue model. However, expanding this model into the case of queueing systems with many servers is a natural extension. The authors conducted such research in another paper as we described in the introduction (see Hassin and Roet-Green, 2013).

Most of the paper dealt with homogenous customers. In section 6, we discussed the case of heterogenous customers when we considered urgent versus non-urgent customers. The results imply that for more parameter sets, as the proportion of urgent customers in the population increases, the revenue is maximized when the queue is observable.

Considering similar models with customer heterogeneity with respect to delay cost or inspection cost could be interesting. These threads are open for future research.

# Appendices

## A Proof of Theorem 1

To prove Theorem 1, we prove Lemma 1 by proving the mapping $f : (P_I, P_B) \to (U_I, U_J)$ is a homeomorphism:

**Definition 1.** *Homeomorphism is a continuous one-to-one map of a topological space $X$ onto a topological space $Y$ such that $f^{-1}$ is also continuous (Kelley 1955, p.87).*

From equations (7) and (9), it follows that $U_I$ and $U_J$ are continuous for any fixed set of parameters, and therefore the mapping $(P_I, P_B) \to (U_I, U_J)$ is continuous.

We use the following lemma (from Ma 2002, p. 39):

**Lemma 10.** *Let $f : X \to Y$ be a continuous bijection. If $X$ is compact, $f$ is a homeomorphism.*

To prove $f$ is bijective, we first show $(P_B, P_I) \to (\xi, \eta)$ is bijective. Then we show $(\xi, \eta) \to (U_I, U_J)$ is bijective. Therefore, $(P_B, P_I) \to (U_I, U_J)$ is bijective.

Because $\xi = (1 - P_B)\rho$ and $\eta = 1 - (1 - P_B - P_I)\rho$, the Jacobian of the transformation $(P_B, P_I) \to (\xi, \eta)$ is

$$J(\xi, \eta) = \begin{vmatrix} \xi'_{P_B} & \xi'_{P_I} \\ \eta'_{P_B} & \eta'_{P_I} \end{vmatrix} = \xi'_{P_B} \cdot \eta'_{P_I} - \eta'_{P_B} \cdot \xi'_{P_I} = -\rho^2, \tag{19}$$

and because $\rho > 0$, the Jacobian is non-zero, and by the inverse function theorem, the mapping is bijective. We now use the same argument to prove $(\xi, \eta) \to (U_I, U_J)$ is bijective. We use the presentation of $U_I, U_J$ in (7) and (9), divide each equation by $\frac{C_W}{\mu}$, and use the definition: $\nu = \frac{R\mu}{C_W}$:

$$U_I = R \cdot \left[ \frac{(1 - \xi^{n_e})\eta}{(1 - \xi^{n_e})\eta + \xi^{n_e}(1 - \xi)} - \frac{1}{\nu} \cdot \frac{(1 - (n_e + 1)\xi^{n_e} + n_e \xi^{n_e+1})\eta}{(1 - \xi)((1 - \xi^{n_e})\eta + \xi^{n_e}(1 - \xi))} \right] - C_I,$$

$$U_J = R \cdot \left[ 1 - \frac{1}{\nu} \left( \frac{(1 - (n_e + 1)\xi^{n_e} + n_e \xi^{n_e+1})\eta}{(1 - \xi)((1 - \xi^{n_e})\eta + \xi^{n_e}(1 - \xi))} + \frac{\xi^{n_e}(n_e \eta + 1)(1 - \xi)}{\eta((1 - \xi^{n_e})\eta + \xi^{n_e}(1 - \xi))} \right) \right].$$

We look at the Jacobian of this transformation:

$$
\begin{aligned}
J(U_I, U_J) &= \begin{vmatrix} {U_I}'_\xi & {U_I}'_\eta \\ {U_J}'_\xi & {U_J}'_\eta \end{vmatrix} = {U_I}'_\xi \cdot {U_J}'_\eta - {U_I}'_\eta \cdot {U_J}'_\xi \\
&= \frac{R^2 \xi^{n_e - 1}}{\eta\nu^2(1-\xi)(\eta - \xi^{n_e}(\xi - 1 + \eta))^3} \cdot \Big( {n_e}^3 (1-\xi)^2 \xi^{n_e} \eta^2 - {n_e}^2 (1-\xi)^2 \xi^{n_e} \eta(\eta\nu - 2) \\
&+ n_e \left( -\xi\eta^2 + \xi^{2n_e+1}[(1-\xi)^2 - \eta^2] + \xi^{n_e}[(1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3\nu] \right) \\
&+ \xi(1-\xi^{n_e})[\xi^{n_e}(\xi - 1 + \eta)[2 + (\xi - 1 - \eta)\nu] + \eta(\eta\nu - 2)] \Big)
\end{aligned}
$$

**Lemma 11.** *For all $R \geq 0$, $\nu > 1$, $\xi > 0$, and $0 < \eta < 1$, $J(U_I, U_J) < 0$.*

*Proof.* For simplicity, we use $n$ in the proof instead of $n_e$. We start by proving the denominator of the Jacobian is always positive. The denominator is a product of $\eta\nu^2 > 0$ and the terms $(1-\xi)$ and $(\eta - \xi^n(\xi - 1 + \eta))^3 = (\eta(1-\xi^n) + (1-\xi)\xi^n)^3$. For $\xi < 1$, the last two terms are positive, and for $\xi > 1$, both terms are negative. Therefore, the product is positive and the denominator is positive. For $\xi = 1$, the denominator is zero, but calculating the limit of the Jacobian is possible, as we show later.

We now wish to determine the sign of the numerator. Denote the numerator as $M$:

$$
\begin{aligned}
M &= R^2 \xi^{n-1} \cdot \Big( n^3(1-\xi)^2 \xi^n \eta^2 - n^2(1-\xi)^2 \xi^n \eta(\eta\nu - 2) \\
&+ n[-\xi\eta^2 + \xi^{2n+1}[(1-\xi)^2 - \eta^2] + \xi^n((1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3\nu)] \\
&+ \xi(1-\xi^n)[\xi^n(\xi - 1 + \eta)[2 + (\xi - 1 - \eta)\nu] + \eta(\eta\nu - 2)] \Big).
\end{aligned}
\tag{20}
$$

Because $R^2\xi^{n-1} > 0$, it is enough to find the sign of the rest of the numerator. For fixed $n$, note $M$ is a linear function of $\nu$ where $n \leq \nu < n+1$. Therefore, it is enough to prove $M$ has the same sign when $\nu = n$ and when $\nu \to n+1$. First we consider the case where $\nu = n$:

$$
\begin{aligned}
\widetilde{M} &= n^3(1-\xi)^2 \xi^n \eta^2 - n^2(1-\xi)^2 \xi^n \eta(\eta n - 2) \\
&+ n[-\xi\eta^2 + \xi^{2n+1}[(1-\xi)^2 - \eta^2] + \xi^n((1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3 n)] \\
&+ \xi(1-\xi^n)[\xi^n(\xi - 1 + \eta)(2 + (\xi - 1 - \eta)n) + \eta(\eta n - 2)] \\
&= 2n^2(1-\xi)^2\xi^n\eta - 2\xi(1-\xi^n)^2\eta + n\xi^n(1-\xi)^2(1 - n(1-\xi) + \xi) - 2\xi^{n+1}(1-\xi^n)(1-\xi).
\end{aligned}
$$

For $n = 1$, $\widetilde{M} = 0$. We now show $\widetilde{M} < 0$ for all $n \geq 2$. We first prove

$$
2n^2(1-\xi)^2\xi^n\eta - 2\xi(1-\xi^n)^2\eta = 2\xi(1-\xi)^2\eta \cdot [n^2\xi^{n-1} - (1 + \xi + \xi^2 + \ldots + \xi^{n-1})^2] < 0. \tag{21}
$$

Because $2\xi(1-\xi)^2\eta > 0$, it is enough to show $n^2\xi^{n-1} < (1+\xi+\xi^2+\cdots+\xi^{n-1})^2$. Both terms are positive and therefore we can look at the square root: $n\xi^{\frac{n-1}{2}} < 1+\xi+\xi^2+\cdots+\xi^{n-1}$. For all $\xi$ and $n > 1$, the function: $f(n) = \xi^{n-1}$ is positive, monotonic decreasing, and convex. If $n$ is even, then from convexity a system of $\frac{n}{2}$ inequalities exists:

$$\begin{aligned}
\frac{1}{2}(1+\xi^{n-1}) &> \xi^{\frac{n-1}{2}} \\
\frac{1}{2}(\xi+\xi^{n-2}) &> \xi^{\frac{n-1}{2}} \\
&\vdots \\
\frac{1}{2}(\xi^{n/2-1}+\xi^{n/2}) &> \xi^{\frac{n-1}{2}}
\end{aligned}$$

If $n$ is odd, a system of $\frac{n-1}{2}$ inequalities and one equality exists:

$$\begin{aligned}
\frac{1}{2}(1+\xi^{n-1}) &> \xi^{\frac{n-1}{2}} \\
\frac{1}{2}(\xi+\xi^{n-2}) &> \xi^{\frac{n-1}{2}} \\
&\vdots \\
\frac{1}{2}(\xi^{\frac{n-3}{2}}+\xi^{\frac{n+1}{2}}) &> \xi^{\frac{n-1}{2}} \\
\xi^{\frac{n-1}{2}} &= \xi^{\frac{n-1}{2}}
\end{aligned}$$

In both cases, when summing all the inequalities in each system, we get

$$n\xi^{\frac{n-1}{2}} < 1+\xi+\xi^2+\cdots+\xi^{n-1}.$$

Next, we prove the remaining elements are negative as well:

$$\begin{aligned}
& n\xi^n(1-\xi)^2(1-n(1-\xi)+\xi) - 2\xi^{n+1}(1-\xi^n)(1-\xi) \\
= \;& \xi^n(1-\xi)^2[n(1-n(1-\xi)+\xi) - 2\xi(1+\xi+\xi^2+\cdots+\xi^{n-1})].
\end{aligned}$$

Because $\xi^n(1-\xi)^2 > 0$, it is enough to show $n[1-n(1-\xi)+\xi] - 2\xi(1+\xi+\xi^2+\cdots+\xi^{n-1}) < 0$. We prove this by induction. For $n = 2$, we get

$$2[1-2(1-\xi)+\xi] - 2\xi(1+\xi) = 2(-1+2\xi-\xi^2) = -2(1-\xi)^2 < 0.$$

Next, we assume the statement is true for $n$ and use it to prove it is also true for $n+1$:

$$\begin{aligned}
& (n+1)(1-(n+1)(1-\xi)+\xi) - 2\xi(1+\xi+\xi^2+\cdots+\xi^n) \\
= \; & n[1-n(1-\xi)+\xi] - 2\xi(1+\xi+\xi^2+\cdots+\xi^{n-1}) - n(1-\xi) + 1 - (\xi+1)(1-\xi) + \xi - 2\xi^{n+1} \\
< \; & -n(1-\xi) + 1 - (n+1)(1-\xi) + \xi - 2\xi^{n+1} = -2n(1-\xi) + 2\xi(1-\xi^n) \\
= \; & -2(1-\xi)[n - \xi(1+\xi+\xi^2+\cdots+\xi^{n-1})] < -2(1-\xi)(n-\xi n) = -2n(1-\xi)^2 < 0,
\end{aligned}$$

which completes the proof of equation (21).

We now show $M$ (Equation (20)) is strictly negative when $\nu \to n+1$:

$$\begin{aligned}
\lim_{\nu \to n+1} M \; = \; & R^2\xi^{n-1} \cdot \Big( n^3(1-\xi)^2\xi^n\eta^2 - n^2(1-\xi)^2\xi^n\eta(\eta(n+1)-2) \\
& + \; n\left[-\xi\eta^2 + \xi^{2n+1}\left((1-\xi)^2-\eta^2\right) + \xi^n\left((1-\xi)^2 + 2\xi\eta^2 - (1-\xi)^3(n+1)\right)\right] \\
& + \; \xi(1-\xi^n)\left(\xi^n(\xi-1+\eta)\left[2+(\xi-1-\eta)(n+1)\right] + \eta(\eta(n+1)-2)\right)\Big).
\end{aligned}$$

Following the same process, we reduce the problem into determining the sign of $\widehat{M}$, where

$$\begin{aligned}
\widehat{M} \; &= \; \widetilde{M} - n^2(1-\xi)^2\xi^n\eta^2 - n\xi^n(1-\xi)^3 + \xi^{n+1}(1-\xi^n)[(1-\xi)^2-\eta^2] + \xi(1-\xi^n)\eta^2 \quad (22) \\
&= \; \widetilde{M} - \eta^2[n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2] - n\xi^n(1-\xi)^3 + \xi^{n+1}(1-\xi^n)(1-\xi)^2.
\end{aligned}$$

We already proved $n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2 < 0$. By adding it to the first two terms of $\widetilde{M}$ (Equation (21)), we get

$$(2\eta-\eta^2)[n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2] = \eta(2-\eta)[n^2(1-\xi)^2\xi^n - \xi(1-\xi^n)^2] < 0.$$

Lastly, we show $-n\xi^n(1-\xi)^3 + \xi^{n+1}(1-\xi^n)(1-\xi)^2 < 0$:

$$\begin{aligned}
& -n\xi^n(1-\xi)^3 + \xi^{n+1}(1-\xi^n)(1-\xi)^2 \\
= \; & \xi^n(1-\xi)^3[-n + \xi(1+\xi+\xi^2+\cdots+\xi^{n-1})] < \xi^n(1-\xi)^3[-n+n\xi] \\
= \; & -n\xi^n(1-\xi)^4 < 0.
\end{aligned}$$

To complete the proof, we need to show $M < 0$ for all $1 < \nu < 2$. Note that when $\nu = 1$, $\widetilde{M} = 0$, but when $\nu = 1$, $R = \frac{C_W}{\mu}$, and the expected utility from inspecting the queue becomes negative, regardless of the strategies of other customers. Therefore, inspecting the queue becomes dominated by the other two actions and can be omitted. As a result, the game becomes a two-action game that has a unique solution due to ATC. Furthermore, because $\widehat{M} = -\xi(1-\xi)^4 < 0$ for $\nu \to 2$, we can deduce that $M < 0$ for all $1 < \nu < 2$.

When $\xi = 1$,

$$\lim_{\xi \to 1} J(U_I, U_J) = -\frac{n(n+1)(-4+6\nu-2n+n(n-1)t(2+nt-t\nu))}{12t(1+nt)^3\nu^2}.$$

The denominator is a product of strictly positive terms and therefore is strictly positive. In the numerator, $n(n+1) > 0$. We want to show the other term in the product is strictly positive. Because $\nu > n \geq 1$, $-4+4\nu > 0$ and $2\nu - 2n > 0$. $2+nt-t\nu = 2-t(\nu-n) > 1$, because both $t < 1$ and $\nu - n < 1$. By combining all these terms, we deduce that the numerator is strictly positive, as is the entire fraction. From the minus sign, we conclude the limit exists and is strictly negative. □

By proving $\widetilde{M}$ and $\widehat{M}$ are strictly negative for all $\nu > 1$ , we conclude the Jacobian of the mapping is non-zero and therefore the continuous mapping is a homeomorphism. Homeomorphism preserves topological properties (Reid and Szendrői 2005, pp.113-118), and thus the topological properties of the triangle $T$ - non-empty simply connected compact set - are preserved in the EUS.

**Proof of existence**. The following analysis is based on the EUS topological properties. We show an equilibrium exists for all possible scenarios. If the point where $P_J = 1$ is in region 1, $P_J = 1$ is a pure equilibrium. Otherwise, $P_J = 1$ has to be in region 2 or 3. First, assume $P_J = 1$ is in region 2. If $P_I = 1$ is also in region 2, $P_I = 1$ is a pure equilibrium. Otherwise, based on Lemma 2, $P_I = 1$ must be in region 1. In that case, the boundary $\{P_I > 0, P_J > 0, P_B = 0\}$ must intersect with line (region) 4, and $\{P_I > 0, P_J > 0, P_B = 0\}$ is a mixed equilibrium.

Assume the point where $P_J = 1$ is in region 3. If $P_I = 1$ is in region 2, $P_I = 1$ is a pure equilibrium. Otherwise, $P_I = 1$ must be in region 1 or 3. If $P_I = 1$ is in region 1, one of the following three cases must occur:

(1) $\{P_I > 0, P_J > 0, P_B = 0\}$ intersects with line 4, and then $\{P_I > 0, P_J > 0, P_B = 0\}$ is a mixed equilibrium (e.g., see Figure 4(c)).

(2) $\{P_I = 0, P_J > 0, P_B > 0\}$ intersects with line 5, and then $\{P_I = 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium (e.g., see Figure 4(d)).

(3) The origin is contained in the EUS, and then $\{P_I > 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium (e.g., see Figure 4(f)).

Otherwise, $P_I = 1$ must be in region 3 and then one of the following three cases must occur:

(1) $\{P_I > 0, P_J = 0, P_B > 0\}$ intersects with line 6, and then $\{P_I > 0, P_J = 0, P_B > 0\}$ is a mixed equilibrium (e.g., see Figure 4(e)).

(2) $\{P_I = 0, P_J > 0, P_B > 0\}$ intersects with line 5, and then $\{P_I = 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium.

(3) The origin is contained in the EUS, and then $\{P_I > 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium.

**Proof of Lemma 2**. From (7) and (9),

$$U_J(U_I) = U_I + C_I - V(U_I), \tag{23}$$

where

$$V(U_I) = \sum_{i=n_e}^{\infty} \pi_i \left( C_W \frac{i+1}{\mu} - R \right) > 0. \tag{24}$$

$V(U_I)$ is the *value of the information*, that is, the expected gain from the balking option when queue length turns out to be $n_e$ or longer. Note that $V(U_I)$ is positive as a sum of positive components.

The outline of the proof is as follows. We show the change in the strategy vector along the boundaries increases the probability of shorter queue lengths. As a result, $U_I$ increases but $V(U_I)$ decreases. Then $V'(U_I) < 0$ and $U_J'(U_I) > 1$. We verify each boundary is increasing with a slope $> 1$:

- When $\{P_I > 0, P_J = 0, P_B > 0\}$, then $P_J = 0$ and $\eta = 1$. Along this boundary, $P_I$ decreases from 1 to 0 while $P_B$ increases from 0 to 1, and therefore $\xi$ decreases. In this case, queue length cannot exceed $n_e$. Therefore, $V(U_I) = \pi_{n_e} \left( C_W \frac{i+1}{\mu} - R \right)$. We show the derivative of $\pi_{n_e} = \frac{\xi^{n_e}}{\sum_{i=0}^{n_e} \xi^i}$ with respect to $\xi$ is positive:

$$\frac{d\pi_{n_e}}{d\xi} = \frac{\xi^{n_e-1}[\sum_{i=0}^{n_e-1}(n_e-1)\xi^i]}{(\sum_{i=0}^{n_e} \xi^i)^2} > 0, \tag{25}$$

and therefore as $\xi$ decreases, $V(U_I)$ decreases while $U_I$ increases, and we conclude this boundary is strictly increasing with a slope $> 1$.

- When $\{P_I > 0, P_J > 0, P_B = 0\}$, then $P_B = 0$ and $\xi$ is fixed and equals $\rho$. Along this boundary, $P_I$ increases from 0 to 1, while $P_J$ decreases from 1 to 0. As a result, $\eta$ increases, as well as $\pi_0$. Because for $0 \leq i \leq n_e - 1$, $\pi_i = \rho^i \pi_0$, the probability of shorter queues increases and, as a result, $U_I$ increases. We prove that in this case, $V(U_I)$ decreases with $\eta$ (see Lemma 12 in Appendix A). As a result, $U_J$ increases even more. Therefore, this boundary is strictly increasing with a slope $> 1$.

- When $\{P_I = 0, P_J > 0, P_B > 0\}$, $P_I = 0$ and $\eta = 1 - \xi$. Along this boundary, $P_B$ increases from 0 to 1, while $P_J$ decreases from 1 to 0. As a result, $\xi$ decreases, $\eta$ increases, $\pi_0 = 1 - \xi$ increases, and so does the probability that the queue length is shorter than $n_e$, which is $\sum_{i=0}^{n_e-1} \xi^i (1-\xi) = 1 - \xi^{n_e}$. As a result, $U_I$ increases. We prove that in this case, $V(U_I)$ decreases along the border (see Lemma 13 in Appendix A). Therefore, we conclude that $U_J$ increases even more and this boundary is strictly increasing with a slope $> 1$.

**Proof of uniqueness.** Using both Lemmas 1 and 2, we show that no more than one of the six scenarios introduced in the existence proof can occur under the assumption $R > \frac{C_W}{\mu}$.

Assume the point where $P_J = 1$ is in region 1; then $P_J = 1$ is a pure equilibrium. From Lemma 2, one can conclude the entire EUS is in region 1, and therefore the equilibrium is unique. See Figure 4(a) for an example. Otherwise, $P_J = 1$ must be in region 2 or 3. First, assume $P_J = 1$ is in region 2. Then $P_I = 1$ can be in region 1 or 2:

(I) If $P_I = 1$ is in region 2, $P_I = 1$ is a pure equilibrium. The EUS boundaries $\{P_I > 0, P_J = 0, P_B > 0\}$ and $\{P_I = 0, P_J > 0, P_B > 0\}$ intersect with line 4, but these intersections do not define another equilibrium (e.g, see Figure 4(a)).

(II) If $P_I = 1$ is in region 1, the boundary $\{P_I > 0, P_J > 0, P_B = 0\}$ must intersect with line 4, and $\{P_I > 0, P_J > 0, P_B = 0\}$ is a mixed equilibrium. The EUS boundary $\{P_I = 0, P_J > 0, P_B > 0\}$ also must intersect with line 4, but this intersection does not define an equilibrium.

Next, assume the point where $P_J = 1$ is in region 3. Then $P_I = 1$ can be in region 1, 2, or 3:

(I) If $P_I = 1$ is in region 2, $P_I = 1$ is a pure equilibrium. In this case, the EUS boundaries $\{P_I > 0, P_J = 0, P_B > 0\}$ and $\{P_I = 0, P_J > 0, P_B > 0\}$ intersect with line 4, and the boundaries $\{P_I > 0, P_J > 0, P_B = 0\}$ and $\{P_I = 0, P_J > 0, P_B > 0\}$ intersect with line 6, but these intersections do not define another equilibrium (see Figure 4(b)).

(II) If $P_I = 1$ is in region 1, one of the following three cases must occur:

(1) $\{P_I > 0, P_J > 0, P_B = 0\}$ intersects with line 4, and then $\{P_I > 0, P_J > 0, P_B = 0\}$ is a mixed equilibrium. No other equilibrium exists here, because the intersections $\{P_I > 0, P_J > 0, P_B = 0\}$ and $\{P_I = 0, P_J > 0, P_B > 0\}$ with line 6 and $\{P_I = 0, P_J > 0, P_B > 0\}$ with line 4 do not define another equilibrium (e.g., see Figure 4(c)).

(2) $\{P_I = 0, P_J > 0, P_B > 0\}$ intersects with line 5, and then $\{P_I = 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium. The intersection of the boundary $\{P_I > 0, P_J > 0, P_B = 0\}$ with line 5 does not yield another equilibrium (e.g., see Figure 4(d)).

(3) The origin is contained in the EUS, and then $\{P_I > 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium. No other equilibrium exists here, because the intersections $\{P_I > 0, P_J > 0, P_B = 0\}$ with line 5 and $\{P_I = 0, P_J > 0, P_B > 0\}$ with line 6 do not define another equilibrium (e.g., see Figure 4(f)).

(III) If $P_I = 1$ is in region 3, one of the following three cases must occur:

(1) $\{P_I > 0, P_J = 0, P_B > 0\}$ intersects with line 6, and then $\{P_I > 0, P_J = 0, P_B > 0\}$ is a mixed equilibrium. The intersection of $\{P_I = 0, P_J > 0, P_B > 0\}$ with line 6 does not define another equilibrium (e.g., see Figure 4(e)).

(2) $\{P_I = 0, P_J > 0, P_B > 0\}$ intersects with line 5, and then $\{P_I = 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium. The intersection of $\{P_I = 0, P_J > 0, P_B > 0\}$ with line 5 does not define another equilibrium.

(3) The origin is in the EUS, and then $\{P_I > 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium. The intersections of $\{P_I = 0, P_J > 0, P_B > 0\}$ with lines 4 and 6, and the intersection of $\{P_I > 0, P_J > 0, P_B = 0\}$ with line 5 do not define another equilibrium.

If the point where $P_J = 1$ is in region 1, $P_J = 1$ is a pure equilibrium. See Figure (4(a)) for an example. Otherwise, $P_J = 1$ must be in region 2 or 3. First, assume $P_J = 1$ is in region 2. If

$P_I = 1$ is also in region 2, $P_I = 1$ is a pure equilibrium (e.g., see Figure 4(b)). Otherwise, based on Lemma 2, $P_I = 1$ must be in region 1. In that case, the boundary $\{P_I > 0, P_J > 0, P_B = 0\}$ must intersect with line (region) 4, and $\{P_I > 0, P_J > 0, P_B = 0\}$ is a mixed equilibrium.

Assume the point where $P_J = 1$ is in region 3. If $P_I = 1$ is in region 2, $P_I = 1$ is a pure equilibrium. Otherwise, $P_I = 1$ must be in region 1 or 3. If $P_I = 1$ is in region 1, one of the following three cases must occur:

(1) $\{P_I > 0, P_J > 0, P_B = 0\}$ intersects with line 4, and then $\{P_I > 0, P_J > 0, P_B = 0\}$ is a mixed equilibrium (e.g., see Figure 4(c)).

(2) $\{P_I = 0, P_J > 0, P_B > 0\}$ intersects with line 5, and then $\{P_I = 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium (e.g., see Figure 4(d)).

(3) The origin is contained in the EUS, and then $\{P_I > 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium (e.g., see Figure 4(f)).

Otherwise, $P_I = 1$ must be in region 3 and then one of the following three cases must occur:

(1) $\{P_I > 0, P_J = 0, P_B > 0\}$ intersects with line 6, and then $\{P_I > 0, P_J = 0, P_B > 0\}$ is a mixed equilibrium (e.g., see Figure 4(e)).

(2) $\{P_I = 0, P_J > 0, P_B > 0\}$ intersects with line 5, and then $\{P_I = 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium.

(3) The origin is contained in the EUS, and then $\{P_I > 0, P_J > 0, P_B > 0\}$ is a mixed equilibrium.

The following Lemmas complete the proof.

**Lemma 12.** *Along the EUS curve $\{P_I > 0, P_J > 0, P_B = 0\}$, $V(U_I)$ decreases.*

*Proof.* Recall that when $P_B = 0$, $\xi = \rho$. We show that in this case, as $\eta$ increases, $V(U_I)$ decreases. First, we evaluate $V(U_I)$ as a function of $\eta$:

$$\begin{aligned} V(U_I) &= \sum_{i=n_e}^{\infty} \pi_i \left( C_W \frac{i+1}{\mu} - R \right) = \sum_{i=n_e}^{\infty} \rho^{n_e} (1-\eta)^{i-n_e} \pi_0 \left( C_W \frac{i+1}{\mu} - R \right) \\ &= \rho^{n_e} (1-\rho) \frac{\eta(\frac{C_W}{\mu} n_e - R) + \frac{C_W}{\mu}}{\eta[(1-\rho^{n_e})\eta + \rho^{n_e}(1-\rho)]}. \end{aligned} \tag{26}$$

Next, we find the derivative of $V(U_I)$ with respect to $\eta$:

$$\frac{\partial V(U_I)}{\partial \eta} = -\rho^{n_e}(1-\rho)(1-\rho^{n_e})\frac{(\frac{C_W}{\mu}n_e - R)\eta + \frac{C_W}{\mu}}{\eta[(1-\rho^{n_e})\eta + \rho^{n_e}(1-\rho)]^2}. \tag{27}$$

Last, we determine the sign of the derivative. $\rho^{n_e}$ is strictly positive, and so is the product $(1-\rho)(1-\rho^{n_e})$. The denominator is also positive. It is left to show that the nominator is also positive:

$$\left(\frac{C_W}{\mu}n_e - R\right)\eta + \frac{C_W}{\mu} > \left(\frac{C_W}{\mu}n_e - R + \frac{C_W}{\mu}\right)\eta = \frac{C_W}{\mu}(n_e + 1 - \nu)\eta > 0. \tag{28}$$

Therefore, $\frac{\partial V(U_I)}{\partial \eta} < 0$, and we conclude $V(U_I)$ decreases as $\eta$ increases along the EUS curve $\{P_I > 0, P_J > 0, P_B = 0\}$. □

**Lemma 13.** *Along the EUS curve $\{P_I = 0, P_J > 0, P_B > 0\}$, $V(U_I)$ decreases.*

*Proof.* We show that when $\eta = 1 - \xi$, $V(U_I)$ decreases as $\xi$ decreases. First, we evaluate $V(U_I)$ as a function of $\xi$:

$$\begin{aligned} V(U_I) &= \sum_{i=n_e}^{\infty} \pi_i \left(C_W \frac{i+1}{\mu} - R\right) = \sum_{i=n_e}^{\infty} \xi^{n_e}\xi^{i-n_e}(1-\xi)\left(C_W\frac{i+1}{\mu} - R\right) \\ &= \sum_{i=n_e}^{\infty} \xi^i(1-\xi)\left(C_W\frac{i+1}{\mu} - R\right) = \xi^{n_e}\left[\frac{C_W}{\mu}n_e - R + \frac{C_W}{\mu}\frac{1}{1-\xi}\right]. \end{aligned} \tag{29}$$

Next, we find the derivative of $V(U_I)$ with respect to $\xi$:

$$\frac{\partial V(U_I)}{\partial \xi} = n_e\xi^{n_e-1}\left[\frac{C_W}{\mu}n_e - R + \frac{C_W}{\mu}\frac{1}{1-\xi}\right] + \xi^{n_e}\frac{C_W}{\mu}\frac{1}{(1-\xi)^2}. \tag{30}$$

Last, we determine the sign of the derivative. In this case, $\eta = 1 - \xi$, and therefore $\xi < 1$. As a result, $\frac{1}{1-\xi} > 1$ and

$$\frac{C_W}{\mu}n_e - R + \frac{C_W}{\mu}\frac{1}{1-\xi} = \frac{C_W}{\mu}\left[n_e - \nu + \frac{1}{1-\xi}\right] > 0, \tag{31}$$

and because $n_e\xi^{n_e-1} > 0$ and $\xi^{n_e}\frac{C_W}{\mu}\frac{1}{(1-\xi)^2}$ are strictly positive as products of positive elements, we conclude the derivative is strictly positive. Therefore, along the boundary that represents $\{P_I = 0, P_J > 0, P_B > 0\}$, when $\xi$ decreases, $V(U_I)$ decreases. □

## B Proofs for section 4

**Proof of Proposition 3**. We first explain how we calculate the thresholds $K_1, K_2, K_3, K_4$, and $K_5$.

$K_1$ is the maximum inspection-cost level for which the equilibrium strategy is still $P_I = 1$. This level is achieved in one of two scenarios: either $U_I = 0 > U_J$ or $U_I = U_J > 0$. When $P_I = 1$, $\xi = \rho$ and $\eta = 1$. Substituting into the definitions of $U_I$ and $U_J$ and solving the corresponding equalities, we get $K_1 = S_1$ for $U_I = U_J > 0$, and $K_1 = S_2$ for $U_I = 0 > U_J$. Moreover, for $\nu > \frac{\sum_{i=0}^{n_e} \rho^i - (n_e+1)\rho^{n_e+1}}{1-\rho^{n_e+1}}$, $S_1 > S_2$; otherwise, $S_1 \leq S_2$.

$K_2$ and $K_3$ are defined as the minimum cost for which the EUS contains the origin that can happen in one of the two following scenarios:

- The cost for which the EUS boundary $\{P_I > 0, P_J > 0, P_B = 0\}$ intersects with the origin where $U_I = U_J = 0$. At that point, $P_B = 0$ and therefore $\xi = \rho$. To find $K_2$, we first find the $\eta$ that solves $U_I = U_J$ (which is the positive root of a quadratic equation in $\eta$). We substitute $\eta$ into $U_I$ and find $\kappa$ such that $U_I = 0$.

- The cost for which the EUS boundary $\{P_I > 0, P_J = 0, P_B > 0\}$ intersects with the origin where $U_I = U_J = 0$. At that point, $P_J = 0$ and therefore $\eta = 1$. To find $K_3$, we first find the $\xi$ that solves $U_I = U_J$. Then we substitute $\xi$ into $U_I = 0$ and find $\kappa$.

$K_4$ is the minimum cost for which $\{P_I = 0, P_J > 0, P_B > 0\}$ and is calculated by finding the cost for which $U_I = U_J = 0$ when $P_I = 0$.

$K_5$ is the minimum cost for which $P_J = 1$ and is calculated by finding the cost for which $U_I = U_J > 0$ when $P_I = 0$.

Next, we prove that given the EUS position when $\kappa = 0$, we can determine which scenario will hold. To do so, we use the properties of the EUS as proven in Lemmas 1 and 2. For fixed $\rho$ and $\nu$, as we increase $\kappa$, the calculation of the $U_J$ component for every given strategy vector does not change; the only change is in $U_I$. If we increase $\kappa$ continuously from 0, the EUS would shift to the left continuously. We use this property in demonstrating the evolution of the equilibrium as a function of $\kappa$.

We start from the EUS when $\kappa = 0$. In this case, the equilibrium strategy is always defined at the point that represents $P_I = 1$. Observing the EUS for these values, we distinguish between three scenarios corresponding to those described in the proof of Lemma 2:

1. The vertex representing $P_J = 1$ has $U_J \geq 0$. As we increase $\kappa$, the EUS shifts to the left. At $\kappa = K_1$, this vertex intersects with the line $U_I = U_J \geq 0$. At $\kappa = K_4$, the vertex representing $P_I = 1$ intersects with the line $U_I = U_J \geq 0$. Between $K_1$ and $K_4$, the equilibrium strategy would be $\{P_I > 0, P_J > 0, P_B = 0\}$; for $\kappa > K_4$, it would be $P_J = 1$. Therefore, we get Scenario 1.

2. The vertex representing $P_J = 1$ has $U_J < 0$, and the vertex representing $P_I = 1$ has $U_J \geq 0$. We increase $\kappa$, and at $\kappa = K_1$, this vertex intersects with the line $U_I = U_J \geq 0$. At $\kappa = K_2$, the line representing $\{P_I > 0, P_J > 0, P_B = 0\}$ intersects with the origin. At $K_3$, the line representing $\{P_I = 0, P_J > 0, P_B > 0\}$ intersects with the origin, corresponding to Scenario 2.

3. The vertex representing $P_J = 1$ has $U_J < 0$, and the vertex representing $P_I = 1$ has $U_J < 0$. As we increase $\kappa$, at $\kappa = K_1$, this vertex intersects with the line representing $U_I = 0$. At $\kappa = K_2$, the line representing $\{P_I > 0, P_J = 0, P_B > 0\}$ intersects with the origin. At $K_3$, the line representing $\{P_I > 0, P_J = 0, P_B > 0\}$ intersects with the origin, corresponding to Scenario 3.

**Proof of Lemma 5**.

1. When $U_I = U_J > 0$, $P_B = 0$ and $\xi = (1 - P_B)\rho = \rho$. From Proposition 3, $P_J$ increases with $\kappa$. Therefore, to show $P_E$ increases with $\kappa$ in this case, it is enough to show $P_E$ increases with $P_J$. We substitute $P_I = 1 - P_J$ and $P_J = \frac{1-\eta}{\rho}$ into the definition of $P_E$:

$$P_E = \sum_{i=0}^{n_e-1} \pi_i P_I + P_J = \frac{1}{\rho}\left[\frac{(1-\rho^{n_e})\eta(\rho+\eta-1)}{(1-\rho^{n_e})\eta + \rho^{n_e}(1-\rho)} + 1 - \eta\right]. \tag{32}$$

We use the chain rule:

$$\frac{dP_E}{dP_J} = \frac{\partial P_E}{\partial \xi}\frac{\partial \xi}{\partial P_J} + \frac{\partial P_E}{\partial \eta}\frac{\partial \eta}{\partial P_J}$$
$$= 0 + (-\frac{(1-\rho)^2\rho^{n_e}}{\rho(\eta - \rho^{n_e}(\eta+\rho-1))^2})(-\rho) = \frac{(1-\rho)^2\rho^{n_e}}{(\eta - \rho^{n_e}(\eta+\rho-1))^2} > 0. \tag{33}$$

2. When $U_I = 0 > U_J$, $P_J = 0$ and $\eta = 1$. Then

$$P_E = \sum_{i=0}^{n_e-1} \pi_i P_I = \frac{1-\xi^{n_e}}{1-\xi^{n_e+1}} P_I. \tag{34}$$

When $\kappa$ increases, $P_B$ increases and therefore $\xi$ decreases. As a result, $\frac{1-\xi^{n_e}}{1-\xi^{n_e+1}}$ decreases. At the same time, $P_I$ decreases, and therefore $P_E$ decreases.

**Numerical analysis for Observation 1**. To analyze the change in $P_E$ at the interval $[K_2, K_4]$ for $(\rho, \nu)$ such that Scenario 2 holds, we took $\kappa^* = K_2, K_2+0.1(K_4-K_2), K_2+0.2(K_4-K_2), \ldots, K_4$. Then we calculated $P_E$ at every $\kappa^*$. By plotting the graph of $P_E$ as a function of $\kappa$ at those points, we verified that $P_E$ decreases with $\kappa$ when $\kappa \in [K_2, K_4]$. We repeated the same process for $(\rho, \nu)$ such that Scenario 3 holds, and verified that within the interval $(K_3, K_4)$, $P_E$ decreases with $\kappa$. The numerical analysis was done systematically by examining pairs of $\rho$ and $\nu$ with high resolution.

**Proof of Lemma 6**. To prove Proposition 6, we rely on the evolution of the equilibrium at each scenario as described in Proposition 3. For $\kappa \leq K_1$, $P_E$ is fixed and equal to $P_E^{\text{obs}}$. Consider now $\kappa > K_1$.

1. When Scenario 1 holds, $U_I = U_J > 0$ on the interval $(K_1, K_5)$. By Lemma 5, $P_E$ is monotonically increasing with $\kappa$ on this interval. Note that as long as $P_I > 0$, $P_E < 1$ by definition (see Equation 15). The maximum is obtained at every $\kappa \geq K_5$ where $P_J = 1$ and $P_E^{\text{unobs}} = 1$.

2. When Scenario 2 holds, by Lemma 5, $P_E$ increases with $\kappa$ on the interval $(K_1, K_2)$ for which $U_I = U_J > 0$. By Observation 1, $P_E$ decreases with $\kappa$ on the interval $(K_2, K_4)$ for which $U_I = U_J = 0$, and stays fixed for every $\kappa \geq K_4$ where $P_E = P_E^{\text{unobs}}$. Therefore, $P_E$ is unimodal and maximized at $K_2$.

3. When Scenario 3 holds, $P_E$ decreases when $\kappa \in (K_1, K_3)$ for which $U_I = 0 > U_J$, and by Observation 1, it continues to decrease on the interval $(K_3, K_4)$ for which $U_I = U_J = 0$. For $\kappa > K_4$, $P_E$ is fixed and equal to $P_E^{\text{unobs}}$. Therefore, $P_E$ decreases with $\kappa$ and reaches its maximum at every $\kappa \leq K_1$ where $P_I = 1$ and $P_E = P_E^{\text{obs}}$.

**Proof of Proposition 7**. When $\kappa \leq K_1$, $SW = \lambda U_I > 0$. Because $U_I$ decreases with $\kappa$, the maximum is obtained when $\kappa = 0$. For $\kappa \geq K_1$, our analysis corresponds to Scenarios 1-3. For each scenario, we show $SW$ is monotonically decreasing with $\kappa$.

**Scenario 1:** For $K_1 \leq \kappa < K_5$, where the equilibrium strategy is $\{P_I > 0, P_J > 0, P_B = 0\}$, as $\kappa$ increases, $P_I$ decreases and the system becomes more congested. Consequently, both $U_J$ and $U_I$ decrease, and as a result, $SW$ also decreases until it reaches $\kappa = K_5$. At this point, $U_J$ reaches its minimum value, and so does $SW$, and they both stay fixed for every $\kappa > K_5$.

**Scenario 2:** For $K_1 \leq \kappa < K_2$, where the equilibrium strategy is $\{P_I > 0, P_J > 0, P_B = 0\}$, by the same argument above, $SW$ decreases with $\kappa$. For $\kappa \geq K_2$, $P_B > 0$ and therefore $SW = 0$.

**Scenario 3:** For $\kappa > K_1$, $P_B > 0$ and therefore $SW = 0$.

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