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MULTITERMINAL XCUT PROBLEMS

Refael HASSIN
School of Mathematics, Tel-Aviv University, Tel-Aviv 69978, Israel

Abstract

An $i–j$ xcut of a set $V = \{1, \ldots, n\}$ is defined to be a partition of V into two disjoint nonempty subsets such that both i and j are contained in the same subset. When partitions are associated with costs, we define the $i–j$ xcut problem to be the problem of computing an $i–j$ xcut of minimum cost. This paper contains a proof that the $\binom{n}{2}$ minimum xcut problems have at most n distinct optimal solution values. These solutions can be compactly represented by a set of n partitions in such a way that the optimal solution to any of the problems can be found in $O(n)$ time. For a special additive cost function that naturally arises in connection to graphs, some interesting properties of the set of optimal solutions that lead to a very simple algorithm are presented.

1. Introduction

Let $V = \{1, \ldots, n\}$ be a given set. A *cut* of V is a partition (S, T) of V into two disjoint nonempty subsets S and T. For each cut (S, T) let $c(S, T)$ be its *cost* (or *value*), where c is an arbitrary real function defined on the cuts of V. In this paper, we assume that c is symmetric, i.e. $c(S, T) = c(T, S)$. For a given pair of distinct elements $i, j \in V$, an *$i–j$ cut* is a cut (S, T) such that either $i \in S$ and $j \in T$, or $i \in T$ and $j \in S$. In this paper, we define an *$i–j$ xcut* to be a cut (S, T) such that either $i, j \in S$ or $i, j \in T$. The *minimum $s–t$ cut (xcut) problem* is to find the $s–t$ cut (xcut) of minimum cost. An example of such a problem is a scheduling problem on two machines with a constraint that a given pair of jobs must be processed on the same machine. A possible objective function is to minimize the makespan. In this case, $c(S, T) = \max\{c(S), c(T)\}$, where $c(X) = \sum_{i \in X} t_i$ and t_i is the processing requirement for job i. A *multiterminal* problem requires solving a set of $i–j$ problems for different $i–j$ pairs. As an example of a multiterminal xcut problem, consider a communication network where an option exists to connect one pair of users by a line of a very high capacity, and it is desired to assess the effects of applying this option to any member of a given set of pairs. For any given realization of the option, the edge connectivity of the resulting network will be equal to the minimum xcut with respect to the pair chosen to be connected by the high-capacity line. To choose the best option, one has to solve a multiterminal xcut problem.

Gomory and Hu [3] analyzed the multiterminal cut problem for a special additive cost function. Since problems with such costs usually arise in connection with graphs, we call them *graphic* problems. Gomory and Hu proved that the $\binom{n}{2}$

graphic cut problems have at most $n-1$ distinct solutions. They also constructed a data structure (*a cut tree*) that stores these values in such a way that allows the determination of the solution to any of the problems in linear time. They proved that a cut tree can be constructed from a set of $n-1$ *noncrossing* cuts, and use this property to compute a cut tree by solving only $n-1$ cut problems. These results were extended by Granot and Hassin [4] to networks with node-capacities and by Gusfield and Naor [5] to a compact representation of *all* of the minimum cuts for each pair of elements. Hassin [6,7], and Cheng and Hu [1] extended Gomory and Hu's results by showing that the multiterminal problem has a most $n-1$ distinct solutions for arbitrary cost functions. Hassin's result comes from a general theorem that also supplies a compact representation to the solutions (called a *maximum solution basis*), generalizing the concept of a cut tree (see section 2).

In this paper, we obtain related results for the multiterminal xcut problem. In section 3, we characterize the structure of the solution bases and show that the number of distinct solutions is at most n. In section 4, we consider multiterminal problems consisting of a mixed set of cut and xcut problems and characterize their solution bases. Finally, in section 5, we consider graphic multiterminal problems and describe a particularly simple algorithm for this special case.

2. Solution bases and the 2-forest matroid

Let c be a real valued function on a set X. Let X_i, $i \in I$, be nonempty subsets of X, where I is an index set with finite cardinality. Define the *cost* (or *value*) of X_i by

$$c_i = \min\{c(x) \mid x \in X_i\}.$$

Let $M(I)$ be the number of distinct values of c_i, $i \in I$. Let $a_i(x) = 1$ if $x \in X_i$ (i.e. x is *i-feasible*) and $a_i(x) = 0$ otherwise.

For our purpose, the set X is the set of candidate solutions to a family of problems, and X_i is the set of i-feasible solutions, $i \in I$. The binary matrix $A = (a_i(x))$, called the *solution matrix*, defines the feasibility relations, and the cost c_i is the minimum cost of an i-feasible solution (i.e. the cost of an *i-optimal* solution).

Call a set $I' \subseteq I$ *dependent* if there exists $I'' \subseteq I'$ such that $\sum_{i \in I''} a_i(x) = 0 \ (mod\ 2)\ \forall x \in X$. Otherwise, I' is *independent*. Let $r(A)$ be the *rank* of A in the binary field, i.e. the maximum cardinality of an independent subset of I.

THEOREM 2.1 [6]

$$M(I) \le r(A).$$

A *solution basis* is a set $I' \subseteq I$ which corresponds to a maximal set of independent rows of A. The cardinality of each solution basis is equal to $r(A)$. Define the *value of a solution basis* I' as $\sum_{i \in I'} c_i$. A *maximum solution basis* is a solution basis with maximum value.

THEOREM 2.2 [6]

Let $I' \subseteq I$ be a maximum solution basis. Let $k \in I \setminus I'$, and let $I'' \subseteq I'$ satisfy $a_k(x) = \sum_{i \in I''} a_i(x)\ (mod\, 2)\ \forall x \in X$. Then, $c_k = \min\{c_i \mid i \in I''\}$ and there exists $p \in I''$ and $y \in X_p \cap X_k$ such that $c(y) = c_k$.

In view of theorem 2.2, a maximum solution basis contains all the information needed to compute c_k for every $k \in I$. It also gives clues to locate a k-optimal solution. As a matter of fact, if the values $c(x)$ are distinct for all $x \in X$, then this solution is the unique element $y \in X$ satisfying $c(y) = \min\{c_i \mid i \in I''\}$. If these values are not distinct, then any solution $y \in X$ with cost c_k which has an odd column-sum in the rows of A corresponding to I'' can be shown to be k-optimal. Hassin [7] describes a procedure that computes a maximum solution basis by solving $r(A) + M(I)$ problems.

Let $G = (V, E)$ be a graph with a vertex set V and an (undirected) edge set E. A *2-forest* of G is a subgraph of G that either has no cycles or has exactly one cycle, and this cycle is of odd cardinality.

THEOREM 2.3 [2]

Let X be the set of 2-forests of G. Then $\mathcal{M}_2 = (E, X)$ is a matroid (the 2-forest matroid). Its rank (i.e. the cardinality of its bases, the maximal 2-forests) is equal to $|V| - p - \alpha$, where $p + 1$ is the number of components of G and $\alpha = 1$ if G is bipartite, 0 otherwise.

We note that if G is a complete graph with $n \geq 3$, then the rank of $\mathcal{M}_2$ is just $|V|$.

3. The multiterminal xcut problem

In the i–j xcut problem, it is required to find a cut (S, T) of V of minimum cost such that $i, j \in S$ or $i, j \in T$. Each cut (S, T) has a cost $c(S, T)$ and $c(S, T) = c(T, S)$. Call a cut *globally minimum* if it has the minimum cost among all of the cuts of V. Let $c_{\min}$ denote the cost of a globally minimum cut.

A simple example demonstrates that a multiterminal xcut problem may have n distinct optimal solutions. Let the cut (S, T) with $S = \{1\}$ be globally minimum. Let the next $n - 1$ least cost cuts have $S = \{1, i\}$ for $i = 2, \ldots, n$. The solution to all of the i–j xcut problems in which $i \neq 1$ and $j \neq 1$ is the cut with $S = \{1\}$. The solution to the 1–i xcut problem is the cut with $S = \{1, i\}$ for $i = 2, \ldots, n$. Thus, altogether there are n distinct solutions.

We will now characterize the solution bases for the multiterminal xcut problem. As we will show, the cardinality of these bases is at most n, and therefore n is also a least upper bound on the number of distinct solutions for this set of problems.

Consider a given multiterminal xcut problem where it is required to solve the set of i–j problems for the (unordered) pairs $(i, j) \in \tilde{E}$. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be an undirected graph with a node set $\tilde{V} = \{1, \ldots, n\}$ and an edge set $\tilde{E}$. Let each edge $(i, j) \in \tilde{E}$ be associated with the minimum $i-j$ xcut problem and the corresponding row in A.

LEMMA 3.1

The edge set of an even cycle of $\tilde{G}$ is associated with a dependent set of rows of the solution matrix.

Proof

Consider a cut (S, T) and an even cycle C. Starting in an arbitrary vertex of C and following its edges, we alternate between vertices in S and vertices in T. We return to the initial vertex after an even number of alternations. This corresponds to an even number of edges of C that are associated with pairs i, j for which the cut (S, T) is not feasible (i.e. for which it is an $i-j$ cut and therefore not an $i-j$ xcut). Since $|C|$ is even, the number of edges for which the cut is feasible is also even. Thus, the sum of the rows in the solution matrix which correspond to the edges of C is zero (*mod* 2) in the column corresponding to the cut (S, T). Since this is true for every cut, it follows that this set of rows is dependent. □

LEMMA 3.2

The union of two odd cycles of $\tilde{G}$ is associated with a dependent set of rows of the solution matrix.

Proof

Similar to the proof of the previous lemma. □

LEMMA 3.3

A 2-forest of $\tilde{G}$ is associated with an independent set of rows of the solution matrix.

Proof

Let F be a 2-forest of $\tilde{G}$. Suppose first that F is a forest. Consider an edge $(i, j) \in F$. F is bipartite and has a bipartition (U, W) such that both i and j are in U and all of the edges in $F \setminus \{(i, j)\}$ have one end in U and the other in W. Thus, of all the problems associated with F, the cut (U, W) is feasible only for the one associated with (i, j). It follows that the rows of A associated with F with the columns corresponding to the above cuts form an identity matrix and hence these rows are independent.

Suppose now that F contains an odd cycle of C. Fix an edge $e \in C$. Let F' be the forest obtained from F by deleting e. Consider an edge $(i, j) \in C$. $F \setminus \{(i, j)\}$ is a forest and thus a bipartite subgraph with two sets of vertices U, W. Since C is an odd cycle, i and j are in the same vertex set, say in U. Thus, the cut $(U, V\setminus U)$ is an $i-j$ xcut, but not a $u-v$ xcut for any $(u, v) \in F\setminus\{(i,j)\}$. Consider now an edge $(i,j) \in F\setminus C$. Consider the forest $F'' = F'\setminus\{(i,j)\}$. Choose for it a bipartition (U, W) such that both i and j are in U. This is possible since i and j are in different components of F''. Moreover, all of the edges in F'' have one end in U and the other in W, and since C is odd, both ends of e will be the same set U or W. Therefore, of all the problems associated with F, the cut (U, W) is feasible only for those associated with (i,j) and e.

Consider the submatrix of the solution matrix composed of the rows corresponding to F (where those contained in C come first), and the columns corresponding to the cuts described above. Permute the columns so that their order will be as that of their corresponding problems. The resulting matrix has a unit main diagonal with zero entries below it. Therefore, it is of a full rank and the corresponding rows are independent as required. □

Combining the above lemmas, we obtain a characterization of the solution bases of a multiterminal xcut problem:

THEOREM 3.4

The set of solution bases for the multiterminal xcut problem corresponds to the set of bases of the 2-forest matroid $\mathcal{M}_2$ of $\tilde{G}$ (i.e. the maximal 2-forests).

COROLLARY 3.5

Let A be the solution matrix corresponding to a set $\tilde{E}$ of xcut problems. Let $p+1$ be the number of components of $\tilde{G}$ and $\alpha = 1$ if $\tilde{G}$ is bipartite, 0 otherwise. Then $r(A) = n - p - \alpha$.

Suppose that we attach to each edge (i,j) of $\tilde{G}$ a weight of c_{ij} equal to the value of a minimum $i-j$ xcut. A maximum solution basis can be computed using the greedy algorithm if all c_{ij} values are known, or using the procedure described in Hassin [7] by solving at most $2r(A)$ problems.

COROLLARY 3.6

If $\tilde{G}$ is bipartite, then a maximum solution basis consists of a maximum spanning forest of $\tilde{G}$. If $\tilde{G}$ is not bipartite, then a maximum solution basis consists of a maximum spanning forest of $\tilde{G}$ and an edge associated with a globally minimum cut.

Proof

The first part of the theorem is immediate from theorem 3.4 and the definition of a 2-forest. Suppose that $\tilde{G}$ is not bipartite. Let (S, T) be an arbitrary cut. The subgraph of $\tilde{G}$ induced by the pairs i, j for which this cut is not an $i-j$ xcut is clearly bipartite. Therefore, any odd cycle of $\tilde{G}$ contains at least one pair for which the cut is an xcut. In particular, if we choose (S, T) to be a globally minumum cut, then each odd cycle contains an edge associated with a pair for which this cut is optimal. This is true also with respect to the odd cycle contained in a maximum maximal 2-forest of $\tilde{G}$. Since $c_{\min}$ is less than any other edge-weight, the subgraph obtained by deleting this edge from the cycle is a maximum spanning forest of $\tilde{G}$. □

Given a maximum solution basis $F_{\max}$ one can easily obtain, through theorem 2.2, the solution to any given problem. Let the problem be the $i-j$ xcut problem. Suppose that (i, j) is in $\tilde{E}$ but not in $F_{\max}$. If $\tilde{G}$ is bipartite, then (i, j) forms a unique cycle with the edges of $F_{\max}$, and this cycle is even. c_{ij} is then equal to the minimum value of an edge in the edges of $F_{\max}$ contained in this cycle. Suppose now that $\tilde{G}$ is not bipartite. Let C be the (odd) cycle in $F_{\max}$. (i, j) forms with the edges of $F_{\max}$ at most one even cycle or a union of two odd cycles. This is so since if it forms a cycle containing edges $L \subset C$ $(0 < |L| < |C|)$, then it also forms exactly one other cycle, with $C \setminus L$ replacing L. However, since C is odd, exactly one of these cycles is even. On the other hand, if (i, j) forms a simple cycle with edges of $F_{\max}$ which is disjoint from C, then either this is an even cycle or its union with C is even (but not both). Again, c_{ij} is computed by taking the minimum value over the values associated with the edges of $F_{\max}$ contained in the even cycle or in the union of two odd cycles. In the latter case, this is just $c_{\min}$ and the problem is solved by a globally minimum cut.

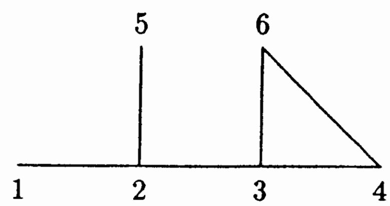

Fig. 1.

Figure 1 illustrates a maximum solution basis for a multiterminal xcut problem. To compute c_{56}, we use the fact that 2–3–6–5–2 is an even cycle and thus $c_{56} = \min\{c_{25}, c_{23}, c_{36}\}$. To compute c_{15}, we note that 1–2–5 is an odd cycle so that to span (1, 5) we also use the cycle 3–4–6–3. Thus, $c_{15} = \min\{c_{12}, c_{25}, c_{34}, c_{46}, c_{36}\} = c_{\min}$.

4. Mixed multiterminal problems

Let E_c and E_x be (not necessarily disjoint) subsets of pairs of elements of V. Consider the multiterminal problem requiring the solution of the $i-j$ cut problems for each $\{i, j\} \in E_c$ and the $i-j$ xcut problems for each $\{i, j\} \in E_x$. The requirements of the problem can be described by means of a multigraph $\hat{G} = (\hat{V}, \hat{E})$ with a vertex set $\hat{V} = \{1, \ldots, n\}$ and an edge set $\hat{E} = E_c \cup E_x$. We denote the edges corresponding to E_c as *c-edges* and those corresponding to E_x as *x-edges*. Note that $\hat{G}$ may have pairs of parallel x- and c-edges.

Let A be the solution matrix corresponding to the set of problems defined by $\hat{E}$. Since $\hat{G}$ may have parallel edges, it should be noted that by a *cycle* of $\hat{G}$ we refer to a sequence of *edges*. We call a cycle *odd* if it has an odd number of x-edges. Otherwise, it is *even*. Note that a pair of parallel x- and c-edges defines an odd cycle. It is quite straightforward to modify the proofs of lemmas 3.1 and 3.2 to the mixed case with the modified definitions of odd and even cycles. Moreover, we can modify the definition of a 2-forest using the modified definition of an odd cycle and show that theorem 2.3 holds for the modified definition as well.

LEMMA 4.1

A 2-forest of $\hat{G}$ is associated with an independent set of rows of A.

Proof

The proof is similar to that of lemma 3.3. Let F be a 2-forest of $\hat{G}$. If F is a forest, then for each $(i, j) \in F$ we can construct a cut (U, W) that is feasible for the problem associated with (i, j) but not for any problem associated with another edge of F. This is done by assigning i and j to U if (i, j) is an x-edge, and assigning i to U and j to W if it is a c-edge. Then we follow paths on F, assigning nodes to U and V in such a way that the resulting cut will not be feasible for any of the corresponding problems.

If F contains a cycle C, then we fix some x-edge $e \in C$. Such an edge exists since C contains an odd number of x-edges. Now, as above, we can construct for any edge in C a cut that is feasible only for the corresponding problem, and for any edge in $F \setminus C$ a cut that is feasible for the corresponding problem and the one associated with e, but not for any other problem associated with an edge of F.

As in the proof of lemma 3.3, lemma 4.1 follows from these observations. □

Combining lemmas 3.1, 3.2, and 4.1, we obtain the following extension to theorem 3.4:

THEOREM 4.2

The set of solution bases for the mixed multiterminal problem corresponds to the set of maximal 2-forests of $\hat{G}$.

Let $\hat{G}_x$ be the graph obtained from $\hat{G}$ by condensing all of its c-edges (i.e. considering the end vertices of each c-edge as a single vertex). Note that a pair of parallel c- and x-edges of $\hat{G}$ generates a loop in $\hat{G}_x$, in which case the latter is not bipartite.

From theorems 2.3 and 4.2, using the modified definition of an odd cycle, we obtain the following:

COROLLARY 4.3

Let A be the solution matrix corresponding to a set $\hat{E}$. Let $p+1$ be the number of components of $\hat{G}$ and $\alpha = 1$ if $\hat{G}_x$ is bipartite, 0 otherwise. Then $r(A) = n - p - \alpha$.

It is interesting to note that the least upper bound on the number of distinct solutions for the mixed problem is n, as it is for the multiterminal xcut problem.

Corollary 3.6 can be modified for the mixed case. In particular, under the modified definition an odd cycle contains a pair for which a cut of value $c_{\min}$ is feasible and thus optimal. A minimal dependent set consists of either a simple even cycle or a union of two simple odd cycles, where again "odd" and "even" refer to the number of x-edges. Each edge not in the basis forms exactly one set of this form with the edges of the basis. The value of a problem is computed by taking the minimum over the cut values associated with the other edges of the minimal independent set generated by the problem.

Figure 2 illustrates a maximum solution basis for a mixed problem. The letters c or x on each edge are used to distinguish the c- and x-edges. In this paragraph, we use c_{ij} for the optimal cut value and x_{ij} for the optimal xcut value. $x_{15} = \min\{x_{12}, c_{25}\}$,

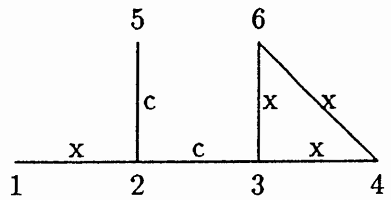

Fig. 2.

while $c_{15} = \min\{x_{12}, c_{25}, x_{34}, x_{36}, x_{46}\} = c_{\min}$. Similarly, $x_{56} = \min\{c_{25}, c_{23}, x_{36}\}$, while $c_{56} = \min\{c_{25}, c_{23}, x_{34}, x_{46}\}$. Note that for a given pair $\{i, j\}$, any given cut is either an i–j cut or an i–j xcut. Therefore, exactly one of this pair of problems is solved by any globally minimum cut.

5. Graphic xcuts

In this section, we consider the problem of constructing a maximum solution basis for the graphic multiterminal problem consisting of the $\binom{n}{2}$ i–j problems. We could solve each of the problems first and then apply a greedy algorithm for computing a maximum basis. Alternatively, we could solve at most $2n$ problems using the procedure described by Hassin [7]. In this section, we prove that for the graphic case one can construct a maximum solution basis by solving just n problems in addition to a preprocessing step that computes a globally minimum cut of the graph and is computationally equivalent to an additional problem. This result is analogous to that of Gomory and Hu for the multiterminal cut problem in the sense that the number of problems to be solved is equal to the cardinality of the solution basis.

We will also describe an algorithm that constructs a very simple data structure from which the solution to any given problem can be easily obtained. Here, in contrast to Gomory and Hu's algorithm, the algorithm does not construct a maximum solution basis, but each vertex of the graph will be associated with a cut. To obtain a solution to an $i-j$ xcut problem, one first needs to check whether the globally minimum cut is feasible for the problem (in which case, it also solves it). If not, then the problem is solved by the better of the cuts associated with its two defining vertices i and j.

Let $G = (V, E)$ be a given (undirected) graph with a vertex set $V = \{1, \ldots, n\}$ and an edge set E. Let σ be a cost function from E to $\mathbb{R}_+$. Define for disjoint subsets $A, B \subset V$, $c(A, B) = \sum_{\{(i,j) \in E \mid i \in A, j \in B\}} \sigma_{ij}$. In particular, the cost of a cut (S, T) of V is $c(S, T)$. The $i-j$ graphic xcut problem is defined with the above special cost function.

DEFINITION 5.1 [3]

Two cuts (S, T) and (S', T') are said to *cross* each other if and only if each of the four sets $S \cap S'$, $S \cap T'$, $T \cap S'$, $T \cap T'$ is nonempty.

The following theorem is analogous to a theorem by Gomory and Hu for the multiterminal cut problem on graphs.

THEOREM 5.2

In a graphic multiterminal xcut problem, there is a set of optimal xcuts that do not cross.

Proof

Let $(X, \bar{X})$ and $(Y, \bar{Y})$ be optimal $i-j$ and $k-l$ xcuts, respectively. Suppose that $i, j \in X$ and $k, l \in Y$. Define $P = X \cap Y$, $Q = X \cap \bar{Y}$, $R = \bar{X} \cap Y$, and $S = \bar{X} \cap \bar{Y}$. Suppose that none of the sets P, Q, R, S are empty, so that $(X, \bar{X})$ and $(Y, \bar{Y})$ cross. We will show that there must be another optimal $k-l$ xcut that does not cross $(X, \bar{X})$.

The optimality of $(X, \bar{X})$ and the feasibility of $(R, V \setminus R)$ for the $i-j$ problem imply

$$c(X, \bar{X}) \le c(R, V \setminus R),$$

and this is equivalent to

$$c(P, S) + c(Q, S) \leq c(R, S).$$

With the nonnegativity of c, this implies that

$$c(Q, S) \leq c(R, S) + c(P, S),$$

and therefore

$$c(Q, V \setminus Q) \leq c(Y, \bar{Y}).$$

Since $(Q, V \setminus Q)$ is a $k-l$ xcut, it follows from the last inequality that it is an optimal solution for this problem (and equality holds there). Moreover, this cut does not cross $(X, \bar{X})$, as required. □

For a set $S \subset V$, we define the $i-S$ xcut problem as the one obtained when the elements of $S \cup \{i\}$ are restricted to be in the same subset of the cut. As before, let $c_{\min} = \min\{c_{ij} | i, j \in V, i \neq j\}$ be the value of a globally minimum cut in G and c_{ij} the solution value for the $i-j$ cut problem.

ALGORITHM 5.3

(Multiterminal graphic xcut algorithm)

(1) Compute a cut (S, T) with value $c(S, T) = c_{\min}$. This is the optimal $i-j$ xcut for all i, j pairs such that $i, j \in S$ or $i, j \in T$.

(2) For every $i \in S$, compute an optimal $i-T$ xcut. Let the value of this cut be c_{iT}. [Execute this step only if $\{i\} \cup T \neq V$, else set $c_{iT} = \infty$.]

(3) For every $j \in T$, compute an optimal $S-j$ xcut. Let the value of this cut be c_{Sj}. [Execute this step only if $\{j\} \cup S \neq V$, else set $c_{Sj} = \infty$.]

(4) For every $i \in S$ and $j \in T$, the optimal $i-j$ xcut has value $c_{ij} = \min\{c_{iT}, c_{Sj}\}$ and an optimal cut is the one producing this value (i.e. the optimal $i-T$ or $S-j$ xcut).

THEOREM 5.4

Algorithm 5.3 correctly computes optimal xcuts for the multiterminal problem.

Proof

The optimality claim in step 1 is obvious. Let $i \in S$, $j \in T$. By theorem 5.2, there exists an optimal $i-j$ xcut (X, Y) with $i, j \in X$ which does not cross (S, T). Thus, either $\{j\} \cup S \subset X$ or $\{i\} \cup T \subset X$. This justifies the optimality claim in step (4). □

Remark 5.5

An $i-j$ xcut problem can be computed by setting σ_{ij} to infinity and computing a globally minimum cut in the resulting network. This cut can be computed, for example, by applying Gomory and Hu's algorithm or the recent algorithm by Nagamochi and Ibaraki [8]. Algorithm 5.3 requires, therefore, $n + 1$ computations of a globally minimum cut.

We note that the number of distinct solutions in a multiterminal problem that consists of all of the $\binom{n}{2}$ xcut problems may be smaller than n even in "nondegenerate" cases (i.e. when distinct cuts have distinct costs), in contrast to the multiterminal cut

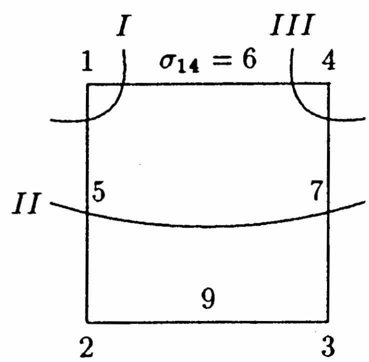

Fig. 3.

problem. This is illustrated by the following graphic example (see fig. 3). The numbers associated with the edges represent the cost σ_{ij}. The cut marked by *I* is an optimal 2–3, 2–4 and 3–4 xcut. Cut *II* is an optimal 1–4 xcut, and cut *III* is an optimal 1–2 and 1–3 xcut. Thus, in this problem $n = 4$ but there are only three distinct solutions. Note that if $\sigma_{3,4}$ is changed to 8.5, then there will be four distinct solutions since the cut ({2}, {1, 3, 4}) will be the 1–3 optimal xcut.

References

[1] C.K. Cheng and T.C. Hu, Maximum concurrent flow and minimum ratio cut, Technical Report CS88-141, Department of Computer Science and Engineering, University of California, San Diego (1988).

[2] M. Conforti and M.R. Rao, Some new matroids on graphs: Cut sets and the max cut problem, Math. Oper. Res. 12(1987)193–204.

[3] R.E. Gomory and T.C. Hu, Multi-terminal network flows, J. SIAM 9(1961)551–570.

[4] F. Granot and R. Hassin, Multi-terminal maximum flows in node capacitated networks, Discr. Appl. Math. 13(1986)157–163.

[5] D. Gusfield and D. Naor, Extracting maximal information on sets of minimum cuts, Technical Report, University of California, Davis (1988).

[6] R. Hassin, Solution bases of multiterminal cut problems, Math. Oper. Res. 13(1988)535–542.

[7] R. Hassin, An algorithm for computing maximum solution bases, Oper. Res. Lett. 9(1990) 315–318.

[8] H. Nagamochi and T. Ibaraki, *Computing Edge-Connectivity in Multiple and Capacitated Graphs, Algorithms*, Lecture Notes in Computer Science 450(1990), pp. 12–20.