Shape-from-Shading by Iterative Fast Marching for Vertical and Oblique Light Sources *

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Abstract. Shape-from-Shading (SfS) is a fundamental problem in Computer Vision. Its goal is to solve the image irradiance equation. One prominent solution is the Fast Marching Method of Kimmel & Sethian. When the light source is oblique, Kimmel & Sethian proposed to rotate the image to the light source coordinate system and then solve an 'almost' Eikonal equation. This paper presents a new, iterative variant of the Fast Marching Method which copes better with images taken under oblique light sources. By avoiding the change of coordinate system, the new method maintains the invariance of the orthographic image irradiance equation to depth translation.

In a comparison on synthetic and real-life images, the suggested method obtained a pronounced improvement which (on the synthetic images) is quantified as lower error rates than the original algorithm.

1. Introduction

Shape-from-Shading (SfS) is one of the fundamental problems in Computer Vision. First introduced by Horn in the 1970s (Horn, 1977), its goal is to solve the image irradiance equation, which relates the reflectance map to image intensity. An efficient way to solve the equation numerically is the Fast Marching Method of Sethian (Kimmel and Sethian, 2001), (Sethian, 1999).

Various methodologies have been proposed since the introduction of the field of Shape-from-Shading by Horn (Horn, 1975), (Horn, 1977), (Horn, 1986) in the 1970s. Horn's book (Horn and Brooks, 1989) reviews the early approaches which include characteristic strips and Calculus of Variations.

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Zhang et al. (Zhang et al., 1999) categorizes Shape-from-Shading techniques by their modus operandi. Namely, minimization approaches: (Zheng and Chellappa, 1991), (Lee and Kuo, 1993); propagation approach: (Bichsel and Pentland, 1992); local approach: (Lee and Rosenfeld, 1985); linear approaches: (Pentland, 1984), (Tsai and Shah, 1994). A newer minimization approach is that of Robles-Kelly & Hancock (Robles-Kelly and Hancock, 2002), which use the Mumford-Shah functional to derive diffusion kernels. Other researchers put topological properties of the surface to use (e.g., Kimmel & Bruckstein (Kimmel and Bruckstein, 1995)) or employ deformable models (e.g., Samaras & Metexas (Samaras and Metaxas, 1999)). These are only examples, as the amount of work in the field of Shape-from-Shading is too large to describe herein.

Of particular relevance to this paper are works which utilize Level-Set and Fast Marching methodologies (see (Sethian, 1999) for a deep insight). These approaches refer to the image irradiance equation as describing the motion of a front ((Osher and Sethian, 1988)). The Fast Marching Method re-orders the computation, to make it a one-pass solution of the Eikonal equation, based on the observation that the upwind difference structure of the numerical approximation allows us to propagate information "one way", that is from smaller values to larger values ((Sethian, 1996a), (Sethian, 1996b)). Sethian (Sethian, 1996a) proves the Fast Marching Method converges to the viscosity solution (see: (Crandall and Lions, 1983), (Lions, 1982) for the definition and properties of viscosity solutions).

Kimmel & Sethian (Kimmel and Sethian, 2001) implemented the Fast Marching Method as an optimal algorithm for surface reconstruction. They referred to the image irradiance equation as an Eikonal equation for vertical light sources. Solution of the equation for oblique light sources is obtained by rotation of the image coordinate system to that of the light source (as inspired by (Lee and Rosenfeld, 1985)).

While the Fast Marching Method is a highly efficient numerical solution to the image irradiance equation for vertical light sources, it is suboptimal for oblique light sources. For non-vertical light sources, the rotation of coordinate system results in an "'almost' Eikonal equation" (terminology borrowed from (Kimmel and Sethian, 2001)). This equation, unlike the orthographic image irradiance equation, is non-invariant to depth translation, so its solution is less robust. This paper presents a new way to employ the Fast Marching Method for oblique light sources as well. The suggested algorithm iteratively applies the Fast Marching Method in the case of an oblique light source. Comparison with the original algorithm (Kimmel and Sethian, 2001) would demonstrate that the new algorithm overcomes the flaws of the original.

The paper is organized as follows. First, we present the notation and basic assumptions (Sect. 2), and review the Fast Marching Method (Sect. 3). We then propose the Iterative Fast Marching Method for improved accuracy in cases where the light source is oblique (Sect. 4). Section 5 compares the original and Iterative Fast Marching Methods on both synthetic and real-life images. Finally, Sect. 6 draws the conclusions.

2. Notation and Assumptions

Let us first describe the notation and assumptions that hold throughout this paper. Photographed surfaces are assumed representable by functions of real-world coordinates. $z(x, y)$ denotes the depth function in a real-world Cartesian coordinate system whose origin is at camera plane. A real-world coordinate $(x, y, z(x, y))$ is projected orthographically onto image point (x, y). The intensity and surface normal at this image point are denoted: $I(x, y)$ and $\vec{N}(x, y)$, respectively. The scene object is Lambertian, and is illuminated by a point light source at infinity whose direction is: $\vec{L} = (p_s, q_s, -1)$.

3. The Fast Marching Method

This section reviews the Fast Marching method of Kimmel and Sethian (Kimmel and Sethian, 2001) for vertical and oblique light sources.

3.1. Fast Marching for Vertical Light Sources

The algorithm of Kimmel and Sethian (Kimmel and Sethian, 2001) stems from the orthographic image irradiance equation:

$$I(x,y) = \vec{L} \cdot \vec{N}(x,y) = \frac{p_s z_x + q_s z_y + 1}{\|\vec{L}\|\sqrt{z_x^2 + z_y^2 + 1}} \tag{1}$$

For a vertical light source, that is $\vec{L} = (0, 0, -1)$, the equation becomes an Eikonal equation which can be written as:

$$p^2 + q^2 = \tilde{F}^2 \tag{2}$$

where $p \stackrel{\text{def}}{=} z_x$, $q \stackrel{\text{def}}{=} z_y$ and $\tilde{F} = \sqrt{(I(x,y))^{-2} - 1}$.

Following (Kimmel and Sethian, 2001), we use the numerical approximation (originally introduced in (Rouy and Tourin, 1992) as a modification of the scheme of (Osher and Sethian, 1988)):

$$\begin{aligned} p_{ij} &\approx \max\{D_{ij}^{-x} z, -D_{ij}^{+x} z, 0\} \\ q_{ij} &\approx \max\{D_{ij}^{-y} z, -D_{ij}^{+y} z, 0\} \end{aligned}$$

where $D_{ij}^{-x}z \stackrel{\text{def}}{=} \frac{z_{ij}-z_{i-1,j}}{\Delta x}$ is the standard backward derivative and $D_{ij}^{+x}z \stackrel{\text{def}}{=} \frac{z_{i+1,j}-z_{ij}}{\Delta x}$, the standard forward derivative in the x-direction ($z_{ij} \stackrel{\text{def}}{=} z(i \cdot \Delta x, j \cdot \Delta y)$). $D_{ij}^{-y}z$ and $D_{ij}^{+y}z$ are defined in a similar manner for the y-direction.

The motivation for employing this numerical scheme is its consistency and monotonicity. For the Eikonal equation, Rouy & Tourin (Rouy and Tourin, 1992) showed that an iterative algorithm based on this scheme with Dirichlet boundary conditions on image boundaries and at all critical points converges towards the viscosity solution with the same boundary conditions. Existence of the viscosity solution was proven in (Lions, 1982) and uniqueness, in (Rouy and Tourin, 1992) and (Ishii, 1987). Sethian (Sethian, 1996a) proved that everywhere, the Fast Marching Method produces a solution that satisfies the discrete version of the Eikonal equation.

Substituting the numerical approximation into Eq. 2, we get the discrete equation:

$$\left(\max\{D_{ij}^{-x}z, -D_{ij}^{+x}z, 0\}\right)^2 + \left(\max\{D_{ij}^{-y}z, -D_{ij}^{+y}z, 0\}\right)^2 = \tilde{F}_{ij}^2$$

where $\tilde{F}_{ij} \stackrel{\text{def}}{=} \tilde{F}(i \cdot \Delta x, j \cdot \Delta y)$. The solution of this equation at every point (i, j) is:

$$z_{ij} = \begin{cases} \min\{z_1, z_2\} + \tilde{F}_{ij}, & \text{if} \quad | z_2 - z_1 | \geq \tilde{F}_{ij} \\ \frac{1}{2}\left(z_1 + z_2 \pm \sqrt{2\tilde{F}_{ij}^2 - (z_1 - z_2)^2}\right), & \text{if} \quad | z_2 - z_1 | < \tilde{F}_{ij} \end{cases} \tag{3}$$

where $z_1 \stackrel{\text{def}}{=} \min\{z_{i-1,j}, z_{i+1,j}\}$ and $z_2 \stackrel{\text{def}}{=} \min\{z_{i,j-1}, z_{i,j+1}\}$.

3.2. Fast Marching in Light Source Coordinates

The solution suggested by Kimmel & Sethian (Kimmel and Sethian, 2001) for the case of an oblique light source (i.e., $\vec{L} \neq (0, 0, -1)$) is to rotate the brightness image to the light source coordinates. This yields an 'almost' Eikonal equation (as (Kimmel and Sethian, 2001) called it), which is solved in a manner similar to the vertical case, but in the new coordinate system.

However, despite the similarity to the vertical case, the rotation of coordinate system breaches an important property of the Eikonal equation: its invariance to translation of the depth function ($z(x, y)$). Thus, following the rotation, $z(x, y)$ and $z(x, y) + c$ (where c is constant) no longer generate an identical image, which contradicts the orthographic model. In practice, the infringement of the invariance contributes to reduced stability of the algorithm for oblique light sources, because two surfaces which create an identical image under the orthographic model may be reconstructed differently by the algorithm of (Kimmel and Sethian, 2001) (for works on the perspective model, see: (Tankus et al., 2003), (Tankus et al., 2004a), (Tankus et al., 2004b)). This would be further demonstrated by experimental results (Sect. 5.2).

4. The Iterative Solution

To overcome the aforementioned flaw of the Fast Marching Method, we base our algorithm on solving Eikonal equations which are approximations to the image irradiance equation. We then successively refine the approximation.

To formulate the approximate equations, we transform the image irradiance equation (Eq. 1) for an oblique light source into the form:

$$p^2 + q^2 = F^2(p, q) \tag{4}$$

where $F(p,q) \stackrel{\mathrm{def}}{=} \sqrt{1 - \left(\frac{p_s p + q_s q + 1}{\|\vec{L}\| I(x,y)}\right)^2}$. A significant difference between the vertical and oblique cases is the dependence of F on p and q.

An important observation described in (Kimmel and Sethian, 2001) is that information always flows from small to large values at points of local minimum of the depth function. Based on this, the Fast Marching Method reconstructs depth by first setting all z values to the correct height values at local minima and to infinity elsewhere. Then, every step extends reconstruction to higher depths. Reconstruction is thus achieved by a single pass.

Nevertheless, a single pass may not be enough to solve the aforementioned formulation of the oblique problem (Eq. 4), because the approximate solution (the right-hand side of Eq. 3) depends on F, which depends on both p and q. Hence, we suggest an iterative method. For each iteration, F is calculated using the depth recovered at the preceding iteration:

$$p_{n+1}^2 + q_{n+1}^2 = F^2(p_n, q_n)$$

where p_n and q_n are the values of p and q at the $n^{\underline{\mathrm{th}}}$ iteration. Based on the approximation $F(p_n, q_n)$, a solution for the new iteration (p_{n+1}, q_{n+1}) is calculated using Eq. 3. We initialize the iterative process by:

$$F_0 = \sqrt{(I(x,y))^{-2} - 1}$$

as done in the vertical light source case. Following each iteration we normalize the depth function $z(x,y)$ (divide by the mean z value) to compensate for the lack of knowledge of grid size $(\Delta x, \Delta y)$.

The iterative process described above results in a series of Eikonal equations, each solved by the Fast Marching Method. Sethian (Sethian, 1996a) showed that the Fast Marching Method produces a solution that everywhere satisfies the discrete version of the Eikonal equation. Therefore, the Fast

Marching solution of each of the equations in the series satisfies the discrete version of that equation. As a result, when the series of solutions to the Eikonal equations converges, convergence is to the correct solution of the discrete version of the original equation (i.e., to the solution of the image irradiance equation with an oblique light source).

One of the properties which results from this convergence (when exists) is invariance to depth translations. This is demonstrated in Sect. 5.2. Empirically, in almost all experiments the series of solutions converged. In fact, very few iterations were necessary to obtain this convergence (i.e., to get close enough to the limit).

5. Experimental Results

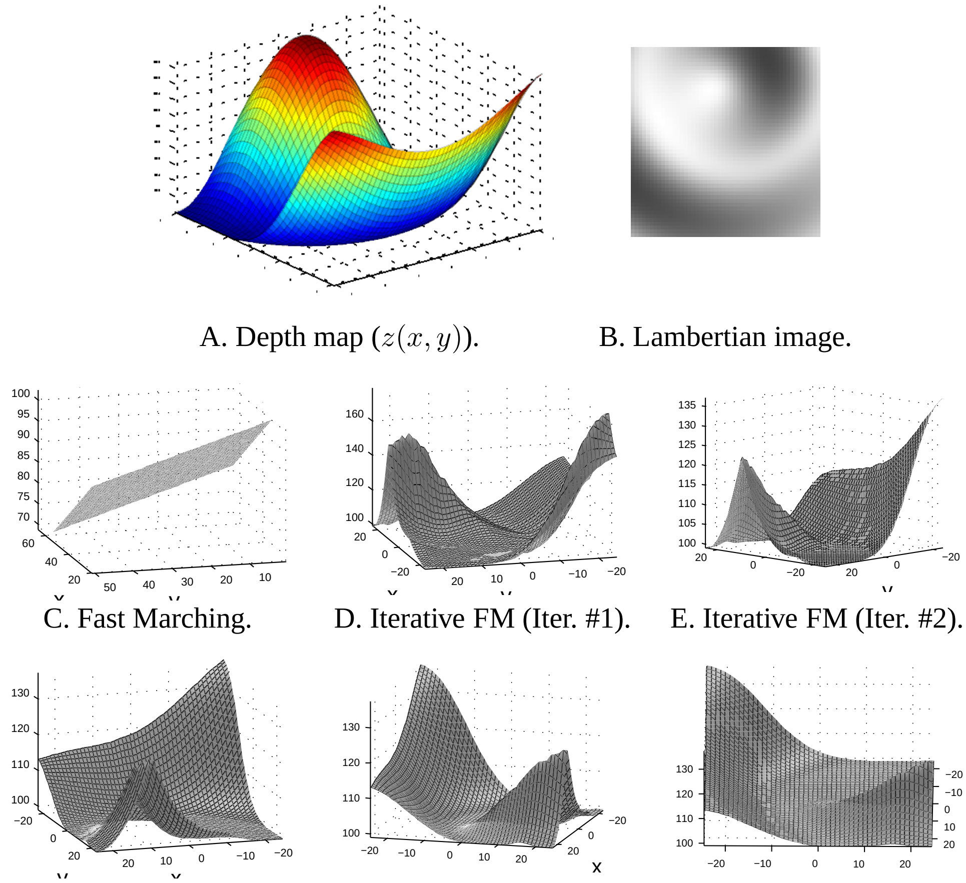
5.1. The Experiments

To evaluate the contribution of the proposed algorithm, we compared it with the original formulation of the Fast Marching (FM) Method (Kimmel and Sethian, 2001). The evaluation involved both synthetic images and real-life images. The synthetic images were produced from a given depth map using the image irradiance equation (Eq. 1). The derivatives in the equation were calculated numerically.

The initialization of the algorithms is based on points of local minima. For synthetic images, these were extracted automatically from the true depth map. For real images, they were located visually in each photograph by a human viewer, and their depths were arbitrarily set to the same constant. To demonstrate the lack of invariance to depth translation by (Kimmel and Sethian, 2001), we ran the algorithms twice for each surface. In the second run, the depth of the original initialization (described above) was translated by a constant. Theoretically, this should merely translate the whole reconstruction along the z-axis by the same constant.

In our comparison we checked five iterations of the Iterative Fast Marching Method for each example. We found out that all iterations (maybe except for the first one) yielded visually-identical images, which implies the suggested algorithm converges very fast. We therefore exploit these iterations to introduce more viewing angles of the reconstructed surface.

To quantitatively evaluate the performance of the algorithms on synthetic data, we adopted three criteria from Zhang et al. (Zhang et al., 1999). These are: mean depth error, standard deviation of depth error, and mean gradient error. For completeness, we also supply the standard deviation of the gradient error, even though it is considered nonphysical.



A. Depth map ($z(x, y)$). B. Lambertian image.

C. Fast Marching. D. Iterative FM (Iter. #1). E. Iterative FM (Iter. #2).

F. Iterative FM (Iter. #3). G. Iterative FM (Iter. #4). H. Iterative FM (Iter. #5).

Figure 1. The Fast Marching Method in two variants: rotation to light source coordinates vs. iterative reconstruction. The original image is of: $z(x, y) = 100 + \cos\left(\sqrt{x^2 + (y-2)^2}\right)$. **A.** Original Depth map. **B.** Lambertian image of (A). **C.** Reconstruction by the Fast Marching Method. **D.–H.** Reconstruction by the Iterative Fast Marching Method. Each image corresponds to a different iteration (and is also from a different point of view). Images (C) and (D) are from the same viewpoint. Lighting is identical for all reconstructions.

5.2. Comparative Evaluation

Fig. 5.2 compares the original Fast Marching Method with the iterative one on the following depth map:

$$z(x, y) \stackrel{\text{def}}{=} 100 + \cos\left(\sqrt{x^2 + (y-2)^2}\right)$$

Table I. *Error rates for the algorithms on* $z(x,y) = 100 + \cos\left(\sqrt{x^2+(y-2)^2}\right)$.

Algorithm:	No. of Iters.:	Mean Depth Error:	Std. Dev. of Depth Error:	Mean Grad. Error:	Std. Dev. of Grad. Error:
FM:	1	0.51697	0.29234	2.24199	1.01400
Iterative FM:	1	0.41667	0.30014	1.30355	1.01143
Iterative FM:	2	0.41615	0.31094	1.37016	0.92723
Iterative FM:	3	0.41558	0.31057	1.36792	0.92512
Iterative FM:	4	0.41540	0.31064	1.36685	0.92459
Iterative FM:	5	0.41492	0.31069	1.36032	0.92422

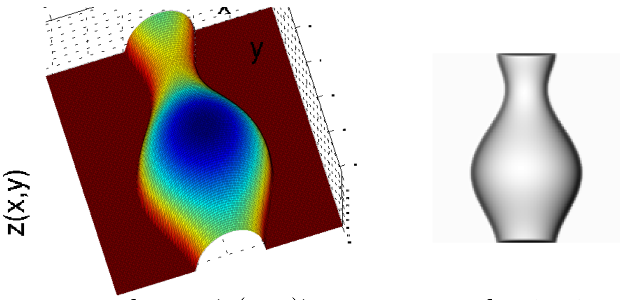
where: $x, y \in [-3.0788, 3.0788]$ (image size: 50×50 pixels). The original Fast Marching Method reconstructed a surface which is close to planar. The iterative method recovered a surface which is notably more similar to the original one than does the method of Kimmel & Sethian. Table 5.2 presents the error rates according to the aforementioned criteria. The iterative algorithm obtained considerably lower error rates according to mean depth error, and mean and standard deviation of gradient error. The standard deviation of depth error is slightly lower for the original Kimmel & Sethian algorithm, but the difference between the two is small (0.0184).

Figure 5.2 shows the famous example of the Vase ($x, y \in [-63.5, 63.5]$; image size: 128×128; background depth: 100). The Fast Marching with rotated coordinate system yielded a step along the side of the vase. In addition, there were two sharp edges from the center of the vase (downward and to the right). That is, the derivatives of the recovered depth ($z(x, y)$) are discontinuous there. The iterative method, on the other hand, reconstructed a smoother object which better fits the original surface. Table 5.2 provides the error rates for the Vase example. Reconstruction by the Iterative Fast Marching Method obtained significantly lower error rates according to all measures.

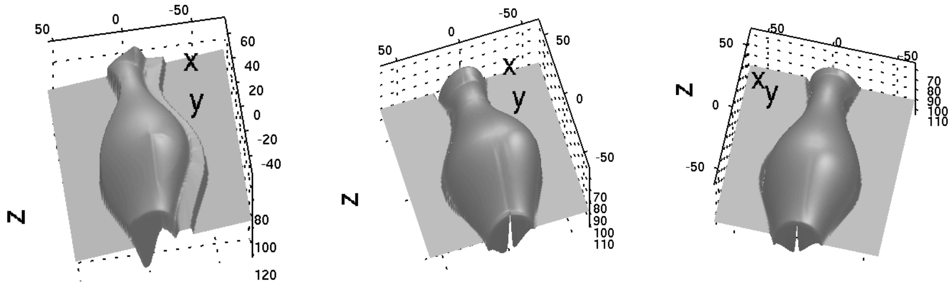
Figure 5.2 introduces a real-world example, taken by endoscopy from the gastric angulus[1] (cropped image size: 64×64). The reconstruction by Kimmel & Sethian's algorithm has a “line of breakage” in the center, which does not exist in the original image. In contrast, the iterative method clearly reconstructs all three gastric folds.

We next demonstrate the lack of invariance to depth translation when rotating the image to the light source coordinate system (Fig. 5.2). We juxtaposed

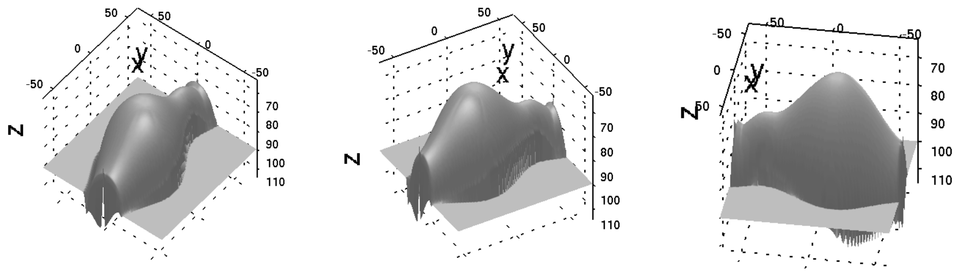
[1] Original is from www.gastrolab.net, courtesy of The Wasa Workgroup on Intestinal Disorders, GASTROLAB, Vasa, Finland.



A. Depth map ($z(x, y)$).   B. Lambertian image.



C. Fast Marching.   D. Iterative FM (Iter. #1).   E. Iterative FM (Iter. #2).



F. Iterative FM (Iter. #3).   G. Iterative FM (Iter. #4).   H. Iterative FM (Iter. #5).

Figure 2. The Fast Marching Method in two variants: rotation to light source coordinates vs. iterative reconstruction. The original image is of the famous Vase. **A.** Original Depth map. **B.** Lambertian image of (A). **C.** Reconstruction by the Fast Marching Method. **D.–H.** Reconstruction by the Iterative Fast Marching Method. Each image corresponds to a different iteration (and is also from a different point of view). Images (C) and (D) are from similar viewpoints. Lighting is identical for all reconstructions.

the reconstructions by the two methods on the Cosine, Vase and Gastric Angulus examples, when depth initializations were translated. One can see that reconstruction by the original Fast Marching Method is subjected to a notable change due to depth translation (cf. Figs. 5.2C, 5.2C, 5.2C), in contrast with the theoretic invariance of the underlying equation. Notwithstanding, the vari-

Table II. *Error rates for the algorithms on the Vase example.*

Algorithm:	No. of Iters.:	Mean Depth Error:	Std. Dev. of Depth Error:	Mean Grad. Error:	Std. Dev. of Grad. Error:
FM:	1	5.64573	4.17373	11.78896	19.72866
Iterative FM:	1	1.78303	2.70073	5.60814	16.64485
Iterative FM:	2	1.88749	2.66494	5.25000	14.13286
Iterative FM:	3	1.88272	2.66171	5.25550	14.16898
Iterative FM:	4	1.88764	2.66592	5.26070	14.18274
Iterative FM:	5	1.88641	2.66449	5.25500	14.15838

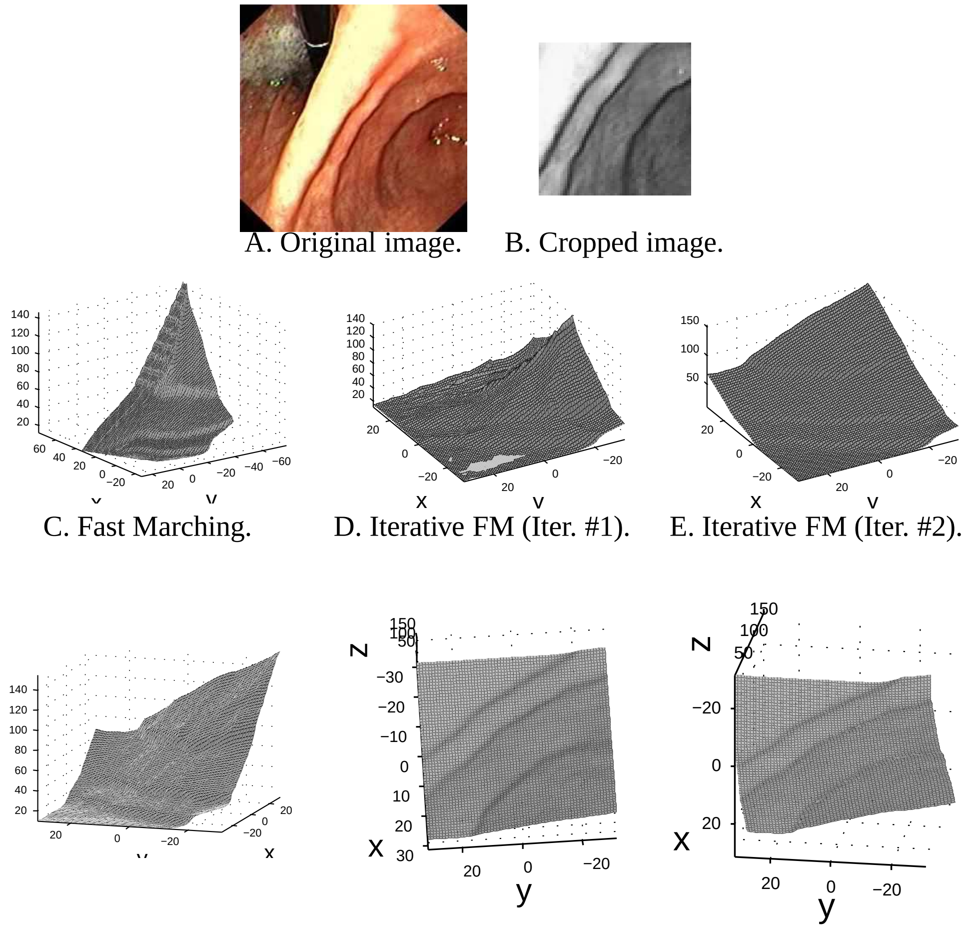
Table III. *Comparison of algorithms on* $z(x,y) = 100 + \cos\left(\sqrt{x^2 + (y-2)^2}\right)$*, with initialization translated by* -90*. Note the significant change in mean gradient error of the original Fast Marching Method with respect to Table 5.2.*

Algorithm:	No. of Iters.:	Mean Depth Error:	Std. Dev. of Depth Error:	Mean Grad. Error:	Std. Dev. of Grad. Error:
FM:	1	0.50952	0.29694	2.03553	1.01126
Iterative FM:	1	0.41666	0.30021	1.30410	1.01180
Iterative FM:	2	0.41546	0.31031	1.36808	0.92218
Iterative FM:	3	0.41044	0.30757	1.32781	0.92801
Iterative FM:	4	0.40885	0.30672	1.31591	0.92827
Iterative FM:	5	0.40834	0.30627	1.31177	0.92782

ation in reconstruction by the Iterative Fast Marching Method[2] is very small (cf. Figs. 5.2G, 5.2D, 5.2E). Quantification of the results in the form of depth and gradient errors appears in Tables 5.2 and 5.2 (for the synthetic examples only). The original Fast Marching Method changed considerably with respect to Tables 5.2 and 5.2, while variations in the Iterative Fast Marching are only minor.

We see, that in all examples, the Iterative Fast Marching Method appears to outrank the original method which rotates the image to the light source coordinate system.

[2] For the Cosine and Vase examples, the images are after 1 iteration. For the Gastric Angulus, after 2 iterations.



A. Original image. B. Cropped image.

C. Fast Marching. D. Iterative FM (Iter. #1). E. Iterative FM (Iter. #2).

F. Iterative FM (Iter. #3). G. Iterative FM (Iter. #4). H. Iterative FM (Iter. #5).

Figure 3. Comparison of the original and Iterative Fast Marching Methods on an endoscopic image from the gastric angulus. **A.** Original image. **B.** Cropped image of (A) Only (B) was used for the reconstruction. **C.** Reconstruction by the Fast Marching Method. **D.–H.** Reconstruction by the Iterative Fast Marching Method. Each image corresponds to a different iteration (and is also from a different point of view). Images (C) and (D) are from the same viewpoint. Lighting is identical for all reconstructions.

When comparing the complexity of the two algorithms, no doubt the original one is faster, by containment. However, as the examples show, the speed in this case is at the expense of accuracy. As the suggested method converges very fast and no more than 2 iterations were ever required, the speed difference turns out to be of secondary importance.

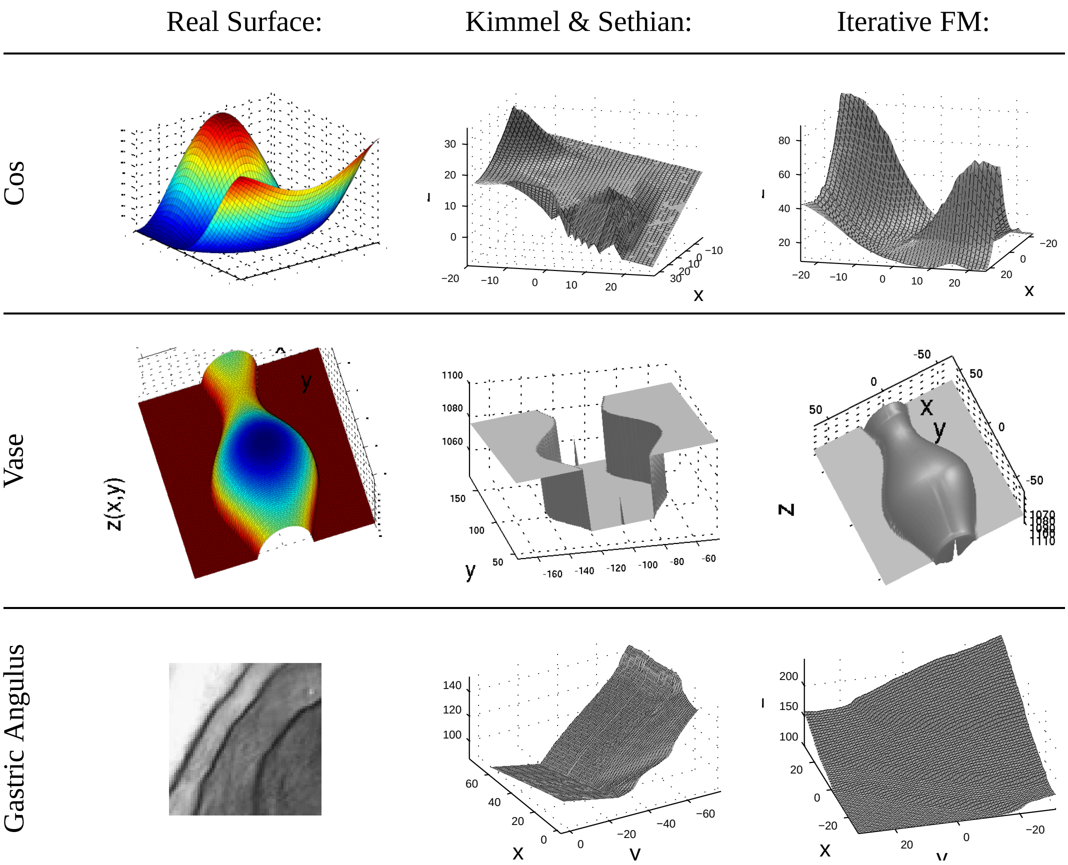


Figure 4. Comparison of the original and Iterative Fast Marching Methods with a translated depth initialization. **Top Row:** The Cosine example (Fig. 5.2). The depth is initialized to the true depth -90. **Middle Row:** The Vase example (Fig. 5.2). The depth is initialized to the true depth $+1000$. **Bottom Row:** The Gastric Angulus example (Fig. 5.2). The depth is initialized to the true depth $+90$.

6. Conclusions

This research proposes an efficient and robust solution to the problem of Shape-from-Shading which handles both vertical and oblique light sources under the orthographic projection model. The suggested solution is a variant of the Fast Marching Method of Kimmel and Sethian (Kimmel and Sethian, 2001). It employs the Fast Marching Method iteratively for oblique light sources. Each iteration solves an approximation to the image irradiance equation. The resultant solution serves for successive refinement of the approximate equation. When this refinement process converges, convergence is to the correct solution of the original equation.

Table IV. *Comparison of algorithms on the Vase example with translated initialization (*$+1000$*). Pay attention to the sharp change in all measures of the original Fast Marching Method with respect to Table 5.2.*

Algorithm:	No. of Iters.:	Mean Depth Error:	Std. Dev. of Depth Error:	Mean Grad. Error:	Std. Dev. of Grad. Error:
FM:	1	5.23793	4.84300	17.40992	32.88627
Iterative FM:	1	1.78336	2.70162	5.60964	16.64280
Iterative FM:	2	1.88275	2.67130	5.25176	14.14409
Iterative FM:	3	1.87339	2.66216	5.25342	14.16180
Iterative FM:	4	1.86905	2.65839	5.25188	14.15634
Iterative FM:	5	1.87500	2.66242	5.25437	14.16484

We compared reconstruction by the original Fast Marching Method and its iterative variant on both synthetic and real-life examples (from endoscopy). We also demonstrated why rotation of the image to light source coordinates, as done in (Kimmel and Sethian, 2001), violates the property of the orthographic image irradiance equation of invariance to depth translation. The Iterative Fast Marching Method outperformed the original Fast Marching Method, and remained invariant to depth translations (due to convergence to the correct solution).

In terms of runtime, indeed the original Fast Marching Method is faster than the suggested one. However, convergence of the suggested variant is very fast; in all examples no more than 2 iterations were ever necessary.

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