Approximation by Cantor Sets Julia Sets

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Abstract

In this note, we show that any compact set in the plane can be approximated by Cantor set Julia sets in the Hausdorff topology.

1 Overview

We show the following:

Theorem 1. *Any compact set can be approximated in the Hausdorff topology by Julia sets of polynomials which are topologically Cantor sets.*

Proof. First, approximate the given compact set by a finite set $\{p_1, p_2, \dots, p_n\}$. Form the polynomial

$$P_1(z) = (z - p_1)(z - p_2) \cdots (z - p_n).$$

Set $P_a(z) := a \cdot P(z)$. We first show that as $|a| \to \infty$, the Julia set $\mathcal{J}(P_a)$ approaches $\{p_1, p_2, \dots, p_n\}$ in the Hausdorff topology. Indeed, any point for which $|P_1(z)| > \epsilon$ iterates to infinity under the dynamics of $P_a :$ in one step, z lies outside the disk $D(0, \epsilon|a|)$ and from then on, it converges to infinity very quickly.

It is left to see that $\mathcal{J}(P_a)$ is a Cantor set. If $\mathcal{J}(P_a)$ is not a Cantor set, then P_a must have an attracting or parabolic orbit or a Siegel disk. This is impossible because on sets of the form $|P_1(z)| < \epsilon$, the derivative $|P_a'(z)|$ is very large (and in particular, greater than 1). $\square$

References

[L] Lindsey, K. A. *Shapes of polynomial Julia sets*, `arxiv:1209.0143v2`, 2013.

[M] Milnor, J. *Dynamics in One Complex Variable*, Third Edition, Annals of Mathematics Studies, 2006.