Scheduling maintenance services to three machines

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We study a discrete problem of scheduling activities of three types under the constraint that at most a single activity can be scheduled to any one period. Applications of such a model are the scheduling of maintenance service to machines and multi-item replenishment of stock. We assume that the cost associated with any given type of activity increases linearly with the number of periods since the last execution of this type. The problem is to specify at which periods to execute each of the activity types in order to minimize the long-run average cost per period. We analyze various forms of optimal solution which may occur, relating them to the combination of the three machine cost constants. Some cases remain unsolved by this method and for these we develop a heuristic whose worst case performance is no more than 1.33\% from the optimal.

**Keywords:** scheduling, maintenance

1. Introduction

We consider a situation in which the performance of a machine deteriorates over time until it receives a maintenance service, and just one machine may be serviced at a time. There are three machines, \(M_1\), \(M_2\) and \(M_3\), to be serviced and each with the same service time of one time unit. We consider a linear structure for the cost of operating each machine. Associated with each machine, \(M_i\), there is a cost constant, \(a_i\). The cost of operating \(M_i\) during a period in which it is serviced is 0, and the cost of operating it in the jth period after its last service is \(ja_i\). There is no cost associated with maintenance service itself. The problem is to find an optimal policy specifying at which periods to service each of the machines in order to minimize the long-run average operating cost.

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We stated the model in terms of "machines" and "servicing" for convenience, but it may be applied equally well in other contexts. One such context is that of multi-item stock replenishment, in which at most one item may be replenished in any one time period. Demand is assumed to be constant, $q_i$ say, for the $i$th item, and the only variable cost is a linear holding cost for each of the items. Let $h_i$ denote the unit holding cost per time period for the $i$th item. The above model then holds for the infinite horizon, discrete time case by considering time to run in the opposite direction and substituting $a_i$ by $h_i q_i$.

The general case of $m$ machines was investigated in an earlier paper by the authors [2]. There, it was shown that there is always a cyclic optimal solution, consisting of repetitions of a subsequence, and a transformation of the problem into one of computing a minimum mean length cycle in a graph is also given. However, the size of the graph grows with the size of the data in a non-polynomial way. The earlier paper also developed an approximation algorithm with a bounded performance guarantee and another heuristic based on a greedy rule that works well in practice. To date it is not known whether the problem considered in [2] is NP-hard.

In [3], Bar Noy et al. obtain a better performance guarantee for the problem considered in [2]. They also consider a more general problem in which several machines can be serviced during any given period and there is a machine dependent service cost.

A variation of our problem is that in which the cost of operating a $M_i$ during its maintenance is $a_i$ and at the $j$th period after the last maintenance this cost is $(j + 1)a_i$. There is no difference between this variation and the one we consider in terms of optimal solutions, since in it merely means increasing the cost per period by $\sum a_i$. However, this increase in cost decreases the ratio between the cost of any pair of solutions and thus better performance guarantees can be obtained (see Remark 6.4 in [2] and the broadcast disk application in [3]).

Other generalizations of the model include a general convex cost function, different lengths of times to service different machines, and service time being dependent upon time since last service. The properties of an optimal policy for the case of two machines are described in [1]. Related problems are treated in [4-6,8,9,11,13]. In [9], there are bounds on the length of intervals between consecutive services to each machine. In [11,13], the service intervals are fixed and the problem is to determine the number of servers needed to form a feasible schedule.

In this paper, our aim is to solve some cases of the three-machine problem to optimality and to deal with the remainder by means of a heuristic for which we present a worst-case performance bound. For convenience, we number the machines so that $a_1 \geq a_2 \geq a_3$.

This paper is organized as follows. In Section 2, we present some general properties of an optimal solution for any number of machines and the form of an optimal solution to the two machine problem, which will be required later in the paper. The structure of an optimal solution for the three-machine problem in which the two
machines with smallest cost constants, $M_1$ and $M_2$, are serviced consecutively, is described in section 3. Such optimal solutions occur when $a_i/a_2$ is small, in particular less than 2. The other cases which arise when $a_i/a_2 < 6$ are then identified in section 4 and an optimal solution prescribed for each case. This leaves the remaining case of $a_i/a_2 \geq 6$, for which a heuristic is presented in section 5. We show that the worst-case performance of the heuristic is no more than 3.333% from the optimal.

2. Preliminaries and properties of optimal solutions

Our problem consists of three machines, $M_1$, $M_2$ and $M_3$. The cost of operating $M_i$ in the $j$th period after the last maintenance of that machine is $a_{ij}$, for $i = 1, 2, 3$ and $j \geq 0$.

A policy $P$ to the $m$-machine problem is a sequence $i_1, i_2, \ldots, i_m$, where $i_k \in \{1, \ldots, m\}$ for $k = 1, 2, \ldots$ denotes the machine scheduled for service during the $k$th period. A policy is said to be cyclic if it consists of repetitions of a finite sequence $i_1, \ldots, i_l$. Such a sequence is said to generate the policy. The minimum length of a generating sequence is denoted by $H(P)$, and any set of $H(P)$ consecutive elements of the sequence is called a basic cycle of $P$. A basic cycle of an optimal cyclic solution is referred to as an optimal basic cycle. For a given machine $M_i$, we shall refer to the time between consecutive services to $M_i$ as the length of this $i$-interval.

Consider for example a cyclic service sequence with a basic cycle 1123. During a basic cycle, $M_1$ is associated with a single interval of length 1 and an interval of length 3. $M_2$ and $M_3$ are associated with intervals of length 4. Therefore, the average cost of the policy is

$$\frac{(a_1 + 2a_3) + (a_2 + 2a_2 + 3a_3) + (a_3 - 2a_2 + 3a_1)}{4} = \frac{3a_1 + 6a_2 + 6a_3}{4}.$$ 

Without loss of generality, we assume that $a_1 \geq a_2 \geq \ldots \geq a_m$. Moreover, we scale the $a_i$ values so that $a_m = 1$. For a policy $P$, let $C(P)$ denote the average cost over periods $1, \ldots, t$. Clearly, we can restrict ourselves to policies with bounded costs and therefore we can define for each such policy $P$ the lim sup of its sequence of average costs,

$$C(P) = \limsup_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} C(i, P_i).$$

A policy is optimal if it minimizes $C(P)$. We let $C^*$ denote the average cost of an optimal policy.

We now quote some results from [2]. The first is a fundamental property of optimal solutions to this problem, which will enable us to restrict our attention to cyclic sequences, the second describes the solution for the two machine problem, and the third gives a lower bound on the solution value.

**Theorem 2.1.** There exists an optimal cyclic solution.
Lemma 2.2. For the two-machine problem, with \( a_1 \geq a_2 \), there is an optimal cyclic solution in which \( M_2 \) is serviced exactly once in a basic cycle and there exists an optimal basic cycle of length \( 2 \tau_2 \), which is the unique integer satisfying

\[
(\tau_2 - 1)\tau_2 \leq 2a_1/a_2 < \tau_2(\tau_2 + 1).
\]

The minimum average cost is given by

\[
C_2 = \frac{a_1(\tau_2 - 1)}{2} + \frac{a_2}{\tau_2}.
\]

Lemma 2.3. A lower bound on the cost of an optimal solution for the \( m \)-machine problem is given by

\[
s \sum_{i=3}^m (i - 1) \omega_i.
\]

In particular, for the 3-machine problem,

\[
C' \geq \omega_1 + 2\omega_2.
\]

Building upon theorem 2.1, we now develop some further properties of an optimal sequence for the general \( m \)-machine problem. We shall use \( C(S) \) to denote the average cost of a cyclic solution with a basic cycle \( S \) and \(|S|\) to denote its length of \( S \).

Lemma 2.4. There exists an optimal basic cycle in which no subsequence which contains at least one service to each machine occurs more than once.

Proof. By contradiction. Take an optimal basic cycle \( S \) of shortest length. Suppose that \( S_0 \) is a subsequence containing at least one service to each machine, which occurs twice in \( S \). Then \( S \) is of the form \( S_0S_1S_2S_3 \) for some, possibly empty, subsequences \( S_1 \) and \( S_2 \). Subsequences \( S_0S_1 \) and \( S_0S_2 \) each contain all the machine indices and are therefore basic cycles to solutions to the problem. Moreover, the average cost \( C(S) \)

\[
C(S) = \frac{|S_0S_1|}{|S_0S_1| + |S_0S_2|} (C(S_0S_1)) + \frac{|S_0S_2|}{|S_0S_1| + |S_0S_2|} (C(S_0S_2)),
\]

is a weighted average of \( C(S_0S_1) \) and \( C(S_0S_2) \), and thus

\[
\min(C(S_0S_1), C(S_0S_2)) \leq C(S) = C'.
\]

Thus, one of \( S_0S_1 \) and \( S_0S_2 \) is a basic cycle shorter than \( S \). \( \square \)

Lemma 2.5. An optimal basic cycle \( S \) has the following properties:

(i) Extending \( S \) by \( k \) periods by inserting extra services to any machine increases the total cost of the basic cycle by at least \( kC' \).
(ii) Removing from S k services cannot reduce the total cost of the basic cycle by more than kC'k.

Proof. For a solution generated by a basic cycle P, denote by K(P) the total cost during a basic cycle of this solution. Let T denote the length of the given optimal basic cycle S. Let S' denote a basic cycle of length T + k derived from S as described in (i). Since S' is optimal, we know that K(S')/(T + k) ≥ K(S)/T = C' and hence K(S') - K(S) ≥ kC', as claimed. Similarly, if S" denotes a basic cycle derived from S in (ii), we have C" = K(S)/T ≤ K(S")/(T - k) and hence K(S) - K(S") ≤ kC".

From now on we focus on the three-machine problem.

Lemma 2.6. In an optimal basic cycle, a 2-interval contains at most two 3's.

Proof. By contradiction. Suppose that there is an optimal basic cycle S in which M2 is serviced in periods t0 and t1, and M3 is serviced in periods t1, t2, t3 in between, i.e. t0 < t1 < t2 < t3 < t1. Without loss of generality, we may take t0 to be 0. Construct a new sequence S' by replacing the service to M2 in period t1 by a service to M1. As a result of this exchange, the total cost due to M2 is not changed, while the total cost due to M3 during T(S) periods is reduced by a1f2(t - t2) and that of M1 is increased by a1f2(t2 - t3). Comparing the average cost of S' to that of S, we obtain

\[ C(S) - C(S') = \frac{a_2f(t - t_2) - a_2f(t_1 - t_2)}{T(S)} \geq \frac{a_2f(t - t_2) - a_2f(t - 1 - t)}{T(S)} \geq a_2f > 0. \]

The first inequality follows from t2 ≥ t - 1 and the last from a2 > 0 and t2 ≥ 2. This contradicts the optimality of S.

Lemma 2.7. An optimal basic cycle does not contain either of the following sub-sequences: (i) 22, (ii) 33.

Proof. (i) Suppose otherwise and let S denote an optimal basic cycle containing sub-sequence 22. Let S' denote the basic cycle got from S by removing one of these 2's. The length of both the 1-interval and the 2-interval spanning 22 in S is at least 3 and each is reduced by 1 in the transformation of the sequence to S'. The decrease in the total cost is therefore at least 2a1 + 2a2. But this is greater than a1 + a2 + a2, the average cost of the cyclic solution with basic cycle 123 and hence greater than the optimal average cost, C'. A contradiction to Lemma part (ii).

(ii) Follows by similar argument to (i), but with the roles of machines M2 and M1 interchanged.
3. Structure of optimal solutions containing subsequence 32

In this section, we explore instances of the 3-machine problem in which it is best to leave the most costly machine, $M_3$, unserviced for more than one period consecutively. First we show that $M_2$ will not be without service for more than two consecutive periods and that during such a two-period gap, each of the two other machines will be serviced. We then explore the structure of an optimal solution which contains the subsequence 1321 (or 231). This leads to a description of the structure and cost of an optimal solution to the 3-machine problem containing the subsequence 32 (or 23) which is given in theorems 3.2 and 3.3.

The following lemma allows us to select certain subsequences from consideration within an optimal cyclic solution containing a 32 (or a 23).

It will be convenient to adopt the following notation. For a given subsequence $S$, let $l_i(c)$ denote the length of the $i$-interval to the left (right) of the leftmost (rightmost) service to $M_i$ in $S$ and let $w_1$ denote $\sum_{i=1}^{d} j = d(d - 1)/2$ so that the total cost of maintenance for machine $M_i$ attributed to an $i$-interval of length $d$ is $w_d$.

Lemma 3.1. For any problem, there exist optimal cyclic policies that do not contain the following subsequences: (i) 323, (ii) 232, (iii) 31321, (iv) 11321.

Proof. (i) Consider an optimal basic cycle $S$ containing 323 and look for a contradiction. Compare its total cost with that of the one obtained by inserting 21, to give 32123. The cost of 323 excluding the first two periods is at least $3a_1 + a_2$, while for 32123 it is exactly $3a_1 + 2a_2 + 5a_3$. The total cost has been increased by at most $a_1 + 5a_3$, which is no more than $2a_2 + 2a_3$ and hence no greater than $2C^*$ by lemma 2.3. Thus, basic cycle $S$ does not satisfy condition (i) of lemma 2.5, producing a contradiction.

(ii) Consider the case when an optimal basic cycle that does not contain 323 (see part (i) of this lemma) does contain the subsequence 232. Then it cannot extend to a 3 and according to lemma 2.7, it cannot extend to a 2, thus it must extend to 12321 within an optimal basic cycle.

Observe that we can assume that $l_2 \leq 3$ for the following reasons: if $l_2 > 3$, then the basic cycle extends to $xyz$12321 where $x$, $y$, and $z$ each take the values 2 or 3. Since repeats of either 2 or 3 may be excluded by lemma 2.7, this leaves either 323, which must occur if $l_2 > 4$, or 232. Our assumption that the basic cycle does not contain 323 means that we are left with the case 123212321, which we may exclude since it cannot be better than 2131.

For the subsequence 12321, $l_2 \geq 2$. We first claim that $l_2 \leq 3$. Suppose that $l_2 > 3$, then swap this 2 and the adjacent 1 to get 21321. Note that this swap does not affect the cycle length and the cost due to $M_3$. Let $l_1$ be defined as above for the sequence 12321. Then the total costs of the affected $M_1$ and $M_3$ cycles are $w_{l_1} + w_3 l_1 + (w_1 + w_2) l_2$ for 12321 and $(w_{l_1} + w_3) l_1 + (w_{l_2} + w_3) l_2$ for 21321. The swap there-
fore costs $(l_1 - 3)a_1 + 13 - l_2)a_2$. Since $l_1 \leq 3$ and $l_2 > 3$, this gives an improvement, which is a contradiction.

If $l_2 = 3$, then we have either 121321 or 2112321. The former subsequence contains two occurrences of subsequence 123. By lemma 2.4, we can exclude the subsequence since there exists a shorter optimal basic cycle. In the latter sequence, 2112321, swapping the second 1 by the second 2 will strictly improve the cost.

This leaves the case when $l_2 = 2$ and $l_1 \geq 3$. Since $l_1 \neq l_2 - 1$ and $l_1 \leq 3$. When $l_1 = 2$ and $l_2 = 2$, we have subsequence 1212321 for which the cost in each period, excluding the first three, is $a_1 + (l_1 - 1)a_2$, $2a_2 + a_3$, $3a_3 + a_2 + 2a_3$. The corresponding costs for the sequence 121321 obtained by omitting a 2 are $a_1 + 2a_2$, $2a_2 + a_3$, $a_2 + 2a_3$. Therefore, the transformation reduces the total cost by at least $3a_3 - a_2 - 4a_3$, which is greater than $a_1 + a_2 + a_3$, an upper bound on the optimal value obtained by costing the basic cycle 123, contradicting lemma 2.3 part (ii).

If $l_2 = 2$ and $l_1 = 3$, the sequence 123212 exists either to 1221321, contradicting lemma 2.7, or to 13212321, giving a repetition of 321, contradicting lemma 2.4.

(iii) The subsequence 31321 may not be extended to the left by a 3 from lemma 1.7. We consider the two cases of the subsequence being extended to the left by a 1 and a 2 separately.

Suppose that an optimal sequence contains 131321. Then $l_3 \geq 5$ and $r_3 \geq 3$. We may exclude the case $r_3 = 3$ (i.e. subsequence 133121) by comparison with 131213, since $(w_2 - w_3 - w_2)a_2 \geq (w_2 - w_3 - w_2)a_2$ is implied by $l_3 \geq 5$. Thus, $r_3 \geq 2$ by comparison with 131213, since $(w_2 - w_3)a_2 + (w_2 + w_3)a_3 \leq (w_2 - w_3)a_2 + (w_2 + w_3)a_3$, implies $(l_3 - 1)a_2 \leq (3 - r_3)a_3$, which combined with $r_3 > 3$ gives $(l_3 - 1)r_3 < 0$. Note that from our subsequence 131321 and our observations that $l_3 \geq 5$, $r_3 \geq 2$ and $r_3 > 3$, we have that the subsequence must be extended to the right by 1 and we obtain 1313321. But these properties imply that 131321 is no more expensive than 1313321 (since $w_2a_2 + (w_2 + w_3)a_3 < 2w_2a_2 + (w_2 + w_3)a_3$ implies $a_2 < (r_2 - r_3 + 1)a_2 \leq a_2$, which gives a contradiction). Thus, when the subsequence 131321 appears in an optimal sequence, we may replace it by 131312.

This leaves the case 231321. Consider how this subsequence may extend to the right. Sequence 2331321 cannot be optimal by comparison with 231321. Consider the sequence 2313211 and compare it to 2312131. The cost of the intervals which are affected is $w_2a_1 + (w_1 + w_3)a_2 + (w_2 + w_3)a_3$ for 2312131, while 2313211 has total cost $3w_2a_1 + (w_1 + w_3)a_2 + (w_2 + w_3)a_3$. Changing to 2313211 therefore gives savings of $a_1 + (3 - r_3)a_2 + (3 - 3a_3)$, which is $a_1 + (3 - r_3)a_2 + (3 - 3a_3)$, i.e. a saving of at least $(a_1 - a_2) + (3 - r_3)a_3$. On the other hand, the sequence 2331321 gives savings of at least $(a_1 - a_2) + (r_2 - 3a_3)$, as it has a total cost of $7w_2a_1 + (w_1 + w_3)a_3$ associated with $M_1$ and $M_2$. We may therefore exclude all but the case of $a_1 = a_2$ and $r_2 = 4$. But then we could renumber the machines by interchanging 1 and 2 and apply lemma 2.7 to exclude this case. Now all cases have been excluded other than 231321.

But the service immediately to the left of this sequence must be a 1 and therefore the sequence must extend to 31231321 applying the above argument to the sequence in
reverse, \(313213213\) cannot be extended to the left or right by 3 according to lemma 3.1. Also, it cannot be extended to the left (right) by 2 to 2312313213 (3132132313) as the subsequence 231 (132) repeats itself, see lemma 2.4. Thus, the extension is \(312313213\). This case may be excluded by comparison with 312312313. To see this, observe that \((1312313213) < (3123132313)\) implies \((w_1 + w_2)a_1 + (w_3 + w_4)b_1 (w_2 + w_3)b_1 < (2w_1 + 2w_2)b_1 + (w_3 + w_4)b_1 + 2w_2a_1\), which in turn implies \((l_2 - 1 + 6 - 1)3a_1 < (12 - 2)3a_1\) so that \((l_2 + 1)3a_1 < 5a_2\), and in view of the fact that \(l_2 \geq 4\), we get a contradiction.

(iv) Suppose that an optimal sequence contains the subsequence 11321. The 2- and 3-intervals have the following properties: \(l_2 \geq 4, l_3 \geq 3\), and \(r_1 > r_2\) by comparison with the subsequence 11321 (since \((11321) < (113213)\) implies \(w_2a_1 + (w_3 + w_4)a_1 < w_2a_1 + (w_3 + w_4)a_1\)). Thus, the next machine to be served in the sequence cannot be 3. Moreover, it cannot be machine 2 (i.e., 113213) as the decrease in cost obtained by swapping the adjacent 3 and 2 to get \(1123213\) is at least \((l_2 - 3)a_1 + (r_2 - l_2 - 1)a_2\), which is positive (since \(l_2 \geq 4, r_2 > l_2, a_1 > 0\) and \(a_2 > 0\)), and hence the swap gives an improvement. This leaves the case 113211.

For 113211 by comparison with costs for 112311, we obtain \((l_2 - 1 - r_2)a_2 \leq (l_2 - r_2 + 1)\), from \((w_1 + w_2)a_1 (w_3 + w_4)a_1, (w_3 + w_4)a_1, (w_3 + w_4)a_1, (w_3 + w_4)a_1\). Now using the property \(r_1 > r_2\), we get \(l_2 \leq r_2 + 1\) (since \((l_2 - 1 - r_2)a_2 \leq (l_2 - r_2 + 1)\)). But comparison with costs for 113211 reveals that \(r_2 < l_2\) (as \((w_1 + w_2)a_1 (w_3 + w_4)a_1, (w_3 + w_4)a_1 < 2w_2a_1 + (w_3 + w_4)a_1)\) implies that \((l_2 - r_2 + 1) < a_2 > 0\). Thus, \(l_2 = r_2 + 1\) and \(2n_2 > a_2\). Now look at the extension of the subsequence 113211 to the left, \(l_1 \leq 3\) by lemma 2.7 and parts (i) and (ii) of this lemma. If \(l_1 = 3\), we have either 123112311 or 123113211. The former cannot be optimal since it represents the sequence generated by 3211 (from part (i) of lemma 2.4) which is always more costly than the one generated by 1312. For the latter sequence 123112311, \(l_1 = 5\), thus \(r_2 > 4\). Therefore, this sequence cannot be extended to the right by 2 as there will be two consecutive 2's. Also, it cannot be extended to the right by 3 as there will be a repetition of 132, see lemma 2.4. Thus, this sequence must be extended as follows: 1231123112. In this sequence, \(r_1 \geq 6\). It is easy to check that 123123112 is less expensive.

This leaves the case when \(l_2 \leq 2\). To get a contradiction as required, it is therefore sufficient to show that the improvement obtained by changing the subsequence 231212 is at least \((l_1 - l_2)a_1 > 2a_2\). Recall that \(l_2 = 1\) and \(2a_2 > 0\). The immediately relevant costs of \(M_2\) and \(M_3\) in the original sequence are \(a_1(w_3 + w_2)\) and \(a_2(w_3 + w_2)\); while in the other sequence, they are \(a_1(w_3 + w_2 + w_3)\) and \(a_2(w_3 + w_1 + w_3)\). The decrease in cost due to the change is therefore \((l_1 - l_2)a_1 + (4l_1 - 4l_2)a_2\), since the difference for \(a_2\) is \((l_1 - l_2) + (4l_1 - 4l_2) - 3 + 6 + r_2 - 1 = 6l_2 - 14\) and the difference for \(a_1\) is \((-l_1 - 1) + 3 + l_1 - 1\). The fact that \(l_2 \geq 4\) now completes the proof of our claim.

Theorem 3.2. Machines \(M_2\) and \(M_3\) need be serviced consecutively in an optimal basic cycle only when there are as many services to machine \(M_1\) as to \(M_2\).
Moreover, in this case machine \( M_3 \) is serviced at regular intervals, \( t_3 \) say, and an optimal basic cycle has one of two forms depending on whether \( t_3 \) is even or odd, namely:

\[
\begin{align*}
321 \ldots t_3 \ldots 21 \\
321 \ldots 123121 \ldots 21
\end{align*}
\]

or in inverse.

**Proof.** Take an optimal basic cycle with \( t_2 \) in it, \( \ldots 32 \delta 32 \delta \ldots \) say. Then both \( \alpha \) and \( \beta \) are 1, since all other possibilities are excluded by lemma 2.7 and parts (i) and (iii) of lemma 3.1. Moreover, \( \gamma \) cannot be 3 or 1 from part (iii) and (iv) of lemma 3.1. Thus, \( \gamma \) is 2 and hence \( \delta \) is either 3 or 1 by lemma 2.7. If \( \delta = 3 \), then the sequence is 31321321 and 321 is a basic cycle, by lemma 2.4, satisfying the theorem. By a similar argument, the theorem holds with basic cycle 213 if \( \delta = 3 \) and we therefore need only examine those solutions which contain a subsequence of the form 1213213, for some subsequence \( S_3 \) of length at least 1 containing no 3.

Note also that \( S_3 \) cannot contain a 22. Subsequence 15, does not contain a 11, since otherwise the whole sequence could be rearranged by moving one of the 1's to between the 3 and 5 to produce a cheaper (or same cost) schedule. Thus, 15, must consist of alternating 1's and 2's.

If \( S_3 \) ends with a 1, then there are two occurrences of 213 in the sequence and hence the sequence 3215, is itself the basic cycle of the sequence by lemma 2.4. It is also of the first form described in the theorem. If this is not the case, then \( S_3 \) ends in a 2 and we may depict the sequence as 35,321213, where \( S_3 \) is a subsequence containing no 3 and ending with 12. Consider the lengths of the 3-intervals, \( l_3 \) and \( r_3 \), between consecutive 3's. If they are not equal, then it would mean no extra cost to swap the middle 3 with either the 1 to its left if \( l_3 > r_3 \geq 4 \), or the 2 to its right if \( l_3 < r_3 \), implying two occurrences of 123 in the sequence, contradicting lemma 2.4.

Thus, we may assume that \( l_3 = r_3 \).

If \( S_3 \) starts with a 2, then there are two occurrences of 321 in the sequence, contradicting lemma 2.4. Thus, \( S_3 \) must start with a 1. We show that \( S_3 \) does not contain two consecutive 1's. If there are any 1 tripples, \( 111 \), then we may swap a 1121 with a 1212 in the right-hand part of the cycle without altering the cost of the schedule. But we have just shown that no such subsequence may occur on the right-hand side, giving the required contradiction. Similarly, if there are two 2112's, then we may put them together and exchange them with 2121212 on the right at no extra cost, again giving a contradiction. This leaves the case of just one occurrence of a 11. But in this case, \( n_3 \) must be odd, which gives the required contradiction since \( l_3 \) and \( r_3 \) are equal. It follows that \( S_3 \) must contain as many 2's as 1's and must begin with a 12. Consider the sequence 35,13221\ldots123, lemma 2.7 and lemma 3.1 part (ii) imply that this sequence continues on the right with 1. Lemma 3.1 parts (iii) and (iv) applied on the reverse subsequence of 1231 at the end of 35,1322\ldots123 imply that this sequence continues on the right with 2. Therefore, we know that the sequence 35,1322\ldots123 continues on the right.
with 12. The subsequence 312 at the beginning and at the end of our sequence indicate that we have a complete basic cycle. Moreover, this sequence is of the second form, completing the proof.

The description of an optimal solution for the case when \( M_2 \) and \( M_3 \) are scheduled consecutively is completed by theorem 3.3. We first need to establish an observation that will be used in the proof of the theorem. Consider the function \( g = (\theta/n) + n \) defined for positive integers \( n \). Let \( \bar{g} \) be the extension of \( g \) to non-negative real numbers. Thus, \( \bar{g} = \theta/x + x \) for \( x \in \mathbb{R}^+ \). Note that \( \bar{g} \) is convex with a minimum at \( x^* = \sqrt{n} \). From (12), the unique integer \( n^* \) satisfying the following inequalities is an integer optimizer of \( g; \sqrt{n^* (n^*) - 1} \leq x^* < \sqrt{n^* (n^*) + 1} \).

Recall that \( \tau^* \), the optimal regular interval length for \( M_2 \), exists from theorem 3.2.

**Theorem 3.3.** The optimal solution which contains 32 has basic cycle of length \( T = \tau^* \) or \( T = 2\tau^* \) as \( \tau^* \) is odd or even, respectively, where the value of \( \tau^* \) is given by the unique integer \( t \) which satisfies

\[
\tau(t - 1) \leq 3(a_1 + a_2)/a_2 < \tau(t + 1).
\]

The optimal average cost is \((a_1 + a_2 - a_3)/2 + 3(a_1 + a_2)/2t + a_3/2\).

**Proof.** If \( \tau^* \) is odd, then a basic cycle is of the form 321...21, the basic cycle is of length \( k = \tau^* \) and has average cost \((a_1 + a_2)(w_1(k - 3)/2 + w_1)/k + a_3w_1/k\). If \( \tau^* \) is even, then a basic cycle is of form 321...312...13 and is of length \( \tau = 2\tau^* \) and has average cost \((a_1 + a_2)(w_1(t - 6)/2 + 2w_1)/T + 2a_3w_1/2t\). This expression is the same as the one above with \( t = 2k \), and is equal to \((a_1 + a_2) + 3(a_1 + a_2)/2k + a_3/2\).

The continuous minimizer of the above function is \( \sqrt{3a_3 + a_2}/a_2 \).

4. **Optimal solutions for \( a_1/a_2 < 6 \)**

In the previous section, we analyzed the special case in which the subsequence 32 (or 23) appears in a basic cycle. We now extend the range of basic cycles for which we solve the problem to optimality. In particular, we enumerate all the cases which may arise for relatively small values of \( a_1/a_2 \), namely \( a_1/a_2 < 6 \).

Let \( d_i \) denote the length of the interval in between two specific occurrences of \( i \) in a sequence.

**Theorem 4.1.** If \( a_1/a_2 < 6 \), then an optimal basic cycle is of one of the following forms:

(i) it contains subsequence 32 (or 23) and is of a form described in theorem 3.2; or else
(ii) it has precisely one occurrence of 3, which occurs in subsequence 21312, and intervals 31...12 in one of the following combinations:
(a) all 212;
(b) one 2112 and the rest 212;
(c) two 212 and the rest 2112;
(d) one 212 and the rest 2112;
(e) all 2112;
(f) one 21112 and the rest 2112; or
(g) two 21112 and the rest 2112.

Proof. Suppose that there is no 32 (or 23) in a basic cycle, then we should consider basic cycles with a 131,...121 (or equivalently 121,...131) with at least one 1 in between the 2 and 3.
If the cycle contains a subsequence 131,...121 with at least two 1’s in between the 2 and the 3, then contrariwise, on the other, hand, with the subsequence obtained by omitting one of the 1’s in between the 2 and the 3, and on the other hand, with the subsequence obtained by inserting a 2 before the 1 preceding 2. Using lemma 2.5, we get
\[ a_3(31 - 1) + a_2(21 - 1) \leq C \leq a_1 - a_2(21 - 2) + a_3, \]
\[ \text{implying } a_1 \geq a_2(23 - 3) - a_1 \geq a_3(23 - 4) \geq 6a_2. \]
The last inequality follows from the fact that 23 \geq 5 and is in contradiction to our assumption that \( a_1/a_2 < 6 \). Thus, there is no need for more than one 1 in between the 3 and the 2. Applying the same argument to the reverse subsequence 121,...131 implies that for \( a_1/a_2 < 6 \), 2 and 3 are separated by at most a single 1. Thus, any 2-interval that contains a single 3 must be of the form 21312 and any 2-interval containing more than one 3 must be of the form 12311,...1321.
Suppose that an optimal solution contains a 2-interval 1213,...1321. By lemma 2.6, the interval does not contain a third 3. Let \( d_i \) (\( d_i \)) denote the length of the 3-interval (2-interval) in the sequence. The total savings due to omitting the subsequence starting at the 1 following the first 3 and ending at the second 3, i.e. a subsequence of length \( d_3 \) of the form 1...3 is
\[ a_1 + a_2(3_4 - 3_{d_3} - 3_{d_3}) + a_3(3_{d_3}), \]
\[ = a_1 + a_2(3_4 - 4 - 4) + 3_4a_3 \]
\[ = a_1 + a_2(d_3(3_4 + 3_4) + 3_4a_3/d_3. \]
By lemma 2.5, this is no greater than \( d_3C \) and hence no greater than \( d_3C(2131) = d_3(2a_1 + 6a_2 + 6a_3)/4. \) Thus,
\[ a_1(d_3 - 2) \geq a_2(d_3 + 4) + a_3d_3(d_3 - 4) > a_2d_3. \]
But this inequality has no solutions for \( a_1/a_2 < 6 \), giving a contradiction.
Therefore, when $a_1/a_2 < 6$, all occurrences of 3 occur either next to a 2 or within a subsequence 21312. The former case, corresponding to part (i) of the theorem, is covered by theorem 3.2. We therefore concentrate on the latter case. That there is only one occurrence of 3 in a basic cycle as claimed in part (ii) now follows from lemma 2.4. It is left to consider the form of such a basic cycle. Observe that 2-intervals within a basic cycle may occur in any order without affecting the cost of the schedule.

Suppose that an optimal solution contains the subsequence 2112212 with at least two 1’s and no 3’s in between the two occurrences of 2. Then by comparison with the sequences 2112122 and 2121222 obtained by inserting a 2 and deleting a 1, respectively, we get \((d_2 - 1)a_2 + (d_2 - 1)a_1 \leq C' \leq a_1 - (d_2 - 2)a_2 + d_2a_1\), which implies $a_1/a_2 \geq 2d_2 - 4$.

Thus, when $a_1/a_2 < 2$, we have $d_2 \leq 2$ and hence $d_2 = 2$ for intervals which do not contain a 3. This is the combination described in part (ii) (a).

When $a_1/a_2 < 3$, we have $d_2 \leq 3$ for 2-intervals which do not contain a 3. But we have no more than one such interval with $d_2 = 3$ since comparing 2112112 and 2121222 gives a saving, as $2w_1a_2 + w_2a_1 > 3w_2a_2 + 2w_2a_1$ when $a_2 < 3w_2$, corresponding to cases (a) or (b).

For $3 \leq a_1/a_2 < 4$, 2-intervals with no 3 have $d_2 \leq 3$. Moreover, since two 2-intervals of length 3 are not more costly than three of length 2, there are at most two 2 intervals with $d_2 = 2$, i.e. 212, and the rest are 2112 (which demonstrates items (c), (d), (e) of part (ii) of the theorem).

Moreover, when $a_1/a_2 \geq 4$, a basic cycle cannot contain two 2-intervals of length 2 as a single 2-interval of length 4 is as cheap, since in this case $w_2a_2 \leq 2w_2a_2 + a_1$ and hence $C(211112) \leq C(211122)$. A basic cycle will not contain one 2-interval of length 2 and another with no 3, of length 4, since two 2-intervals of length 3 are always cheaper. Moreover, since $a_1/a_2 < 6$, $4w_2a_2 + 3a_1 < 3w_2a_2 + 2a_1$, and hence $C(211211221122) < C(2111121121122)$. It follows that four 2-intervals of length 3 are cheaper than three 2-intervals with no 3, each of length 4. This demonstrates when the cases in items (f) and (g) of part (ii) of the theorem arise.

\[\square\]

Remark 4.2. The categories in theorem 4.1 are not mutually exclusive.

Remark 4.3. The above proof gives more information about the relationship between the value of $a_1/a_2$ and each type of basic cycle that appears in the statement of the theorem.

Recall the function $g = (\theta/n) + n$ defined for positive integers $n$. As before, let $\tilde{g}$ be the extension of $g$ to non-negative real numbers. Thus, $\tilde{g} = \theta(x) + x$ for $x \in \mathbb{R}$ is minimized by $x^* = \tilde{g}'$. Also, for any $\gamma > 0$,

$$\tilde{g}(\gamma x') = \tilde{g}(x' / \gamma) = \tilde{g}(x') / \gamma.$$
where $e(y) = 2/(y + y^{-1})$ is quasi-concave with a maximum at $y = 1$. Now, suppose that one wants to minimize $g(n)$ under the constraint that $n \in L$, where $L$ is a subset of the positive integer numbers. By using the above properties of the function $e$, the above problem can be solved by finding the member of $L$ which is the closer to $x^*$ to the left of $x^*$ (if any) and to the right of $x^*$ (if any). If $L$ is non-empty, then we get one or two values which are candidates as minimizers of the given problem. If there is only one candidate, it means that either $x^*$ is large (smaller) than all the numbers in $L$ or $x^* \in L$. In either of these cases, the candidate is the minimizer. Otherwise, we have two values, one to each side of $x^*$. Denote these values by $n_1$ and $n_2$. The best of the two candidates is the one that maximizes $e(x^*/n)$. From [12], the unique integer $n^*$ satisfying the following inequalities is an integer optimizer of $g$: $\sqrt{n^*(n^*-1)} \leq x^* < \sqrt{n^*(n^*+1)}$. We are now ready to prove the following theorem.

**Theorem 4.4.** The optimal basic cycles of the types described in part (ii) of theorem 4.1 are of length $l$, where $l$ is the closest integer to $x^*$ in the set $L$, where $x^*$ and $l$ are defined below:

(a) $x^* = \sqrt{2a_2/a_3}$ and $L = \{4 + 2k : k = 0, 1, 2, \ldots\};$
(b) $x^* = \sqrt{(1a_2 - a_3)/a_3}$ and $L = \{7 + 2k : k = 0, 1, 2, \ldots\};$
(c) $x^* = \sqrt{5a_1/3a_3}$ and $L = \{\beta + 3k : k = 0, 1, 2, \ldots\};$
(d) $x^* = 2(a_1 + a_3)/a_3$ and $L = \{6 + 3k : k = 0, 1, 2, \ldots\};$
(e) $x^* = \sqrt{3(a_1 + 2a_2)/3a_3}$ and $L = \{4 + 3k : k = 0, 1, 2, \ldots\};$
(f) $x^* = 2(a_1 + 3a_2)/(3a_3)$ and $L = \{\beta + 3k : k = 0, 1, 2, \ldots\};$ or
(g) $x^* = \sqrt{12a_1/a_3}$ and $L = \{12 + 3k : k = 0, 1, 2, \ldots\}.$

**Proof.** The average cost functions, $C(t)$, for the cases are, respectively:

(a) $(a_1 + a_2 - a_3)/2 + 4a_2/\pi + \phi a_3/2;$
(b) $(a_1 + a_2 - a_3)/2 + (11a_2 - a_3)/2 + \phi a_3/2;$
(c) $(a_1/3 + a_2 - a_3)/2 + 4a_3/\pi + \phi a_3/2;$
(d) $(a_1/3 + a_2 - a_3)/2 + (a_1 + a_2)/\pi + \phi a_3/2;$
(e) $(a_1/3 + a_2 - a_3)/2 + (a_1 + 3a_2)/3 + \phi a_3/2;$
(f) $(a_1/3 + a_2 - a_3)/2 + (a_1 + 12a_3)/3 + \phi a_3/2; and
(g) $(a_1/3 + a_2 - a_3)/2 + 6a_3/\pi + \phi a_3/2.$

These costs are minimized by the values $x^*$ given in the theorem. The feasible lengths in each case are computed by considering the structure of the respective sequence.
Since each of the cost functions is convex the minimum feasible cost will be given by one of the integers in \( L \) on either side of \( \epsilon^* \) and the proof is completed by the observation preceding this theorem.

\[ \square \]

Corollary 4.5. For \( a_i/a_j < 6 \), we may determine the optimal solution in constant time.

5. A heuristic and its performance

We now describe a heuristic for finding a solution whose value is within 3.33% of the optimal. In many cases, we may use the results proved above to find the optimal solution itself. For the other cases, we develop a heuristic based on the following relaxation of the problem into independent 2-machine problems.

Let \( R^2 \) denote the relaxation of the problem in which we allow machines \( M_2 \) and \( M_3 \) to be serviced simultaneously. However, we do not relax the condition on the total number of services and therefore every time machines \( M_2 \) and \( M_3 \) are serviced simultaneously, there must be a corresponding gap with no service somewhere in the schedule. We cost maintenance for machines \( M_2 \) and \( M_3 \) in the usual way but for machine \( M_j \), a cost of \( a_j \) is incurred with each service to machines \( M_2 \) and \( M_3 \). In this way, the positioning of the gaps in the maintenance schedule becomes immaterial and consecutive services of machines \( M_2 \) and \( M_3 \) incur no more cost for maintenance of machine \( M_j \) than separate services.

Let \( C(\mathcal{R}^2) \) denote the average reduced cost of an optimal solution to the relaxed problem \( R^2 \). This cost provides a lower bound to the average cost of the original problem, i.e., \( C(\mathcal{R}^2) \leq C \). Let \( \bar{T}_j \) and \( C_{ij} \) denote the basic cycle length and the average cost of an optimal solution to the 2-machine problem involving \( M_i \) and \( M_j \).

**Lemma 5.1.** An optimal solution to \( R^2 \) involves regular services to machine \( M_i \) at interval \( \bar{T}_i \), for \( i = 2, 3 \), and \( C(\mathcal{R}^2) = C_{12} + C_{13} \).

**Proof.** Under \( R^2 \), \( M_2 \) and \( M_3 \) are independent of each other. For a given \( M_i \), it must be decided at what periods to schedule it, taking into account a fixed cost of \( a_i \) per scheduled period plus the maintenance costs associated with it. However, we have seen that the 2-machine problem involving \( M_1 \) and \( M_2 \) has a solution in which \( M_1 \) is never served in two consecutive periods. Hence, the actual cost structure is as in the relaxed problem and the minimum average cost is again \( C_{12}, R^2 \) is therefore equivalent to the amalgamation of 2 independent 2-machine problems for \( M_i \) and \( M_j \) with average minimum cost \( C_{12} + C_{13} \). \[ \square \]

**Algorithm 5.2 (3-machine heuristic, H3)**

A. If \( a_i/a_j < 6 \).

Find the optimal solution as described in theorems 4.1 and 4.4,
B. If $a_1/a_2 \geq 6$.

Step 1. Construct a basic cycle (possibly not feasible), $S_2$, which is optimal for the relaxation $R2$.

- Schedule services to $M_2$ at regular intervals of length $\bar{T}_1$ starting at period 1.
- Schedule services to $M_2$ at regular intervals of length $\bar{T}_2$ starting at period 3.
- Schedule services to machine $M_1$ in all the gaps except the ones before a period in which $M_2$ and $M_3$ are served simultaneously.

Step 2. Modify $S_2$ to make a feasible schedule $S_3$.

- In periods where $M_2$ and $M_3$ are both serviced simultaneously, move the service to $M_2$ one period to the left and exchange the service to $M_3$ with the service to $M_1$ on its right.
- Where 231 occurs, swap the 3 with the 1 to its right.
- Where 132 occurs, swap the 3 with the 1 to its left.

Let $C(H3)$ denote the cost of the solution produced by heuristic $H3$.

Lemma 5.3. When $a_1/a_2 \geq 6$,

$$C(H3) \leq C(R2) + 2(a_1 + a_2)/\bar{T}_1\bar{T}_2$$

Proof. By construction, the average cost of the basic cycle, $S_2$, developed in step 1 is $C(R2)$. Moreover, $S_2$ contains at most one occurrence of 2 and 3 overlapping, one of 1231 and one of 1132, and it is of length $T = \text{lcm}(\bar{T}_1, \bar{T}_2)$. Substituting $6 \leq a_1/a_2$ in Lemma 2.2, we conclude that $\bar{T}_2 \geq 4$. We distinguish between two cases:

(a) $\text{lcm}(\bar{T}_2, \bar{T}_3) = \bar{T}_2\bar{T}_3$. In this case, $T = \bar{T}_2\bar{T}_3$. The schedule may be made feasible by moving the 2 and the 3 which overlap each one place at a cost of $a_1 + a_2$. Moreover, modifying 1132 and 2311 to 1132 and 2311 adds a cost of at most $2a_2 \leq a_2 + a_3$. We have therefore produced a schedule with average cost of $2(a_1 + a_2)/\bar{T}_2\bar{T}_3$ above that of the relaxed problem.

(b) $\text{lcm}((\bar{T}_2, \bar{T}_3)) \leq \bar{T}_2\bar{T}_3$. In this case, there exist integers $p > 1$ and $\bar{T}_1, \bar{T}_2$ such that $\bar{T}_1 = \bar{T}_1p$ for $i = 2, 3$ and gcd$(\bar{T}_1, \bar{T}_2) = 1$. Thus, $T = \bar{T}_2\bar{T}_3/p = p\bar{T}_2\bar{T}_3$.

We first show that the basic cycle generated in step B1 cannot contain more than one of the three possible occurrences that require a modification by step B2. If 2 and 3 overlap in the schedule, then there exist integers $k_1 \geq 1$ and $l_1 \geq 1$ such that $1 + k_1\bar{T}_1 = 3 + l_2\bar{T}_2$, i.e., $k_1\bar{T}_1 - l_2\bar{T}_2 = 2$, which implies $p = 2$. If the schedule contains 1132, then there exist integers $k_2 \geq 1$ and $l_2 \geq 1$ such that $1 + (l + k_2\bar{T}_2) = 3 + l_2\bar{T}_2$, i.e., $k_2\bar{T}_2 - l_2\bar{T}_2 = 1$, which contradicts the assumption that $p > 1$. If the schedule contains 1231, then there exist integers $k_3 \geq 1$ and $l_3 \geq 1$ such that $1 + (1 + k_2\bar{T}_2) = 3 + l_3\bar{T}_2$, i.e., $k_2\bar{T}_2 - l_3\bar{T}_2 = 3$, implying that $p = 3$.

Thus, the modification to the schedule in step B2 incurs an additional cost only if $p = 2$ or 3. In the former case, $T = \bar{T}_1\bar{T}_2/2$ and the additional cost is $a_2 + a_3$, result-
ing in an increase in the average cost of the schedule relative to the relaxed problem of \((a_1 + a_2)/T = 2(a_1 + a_2)/2T_2 T_3\). In the later case, \(T = T_2 T_3/3\) and the cost associated with the modification of the schedule is \(a_3\). Thus, the increase in the average cost is \(S_T\) relative to \(S_T\) is \(a_3/T = 3a_3/2T_2 T_3 \leq 2(a_3 + a_2)/2T_2 T_3\).

Theorem 5.4.

\[
\frac{C_{(H3)}}{C^*} \leq 1 + \frac{1}{30}
\]

Proof. When \(3a_1/a_2 < 6\), heuristic \(H3\) determines an optimal solution.

When \(a_1/a_2 \geq 6\), from the observation that \(C_{(R2)} < C^*\) and from lemma 5.3,

\[
\frac{C_{(H3)}}{C^*} \leq \frac{C_{(H2)}}{C_{(R2)}} \leq 1 + \frac{2(a_1 + a_2)}{C_{(R2)} T_2 T_3} \leq 1 + \frac{2a_1}{C_{(R2)} T_2 T_3}
\]

Using the value of \(C_{(R2)}\) from lemma 5.1, the inequality \((T_2 - 1)/2 \leq a_1/a_2\) for \(i = 2, 3\) from lemma 2.2, and \(T_2 \leq T_3\), we obtain the following inequalities:

\[
f_2 T_3 C_{(R2)} = T_3 (a_1 + T_2 (T_2 - 1)/2) + T_2 (a_1 + T_2 (T_2 - 1)/2)
\]

\[
\geq T_2 T_3 (T_2 - 1) + T_3 \alpha \gamma \gamma (T_2 - 1)
\]

\[
\geq (a_1 + a_2) (T_2 - 1)/2 T_3
\]

Therefore, \(C_{(H3)}/C^* \leq 1 + 1/(T_2 - 1)/2 T_3\). Now \(T_2 \geq 4\), since \(6 \leq a_1/a_2 \leq T_2 (T_2 + 1)/2\), and may exclude the case \(T_2 = 4\), since then \(H3\) gives the optimal solution. Thus, the largest possible value of the above expression may occur is \(1 + 1/8 = 1 + 1/8\). \(\Box\)

Acknowledgements

The authors are grateful to an anonymous referee for helpful advice which helped to improve the presentation in this paper.

References


