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SOLUTION BASES OF MULTITERMINAL CUT PROBLEMS*

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Gomory and Hu proved that the $\binom{n}{2}$ maxflow problems in an undirected network have at most $n-1$ distinct solution values. We generalize their theorems in several ways. An interesting special case is that the $O(n^4)$ 2-commodity maxflow problems on an undirected network have at most $\binom{n}{2}$ distinct solution values. These results follow from general theorems on the number of distinct solution values to sets of optimization problems. We then show how the solutions can be compactly represented by a basis of the *solution matrix* A, where $a_{ij} = 1$ if solution j is feasible for problem i, and $a_{ij} = 0$ otherwise. For the special case of the 2-cut problem we also show how to construct such a representation efficiently.

1. Introduction. Each pair of nodes in an undirected network with edge capacities can serve as the set of terminals in a minimum cut problem, defined on this network. The set of the $\binom{n}{2}$ problems defined this way is the *multiterminal problem*. Gomory and Hu (1961) proved that this multiterminal problem may have at most $n-1$ distinct optimal solution values. Their theorem was extended by Granot and Hassin (1986) to networks with node capacities. In both cases the max-flow min-cut theorem of Ford and Fulkerson (1956) makes the theorems applicable also for the related max-flow problems.

In this paper we generalize Gomory and Hu's theorem in several ways. We define cuts as partitions of a set of elements, where each cut may have an arbitrary value. The cut problem is to locate the cut with minimum value. In a k-cut problem, k elements are given, and only partitions to k subsets each containing exactly one of the given elements are allowed. Thus there are $\binom{n}{k}$ such problems on a given set of n elements. We show that these problems may have at most $\binom{n-1}{k-1}$ distinct solution values. In a k-pair problem, a partition of the set of elements into two subsets separating k given pairs of elements is sought. For a fixed value of k there are $O(n^{2k})$ such problems and we show that they may have only $O(n^k)$ distinct solution values. For the special case of 2-pairs, the $\binom{\binom{n}{2}}{2}$ problems have at most $\binom{n}{2}$ values. With the max-flow min-cut theorem of Hu (1963) it follows that this bound also applies to the associated multiterminal 2-commodity max flow problem.

We obtain these results by applying general theorems concerning bounds on the number of distinct solution values that a set of problems may have. We also show how to represent these solutions compactly. For the 2-cut problem, we also present an algorithm to construct this compact representation efficiently.

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2. Solution bases. Let v be a real valued function on a set X. X_i, $i \in S$, be nonempty subsets of X, where S is an index set with finite cardinality. Define the *value* of X_i by $V_i = \min\{v(x) | x \in X_i\}$. Let $M(S)$ be the number of distinct values in V_i, $i \in S$. In this paper, X will be the set of solutions of a family of combinatorial problems, and the subsets X_i will be the subsets of solutions that are feasible for the ith problem.

Let $a_i(x) = 1$ if $x \in X_i$ and $a_i(x) = 0$ otherwise. Call a set $S' \subseteq S$ *unary* if for every nonempty subset $S'' \subseteq S'$ there exists some $x \in X$ for which $(a_i(x), i \in S'')$ is a unit vector. Let $R(S)$ be the maximum cardinality of a unary set.

THEOREM 2.1. $M(S) \leqslant R(S)$.

PROOF. Let $S' \subseteq S$ have $|S'| > R(S)$. It suffices to prove that S' cannot have $|S'|$ distinct values. By the definition of $R(S)$ there exists a nonempty subset $S'' \subseteq S'$ such that $a_i(x)$, $i \in S''$, is not a unit vector for any $x \in X$. Let $j \in S''$ and $x \in X_j$ satisfy $v(x) = \min\{V_i | i \in S''\}$, then $x \in X_k$ for some $k \neq j$ in S'', and therefore $V_k = v(x) = V_j$. This proves our assertion. ∎

Call a set $S' \subseteq S$ *dependent* if there exists $S'' \subseteq S'$ such that $\sum_{i \in S''} a_i(x) = 0 \pmod 2$ $\forall x \in X$. Otherwise S' is *independent*. Let $r(S)$, the *rank* of S, be the maximum cardinality of an independent subset of S. Clearly a unary set is independent, so that $R(S) \leqslant r(S)$. With Theorem 2.1 we obtain the following:

COROLLARY 2.2. $M(S) \leqslant r(S)$.

In the next section we apply Corollary 2.2 to obtain upper bounds on the number of distinct solution values to families of combinatorial problems. The set X is then a finite set of possible solutions to the problems, and X_i is the set of feasible solution to problem i, $i \in S$. The binary matrix $A = (a_{ij})$ defines the feasibility relations, where $a_{ij} = 1$ if and only if solution x_j is feasible for problem i, and the value V_i is the minimum value of a feasible solution to problem i.

Let $S' \subseteq S$ correspond to a maximal set of independent rows of A, then we call S' a *solution basis*. The cardinality of each solution basis is equal to $r(A)$, the rank of A in the binary field. Define the *value of a solution basis* S' as $\sum_{i \in S'} V_i$. A *maximum solution basis* is a solution basis with maximum value.

THEOREM 2.3. *Let $S' \subseteq S$ be a maximum solution basis. Let $k \in S \setminus S'$, and let $S'' \subseteq S'$ satisfy $a_k(x) = \sum_{i \in S''} a_i(x) \pmod 2$ $\forall x \in X$. Then $V_k = \min\{V_i | i \in S''\}$ and there exist $p \in S''$ and $y \in X_p \cap X_k$ such that $v(y) = V_k$.*

PROOF. By construction, for every $x \in X_k$ there exists $p \in S''$ such that $x \in X_p$. Therefore $V_k \geqslant \min\{V_i | i \in S''\}$. This inequality cannot be strict since if $V_k > V_p$ for some $p \in S''$ then $(S' \cup \{k\}) \setminus \{p\}$ is a solution basis with the value larger than that of S'. Therefore equality holds and there exist $p \in S''$ and $y \in X_k \cap X_p$ as claimed. ∎

In view of Theorem 2.3, a maximum solution basis contains all the information needed to compute V_k for every $k \in S$. It also gives clues to locate the element of X_k with the value V_k. As a matter of fact, if the values $v(x)$ are distinct for all $x \in X$ then this element is the unique element $y \in X$ satisfying $v(y) = \min\{V_i | i \in S''\}$.

When all of the values V_i, $i \in S$, are given, a maximum solution basis can be computed by the greedy algorithm: Suppose $V_1 \geqslant V_2 \geqslant, \ldots, V_{|S|}$. Insert $i \in S$ to the basis if it does not form a minimal dependent set with any subset $S' \subseteq \{1, \ldots, i-1\}$.

However, in some cases we need not compute directly all of the values V_i, $i \in S$, to form a maximum solution basis. When distinct solutions are known to have distinct values, we can propose the following solution procedure. Suppose that an arbitrary basis S' has been computed. We know that every minimal dependent set contains its minimum value an even number of times. Consider $k \in S \setminus S'$. Let S'' be the subset of S' which forms with k a minimal dependent set. Let $V^* = \min\{V_i | i \in S''\}$. If S'' has an odd number of elements with value V^* then clearly $V_k = V^*$ and V_k need not be computed directly. Otherwise, $V_k > V^*$ and V_k must be computed and then replace an element $j \in S''$ with $V_j = V^*$ to form a new basis with a larger value than S'.

While the above procedure may save computations, no specific amount of saving can be guaranteed in the general case. Specific algorithms can be constructed however for certain families of problems, as we demonstrate in §4.

3. Multiterminal cut problems. We now apply Corollary 2.2 to a class of problems which are often considered in connection with network flows. A *cut* is a partition of $N = \{1, \ldots, n\}$. A *cut problem* consists of a feasibility rule and an objective function that assigns costs to the feasible cuts. A cut problem requires computing an *optimal* cut, that is a cut of minimum cost among all feasible cuts.

In a *k-cut problem*, Q_T, a set $T = (t_1, \ldots, t_k) \subseteq N$ is given, and it is required to find a partition $I_1, \ldots, I_k \subseteq N$ of minimum cost such that $t_j \in I_j$, $j = 1, \ldots, k$:

$$(Q_T) \qquad \begin{aligned} &\min c(x) \\ &\text{s.t.} \\ &x \text{ is binary}, \\ &\sum_{j=1}^{k} x_{ij} = 1, \qquad i = 1, \ldots, n, \\ &x_{t_j, j} = 1, \qquad j = 1, \ldots, k. \end{aligned}$$

Here $x_{ij} = 1$ if element i of N is assigned to I_j. The constraints assure that the elements of T are assigned to distinct subsets of the partition.

We call a feasible solution a k-cut, and the problem of solving Q_T for all $T \subseteq N$ with $|T| = k$ the *multiterminal k-cut problem*. A 2-cut with $T = \{s, t\}$ is called an *s-t* cut.

Gomory and Hu (1961) proved the following theorem on multiterminal 2-cuts:

THEOREM 3.1. *Let $D = (d_{kl})_{k, l \in N}$ be a nonnegative symmetric matrix. For $I \subseteq N$ define $d(I) = \sum_{k \in I, l \in N \setminus I} d_{kl}$. For $i, j \in N$ define*

$$v_{ij} = \min\{d(I) | I \subseteq N, i \in I, j \in N \setminus I\}.$$

Then $\{v_{ij} | i \neq j, i, j \in N\}$ has at most $n - 1$ distinct values.

A simple example demonstrates that a multiterminal k-cut problem may have $M(S) = \binom{n-1}{k-1}$. Let $D = (d_{ij}\ i = 1, \ldots, n,\ j = 1, \ldots, n)$ be a symmetric matrix with $d_{in} = 2^i$ $i = 1, \ldots, n-1$ and $d_{ij} = 0$ otherwise. Let the value of a k-cut be the sum of the d_{ij} values of all pairs i, j that are separated by the cut. For $T = \{t_1, \ldots, t_{k-1}, n-1\}$ with $t_i \neq n$, $i = 1, \ldots, k-1$, the optimal k-cut is $I_j = \{t_j\}$, $j = 1, \ldots, k-1$, and $I_k = N - \{t_1, \ldots, t_{k-1}\}$, with value $\sum_{j \in T \setminus \{n-1\}} 2^j$. For $T = \{t_1, \ldots, t_{k-2}, n-1, n\}$ the optimal k-cut is $I_j = \{t_j\}$, $j = 1, \ldots, k-2$, $I_{k-1} = \{n-1\}$, and $I_k = N - \{t_1, \ldots, t_{k-2}, n-1\}$ with value $\sum_{j \in T \setminus \{n\}} 2^j$. Altogether we have $\binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$ distinct solution values. Theorem 3.2 and Corollary 3.3 prove that this is the maximum number of distinct solution values that a multiterminal k-cut problem may have.

THEOREM 3.2. *For the multiterminal k-cut problem* $r(A) = \binom{n-1}{k-1}$.

PROOF. We prove the theorem by showing that the submatrix B of A corresponding to the problems $\{Q_T \mid |T| = k, 1 \in T\}$ constitutes a solution basis.

1. Independence of the rows of B follows from the following observation:

Let $T = \{t_1, \ldots, t_{k-1}, 1\}$. Then the k-cut $I_j = \{t_j\}$, $j = 1, \ldots, k-1$, and $I_k = N \setminus \{t_1, \ldots, t_{k-1}\}$ is feasible for Q_T but not for any other problem corresponding to a row of B.

2. To prove that B is a basis, we must show that each other row of A not in B forms a dependent set with a subset of rows of B. Let $T = \{t_1, \ldots, t_k\} \subseteq N$, with $1 \in N \setminus T$ and let $T(j) = T \setminus \{t_j\} \cup \{1\}$. We claim that the sum (mod 2) of the k rows corresponding to $Q_{T(j)}$, $j = 1, \ldots, k$, is the row of T. This is implied by the following observations:

(i) Consider any feasible k-cut $I_1, \ldots, I_k$ of Q_T where $t_j \in I_j$, $j = 1, \ldots, k$. This cut is feasible to $Q_{T(j)}$ if and only if $1 \in I_j$. Therefore, every "1" in the Q_T-row appears in the same column in exactly one $Q_{T(j)}$-row.

(ii) Consider a k-cut $I_1, \ldots, I_k$ with $t_j \in I_j$, $j \neq i$, and $1 \in I_i$, which is feasible to problem $Q_{T(i)}$ but not for Q_T. Suppose $t_i \in I_l$, then of all problems $Q_{T(j)}$ this solution is feasible only for $Q_{T(i)}$ and $Q_{T(l)}$. Therefore, if the Q_T-row has "0" in a certain column then this column has "1" in either two or none of the $Q_{T(j)}$-rows. ∎

COROLLARY 3.3. *In the multiterminal k-cut problem* $M(S) \leqslant \binom{n-1}{k-1}$.

In a *k-pair cut problem*, k distinct pairs $\{s_i, t_i\}$ such that $s_i \neq t_i$, $i = 1, \ldots, k$ are given: A k-pair cut is a partition $(I, N \setminus I)$ which is an $s_i - t_i$ cut for all $i = 1, \ldots, k$. The problem is to obtain the k-pair cut of minimum cost:

$$\min c(x)$$

$$\text{s.t.}$$

$$x_{s_i} + x_{t_i} = 1, \qquad i = 1, \ldots, k,$$

$$x \text{ is binary.}$$

We note that for $k \geqslant 3$, the problem will have no feasible solution if the pairs $\{s_i, t_i\}$ include an odd permutation of elements of N. For example $\{1,2\}$, $\{2,3\}$, and $\{3,1\}$.

The *multiterminal k-pair problem* requires solving all of the m-pair problems with $m \leqslant k$. Theorem 3.4 and Corollary 3.5 bound the value of $M(S)$ for this problem. For example, it shows that the $\frac{1}{2}\binom{n}{2}[\binom{n}{2} + 1]$ 2-pair problems may have at most $(n-1) + \binom{n-1}{2} = \binom{n}{2}$ distinct solution values. The number of distinct pairs $\{i, j\}$ is $\binom{n}{2}$. Therefore, for $m \leqslant \binom{n}{2}$, the number of m-pair problems is $\binom{\binom{n}{2}}{m}$, and the multiterminal k-pair problem consists of $\sum_{m=1}^{k} \binom{\binom{n}{2}}{m}$ problems.

The following simple example demonstrates that a multiterminal k-pair problem, with $k < n$ may have $\sum_{m=1}^{k} \binom{n-1}{m}$ distinct solution values. Let $D = (d_{ij}, i = 1, \ldots, n, j = 1, \ldots, n)$ be a symmetric matrix where $d_{ij} = 0$ for all i, j except for $d_{i,i+1}$, $i = 1, \ldots, n-1$. Let the cost of a cut (I, I') be the sum of the values $\{d_{ij} \mid i \in I, j \in I'\}$. For $s_1 < s_2 < , \ldots, < s_m < n$ and $t_i = s_i + 1$ the optimal cut has $I =$

TABLE 1

$\{t_i\}/I$	2	3	4	2,3	2,4	3,4	2,3,4
2	1	0	0	1	1	0	1
3	0	1	0	1	0	1	1
4	0	0	1	0	1	1	1
2,3	0	0	0	1	0	0	1
2,4	0	0	0	0	1	0	1
3,4	0	0	0	0	0	1	1
2,3,4	0	0	0	0	0	0	1

$\{1,\ldots,s_1,t_2,\ldots,s_3,t_4,\ldots,s_5,\ldots,\}$ with cost $\sum_{i=1}^{m} d_{s_i,t_i}$. For $m \leqslant n-1$ fixed, this may contribute up to $\binom{n-1}{m}$ distinct values, and the multiterminal k-pair problem with $k < n$ may then have up to $\sum_{m=1}^{k}\binom{n-1}{m}$ distinct values.

Theorem 3.4 shows that this is also an upper bound on the number of distinct values. We note that for $k \geqslant n-1$ the theorem implies $r(A) = 2^{n-1} - 1$, which is the number of possible partitions of n elements into two nonempty sets.

THEOREM 3.4. *The multiterminal k-pair problem has* $r(A) = \sum_{m=1}^{\min(k,n-1)}\binom{n-1}{m}$.

PROOF. Let B be the set of rows of A corresponding to m-pair problems with $m \leqslant k$ and $s_i = 1$, $i = 1,\ldots,m$. There are $\binom{n-1}{m}$ such problems with exactly m distinct elements $t_i \neq 1$, $i = 1,\ldots,m$, where $m \leqslant n-1$. Therefore, $|B| = \sum_{m=1}^{\min(k,n-1)}\binom{n-1}{m}$, and it suffices to prove that B is a solution basis.

1. Independence of B follows from the following observation: A cut (I, I') with $|I| = m$ and $1 \in I'$ solves, of all the problems corresponding to B with *at least* m pairs, only the one with $\{t_1,\ldots,t_m\} = I$. This property holds for $m = 1,\ldots,$ $\min(k, n-1)$. Therefore, a triangular matrix can be obtained from the rows of B and the columns corresponding to cuts (I, I') with $1 \in I'$ and $|I| \leqslant \min(k, n-1)$. Table 1 shows this matrix for $n = 4$ and $k = 3$.

2. That B is a basis of A follows from the following observation:

Let Q be an m-pair problem, with pairs $\{s_i, t_i\}$, $i = 1,\ldots,m$. Let S' be the subset of B corresponding to all m-pair problems with pairs $\{1, q_i\}$, $i = 1,\ldots,m$, where $q_i \in \{s_i, t_i\}$, $q_i \neq 1$. We claim that the sum of rows of S' is equal to the row of Q (mod 2).

(i) Let (I, I') be feasible for Q, and without loss of generality, suppose $1 \in I$. Let $R = \{r_1,\ldots,r_m\} = I' \cap \{s_1,\ldots,s_m,t_1,\ldots,t_m\}$. Then of all problems corresponding to S', (I, I') is feasible only for the one with pairs $\{1, r_i\}$, $i = 1,\ldots,m$.

(ii) Suppose that (I, I') is not feasible for Q but feasible for some Q' in S'. Without loss of generality, suppose that Q' has pairs $\{1, s_i\}$, $i = 1,\ldots,m$, and that $1 \in I$. Thus $s_1,\ldots,s_m \in I'$, and not all of $t_1,\ldots,t_m$ are in I (otherwise (I, I') is feasible for Q). Suppose that $t_1,\ldots,t_r$ are in I' where $1 \leqslant r \leqslant m$. In this case, (I, I') is feasible for the 2^r problems of S' with pairs $\{1, q_i\}$, $i = 1,\ldots,r$, $q_i \in \{s_i, t_i\}$, and $\{1, s_i\}$, $i = r + 1,\ldots,m$. Therefore, the column corresponding to (I, I') has an even number of "1" elements in rows of S' summing to zero (mod 2) in accordance with the claim. ∎

COROLLARY 3.5. *For the multiterminal k-pair problem* $M(S) \leqslant \sum_{m=1}^{\min(k,n-1)}\binom{n-1}{m}$.

We note that Corollaries 3.3 and 3.5 still hold if certain sets of cuts are excluded from consideration. This can be done either by substituting high values for the forbidden cuts or by eliminating the associated columns from the solution matrix. This operation may only decrease $r(A)$ so that the corollaries still hold. For example, we may consider only cuts (I, I') where $a \leqslant |I| \leqslant b$. In another example, a network may be given where each edge has both a capacity and a cost, and only cuts whose total cost does not exceed a given budget are considered.

Finally, the theorem of Granot and Hassin (1985), generalizing Theorem 3.1 to networks with both node and edge capacities, is not a special case of either Corollary 3.3 or Corollary 3.5. However, it can also be obtained from Corollary 2.2 by identifying an appropriate solution basis.

4. Maximum solution bases of the multiterminal 2-cut problem. Let $G = (N, E)$ be a complete undirected graph with a node set $N = \{1, \dots, n\}$ and an edge set E. Consider a given multiterminal 2-cut problem. Let each edge $(i, j) \in E$ be associated with the minimum $i - j$ cut problem, and assign to it a weight V_{ij}, equal to the value of the minimum $i - j$ cut.

LEMMA 4.1. *A set of edges of G is associated with a minimal dependent set of the solution matrix A if and only if it is a simple cycle of G.*

PROOF. (i) Consider a forest F of G. Let (p, r) be an edge of a tree T in this forest, and let T_1, T_2 be the set of nodes of the components of T that are formed when (p, r) is deleted from F. Then the cut $(T_1, N \setminus T_1)$ is a $p - r$ cut but not an $i - j$ cut for any $(i, j) \in F$ other than (p, r). Therefore, a forest of G is associated with an independent set of A.

Consider now a simple cycle C of G. It is easily seen that each 2-cut either contains C in one of its subsets, or it separates C to an even number of simple paths. Thus a 2-cut is feasible for an even number of $i - j$ cut problems corresponding to edges of C, namely the edges where C is disconnected to form the paths. Thus the set of rows in A corresponding to C is dependent, and in view of the previous argument it is a minimal dependent set.

(ii) A dependent set S corresponds to a set of edges in G containing a simple cycle (otherwise it is a forest and cannot be dependent). Therefore a minimal dependent set is a simple cycle. ∎

COROLLARY 4.2. *A set of 2-cut problems is a maximum solution basis of the multiterminal 2-cut problem if and only if it is associated with a maximum spanning tree (MST) of G.*

We could solve the $\binom{n}{2}$ 2-cut problems of the multiterminal problem and then apply any algorithm for computing an MST. However, each problem may be hard to solve and therefore we pose the following question: Can we compute an MST of G without actually solving all $\binom{n}{2}$ problems?

For the case of Theorem 3.1, Gomory and Hu (1961) showed how an MST can be constructed by solving exactly $n - 1$ 2-cut problems. They use there a noncrossing property of the optimal 2-cuts which does not hold in general even if cut values are defined as in Theorem 3.1 (for example if we require 2-cuts $(I, N \setminus I)$ where $|I| = n/2$, or if we consider the 2-cut problems defined by 2-pair cuts where one pair is fixed for all of the problems). Granot and Hassin (1986) show that it is also possible to solve the multiterminal problem with node capacities by solving exactly $n - 1$ 2-cut problems.

In the following we show, under Assumption 4.3, that $O(n \log n)$ optimal 2-cut computations are sufficient.

ASSUMPTION 4.3. *Distinct 2-cuts have distinct values.*

We note that this property can be guaranteed by appropriate perturbation of the value function (cf. Eaves and Rothblum 1985).

LEMMA 4.4. *Let T be an MST of G. Let P_{ij} be the path connecting i and j on T. Let $k \neq i, j$ be on P_{ij}. Then $V_{ik} \neq V_{jk}$.*

PROOF. $V_{ik}(V_{jk})$ is equal, by Theorem 2.3 and Lemma 4.1, to the minimum weight of an edge of the subpath of P_{ij} connecting i and k (j and k), while V_{ij} is equal to the minimum of all edge weights on P_{ij}. Thus, $V_{ik} = V_{jk}$ implies $V_{ik} = V_{jk} = V_{ij}$. Since an $i - j$ cut cannot be both an $i - k$ and a $j - k$ cut, this means that there are two cuts with identical values, in contradiction to Assumption 4.3. ∎

In the following we denote by G_m the subgraph of G induced by nodes $1, \ldots, m$.

LEMMA 4.5. *Let T be an MST of G_{k-1}, where $3 \leqslant k \leqslant n - 1$. Suppose that $V_{k,r} = \max\{V_{k,i} | i = 1, \ldots, k-1\}$, where $r \in \{1, \ldots, k-1\}$. Then $T \cup \{(k, r)\}$ is an MST of G_k.*

PROOF. Consider an MST T' of G_k, such that $(k, r) \notin T'$. Then k and r are connected in T' by a path containing an edge (k, p), $p \in \{1, \ldots, k-1\}$. The tree obtained from T' by replacing (k, p) with (k, r) is therefore an MST of G_k.

Consider now an MST, T' of G_k containing (k, r) and at least one additional edge (k, p), $p \neq r$. Then by Theorem 2.3 and Lemma 4.2, $V_{rp} = \min\{V_{kr}, V_{kp}\}$ so that the tree obtained from T' by replacing (k, p) by (r, p) is also an MST of G_k. Repeating this step we end with an MST of G_k consisting of (k, r) and an MST of G_{k-1}. Since every MST of G_{k-1} has the same sum of weights, this tree has the same weight as $T \cup \{(k, r)\}$. Therefore, $T \cup \{(k, r)\}$ is an MST of G_k. ∎

Consider a tree T with k nodes. A *centroid* of T is a node r of T, such that if r and the edges incident with it are deleted from T, none of the resulting components contains more than $k/2$ nodes. A centroid always exists although it need not be unique.

THEOREM 4.6. *An MST of G can be constructed by computing $O(n \log n)$ optimal 2-cuts.*

PROOF. We show how an MST of G_k can be obtained from an MST T of G_{k-1} by computing $O(\log n)$ optimal 2-cuts. By Lemma 4.5, all we need is to locate a node r in G_{k-1} such that $V_{kr} = V^*$ where $V^* = \max\{V_{ki} | i = 1, \ldots, k-1\}$.

Let p be a centroid of T. Compute V_{kp}. Let M be the set of nodes of G_{k-1} with $V_{mp} = V_{kp}$, $m \in M$.

By Lemma 4.4, M is contained in a single component of the forest obtained by deleting p and the edges incident to it from T. Since p is a centroid of T, $|M| \leqslant (k-1)/2$.

Consider a node $q \in \{1, \ldots, k-1\} \setminus M$. Suppose that $V_{kq} = V^*$. Then $T \cup \{(k, q)\}$ is an MST of G_k and thus $V_{kp} = \min\{V_{kq}, V_{qp}\}$. Since $q \notin M$ then $V_{kp} \neq V_{pq}$ and thus $V_{kp} = V_{kq} = V^*$. We conclude that if $V_{kp} \neq V^*$ then only edges (k, m), $m \in M$, can have $V_{km} = V^*$.

We repeat the above step (by computing the centroid of the component containing M, etc.) till we reach a situation where M is empty, in which case the edge with the longest value computed in this process must have value V^*. Clearly, the number of steps until M becomes empty is $O(\log n)$. ∎

5. Concluding remarks. We presented some general results in §2 and applied them to special cases in §3, to obtain bounds on the number of distinct solution values in sets of problems. We showed through examples where $M(N) = r(N)$ that these bounds cannot be improved in the special cases discussed in §3. On the other hand the following example illustrates that $M(N)$ may be strictly less than $r(N)$ in other cases. Consider the 2-cut problem with $n = 4$, and costs $c(x) = 1$ for $x = (1,1,0,0)$, $c(x) = 2$

for $x = (1,0,1,0)$, and $c(x) > 2$ for other solutions. Then, the minimum $1 - 2$ and $3 - 4$ cuts have value 2 while all other minimum $i - j$ cuts have value 1, so that $M(N) = 2$, while $r(N) = 3$.

It is interesting to note that when distinct cuts have distinct values (Assumption 4.3), the bound in Theorem 3.1 holds with equality, i.e., $M(N) = r(N) = n - 1$. This follows from the noncrossing property of the solutions, which does not hold in the general 2-cut problem. The question of under what conditions Assumption 4.3 guarantees that $M(N) = r(N)$ is open. We conjecture that this is the case at least for the 2-commodity multiterminal flow problem (Hu 1963), which is associated with the 2-pair cut problem on undirected networks.

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