The Bierstone Lectures on *Complex Analysis*

Scribed by Oleg Ivrii during Spring 2007

Preface

These notes of my lectures on Complex Analysis at the University of Toronto were written by Oleg Ivrii on his own initiative, after he took my course in the spring, 2007. The course is cross-listed as MAT1001S, one of the Core Courses in our Graduate Program, and as MAT454S, a fourth-year undergraduate course which is the second part of a two-term sequence in complex analysis in our Specialist Program in Mathematics. Oleg took this course as a second-year undergraduate.

My lectures were based on the classical textbooks of Lars Ahlfors (Complex Analysis, Third Edition, McGraw-Hill 1979) and Henri Cartan (Elementary Theory of Analytic Functions of One or Several Variables, Dover 1994). My exposition closely follows these texts. Oleg has included the problems that I assigned as homework; most of these can be found in Ahlfors, Cartan or other textbooks. Oleg has added a concluding chapter as an introduction to several fundamental further results. I have included some of these topics when giving the course on earlier occasions, and they improve my lectures as presented here!

During my thirty-five years of teaching at the University of Toronto, I have been truly fortunate to have had so many inspiring students like Oleg. They are passionate about mathematics and they care for each other and the world around them. I am grateful to Oleg that he thought my lectures were interesting enough to want to write them up, and I give him my warmest wishes for a very happy and fruitful mathematical career.

March 2008 — Edward Bierstone

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Chapter 1

Space of Holomorphic Functions

1.1 Topology on the Space of Holomorphic Functions

Suppose Ω is open in $\mathbb{C}$, let $C(\Omega)$ be the ring of continuous $\mathbb{C}$–valued functions on Ω and let $H(\Omega)$ be the subring of holomorphic functions.

We can topologize $C(\Omega)$: a sequence of functions $\{f_n\}$ *converges uniformly on compact sets* if for every compact set $K \subset \Omega$, $\{f_n|_K\}$ converges uniformly.

Uniform convergence on compact sets defines the so-called *compact-open topology.*

The fundamental system of open neighbourhoods at 0 is

$$V(K, \epsilon) = \{f : |f(z)| < \epsilon, z \in K\}.$$

This is because $f_n \to f$ uniformly on compact sets if and only if $f - f_n$ lies in $V(K, \epsilon)$ for n large enough for any given K, ϵ.

In fact, $C(\Omega)$ is metrizable (even by a translation-invariant metric).

Write $\Omega = \bigcup_i K_i$ and let $d(f) = \sum_i \frac{1}{2^i} \cdot \min(1, M_i(f))$ where $M_i(f) = \max_{z \in K_i} |f(z)|$.

$C(\Omega)$ is complete: if $\{f_n\}$ is a sequence of continuous functions which converge uniformly on compact sets, $f = \lim f_n$ is continuous.

We give $H(\Omega)$ the subspace topology.

Theorem. *(1). $H(\Omega)$ is a closed subspace. (2). Furthemore, the map $f \to f'$ is continuous.*

(1) means that if $\{f_n\} \subset H(\Omega)$ converges uniformly on compact sets, $f = \lim f_n \in H(\Omega)$. (2) means that if $\{f_n\} \subset H(\Omega)$ converges uniformly on compact sets and if $f = \lim f_n$, then $\{f_n'\}$ converges uniformly on compact sets to f'.

Proof of (1). We have to show that $f(z)dz$ is closed. By Morera's theorem, this is equivalent to showing that $\int_\gamma f(z)dz = 0$ for any small loop γ. As $f_n(z)$ are holomorphic, $\int_\gamma f_n(z)dz = 0$ holds for all n. Then, $\int_\gamma f(z) = \lim_{n\to\infty} f_n(z)dz$ because f_n converges to f uniformly on the image of γ.

Proof of (2). It is enough to show that f_n' converges to f' uniformly on a closed disk $D(z,r)$ in Ω. Let γ be a positively oriented loop around a disk of slightly larger radius. By Cauchy's integral formula,

$$\lim_{n\to\infty} f_n'(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta = f'(z).$$

The limit above converges uniformly as $\zeta - z$ is bounded away from 0.

Corollary. *If a series of holomorphic functions $\sum_{n=1}^\infty f_n(z)$ converges uniformly on compact sets of Ω, the sum is holomorphic and can be differentiated term-by-term.*

Proposition (Hurwitz)**.** *If Ω is a domain (i.e connected), if $\{f_n\} \subset H(\Omega)$ converges uniformly on compact sets and each f_n vanishes nowhere in Ω, then $f = \lim f_n$ is never 0 or identically 0.*

Suppose f was not identically 0. Then its zeros are isolated. In particular around any point z, there is a small loop γ around z for which f does not vanish on the image of γ. But as the image of γ is compact, f is bounded away from 0. Then,

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz.$$

The left hand side counts the number of zeros of f_n enclosed within γ, which is 0. Hence, $f(z)$ also does not possess any zeros inside γ, i.e $f(z) \neq 0$. As z was arbitrary, $f(z)$ vanishes nowhere in Ω.

Proposition. *If Ω is a domain and $\{f_n\} \subset H(\Omega)$ are one-to-one and converges uniformly on compact sets, then $f = \lim f_n$ is either one-to-one or a constant.*

Suppose f was not identically a constant, but nevertheless $f(z_0) = f(z_1) = a$. Choose disjoint neighbourhoods U and V of z_0 and z_1 respectively. Then $f(z) - a$ vanishes at a point in U, so we can extract a subsequence of $\{f_n\}$ for which $f_n(z) - a$ vanishes at a point in U. From this subsequence, we extract another subsequence for which $f_n(z) - a$ vanishes at a point in V. But then, $f_n(z)$ would fail to be injective.

1.2 Series of Meromorphic Functions

A series of meromorphic functions $\sum_{n=0}^{\infty} f_n$ on an open set $\Omega \subset \mathbb{C}$ *converges uniformly on compact subsets* of Ω if for every compact $K \subset \Omega$, all but finitely many terms have no poles in K and form a uniform convergent series.

We define the sum on any relatively compact (precompact) open U as $\sum_{n<p} f_n + \sum_{n\geq p} f_n$ with each function in the second sum having no pole in $\overline{U}$.

Theorem. *If $\sum_{n=0}^{\infty} f_n$ is a series of meromorphic functions that converges on compact subsets of Ω, then the sum f is a meromorphic function on Ω and $\sum_{n=0}^{\infty} f'_n$ converges uniformly on comapct subsets of U.*

The poles of f, $P(f)$ are contained are in $\bigcup_n P(f_n)$.

Example 1

The series $f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ converges uniformly and absolutely on compact subsets of $\mathbb{C}$. In fact the series converges uniformly and absolutely in each vertical strip $x_0 < x < x_1$ (we write $z = x + iy$).

To see this, note that in $\sum_{n<x_0} \frac{1}{(z-n)^2}$, each term is bounded by $\frac{1}{(x_0-n)}$; while in $\sum_{n>x_1} \frac{1}{(z-n)^2}$, each term is bounded by $\frac{1}{(x_1-n)^2}$.

The function $f(z)$ is meromorphic on $\mathbb{C}$ and enjoys three properties: (1) it is periodic, $f(z+1) = f(z)$, (2) it has poles $z = n$, each with principal part $\frac{1}{(z-n)^2}$ and (3) $f(z) \to 0$ as $y \to \infty$ uniformly with repect to x, i.e for all $\epsilon > 0$, there is a b such that $|f(z)| < \epsilon$ whenever $|y| > b$.

The function $g(z) = \left(\frac{\pi}{\sin \pi z}\right)^2$ also enjoys these properties. The first property is obvious.

To check the second property, we first check that $\sin \pi z$ vanishes only on the integers. By periodicity, it suffices to check that the principal part of $g(z)$ at the origin is $\frac{1}{z^2}$; for a proof, see the calculation below.

Easiest way to see property (3) is by means of the identity $|\sin\pi z|^2 = \sin^2\pi x + \sinh\pi y$, as y goes to ∞, $\sin\pi z \to \infty$ and hence $g(z) \to 0$.

Hence, $f-g$ is holomorphic in $\mathbb{C}$ because f and g have the same poles and same principal parts; $f-g$ is bounded, so $f-g$ is a constant by Liouville's theorem, and by the third property, the constant is 0.

We derive the formula $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Consider the function $\left(\frac{\pi}{\sin\pi z}\right)^2 - \frac{1}{z^2} = \sum_{n\neq 0}\frac{1}{(z-n)^2}$. It is holomorphic in a neighbourhood of 0 and as $z\to 0$, it tends to

$$\frac{\pi^2}{(\pi z - \frac{1}{6}\pi^3 z^3 + \dots)^2} - \frac{1}{z^2} = \frac{1}{z^2(1-\frac{1}{3}\pi z^2 + \dots)} - \frac{1}{z^2} = \frac{\pi^2}{3}.$$

But when $z=0$, the function is $2\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Example 2

The series $f(z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n\neq 0}\frac{z}{n(z-n)}$ converges uniformly and absolutely on compact subsets of $\mathbb{C}$ by comparison with $\sum_{n\neq 0}\frac{1}{n^2}$.

Thus $f(z)$ is a meromorphic function with poles $z=n$ and principal parts $\frac{1}{z-n}$ (i.e simple poles with residue 1).

We notice:

$$f'(z) = -\frac{1}{z^2} - \sum_{n\neq 0}\frac{1}{(z-n)^2} = -\left(\frac{\pi}{\sin\pi z}\right)^2 = \frac{d}{dz}(\pi\cot\pi z).$$

Hence, $f(z) - \pi\cdot\cot\pi z$ is constant, which is 0 as both functions are odd (to see that $f(z)$ is odd, group terms with $+n$ and $-n$). So, $\frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2} = \pi\cot\pi z$.

1.3 The Weierstrass $\wp$-function

Let Γ be a discrete subgroup of $\mathbb{C}$ with generators e_1, e_2 linearly independent over $\mathbb{R}$, i.e $\Gamma = \{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\}$.

Vectors e_1', e_2' are another set of generators for Γ if and only if e_1', e_2' are are linear combination of e_1, e_2 with determinant of matrix of coefficients invertible in the ring of integers, i.e ± 1.

A function $f(z)$ has Γ as a group of periods if $f(z+n_1e_1+n_2e_2) = f(z)$ for all $n_1, n_2 \in \mathbb{Z}$.

We define

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum_{w\in\Gamma, w\neq 0} \Big\{ \frac{1}{(z-w)^2} - \frac{1}{w^2} \Big\}.$$

Note that the Weierstrass $\mathfrak{p}$ function depends on the choice of Γ.

We check that it converges uniformly and absolutely on compact subsets, for this we estimate:

$$\Big| \frac{1}{(z-w)^2} - \frac{1}{w^2} \Big| = \Big| \frac{2zw - z^2}{w^2(z-w)} \Big| = \dots$$

We can restrict our attention to the compact subset $|z| \leq r$. As finitely many terms don't matter, we only need to estimate when w is sufficiently large, say $|w| \geq 2r$. We continue:

$$\dots = \Big| \frac{z(2-\frac{z}{w})}{w^3(1-\frac{z}{w})^2} \Big| \leq \Big| \frac{r \cdot \frac{5}{2}}{w^3 \cdot \frac{1}{4}} \Big| = \frac{10r}{|w|^3}.$$

We want to show that $\sum_{w\in\Gamma, w\neq 0} \frac{1}{|w|^3} < \infty$. We define

$$P_n = \{ n_1e_1 + n_2e_2 : \max(|n_1|, |n_2|) \}.$$

We let k be the distance between the origin to the closest point of P_1. As the number of points in P_n is $8n$, we see that

$$\sum_{w\in\Gamma, w\neq 0} \frac{1}{|w|^3} = \sum_{n=1}^{\infty} \sum_{w\in P_n} \frac{1}{|w|^3} < \sum_{n=1}^{\infty} \frac{8n}{k^3n^3} = \frac{8}{k^3} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

We have thus shown that $\mathfrak{p}(z)$ is a meromorphic function with pole $\frac{1}{(z-w)^2}$ at $w \in \Gamma$. Also, $\mathfrak{p}(z)$ is even.

The derivative $\mathfrak{p}'(z) = -2\sum_{w\in\Gamma} \frac{1}{(z-w)^3}$ is doubly periodic. To show that $\mathfrak{p}(z+e_i) = \mathfrak{p}(z)$ for $i = 1, 2$, we consider $\mathfrak{p}'(z+e_i) - \mathfrak{p}'(z) = 0$, so $\mathfrak{p}(z+e_i) - \mathfrak{p}(z)$ is constant.

But as $\mathfrak{p}(z)$ is even, plugging in $z = -\frac{e_i}{2}$, we see that the constant is 0, so $\mathfrak{p}(z)$ also has Γ as a group of periods.

As $\mathfrak{p}(z) - \frac{1}{z^2} = g(z)$ is a holomorphic even function in a neighbourhood of the origin with $g(0) = 0$, the Laurent expansion at 0 of $\mathfrak{p}(z)$ is $\frac{1}{z^2} + a_2z^2 + a_4z_4 + \dots$

The $2n$-th derivative of $\frac{1}{(z-w)^2}$ is $(2n+1)!\frac{1}{(z-w)^{2n+2}}$. Hence $a_{2n} = (2n+1)\sum_{w\neq 0} \frac{1}{w^{2n+2}}$, in particular $a_2 = 3\sum \frac{1}{w^4}$, $a_4 = 5\sum \frac{1}{w^6}$.

We now find a differential equation for $\mathfrak{p}'(z)$:

$$\begin{aligned}
\mathfrak{p}(z) &= \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \dots \\
\mathfrak{p}'(z) &= -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \dots \\
\mathfrak{p}'(z)^2 &= \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + \dots \\
\mathfrak{p}(z)^3 &= \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + \dots
\end{aligned}$$

So, $\mathfrak{p}'(z)^2 - 4\mathfrak{p}(z)^3 = -\frac{20a_2}{z^2} - 28a_4 + \dots$

Hence, $\mathfrak{p}'(z)^2 - 4\mathfrak{p}(z)^3 + 20a_2\mathfrak{p}(z) + 28a_4$ is a holomorphic function vanishing at the origin. As it is periodic, it is bounded and by Liouville's theorem is constant, so it must be identically 0.

Thus, $(x, y) = (\mathfrak{p}(z), \mathfrak{p}'(z))$ parametrizes the algebraic equation

$$y^2 = 4x^3 - 20a_2 x - 28a_4.$$

1.4 The Weierstrass $\mathfrak{p}$-function II

Suppose Γ is a discrete subgroup of $\mathbb{C}$ given by vectors e_1, e_2 (linearly dependent over $\mathbb{R}$). Let $X \subset \mathbb{C}^2$ be the collection of pairs (x, y) satisfying the polynomial equation

$$y^2 = 4x^3 - 20a_2 x - 28a_4$$

with $a_2 = 3\sum \frac{1}{w^4}$ and $a_4 = 5\sum \frac{1}{w^6}$. The goal of this lecture it to show that the RHS has three distinct roots and any $(x, y) \in X$ is given by $x = \mathfrak{p}(z), y = \mathfrak{p}'(z)$ for a unique $\mathbb{C}/\Gamma$.

Proposition. *If $f(z)$ is a non-constant meromorphic function having Γ as a group of periods, then the number of zeros of $f(z)$ in a period parallelogram is equal to the number of poles (counting with multiplicity).*

Let γ be the positively oriented loop around the period parallelogram. The number of zeros minus the number of poles enclosed with in γ is calculated by the integral $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz$. However, as γ is doubly-periodic, the opposite sides of γ cancel each other and the integral reduces to 0.

Proposition. *Suppose $f(z)$ is above. Let its zeros inside the fundamental parallelogram be $\{\alpha_i\}$ and its poles be $\{\beta_j\}$. Then, $\sum_i \alpha_i = \sum_j \beta_j \pmod{\Gamma}$.*

The value $\sum_i \alpha_i - \sum_j \beta_j$ is counted by the integral $\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}dz$. If γ_1 and γ_2 are two sides of γ along the vectors e_1 and e_2 starting at the origin respectively, then

$$\sum_i \alpha_i - \sum_j \beta_j = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}dz = -\frac{1}{2\pi i}\int_{\gamma_1} \frac{e_1 f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{\gamma_2} \frac{e_2 f'(z)}{f(z)}dz.$$

However, $\frac{1}{2\pi i}\frac{f'(z)}{f(z)}dz$ is an integer because it is a difference between two determinations of $\log f(z)$ with same value of $f(z)$.

Remark. *The integral formulae used above may be obtained from the Residue theorem as follows: suppose a_i is a zero; we write $f(z) = (z-\alpha_i)^k(c+\dots)$, $f'(z) = k(z-\alpha_i)^{k-1}(c+\dots)$ and $z = \alpha_i + (z-\alpha_i)$. Then*

$$\frac{f'(z)}{f(z)} = \frac{k}{z-\alpha_i}(1+\dots), \qquad \frac{zf'(z)}{f(z)} = \frac{k}{z-\alpha_i}(1+\dots)(\alpha_i+\dots).$$

The considerations for poles $\{\beta_j\}$ are analogous.

The zeros of $\mathfrak{p}'(z)$ are $\frac{e_1}{2}, \frac{e_2}{2}$ and $\frac{e_1+e_2}{2}$ (the points z such that $2z \in \Gamma$ but $z \notin \Gamma$). The fact that they actually are zeros follows from $\mathfrak{p}'(z)$ being doubly-periodic and odd. But as $\mathfrak{p}'(z)$ has a unique pole of order 3 inside a period parallelogram, these zeros are simple and the only ones there.

Now we examine the zeros of $\mathfrak{p}(z) - a$ (i.e roots of $\mathfrak{p}(z) = a$). For any value of $a \in \mathbb{C}$, there are exactly two roots in a period parallelogram. Write $a = \mathfrak{p}(z_0)$. Clearly, $z_0 \notin \Gamma$ as Γ is the set of poles of $\mathfrak{p}$. We have two possibilities:

(1) If $2z_0 \in \Gamma$, then $\mathfrak{p}(z) - \mathfrak{p}(z_0)$ has a unique zero of order 2 as $\mathfrak{p}'(z_0)$ vanishes.

(2) If however $2z_0 \notin \Gamma$, then $\mathfrak{p}(z_0) = \mathfrak{p}(-z_0)$ are the two non-congruent zeros.

With these preparations, we can return back to our equation $y^2 = 4x^3 - 20a_2x - 28a_4$. If $y = 0$, we may choose $x = \mathfrak{p}(\frac{e_1}{2}),\ \mathfrak{p}(\frac{e_2}{2}),\ \mathfrak{p}(\frac{e_1+e_2}{2})$. These are distinct as the values they take are unique by (1).

If $(x, y) \in X$, but $y \neq 0$, then $x = \mathfrak{p}(z), y = \mathfrak{p}'(z)$ and $x = \mathfrak{p}(-z), y = \mathfrak{p}'(-z)$ are the two points which map to the same value of x by (2), but they map to different values of y as $-\mathfrak{p}'(z) = \mathfrak{p}'(-z)$.

1.5 Complex Projective Space

The set $X \subset \mathbb{C}^2 : y^2 = 4x^3 - 20a_2x - 28a_4$ is a smooth curve as it is locally a graph of a function $y = f(x)$ or $x = g(y)$. In other words, X is a one-dimensional complex submanifold.

If $(x_0, y_0) \in X, y_0 \neq 0$, then $y = f(x)$ near (x_0, y_0), i.e x is a local coordinate.

On the other hand, if $y_0 = 0$, then $\mathfrak{p}'(x_0) \neq 0$ since $\mathfrak{p}(x)$ has 3 distinct roots, so $X = g(y)$ near (x_0, y_0), i.e y is a local coordinate.

Remark. *These statements follow from the implicit function theorem, but in classical form, the implicit function theorem only gives that locally $y = f(x)$ and $x = g(y)$ are differentiable as functions of two real variables, whereas we want to imply that the mappings are actually holomorphic. We return to this subtle point in a later lecture.*

The n-dimensional complex projective space $P^n(\mathbb{C})$ is defined to be $\mathbb{C}^{n+1} \setminus \{0\}$ modulo the equivalence relation $(x_0, \dots x_n) \sim (x'_0, \dots x'_n)$ if $(x'_0, \dots x'_n) = (\lambda x_0, \dots \lambda x_n)$ for some $\lambda \neq 0$ in $\mathbb{C}$.

We write $[x_0, \dots x_n]$ for the equivalence class of $(x_0, \dots x_n)$. These are called *homogenous coordinates* for we are using $n+1$ "coordinates" to represent an n-dimensional object.

The complex projective space may be covered by coordinate charts

$$U_i = \{[x_0, \dots x_n] \in P^n(\mathbb{C}) : x_i \neq 0\}.$$

Each chart U_i is homeomorphic to $\mathbb{C}^n$ by map $[x_0, \dots x_n] \to (\frac{x_0}{x_i}, \dots \frac{\hat{x_i}}{x_i}, \dots \frac{x_n}{x_i})$ where the hat notation as usual means that the variable is omitted. The inverse map is $(y_1, y_2, \dots y_n) \to [y_1, \dots y_{i-1}, 1, y_i, \dots y_n]$.

We thus get a complex (algebraic) manifold structure (actually, we have to check that the transition maps are analytic, they even turn out to be rational, this is left as an exercise).

We can think of $P^n(\mathbb{C})$ as the chart $U_0 \cong \mathbb{C}^n$ together with the set $\{x_0 = 0\} \cong P^{n-1}(\mathbb{C})$ which is "the hyperplane at infinity".

It is easy to see that $P^1(\mathbb{C})$ is just the Riemann sphere S^2. Right now, we are interested in $P^2(\mathbb{C}) = [x, y, t]$. We compactify $X \subset \mathbb{C}^2$ to $X' \subset P^2(\mathbb{C})$ by rewiting $y^2 = 4x^3 - 20a_2x -$

$28a_4$ in homogeneous coordinates

$$\left(\frac{y}{t}\right)^2 = 4\left(\frac{x}{t}\right)^3 - 20a_2\left(\frac{x}{t}\right) - 28a_4$$

and allowing t to be 0. The equation simplifies to $y^2t = 4x^3 - 20a_2xt^2 - 28a_4t^3$.

Thus, X' is X together with an added point at infinity: if $t = 0$, the equation reduces to $4x^3 = 0$ from which we see $x = 0$. The variable y can be any complex number with the exception of 0, but all these options are equivalent to the point $[0, 1, 0]$.

1.6 The Elliptic Integral

Let $\Gamma \subset \mathbb{C}$ be a discrete subgroup and $X \subset \mathbb{C}^2$ be the set of points $y^2 = 4x^3 - 20a_2x - 28a_4$. Recall that $X' \subset P^2(\mathbb{C})$ is the set of points $y^2t = 4x^3 - 20a_2xt^2 - 28a_4t^3$, in which X naturally embeds by $(x, y) \to [x, y, 1]$. Then, $X' = X \cup [0, 1, 0]$.

The point $[0, 1, 0]$ lies in the chart $\{[x, y, t] : y \neq 0\}$. Let $(x', t') = (\frac{x}{y}, \frac{t}{y})$ be the affine coordinates in this chart. In these coordinates, the equation of X' is

$$t' = 4x'^3 - 20a_2x't'^2 - 28a_4t'^3.$$

The implicit function theorem tells us that in some neighbourhood of $(x', t') = (0, 0)$, the point at infinity, t' is a holomorphic function of x'.

If we let $t' = c_0 + c_1x' + c_2x'^2 + \dots$ then $t' = 4x'^3 - 320a_2x'^7 + \dots$

Indeed, $c_0 = 0$ as $t'(0) = 0$. By comparing coefficients, we see that $c_1 = c_2 = 0$ as well and $c_3 = 4x'^3$. The next non-zero coefficient comes from the term $-20a_2x't'^2$ and is $-20a_2x'(4x'^3)^2 = -320a_2x'^7$.

In a neighbourhood of the point at infinity, we can take x' as a local coordinate; this concludes the proof that X' has complex manifold structure.

In Section 1.4, we have seen that the map $\mathbb{C}/\Gamma \to X'$ given by

$$z \to \underbrace{[\mathfrak{p}(z), \mathfrak{p}'(z), 1]}_{\text{in usual coordinates}} = \underbrace{\left[\frac{\mathfrak{p}(z)}{\mathfrak{p}'(z)}, 1, \frac{1}{\mathfrak{p}'(z)}\right]}_{\text{in coords at infinity}}.$$

is a bijection of sets, but more can be said. The map is holomorphic: if $z \neq 0$, the coordinate functions $\mathfrak{p}(z), \mathfrak{p}'(z), 1$ are holomorphic; while near $z = 0$, the functions $\frac{\mathfrak{p}(z)}{\mathfrak{p}'(z)}, 1, \frac{1}{\mathfrak{p}'(z)}$ are holomorphic.

Thus $\mathbb{C}/\Gamma \to X'$ is a holomorphic bijection from a compact space and a Hausdorff space; so in particular, a homeomorphism. In fact, the inverse function theorem guarantees us that the inverse is also holomorphic.

From the parametrization $x = \mathfrak{p}(z)$, $y = \mathfrak{p}'(z)$, we see that $dx = ydz$; so $dz = \frac{dx}{y}$ whenever $y \neq 0$. Differentiating the equation $y^2 = 4x^3 - 20a_2x - 28a_4$, we find that $(12x^2 - 20a_2)dx = 2ydy$ so $\frac{dx}{y} = \frac{dy}{6x^2-10a_2}$.

By "dz" we mean the holomorphic differential form on X' which is

$$dz = \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}} \quad \text{on } y \neq 0 \quad \text{and} \quad \frac{dy}{6x^2 - 10a_2} \quad \text{on } y' \neq 0.$$

Remark. *Thus dz is well-defined on all of X': functions $y(x)$ and $y'(x)$ cannot vanish simultaneously, otherwise $y(x)$ would have multiple roots but we have seen that it is not the case.*

Inverse defines z as a *multi-valued* holomorphic function on X' where 2 branches differ by a constant in Γ:

$$z = \underbrace{\int_0^z dz}_{\text{in } \mathbb{C}} = \int_{[0,1,0]}^{\mathfrak{p}(z)} \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}.$$

The form dz is closed but not exact, so a particular value of z not only determines $\mathfrak{p}(z)$ but also the choice of path of integration from $[0,1,0]$ to $\mathfrak{p}(z)$.

We say that the Weierstrass $\mathfrak{p}$ function is given by "inversion" of the *elliptic integral.*

Trigonometric Functions

For convenience, we repeat our considerations with $X \subset \mathbb{C}^2$ given by $x^2 + y^2 = 1$. We can parametrize X by $(x, y) = (\cos\theta, \sin\theta)$.

We can differentiate the equation $x^2 + y^2 = 1$ to obtain $xdx + ydy = 0$; so, $\frac{dy}{x} = -\frac{dx}{y}$. But, we can also differentiate the parametrization: $\cos\theta d\theta = d(\sin\theta)$ or $xd\theta = dy$ to obtain $d\theta = \frac{dy}{x} = \frac{dy}{\sqrt{1-y^2}}$.

Thus, by "$d\theta$", we mean the form which is $-\frac{dx}{y}$ when $y \neq 0$ and $\frac{dy}{\sqrt{1-y^2}}$ when $y \neq \pm 1$. We then invert the integral $\int \frac{dy}{x} = \int \frac{dy}{\sqrt{1-y^2}}$ by defining $\sin\theta$ by the formula

$$\theta = \int_{(1,0)}^{(\cos\theta, \sin\theta)} \frac{dy}{x} = \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}}.$$

The form $d\theta$ is closed but not exact, so θ is a multi-valued function with branches whose difference is a multiple of 2π. Put another way, a particular value of θ determines $(\cos\theta, \sin\theta)$ as well as the winding number of the curve from $(1, 0)$ to $(\cos\theta, \sin\theta)$.

1.7 Implicit Function Theorem

We say that a function $f(z_1, z_n, \ldots z_n)$ in an open set $\Omega \subset \mathbb{C}^n$ is *holomorphic* if $f \in C^1$ and $df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$. This is equivalent to:

(1) f is continuous and holomorphic in each variable separately,

(2) f is analytic (locally represented by convergent power series).

Actually, it is even sufficient just to say "holomorphic in each variable separately," but showing that this condition is equivalent to the previous conditions is rather difficult.

We discuss the implicit function theorem for 2 variables: suppose $f(x, y)$ is a holomorphic function with $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ and we wish to solve the equations $z = f(x, y), \overline{z} = \overline{f}(x, y)$ for $y, \overline{y}$ near (x_0, y_0).

Write $x = x_1 + ix_2, y = y_1 + iy_2$ and $z = z_1 + iz_2$. A simple calculation shows that $dx \wedge d\overline{x} = -2i dx_1 \wedge dx_2$ (we will use this equation for variables y and z in place of x).

For a fixed x, $dz = \frac{\partial f}{\partial y} dy$ and $d\overline{z} = \frac{\partial \overline{f}}{\partial \overline{y}} d\overline{y} = \overline{\frac{\partial f}{\partial y}} d\overline{y}$, so $dz \wedge d\overline{z} = \left|\frac{\partial f}{\partial y}\right|^2 dy \wedge d\overline{y}$. By the above equation, this is the same as $dz_1 \wedge dz_2 = \left|\frac{\partial f}{\partial y}\right|^2 dy_1 \wedge dy_2$, i.e the Jacobian determinant

$$\det \frac{\partial(z_1, z_2)}{\partial(y_1, y_2)} = \left|\frac{\partial f}{\partial y}\right|^2 > 0.$$

By the implicit function theorem (for real-valued functions), we see that we can solve y_1, y_2 as C^1 functions in terms of $z_1, z_2; x_1, x_2$ near $(x_0, y_0, 0)$.

The equation $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ shows that that dy is a linear combination of dx, dz with holomorphic coefficients (here we are using that $\frac{\partial f}{\partial y} \neq 0$). This means the solution is holomorphic.

Chapter 2

Interpolation

2.1 Meromorphic functions with prescribed singularities

We now shift gears to constructing meromorphic functions with prescribed singularities, i.e with given poles and their singular parts.

Over the Riemann sphere, the only meromorphic functions are rational functions; but over the complex plane, we have a wealth of meromorphic functions such as $\tan z$ or $\sec z$ but infinity is a limit of poles.

Theorem (Mittag-Leffler)**.** *Suppose* $\{b_k\} \in \mathbb{C}$ *are given and distinct with* $\lim_{k\to\infty} b_k = \infty$ *and* $P_k(z)$ *be polynomials without constant term. Then there is a meromorphic function in* $\mathbb{C}$ *with poles* b_k *and principal parts* $P_k(\frac{1}{z-b_k})$*. The most general meromorphic function with these poles and primitives parts is*

$$f(z) = \underbrace{\sum_{k=1}^{\infty}\left(P_k\Big(\frac{1}{z-b_k}\Big) - p_k(z)\right)}_{\diamondsuit} + g(z)$$

where $p_k(z)$ *are polynomials chosen such that the series convergences and* $g(z)$ *is an entire function.*

We can assume that no $b_k = 0$. The function $P_k(\frac{1}{z-b_k})$ is holmorphic in $|z| < b_k$ so we can expand it as a convergent power series in that disk.

Let $p_k(z)$ be the sum of the terms of order up to m_k where m_k is chosen such that

$$\left|P_k\Big(\frac{1}{z-b_k}\Big) - p_k(z)\right| \leq \frac{1}{2^k} \qquad \text{on} \qquad |z| \leq \frac{|b_k|}{2}$$

We show $\diamondsuit$ converges uniformly and absolutely on $|z| \leq r$ for any r. Chosen n such that $|b_k| \geq 2r$ if $k > n$. It suffices to compare

$$\sum_{k=n+1}^{\infty} P_k\Big(\frac{1}{z-b_k}\Big) - p_k(z)$$

with $\sum \frac{1}{2^k}$. Thus $\diamondsuit$ is meromorphic in $\mathbb{C}$ with poles $\{b_k\}$ and principle parts $P_k(\frac{1}{z-b_k})$. Given any other function with prescribed singularities, the difference of it and $\diamondsuit$ is holomorphic and conversely, given any holomorphic function $g(z)$, the function $\diamondsuit + g(z)$ also has the desired singularities.

2.2 Infinite products

We say that $\prod_{n=1}^{\infty} b_n$ *converges* to $p \in \mathbb{C}$ if no b_n are 0 and the partial products $p_n = b_1 b_2 \dots b_n$ have a non-zero limit p. Actually, this is too restrictive: we should allow a finite number of terms to be 0, but after removing these few terms, the partial products should have a non-zero limit.

A necessary condition for convergence is $b_n = \frac{p_n}{p_{n-1}} \to 1$. We will instead write the product as (P) $\prod_{n=1}^{\infty}(1+a_n)$, so the condition requires that $a_n \to 0$. To the product, we shall associate the sum (S) $\sum_{n=1}^{\infty} \log(1+a_n)$ where we take the principal branch of log, defined on the complement of the negative real ray with $\log 1 = 0$.

Proposition. *The product (P) and the sum (S) converge or diverge simultaneously. Just as p_n is the n-th partial product of (P), we will write s_n for the n-th partial sum of (S).*

If s_n converges, then so does p_n as $p_n = e^{s_n}$. Conversely, suppose p_n converges to some $p \neq 0$. Fix a determination of $\log p = \log|p| + i \cdot \arg p$. Choose determinations of $\log p_n = \log|p_n| + i \cdot \arg p_n$ such that $\arg p - \pi < \arg p_n \leq \arg p + \pi$.

Then $s_n = \log p_n + 2\pi i \cdot k_n$ for integers k_n. For s_n to converge, we need to know that k_n stabilize. Indeed,

$$\begin{aligned} 2\pi i(k_{n+1} - k_n) &= \log(1+a_{n+1}) - \log p_{n+1} + \log p_n, \\ &= \log(1+a_{n+1}) - (\log p_{n+1} - \log p) + (\log p_n - \log p). \end{aligned}$$

The three terms $\log(1+a_{n+1})$, $\log p_{n+1} - \log p$, $\log p_n - \log p$ are small when n is large. As the left hand side can take upon a discrete set of values, it must be 0.

We say that the product (P) *converges absolutely* if (S) does, but the latter is equivalent to (S') $\sum_{n=1}^{\infty} a_n$ converging absolutely as $\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$.

Now, suppose that $f_n(z)$ are continuous complex-valued functions defined on an open set $\Omega \subset \mathbb{C}$. We say that the product $\prod_{n=1}^{\infty} f_n(z)$ converges uniformly (or absolutely) on a compact set $K \subset \Omega$ if $\sum \log f_n(z)$ does.

Theorem. *Suppose $f_n(z)$ are holomorphic in Ω and $\prod_{n=1}^{\infty} f_n(z)$ converges uniformly and absolutely on compact subsets of Ω. Then $f(z) = \prod_{n=1}^{\infty} f_n(z)$ is holomorphic on Ω and $f = f_1 f_2 \dots f_q \prod_{n>q} f_n(z)$.*

If $Z(f)$ denotes the zeros of function f, then $Z(f) = \bigcup Z(f_n)$. Additionally, the multiplicity of a zero of f is the sum of that of f_n.

Corollary. *Under the same hypothesis, the series of meromorphic functions $\sum_{n=1}^{\infty} \frac{f_n'}{f_n}$ converges uniformly on compact subsets of Ω and limit is $\frac{f'}{f}$.*

Let U be a relatively compact subset of Ω and $g_q(z) = \exp\Big(\sum_{n>q} \log f_n(z)\Big)$. By the product rule:

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{q} \frac{f_n'(z)}{f_n(z)} + \frac{g_q'(z)}{g_q(z)} = \sum_{n=1}^{q} \frac{f_n'(z)}{f_n(z)} + \sum_{n>q} \frac{f_n'(z)}{f_n(z)}.$$

The latter equality is justified as $\log_{n>q} \log f_n$ converges uniformly to a (branch of) $\log g_q$ and thus can be differentiated term-by-term (take q large enough so that f_n have no zeros in $\overline{U}$ for $n > q$).

Example

We develop an infinite product for $\sin \pi z$. The product $f(z) = z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$ converges uniformly and absolutely on compact subsets of $\mathbb{C}$ (because $\sum \frac{z^2}{n^2}$ does).

Differentiating logarithmically, we find that

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n^2}}{1 - \frac{z^2}{n^2}} = \frac{1}{z} + \sum \frac{2z}{z^2 - n^2} = \pi \cot \pi z = \frac{\sin' \pi z}{\sin \pi z}$$

by Example 2 in Section 1.2. It follows that $f(z) = c \sin \pi z$. But as $\lim_{z\to 0} \frac{f(z)}{z} = 1$ and $\lim_{z\to 0} \frac{\sin \pi z}{z} = \pi$, we see that $c = \frac{1}{\pi}$. Hence, $\sin \pi z = \pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$.

2.3 Entire functions with prescribed zeros

An entire function $f(z)$ with no zeros is of the form $e^{g(z)}$ for some entire function $g(z)$. Indeed, the form $\frac{f'(z)}{f(z)}dz$ counts the number of zeros, but since there aren't any, it is exact. So, it is the derivative of an entire function $g(z)$. Then

$$\frac{d}{dz}\Big(f(z)\cdot e^{-g(z)}\Big) = f'(z)\cdot e^{-g(z)} - f(z)\cdot\frac{f'(z)}{f(z)}e^{-g(z)} = 0,$$

so $f(z) = \text{const}\cdot e^{-g(z)}$ but the constant can be absorbed into $g(z)$.

An entire function with finitely many zeros $\{a_k\}$ (these not be necessarily distinct) can be written as

$$f(z) = z^m e^{g(z)}\prod_{k=1}^{n}\Big(1-\frac{z}{a_k}\Big).$$

Here, we put the zeros at $z=0$ separately in the factor z^m. Now, we examine the case when an entire function has infinitely many zeros.

Theorem (Weierstrass)**.** *Suppose $\{a_k\}\in\mathbb{C}$, $\lim_{k\to\infty} a_k = \infty$, a_k not necessarily distinct. There exists an entire function with prescisely $\{a_k\}$ as zeros. Any such function is of the form*

$$f(z) = z^m e^{g(z)}\prod_{k=1}^{\infty}\Big[\Big(1-\frac{z}{a_k}\Big)\cdot p_k(z)\Big].$$

Here, $p_k(z)$ are factors which guarantee that the product converges. The product converges if the sum of logarithms of its terms converge, so we look at the expansion

$$\log\Big(1-\frac{z}{a_k}\Big) = -\frac{z}{a_k} - \frac{1}{2}\cdot\frac{z^2}{a_k^2} - \frac{1}{3}\cdot\frac{z^3}{a_k^3} - \dots$$

Thus, a good form for

$$p_k(z) = \exp\Big(\frac{z}{a_k} + \frac{1}{2}\cdot\frac{z^2}{a_k^2} + \dots + \frac{1}{m_k}\cdot\frac{z^{m_k}}{a_k^{m_k}}\Big).$$

We need to show we can choose integers m_k to make that the product converge, i.e we need to make $\sum_k\Big\{\log\Big(1-\frac{z}{a_k}\Big) + \frac{z}{a_k} + \frac{1}{2}\cdot\frac{z^2}{a_k^2} + \dots + \frac{1}{m_k}\cdot\frac{z^{m_k}}{a_k^{m_k}}\Big\}$ converge. We denote the general term of this sum by $g_k(z)$.

Actually, we would need that $-\pi < \operatorname{Im} g_k(z) < \pi$, so we know that we took the principal branch of logarithm and not some other branch (this would be automatic because we need g_k to tend to 0 for convergence).

Suppose $r > 0$, for $|z| \leq r$, we consider terms with $|a_k| > r$:

$$g_k(z) = -\frac{1}{m_k+1}\Big(\frac{z}{a_k}\Big)^{m_k+1} - \frac{1}{m_k+2}\Big(\frac{z}{a_k}\Big)^{m_k+2} - \dots$$

Thus,

$$|g_k(z)| \leq \frac{1}{m_k+1}\Big(\frac{r}{|a_k|}\Big)^{m_k+1}\Big(1 - \frac{r}{|a_k|}\Big)^{-1}$$

It suffices to choose $\{m_k\}$ so that $\sum(\frac{r}{|a_k|})^{m_k+1}$ converges for *every* $r > 0$ (note; however, that the sum has less terms with greater r); then the product converges uniformly and absolutely in $|z| \leq r$. Clearly, $m_k = k$ suffices.

Corollary. *Every meromorphic function $F(z)$ defined on the entire complex plane is a quotient of entire functions.*

Indeed, let $g(z)$ be the entire function with the poles of $F(z)$ as its zeroes. Then, $F(z)g(z)$ is an entire function $f(z)$, so $F(z) = \frac{f(z)}{g(z)}$.

Corollary. *Suppose $a_k \in \mathbb{C}$, $\lim_{k\to\infty} a_k = \infty$, $b_k \in \mathbb{C}$, $m_k \in \mathbb{N}$. Then there exists an entire function $f(z)$ such that a_k is a root of order m_k of $f(z) = b_k$.*

Let $g(z)$ be an entire function with zero of order m_k at a_k. Write:

$$f(z) = g(z)h(z) = b_k + g(z)\cdot\Big(h(z) - \frac{b_k}{g(z)}\Big)$$

Now choose $h(z)$ to be a meromorphic function with poles $\{a_k\}$ such that at each a_k, the principal part of $h(z)$ was the principal part of $\frac{b_k}{g(z)}$.

At $z = a_k$, $g(z)$ has a zero of order m_k, $h(z) - \frac{b_k}{g(z)}$ is holomorphic which shows that $f(z) = b_k$ has a root of order at least m_k at a_k.

To make sure that $f(z) = b_k$ has a root of order exactly m_k at a_k, we make two changes: (1) we change the order of $g(z)$ at a_k from m_k to $m_k + 1$ and (2) we change the principal part of $h(z)$ at a_k to be $\frac{b_k}{g(z)} + \frac{1}{z-a_k}$.

2.4 The Gamma Function

Consider the infinite product:

$$g(z) = \underbrace{z(1+z)}_{f_1(z)} \prod_{n=2}^{\infty} \Big[\underbrace{\Big(1+\frac{z}{n}\Big)\Big(1-\frac{1}{n}\Big)^z}_{f_n(z)} \Big].$$

We want to show it converges uniformly and absolutely on compact subsets of $\mathbb{C}$. For this purpose, we examine the sum of

$$\begin{aligned} \log f_n(z) &= \log\Big(1+\frac{z}{n}\Big) + z\log\Big(1-\frac{1}{n}\Big). \\ &= \Big(\frac{z}{n} - \frac{z^2}{2n^2} + \frac{z^3}{3n^3} - \dots\Big) + z(-\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \dots\Big) \end{aligned}$$

If $|z| \le r$ and $n > 2r$, we have the estimate

$$\Big|\log f_n(z)\Big| \le 2 \cdot \frac{r^2}{n^2} \cdot \Big(1-\frac{r}{n}\Big)^{-1} < 4r^2 \cdot \frac{1}{n^2}.$$

Hence the associated sum converges by comparison with $\sum \frac{1}{n^2}$.

Let $g_n(z) = \prod_{k=1}^n f_k(z)$. As $\Big(1-\frac{1}{2}\Big)\Big(1-\frac{1}{3}\Big)\dots\Big(1-\frac{1}{n}\Big) = \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n} = \frac{1}{n}$, we see

$$g_n(z) = z(1+z)\Big(1+\frac{z}{2}\Big)\dots\Big(1+\frac{z}{n}\Big)\cdot n^{-z} = \frac{z(z+1)\dots(z+n)}{n!}n^{-z}$$

The function $g(z)$ has zeros at $0, -1, -2, \dots$, all of which are simple.

Also, $\frac{g(z)}{g(z+1)} = z$. Indeed, $g_n(z+1) = \frac{(z+1)\cdots(z+n+1)}{n!}n^{-z-1}$, so $\frac{g_n(z)}{g_n(z+1)} = \frac{zn}{z+n+1} \to z$ as $n \to \infty$. Additionally, $g(1) = \lim_{n\to\infty} g_n(1) = \lim_{n\to\infty}\frac{n+1}{n} = 1$.

Define the *Euler gamma function* $\Gamma(z)$ as $\frac{1}{g(z)}$. It is meromorphic with simple poles at $0, -1, -2, \dots$ and satisfies $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1) = 1$, from which we can see $\Gamma(n+1) = n!$.

We now prove the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$:

$$\begin{aligned} g_n(z)g_n(1-z) &= \Big\{\frac{z(z+1)\dots(z+n)}{n!}\cdot n^{-z}\Big\}\Big\{\frac{(1-z)(2-z)\dots(n+1-z)}{n!}\cdot n^{z-1}\Big\} \\ &= \frac{n+1-z}{n}\cdot z\prod_{k=1}^{n}\Big(1-\frac{z^2}{k^2}\Big). \end{aligned}$$

By the Example in Section 2.2, as $n \to \infty$, the infinite product converges to $\frac{\sin \pi z}{\pi}$ and $\frac{n+1-z}{n} \to 1$, hence $g(z)g(1-z) = \frac{\sin \pi z}{z}$.

We now give an alternate infinite product for $\frac{1}{\Gamma(z)}$. We write

$$g_n(z) = z\prod_{k=1}^{n}\left\{\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\right\}e^{z(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)}.$$

This suggests that

$$g(z) = z\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\right\}\cdot e^{z\gamma}.$$

The convergence of the infinite product can be established as before (we can again compare with $\sum \frac{1}{n^2}$). This alternate expansion shows that $1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n$ converges; its limit is γ, the *Euler gamma constant.*

The logarithmic derivative of $\Gamma(z)$ is

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty}\left\{\frac{1}{n} - \frac{1}{z+n}\right\}.$$

Taking derivatives again, we find $\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$. This is positive when z is real and positive, so $\log\Gamma(z)$ is convex.

A theorem of Bohr and Mollerup says that Γ is the only meromorphic function which is logarithmically convex on the positive real axis which satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1) = 1$.

2.5 Normal families and compact subsets of $\mathbb{C}(\Omega)$

Suppose $\Omega \subset \mathbb{C}$ is open. We have seen that $C(\Omega)$ is metrizable and that $H(\Omega)$ is a closed subset of $C(\Omega)$ with the induced metric.

Recall that a metric space is compact if and only if every sequence has a convergent subsequence.

We say that $\mathcal{S} \subset H(\Omega)$ is a "normal family" if every sequence in $\mathcal{S}$ has a subsequence that converges uniformly on compact subsets of Ω; in other words, $\mathcal{S}$ is normal if its closure is compact.

A set $\mathcal{S} \subset H(\Omega)$ or of $C(\Omega)$ is *bounded uniformly on compact subsets* of Ω if for every compact $K \subset \Omega$, there is a constant $M(K)$ such that $|f(z)| \le M(K)$ when $z \in K$ for all $f \in \mathcal{S}$.

Proposition. *If $\mathcal{S} \subset H(\Omega)$ or of $C(\Omega)$ is compact, then $\mathcal{S}$ is closed and bounded uniformly on compact sets.*

The fact that $\mathcal{S}$ is closed follows from $H(\Omega)$ being Hausdorff. For a fixed compact set K, the map $H(\Omega) \to \mathbb{R} : f \to \max_{z \in K} |f(z)|$ is continuous and hence it takes K to some bounded set of $\mathbb{R}$.

The converse is only true in $H(\Omega)$. It follows from the following theorem:

Theorem (Montel)**.** *A set of functions $S \subset H(\Omega)$ is a normal family if and only if it is bounded uniformly on compact sets.*

We prove it using two lemmas.

Lemma. *The mapping $H(\Omega) \to H(\Omega) : f \to f'$ takes a subset that is bounded uniformly on compact sets to another subset bounded uniformly on compact sets.*

It suffices to show that for any $z_0 \in \Omega$, there is a small disk $D \subset \Omega$ centered at z_0 such that $\mathcal{S}'$ is bounded uniformly on D. Choose D with radius $r/2$ such that the disk $\tilde{D}$ centered at z_0 with radius r is contained in Ω. Let γ be the positively oriented boundary of $\tilde{D}$.

Suppose $z \in D$. By Cauchy's integral formula, $f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$. Suppose M is such that $|f(z)| \leq M$ on γ for all $f \in S$. As $|\zeta - z| \geq r/2$, $|f'(z)| \leq \frac{M}{2\pi} \cdot \frac{4}{r^2} \cdot 2\pi r = \frac{4M}{r}$.

Lemma. *Let D be the open disk with center z_0 and radius R and $\mathcal{S} \subset H(\Omega)$ be bounded uniformly on compact subsets of D. Then a sequence $\{f_n\} \subset \mathcal{S}$ converges uniformly on compact sets if and only if for every k, the sequence of derivatives at z_0, $\{f_n^{(k)}(z_0)\}$ converges.*

For the "only if" direction, we simply need to know that the map $f \to f'$ is continuous (we don't need the hypothesis).

Now we prove the "if" direction. It suffices to show that for any $z_0 \in \Omega$, there is a small disk D centered at z_0 such that for any $f \in S$, $\{f_n\}$ converges uniformly on D. Let $D(z_0, R)$ be a closed disk in Ω. We pick $D = D(z_0, r)$ with radius $r < R$.

Let $r < r_0 < R$. Choose M so that $|f_n(z)| \leq M$ for all n whenever $|z - z_0| \leq r_0$. We can write $f_n(z)$ as

$$\sum_{k=0}^{\infty} \frac{f_n^{(k)}(z_0)}{k!} \cdot (z - z_0)^k.$$

If we let the coefficients $\frac{f_n^{(k)}(z_0)}{k!}$ be a_{nk}, then by Cauchy's estimate, $|a_{nk}| \leq \frac{M}{r_0^k}$. It follows that

$$|f_n(z) - f_m(z)| \leq \sum_{k=0}^{l} |a_{nk} - a_{mk}| \cdot r^k + 2M \sum_{k>l} \left(\frac{r}{r_0}\right)^k.$$

As the geometric series converges, we can choose l large enough so the second term is small. We then choose n_0 large so that when $m, n > n_0$, $|a_{nk} - a_{mk}|$ are simultaneously small for all $0 \leq k \leq l$. Then $\{f_n(z)\}$ converges uniformly as desired.

We are now ready to prove Montel's theorem. Suppose $\mathcal{S} \subset H(\Omega)$ is bounded uniformly on compact sets and $\{f_n\} \subset \mathcal{S}$. We wish to show $\{f_n\}$ has a convergent subsequence.

Take a countable cover of Ω by disks $D(z_i, r_i) \subset \Omega$. Let $\lambda_i^j : H(\Omega) \to \mathbb{C}$ take $f \to f^{(k)}(z_i)$. It suffices to find a subseqence $N \subset \mathbb{N}$ such that $\lim_{n \in N} \lambda_i^k(f_n)$ exists for all k, i.

We construct N as follows: We reindex λ_i^k as μ_l. By the first lemma, for each l, $\mu_l(f_n)$ is bounded. Then we diagonalize, i.e we first choose $N_1 \subset \mathbb{N}$ so that the $\lim_{n \in N_1} \mu_1(f_n)$ converges; next we choose $N_2 \subset N_1$ so that $\lim_{n \in N_2} \mu_2(f_n)$ converges, then $N_3 \subset N_2$ so $\lim_{n \in N_3} \mu_3(f_n)$ converges and so on.

We then take N to be the diagonal sequence: $N_{11}, N_{22}, N_{33}, \ldots$

Chapter 3

Automorphisms

3.1 Local Geometry of Holomorphic functions

We say that a function f is *conformal* (*biholomorphic*) if f is holomorphic and has a holomorphic inverse. If f is a holomorphic function not identically constant, either of the two possibilities hold:

(1) Suppose $w_0 = f(z_0)$ and $f'(z_0) \neq 0$. By the inverse function theorem, f^{-1} exists and is holomorphic in a neighbourhood of w_0 and so f is conformal at z_0.

(2) Suppose instead $f'(z_0) = 0$. We may assume $z_0 = 0$, $f(z_0) = 0$; otherwise we work with $f(z + z_0) - f(z_0)$. We may write $w = z^p f_1(z)$ with $f_1(0) \neq 0$ for some $p \geq 2$. Then, $w = (zg(z))^p$ where $g(z)$ is some determination of the p-th root of f_1. Then $w = \zeta^p$ where $\zeta = zg(z)$ is a *holomorphic coordinate change*, i.e a map of type (1).

Thus up to a conformal change of coordinates, every holomorphic function is of form $\zeta^p + f(z_0)$. The number $p - 1$ is called the *ramification index* of f at z_0.

Lemma. *A non-constant holomorphic mapping is open.*

We know that in a small neighbourhood of $f(z_0)$, every value sufficiently close to $f(z_0)$ appears exactly p times (as this is true for $\zeta^p + f(z_0)$), so f is open.

Corollary. *If $f \in H(\Omega)$ and injective, then f is a homeomorphism. Furthermore, f^{-1} is holomorphic.*

Since (2) cannot be injective, (1) must always be the case, hence f is a local homeomorphism. But as f is injective, it is a genuine homeomorphism.

Remark. *Even if $f'(z) \neq 0$ for all $z \in \Omega$, this not guarantee that f is injective. For instance, $f(z) = e^z$ is one-to-one on any strip $a < Im\ z < b$ with $b - a < 2\pi$.*

Remark. *While the complex plane $\mathbb{C}$ and the unit disk $D = \{z : |z| < 1\}$ are homeomorphic; they are not biholomorphic, for any function $f : \mathbb{C} \to D$ is constant by Liouville's theorem.*

Notice that if $f, g : \Omega \to \Omega'$ are two biholomorphic maps, then $g^{-1} \circ f : \Omega \to \Omega$ is biholomorphic. Hence all biholomorphic maps $\Omega \to \Omega'$ are given by composing a fixed holomorphic map with an automorphism of Ω.

Examples

The automorphisms of the Riemann sphere are the fractional linear transformations $z \to \frac{az+b}{cz+d}$. The automorphisms of the complex plane are affine maps $z \to az + b$.

The automorphisms of the upper half plane are fractional linear transformations with real coefficients and $ad - bc > 0$.

3.2 Automorphisms

We classify automorphism groups of some domains in $\mathbb{C}$.

Theorem. $\operatorname{Aut}\mathbb{C}$ *are affine maps $w = az + b$ with $a \neq 0$.*

Suppose $w = f(z)$ is an automorphism of $\mathbb{C}$. Either, (1) $f(z)$ has an essential singularity at infinity, or (2) it is a polynomial (being a rational function with only pole at infinity).

The first case cannot happen: on one hand, $\mathrm{Im}\{|z| < 1\}$ and $\mathrm{Im}\{|z| > 1\}$ should be disjoint (as $f(z)$ is injective), but as f is open, the first set is open; but the essential singularity at infinity implies the second set is dense which is impossible.

We investigate the second case further. Suppose f is a polynomial of degree n. Then, $f(z) = w$ has n distinct roots except for special values of w (namely, when $f'(z) = 0$). As f is injective, we see that $n = 1$.

We conclude $f(z) = az + b$. Obviously, $a \neq 0$. If $a = 1$, $f(z)$ is a translation; if $a \neq 1$, f has a unique fixed point $z = \frac{b}{1-a}$.

Theorem. Aut S^2 *are fractional linear transformations* $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ *(the inverse is* $z = \frac{dw-b}{-cw+a}$*). (Note:* w *takes infinity to* $\frac{a}{c}$ *if* $c \neq 0$ *and to infinity if* $c = 0$*).*

Obviously, the fractional linear transformations form a *subgroup* of $G \subset \operatorname{Aut} S^2$. We want to show that actually $G = \operatorname{Aut} S^2$.

The subgroup of Aut S^2 which fixes the point at infinity is precisely Aut $\mathbb{C}$. As fractional linear transformations act transitively on the Riemann sphere, for $T \in \operatorname{Aut} S^2$, we could consider a fractional linear transformation S which takes $T(\infty) \to \infty$. Then $S \circ T$ must be affine which shows that T is a fractional linear transformation.

Let $\mathbb{H}^+$ be the upper half-plane and D the unit disk. The two are biholomorphic: indeed, the map $z \to \frac{z-i}{z+i}$ which takes $\mathbb{H}^+$ to D: it maps i to the origin and points $0, 1, \infty$ to $-1, -i, 1$ respectively.

Theorem. Aut D *are fractional linear transformations of the form*

$$w = e^{i\theta} \cdot \frac{z - z_0}{1 - \overline{z_0} z}$$

with $|z_0| < 1$.

The above transformations are automorphisms, they take the unit disk into itself and their inverse is of the same form. We now show that there are no other automorphisms.

Suppose $T \in \operatorname{Aut} D$. Let $S = e^{i\theta} \cdot \frac{z-z_0}{1-\overline{z_0}z}$ with $z_0 = T^{-1}(0)$ and $\theta = \arg T'(z_0)$. Let $f(z) = S \circ T^{-1}(z)$. Then $f(0) = 0$ and $|f(z)| < 1$ when $|z| < 1$. By Schwarz's lemma, $|f(z)| \leq |z|$.

Applying the same argument to the inverse of f, we see that $z \leq |f(z)|$, so $|f(z)| = |z|$. By the equality condition of Schwarz's lemma, we see that $f(z) = e^{i\alpha} \cdot z$, so $S(z) = e^{i\alpha} \cdot T(z)$. We now differentiate and set $z = z_0$: $S'(z_0) = e^{i\alpha} \cdot T'(z_0)$; from which follows that $\alpha = 0$ or $S = T$ as desired.

Theorem. Aut $\mathbb{H}^+$ *are fractional linear transformations with real coefficients and positive determinant.*

Conjugation by $z \to \frac{z-i}{z+i}$ shows that Aut $\mathbb{H}^+$ is composed solely of fractional linear transformations. If a fractional linear transformation $w(z)$ is to take the real axis to itself, it takes $1, 0, \infty$ to three real numbers z_1, z_0, z_∞; so $w(z) = \frac{z-z_0}{z-z_\infty} \cdot \frac{z_1-z_\infty}{z_1-z_0}$.

3.3 Riemann Mapping Theorem

Our next goal is to show that any simply-connected subset Ω of $\mathbb{C}$ (except for $\mathbb{C}$ itself) is biholomorphic to the unit disk $D = \{|z| < 1\}$.

We first reduce to the case of bounded domain. If Ω isn't the entire complex plane, then there is some point $a \notin \Omega$. As Ω is simply-connected, $\frac{1}{z-a}$ has a primitive $g(z)$ in Ω. Then, $z - a = e^{g(z)}$.

As $G(\Omega)$ is open, it contains a disk E centered at $g(z_0)$. As $e^{g(z)}$ is injective, E and its translate $E + 2\pi i$ are disjoint. Then, $\frac{1}{g(z)-[g(z_0)+2\pi i]}$ is a biholomorphism of Ω onto a bounded domain.

By translating and scaling if necessary, we can assume that Ω contains the origin and lies inside the unit disk.

Let $\mathcal{A} = \{f \in H(\Omega) : \text{ injective}, f(0) = 0, |f(z)| < 1\}$.

Lemma. *Then $g(\Omega) = D$ if and only if $|g'(0)| = \sup_{f\in\mathcal{A}} |f'(0)|$.*

Assuming the lemma, it suffices to show that $\sup_{f\in\mathcal{A}} |f'(0)|$ is attained. For this purpose, we introduce $\mathcal{B} = \{f \in A : |f'(0)| \geq 1\}$. $\mathcal{B}$ is non-empty as $f(z) = z$ is in it. It suffices to show that $\mathcal{B}$ is compact.

By Montel's theorem, $\mathcal{B}$ is normal as it is bounded uniformly on compact sets (its actually bounded everywhere by 1). To show that $\mathcal{B}$ is closed, suppose $f = \lim f_n$ with $f_n \in \mathcal{B}$ (as usual, we assume that the sequence converges uniformly on compact sets).

We know that $f(0) = \lim f_n(0) = 0$, clearly $|f'(0)| \leq 1$. Injectivity follows from Hurwitz's theorem (see Section 1.1), the functions are non-constant as $|f_n'(0)| \geq 1$ (from which follows that $|f'(0)| \geq 1$, recall that the map $f \to |f(0)|$ is continuous).

It is easy to see that $|f(z)| \leq 1$, but if the value 1 is attained, by the maximum-modulus principle, f reduces to a constant which is impossible.

The remaining lemma follows from *Schwarz lemma* which says that if $f \in H(D), f(0) = 0, |f(z)| < 1$ then $|f(z)| \leq |z|$ for all $z \in D$. Indeed, applying the maximum-modulus principle to $g(z) = \frac{f(z)}{z}$, we see $|g(z)| \leq \frac{1}{|z|} \leq 1$ and $|f'(0)| \leq 1$.

If additionally $|f(z_0)| = |z_0|$ for some $z_0 \in D$, then $f(z) = \lambda z$ with $|\lambda| = 1$: by the maximum-modulus principle, g must be constant; and at z_0, λ has absolute value 1.

3.4 Riemann Mapping Theorem II

We defined family $\mathcal{A} = \{f \in H(\Omega) :$ injective, $f(0) = 0, |f(z)| < 1\}$. We want to show that $g(\Omega) = D$ if and only if $|g'(0)| = \sup_{f\in\mathcal{A}} |f'(0)|$.

The "only if" direction is quite clear: suppose $g \in \mathcal{A}$ with $g(\Omega) = D$ and $f \in \mathcal{A}$ is some other holomorphic map. Then $h = f \circ g^{-1}$ takes the disk to $f(\Omega)$ and takes the origin to itself. By Schwarz's lemma, $|h'(0)| \le 1$, differentiating $f = g \circ h$, we see that $|f'(0)| \le |g'(0)|$.

Suppose $f \in \mathcal{A}$ and a is some point in the disk but not in $f(\Omega)$. We want to show that there is some $g \in \mathcal{A}$ with $|g'(0)| > |f'(0)|$.

Let $\varphi(\zeta) = \frac{\zeta - a}{1-\overline{a}\zeta}$ be an automorphism of the disk which sends a to 0. Then, $\varphi \circ f(z) = \frac{f(z)-a}{1-\overline{a}f(z)}$ is non-vanishing. Since Ω is simply-connected, it has a single-valued holomorphic square root $F(z)$, i.e if $\theta(w) = w^2$, we may write $\varphi \circ f(z) = \theta \circ F(z)$.

Then

$$f = \varphi^{-1} \circ \theta \circ F = \underbrace{\varphi^{-1} \circ \theta \circ \psi^{-1}}_{h(z)} \circ \underbrace{\psi \circ F}_{g(z)}$$

where $\psi(\eta) = \frac{\eta - F(0)}{1-\overline{F(0)}\eta}$. Then h take the disk to itself and fixes the origin.

This allows us to apply Schwarz's lemma: $|h'(0)| \le 1$. We don't win unless we see that $|h'(0)|$ is actually less than 1. But if $|h'(0)| = 1$, the equality condition of Schwarz would imply that h is a rotation, but it isn't.

Boundary Behaviour

Having shown that any two simply connected domains in the complex plane are biholomorphic, we wish to see if the mapping extends a homeomorphism of closed domains. We will do this in the special case of a polygon (although a theorem of Carathéodory shows that this is true for any domains with Jordan curve boundary).

Suppose Ω is a simply-connected open set whose boundary is a polygonal curve with vertices $z_1, z_2, \dots z_n; z_{n+1} = z_1$. Let α_k be the inner angle at z_k, so $0 < \alpha_k < 2$ (more precisely, α_k is the value of the argument of $\frac{z_{k-1}-z_k}{z_{k+1}-z_k}$ between 0 and 2).

Let $\beta_k\pi$ be the outer angle at z_k, so $\beta_k = 1 - \alpha_k$, i.e $-1 < \beta_k < 1$ and $\beta_1 + \dots + \beta_n = 2$. It is easy to see that Ω is convex if and only if all $\beta_k > 0$.

We will need a non-standard version of the *Schwarz reflection principle*:

Lemma (Schwarz)**.** *Suppose $\Omega = \Omega^+ \cup \Omega^-$ is a domain symmetric about the real axis and $f(z)$ is a holomorphic function defined on Ω^+ whose imaginary part tends to zero as z approaches the real axis. Then $f(z)$ can be completed to a holomorphic function defined on all of Ω by setting $f(\overline{z}) = \overline{f(z)}$.*

The difficulty here is that a priori the real part of $f(z)$ may not extend continuously to the real axis. We prove that this is indeed the case. For this purpose, we assume that Ω is simply-connected (this is a local question, if $x \in \mathbb{R}$, we can restrict to a small disk centered at x).

The function $v(z) = \operatorname{Im} f(z)$ is harmonic. We can extend it to a function on Ω by setting $v(\overline{z}) = -v(z)$. It is harmonic as it satisfies the mean value principle. We take $u(z)$ to be the "harmonic conjugate" which agrees with $\operatorname{Re} f(z)$ on Ω^+. Then $u + iv$ is *an* extension of $f(z)$. But if $f(z)$ is to admit any extension beyond the real axis, it must extend continuously up to the real axis.

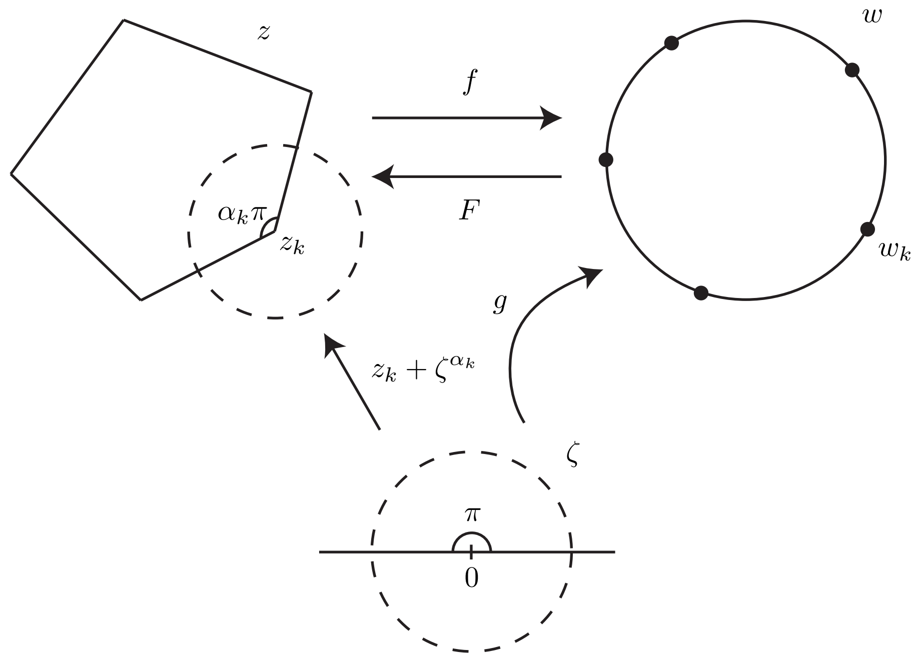
We now know that the formula $f(\overline{z}) = \overline{f(z)}$ defines a function which is holomorphic on $\Omega^+ \cup \Omega^-$ and is continuous on the real axis. By a corollary of Morera's theorem, $f(z)$ must be holomorphic on the real axis as well.

We now return to the question of boundary behaviour of conformal mapping of a polygon Ω. Suppose $f : \Omega \to D$ is conformal. We first show that f extends continuously at the *sides* of Ω. We even show more: namely that f extends beyond each side as a holomorphic function.

This is easy. Simply compose f with the fractional linear transformation from $\varphi : D \to \mathbb{H}^+$ given by $\varphi(w) = \frac{w-i}{w+i}$ and apply the Schwartz reflection principle (exercise: φ does extend to a continuous map from $\overline{D} \to \overline{H}$).

At the angles, the situation is a little more tricky. We straighten the angle at vertex z_k of Ω. For this purpose, we consider the function $g(\zeta) = f(z_k + \zeta^{\alpha_k})$ defined in the neighbourhood of the origin in the upper half plane (see picture).

By arguments with the Schwartz reflection principle, $g(\zeta)$ extends to a holomorphic function in small disk centered at the origin and as $f|_{\partial\Omega} = g|_{\mathbb{R}}$, f extends continuously at the angles as well.



Thus we have shown that the map $f : \partial\Omega \to \partial D$ is continuous. The above reasoning also allows us to conclude that f is locally one-to-one. However, we really want to see that f is genuinely one-to-one on the boundary.

For this purpose, take a positively oriented curve $\gamma \subset \Omega$ which is close and homotopic to $\partial\Omega$. Then $f(\gamma)$ is a simple closed curve which goes around the origin with winding number +1. So $f(\partial\Omega)$ must wind around once counter clockwise around the origin (details are left as an exercise).

3.5 Christoffel-Schwarz Formula

We can even find an explicit formula of a Riemann mapping from a polygon to the unit disk. Actually, the formula is not for $f(z)$, but rather its inverse, which we denote F.

Theorem (Christoffel-Schwarz)**.** *F is of the form $F(w) = c\int_0^w \prod_{k=1}^n (w - w_k)^{-\beta_k} dw + c'$ where are c, c' are constants.*

We denote the image of z_k under f by w_k. Any 3 of the w_k's can be picked arbitrarily but the remainder are uniquely determined (exercise).

Remark. *The formula always takes $\{|z| = 1\}$ to a closed polygonal curve, but the curve may have self intersections. If the curve does not have self-intersections, then F takes the unit disk biholomorphically onto the interior of the curve.*

Fix a vertex z_k of Ω and consider its associated function $g(\zeta)$ constructed previously. As $w = g(\zeta)$ is locally one-to-one at the origin, it is invertible and so exists a power series expansion $\zeta = \sum_{m=1}^{\infty} b_m(w - w_k)^m$ with $b_1 \neq 0$.

Raising to α_k-th power we see that $\zeta^{\alpha_k} = F(w) - z_k = (w - w_k)^{\alpha_k} G(w)$ where $G(w)$ is holomorphic and non-zero near w_k. Differentiating, we find that $F'(w)(w - w_k)^{\beta_k} = \alpha_k G(w) + (w - w_k)G'(w)$ is holomorphic and non-zero at w_k. We can now forget the auxiliary functions g and G.

Set $H(w) = F'(w)\prod_{k=1}^{n}(w - w_k)^{\beta_k}$. By the discussion above, it is holomorphic in the unit disk, extends continuously to the boundary and does not vanish in the closed unit disk. We claim that in fact $H(w)$ is *constant.*

As $H(w)$ does not vanish, it suffices to show that $\arg H(w)$ is constant on $\{|w| = 1\}$ (recall arg is the imaginary part of log). To this end, we show that $\arg H(w)$ is constant for $w = e^{i\theta}$ with $\arg w_k < \theta < \arg w_{k+1}$ and by continuity, $\arg H(w)$ will be constant on the whole unit circle.

Taking arguments, we see that

$$\arg H(w) = \arg F'(w) + \beta_k \sum \arg(w - w_k)$$

The chain rule tells us that $\arg F'(w) = \arg(F \circ w)'(\theta) - \arg w'(\theta)$. The argument of $(F \circ w)'(\theta)$ measures the direction of the tangent to the graph of F which is constant as F traces a side of Ω. On the other hand, $w'(\theta)$ represents the direction of the tangent to circle at $w = e^{i\theta}$, i.e $w'(\theta) = \theta + \pi/2 = \theta + \text{const}$.

By elementary geometry, we see that $\arg(w - w_k) = \theta/2 + \text{const}$ (exercise, hint: by rotation, we can assume that $w_k = 1$). Continuing our calculation, we see that

$$\arg H(w) = -\theta + \left(\sum \beta_k\right)\frac{\theta}{2} + \text{const} = \text{const}.$$

If $H(w) = c$ (identically), then $F'(w) = c\prod_{k=1}^{n}(w - w_k)^{-\beta_k}$ and we obtain the Schwarz-Christoffel formula by integration.

Chapter 4

Complex Geometry

4.1 Complex Manifolds

A *manifold* is a paracompact Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$, i.e we can cover M by open sets U_i for which exist homeomorphisms (charts) $\varphi_i : U_i \to V_i$ with V_i open in $\mathbb{R}^n$.

For *complex structure*, we replace $\mathbb{R}^n$ with $\mathbb{C}^n$ and insist that the transition maps $\varphi_j \circ \varphi_i^{-1} : V_i \to V_j$ be holomorphic for all i, j. A *complex manifold* is a manifold with complex structure.

A function on a complex manifold M is *holomorphic* if it is holomorphic in local coordinates (that is, for any coordinate chart $\varphi : U_i \to V_i$, $f \circ \varphi_i^{-1}$ is holomorphic on V_i).

A one-dimensional complex manifold is called a *complex curve* or an *abstract Riemann surface.*

Example: the Riemann sphere. As charts, we take $U = S^2 \setminus \{N\}$ with $\varphi_U : (x, y, t) \to \frac{x+iy}{1-t}$ (stereographic projection) and $V = S^2 \setminus \{S\}$ with $\varphi_V : (x, y, t) \to \frac{x-iy}{1+t}$ (stereographic projection followed by conjugation).

The transition map $\varphi_U \circ \varphi_V^{-1} : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ takes z to $\frac{1}{z}$. In local coordinates, $z_U = z \circ \phi_U$, $z_V = z \circ \phi_V$, we have $z_U z_V = 1$. It is an exercise to the reader to verify that S^2 is biholomorphic to $P^1(\mathbb{C})$.

Given two complex manifolds M and N, a function $f : M \to N$ is *holomorphic* if it is in local coordinates, i.e if $\psi_j \circ f \circ \varphi_i^{-1}$ is holomorphic for all coordinates charts φ_i of M and ψ_j of N.

Given manifolds M and N with complex structures we say that $f : M \to N$ is an "isomorphism" (or a biholomorphism) if f is a homeomorphism of the underlying topological spaces and f, f^{-1} are holomorphic.

Two complex structures on M are *equivalent* if the identity mapping is an isomorphism. Thus a complex manifold is really a paracompact Hausdorff space with an equivalence class of complex structures.

Properties of holomorphic functions extend to manifolds

(1) If M and N are complex manifolds with M connected and $f, g : M \to N$ are holomorphic and coincide on a set with a limit point, then $f = g$ on M.

Consider the set of points in which f, g coincide in a neighbourhood. It is open (automatic). It is closed (given a sequence $\{z_k\}$, its tail lies in one chart). It is not empty, for it contains the limit point; so f, g must coincide everywhere on M.

(2) Suppose M is connected and f is holomorphic on M, if $|f|$ has a relative maximum, it is constant.

If $|f|$ has a relative maximum, in a neighbourhood, it coincides with the constant function, use part (1). In fact, any holomorphic function on a compact connected manifold is constant.

4.2 Complex Manifolds II

We have seen that holomorphic functions on compact complex manifolds are constant. We are more interested in *meromorphic functions*; these are holomorphic maps into the Riemann sphere.

For example, $\mathfrak{p} : \mathbb{C} \to \mathbb{C}/\Gamma : f \to f \circ \mathfrak{p}$ is a bijection between meromorphic functions on $\mathbb{C}/\Gamma$ and meromorphic functions on $\mathbb{C}$ with Γ as a group of periods.

Suppose $f : M \to N$ is holomorphic, $a \in M$ and $b = f(a)$. Choose local coordinates z, w vanishing at a, b respectively. We define the *ramification index* to be the multiplicity of the root of $w \circ f \circ z^{-1}$.

The multiplicity does not depend on the choice of local coordinates, for coordinates charts are homeomorphisms; in particular, they are locally one-to-one.

If the ramification index is k, we can choose local coordinates in which $w = z^k$.

A *holomorphic differential form* ω on manifold M with atlas (U_i, z_i) is a set of forms $\omega = f_i(z_i)dz_i$ which transform properly, i.e $dz_i = f'_{ij}(z_j)dz_j$.

In a neighbourhood of any point, a holomorphic differential form has a primitive g, i.e a holomorphic function for which $\omega = dg$ (by definition dg is the form $g'(z_i)dz_i$). If M is simply-connected, then ω has a global primitive.

Complex manifolds have a natural orientation as the transition maps are orientation-preserving. This allows us to consider the oriented boundary of compact set. In particular, if ω is a holomorphic differential form and Γ is the oriented boundary of a compact set then $\int_\Gamma w = 0$.

Let ω be a differential form holomorphic in the complement of a discrete set E. If $a \in E$ and z is a local coordinate in a neighbourhood of a, we may write $\omega = \omega_1 + (\frac{c_1}{z} + \frac{c_2}{z^2} + \dots)dz$ where ω_1 is a differential form holomorphic near a.

We define $\mathrm{Res}(\omega, a) = \frac{1}{2\pi i}\int_\gamma \omega$ where γ is a small closed curved around a with winding number 1. The Residue theorem carries over to manifolds (with same proof):

Theorem. *Given a compact set $K \subseteq M$ with piecewise smooth oriented boundary Γ, with Γ containing no points of E, we have $\frac{1}{2\pi i}\int_\Gamma \omega = \sum_{a\in K} \mathrm{Res}(\omega, a)$.*

Let Y be a complex curve. By a *Riemann surface* over Y, we mean a connected complex curve X together with a non-constant holomorphic map $\varphi : X \to Y$ whose ramification points are isolated and the fibers $\varphi^{-1}(y)$ are discrete.

4.3 Examples of Riemann Surfaces

Example 1: $y = \log x$

Let $Y = \mathbb{C}^*$ be the complex plane without the origin. The map $\log z : Y \to \mathbb{C}$ is multi-valued. We consider $X = \mathbb{C}$, $\varphi : X \to Y$ given by $\varphi(t) = e^t$. Then t is a holomorphic function on X which is locally a branch of log, i.e $t = \log \circ \varphi$.

We made the function $\log z$ single-valued by introducing the Riemann surface.

Here $\varphi : X \to Y$ is a covering space or an *unramified* covering space, i.e any point of Y has an open neighbourhood V such that $\varphi^{-1}(V)$ is a disjoint union of open sets U_i each mapped homeomorphically onto V.

Example 2: $y = x^{1/2}$

We now construct a Riemann surface for the function $y = x^{1/2}$ over $Y = \mathbb{C}$. Let $X \subset \mathbb{C}^2$ be the points (x, y) for which $y^2 = x$.

The origin is a ramification point: X lies in two sheets over Y everywhere except at the origin.

To construct a model of X, take two copies of the complex plane, cut out their positive real rays and glue the top of one to bottom of the other and vica versa.

Example 3: $y = (1 - x^3)^{1/3}$

Let $X \subset \mathbb{C}^2$ be the set of points (x, y) for which $x^3 + y^3 = 1$. It is a complex manifold, near (x_0, y_0) with $y_0 \neq 0$, we may take x as a local coordinate, $x \to (x, \sqrt[3]{1 - x^3})$ with the branch of cube root corresponding to (x_0, y_0). Similarly, at $x_0 \neq 0$, $y \to (\sqrt[3]{1 - y^3}, y)$ is a local coordinate.

To see that the two charts are compatible, if $x_0 \neq 0$ and $y_0 \neq 0$, $\sqrt[3]{1 - x^3}$ has a branch equal to y_0 at $x = x_0$ and $\sqrt[3]{1 - x^3}$ has a branch equal to x_0 at $y = y_0$.

As functions on X, both x and y are holomorphic: e.g to see that x is holomorphic, near (x_0, y_0) with $y_0 \neq 0$, x is the local coordinate, $x = x$, the identity map; near (x_0, y_0) with $x_0 \neq 0$, y is the local coordinate, $x = \sqrt[3]{1 - y^3}$.

X has 3 points over every point of $\mathbb{C}$ except at the cube roots of unity, $1, j, j^2$ where $j = e^{2\pi i/3}$.

Consider the differential form $\omega = \frac{dx}{(1-x^3)^{1/3}} = \frac{dx}{y}$ on X. The equation defines ω when $y \neq 0$. Differentiating the equation $x^3 + y^3 - 1 = 0$, we see that $3x^2dx + 3y^3dy = 0$ or $\frac{dx}{y} = -\frac{y}{x^2}dy$ when both $x \neq 0$ and $y \neq 0$. Thus, ω is implicitly defined to be $-\frac{y}{x^2}dy$ when $x \neq 0$.

We made the differential form $\frac{dx}{\sqrt{1-x^3}}$ single-valued by introducing the Riemann surface.

We can extend X to a Riemann surface over $S^2 = P^1(\mathbb{C})$. Let $X' \subset P^2(\mathbb{C})$ be the set of points $[x, y, z]$ given by $x^3 + y^3 = z^3$ formed by homogenizing $(\frac{x}{z})^3 + (\frac{y}{z})^3 = 1$.

X' is composed of X together with 3 points at infinity (i.e where $z = 0$), namely $[1, -1, 0]$, $[j, -1, 0]$ and $[j^2, -1, 0]$.

4.4 Example: $y = \sqrt{P(x)}$

Let $P(x) = (x - a_1)(x - a_2)\ldots(x - a_d)$ be a polynomial with distinct roots. Suppose $X \subset \mathbb{C}^2$ is the set of points (x, y) for which $y^2 = P(x)$. We complete it to a Riemann surface $X' \subset P^2(\mathbb{C})$ over $P^1(\mathbb{C})$ by homogenizing the equation:

$$y^2z^{d-2} = (x - a_1z)(x - a_2z)\ldots(x - a_nz).$$

We now investigate this in detail, paying attention to the branch points and behaviour at infinity.

Case $d = 1$:

The set $X : \{y^2 = x - a\}$ is a parabola and hence a manifold; so $X' : \{y^2 = xz - a_1z^2\}$ is smooth at least in the finite part of $P^2(\mathbb{C})$. At infinity, X' has one point: if we plug $z = 0$ into the equation for X', we get $y^2 = 0$ yielding the point $[1, 0, 0]$.

The equation for X' has multiple roots at a and infinity, which are the *branch points*. One can form a model of X' by taking two copies of the Riemann sphere, making cuts in each of them between a and ∞, gluing the top bank of the first sphere to the bottom bank of the second sphere and vica versa. This construction shows that X' is topologically a sphere.

Case $d = 2$**:**

The equation for X is $\{y^2 = (x-a)(x-b)\}$. Homogenizing, we obtain the equation for X': $\{y^2 = (x-az)(x-bz)\}$. If we set $z = 0$, we get the equation $y^2 = x^2$, so X' has two points at infinity, namely $[1,1,0]$ and $[1,-1,0]$. In coordinates (y,z) at infinity,

$$y = \pm(1-az)^{1/2}(1-bz)^{1/2}.$$

So, X' is a manifold and has branch points are a and b. Actually, this case is pretty much the same as the previous case: if we take the equation $y^2 = (x-az)(x-bz)$ and perform a linear change of coordinates to make it $y^2 = uv$, after we dehomogenize with respect to v, we are left with the equation $y^2 = u$.

Case $d = 3$**:**

Now, the equation for X is $y^2 = (x-a)(x-b)(x-c)$ and for X', $y^2 z = (x-az)(x-bz)(x-cz)$. If we set $z = 0$, we obtain the equation $x^3 = 0$, so there is one point at infinity, namely $[0,1,0]$.

X' has four branch points: a, b, c, ∞. To form a model for X', we take two copies of the Riemann sphere, pair up the points, say a, b and c, d, make cuts between a and b and between c and d in both spheres and glue the banks of $a-b$ and $c-d$ to each other. This shows that X' is topologically a torus.

Case $d \geq 4$**:**

The equation for X' is $y^2 z^{d-2} = \prod_{i=1}^{d}(x - a_i z)$. There is only one point at infinity, $[0,1,0]$. We must choose (x,z) to be the coodinates at infinity:

$$z^{d-2} = \prod (x - a_i z).$$

Unfortunately, this is not smooth at the origin, for it is not true that either $\frac{\partial}{\partial x} = 0$ or $\frac{\partial}{\partial z} = 0$, so the implicit function theorem does not apply.

4.5 Abel's Theorem

Let $X \subset \mathbb{C}^2$ be given by the equation $y^2 = P(x)$ with $P(x) = 4x^3 - 20a_2x - 28a_4$ having 3 distinct roots. Let $X' \subset P^2(\mathbb{C})$ be given by the equation $y^2t = 4x^3 - 20a_2xt - 28a_4$. Then, X embeds into $X' \subset P^2(\mathbb{C})$ by taking (x, y) to $[x, y, 1]$.

We have seen that X' has one point at infinity, namely $[0, 1, 0]$. If $(x', t') \to [x', 1, t']$ are the coordinates at $[0, 1, 0]$, then $t' = 4x'^3 - 20a_2x't'^2 - 28a_4t'^3$. This tells us that $t' = 4x'^3 - 320a_2x'^7 + ...$ (see Section 1.6 for details).

The projection map $\varphi : X \to \mathbb{C}$ given by $\varphi(x, y) \to x$ gives X the structure of a Riemann surface over $\mathbb{C}$. We extend it to φ' which makes X' a Riemann surface over $P^1(\mathbb{C}) \cong S^2$:

$$\varphi' : [x, y, 1] \to [x, 1], \qquad \varphi' : [x', 1, t'] \to [x', t'] \text{ in } P^1(\mathbb{C}),$$

i.e to $\frac{x'}{t'} \in \mathbb{C}$ in coordinates at 0 or to $\frac{t'}{x'}$ in coordinates at infinity.

The form $\omega = \frac{dx}{y} = \frac{dy}{6x^2-10a_2}$ has a primitive in a neighbourhood of each point of X; globally the primitive is a multi-valued holomorphic function $z = z(x, y)$. Each branch of z in the neighbourhood of any point $(x_0, y_0) \in X$ is a local coordinate:

(1) if $y_0 \neq 0$, using $dz = \frac{dx}{y}$, we see that $\frac{dz}{dx} = \frac{1}{y} \neq 0$,
(2) if $y_0 = 0$, using $dz = \frac{dy}{6x^2-10a_2}$, we see that $\frac{dz}{dy} = \frac{1}{6x^2-10a_2} \neq 0$.

Near the point at infinity in X', $[0, 1, 0]$, we have $[x, y, 1] = [\frac{x'}{t'}, \frac{1}{t'}, 1]$, so

$$\omega = \frac{dx}{y} = t' \cdot d\Big(\frac{x'}{t'}\Big) = dx' - \frac{x'}{t'}dt' = dx' - \frac{12x'^2 - 2240a_2x'^6 + \dots}{4x'^2 - 320a_2x'^6 + \dots} \cdot dx' = -2dx'(1 + g(x')).$$

with $g(x')$ holomorphic near $x' = 0$ and $g(0) = 0$.

Suppose that a_2, a_4 come from a discrete subgroup Γ, i.e $a_2 = 3\sum_{w\neq 0} \frac{1}{w^4}$, $a_4 = 5\sum_{w\neq 0} \frac{1}{w^6}$. We have seen that the meromorphic transformation $x = \mathfrak{p}(z), y = \mathfrak{p}'(z)$ defines a biholomorphism from $\mathbb{C}/\Gamma \to X'$. Recall that the inverse map is $z(p) = \int_{[0,1,0]}^{p} \omega$ where the integral is determined only modulo Γ.

We are interested in the converse:

Theorem (Abel)**.** *Given a_2, a_4 such that $p(x) = 4x^3 - 20a_2x - 28a_4$ has 3 distinct roots, there is a discrete subgroup Γ of $\mathbb{C}$ generated by e_1, e_2 linearly independent over $\mathbb{R}$ such that $a_2 = 3\sum_{w\neq 0} \frac{1}{w^4}$, $a_4 = 5\sum_{w\neq 0} \frac{1}{w^6}$.*

We will prove it using two lemmas:

Lemma (1)**.** *The difference of two branches of z belongs to some discrete group Γ.*

We can then consider the elliptic curve associated to Γ, i.e X'' given by equation $y^2 = 4x^3 - 20b_2x - 28b_4$ where $b_2 = 3\sum_{w\neq 0}\frac{1}{w^4}$ and $b_4 = 5\sum_{w\neq 0}\frac{1}{w^6}$. We can consider the composition:

$$\begin{array}{ccccc} X' & \xrightarrow{\quad\quad ? \quad\quad} & \mathbb{C}/\Gamma & \xrightarrow{\quad\quad \cong \quad\quad} & X'' \\ [x,y,1] & & z & & [(\mathfrak{p}(z), \mathfrak{p}'(z), 1] \end{array}$$

Abel's theorem says that this composition is in fact the "identity mapping" , i.e $[\mathfrak{p}(z), \mathfrak{p}'(z), 1]$ is nothing but the coordinate mapping $[x, y, 1]$. And therefore we must have $a_2 = b_2$ and $a_4 = b_4$. It will turn out *necessary* to show first the weaker statement first:

Lemma (2)**.** *The map z is a biholomorphism.*

4.6 Abel's Theorem II

Lemma 1

Let $p_0 = [0, 1, 0]$ be the point at infinity, then $z(p) = \int_{p_0}^{p}\omega$ is globally well-defined up to a period of ω, i.e to $\pi(\gamma) = \int_\gamma \omega$ where γ is a loop.

We can extend π to 1-cycles by linearity. By Stokes theorem, the map π vanishes on boundaries: $\int_{\partial\sigma}\omega = \int_\sigma d\omega = 0$, so π descends to a map from $H_1(X', \mathbb{Z}) \to \mathbb{C}$.

We now need two facts not from this course: first, X' is a torus, say by the degree genus formula $g = \frac{(d-1)(d-2)}{2} = \frac{(3-2)(3-1)}{2} = 1$ and secondly, $H_1(X', \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

We claim that Γ is lattice. If not, Γ is contained within a straight line. Take α so that $\Re(\alpha\pi(\gamma)) = 0$. But then $\Re(\alpha z)$ defines a *single-valued* function harmonic on X'. As X' is compact, by maximum-modulus, we see that it is constant. But z is not constant as ω does not vanish identically.

Lemma 2

First, we establish that z is surjective: as branches of z are local coordinates at each point of X', z is a local homeomorphism, in particular open, so has open image. But the image is also closed as X' is compact.

Thus X' is a covering space of $\mathbb{C}/\Gamma$ and so it must be biholomorphic to $\mathbb{C}/\Gamma'$ for some subgroup Γ' of Γ (and z is just the quotient map). We show that in fact $\Gamma = \Gamma'$, i.e we need to prove the other inclusion.

The universal covering space of $\mathbb{C}/\Gamma$ is $\tilde{X'} \cong \mathbb{C}$. The map $z : X' \to \mathbb{C}/\Gamma$ lifts to a map $\tilde{z}$ between the covering space $\tilde{X'}$ and $\mathbb{C}/\Gamma$. We have the following commutative diagram:

$$\begin{tikzcd} \tilde{X'} \cong \mathbb{C} \arrow[r, "\tilde{z}"] \arrow[d, "p"'] & \mathbb{C} \arrow[d, "q_\Gamma"'] \\ X' \cong \mathbb{C}/\Gamma' \arrow[r, "z"] & \mathbb{C}/\Gamma \end{tikzcd}$$

We explain the map $\tilde{z}$. Let $q_0 \in \tilde{X'}$ be a point above p_0. Given a point $q \in \tilde{X'}$, let $\tilde{\gamma}$ be an arbitrary path joining q_0 to q. If $\gamma = p(\tilde{\gamma})$ then $\tilde{z}(q_0) = \int_\gamma \omega$. Note that it is no longer necessary to quotient out by Γ to make the map well-defined.

Clearly $\tilde{z}$ takes Γ' into Γ for if $q \in \Gamma'$, when we project the path connecting q_0 to q on X', it becomes a loop and we know periods of ω lie in Γ. Denote the restriction $\tilde{z}|_{\Gamma'}$ by I.

As every element of Γ is a periodic, the map I is surjective. This forces I to be injective (its a map from $\mathbb{Z}^2 \to \mathbb{Z}^2$) and therefore bijective so the covering map $z : \mathbb{C}/\Gamma' \to \mathbb{C}/\Gamma$ is trivial, i.e z is a biholomorphism.

Conclusion

The function x is meromorphic on X' with a pole of order 2 at infinity; indeed

$$x = \frac{x'}{t'} = \frac{x'}{4x'^3 - 320a_2x'^7 + \ldots}.$$

By lemma 2, we can think of x as a meromorphic function on $\mathbb{C}$ of z with Γ being the group of periods and having a double pole at $z = 0$. Write:

$$\begin{aligned} x(z) &= \frac{c}{z^2} + \frac{d}{z} + e + fz + \ldots \\ x'(z) &= -\frac{2c}{z^3} - \frac{d}{z^2} + f + \ldots \end{aligned}$$

As $dz = \frac{dx}{y}$, $x'(z) = y$ i.e $x'(z)^2 = y^2 = 4x^3 - 20a_2x - 28a_4$. Equating coefficients, we see that $4c^2 = 4c^3$, $4cd = 12c^2d$ and $d^2 = 12c^2e + 12cd$. It follows $c = 1, d = 0, e = 0$.

Hence $x(z) = \mathfrak{p}(z)$ as the difference is holomorphic and doubly periodic and vanishing at the origin. Differentiating, we also see that $y(z) = \mathfrak{p}'(z)$. Abel's theorem is now completely proved.

Chapter 5

A Modular Function

5.1 Analytic Continuation

Given a holomorphic function f defined on a domain U, we wish to find the largest connected open set $V \supseteq U$ to which we can extend f.

Suppose (f_0, D_0) is a *function element*, that is f_0 is a holomorphic function defined on the disk D_0. We say that (f_0, D_0) may be *analytically continued* along a chain of disks $\mathcal{C} = \{D_0, D_1, \ldots D_n\}$ ($D_i \cap D_{i+1} \neq \emptyset$) if there exist function elements (f_i, D_i) such that $f_i = f_{i+1}$ on $D_i \cap D_{i+1}$.

We say that (f_n, D_n) is an *analytic continuation* of (f_0, D_0) along $\mathcal{C}$ and that (f_i, D_i), (f_{i+1}, D_{i+1}) are *direct analytic continuations* of each other.

Suppose that $f \in H(D)$ where D is a disk centered at a. Suppose $\gamma : [0,1] \to \mathbb{C}$ is a curve with $f(0) = a$. We say that (f, D) can be analytically continued along γ if it can be continued along a chain of open disks $\{D = D_0, D_1, \ldots D_n\}$ which covers γ.

This means we have a partition $0 = t_0 < t_1 < \cdots < t_{n+1} = n$ such that $\gamma([t_i, t_{i+1}]) \subset D_i$ and $\gamma(1)$ being the center of the last disk.

Lemma. *A function element (f, D) has at most one analytic continuation along γ.*

Suppose we can continue (f, D) along chains $\mathcal{C}_1 = \{D = D_0, D_1, \ldots D_m\}$ and $\mathcal{C}_2 = \{D = E_0, E_1, \ldots, E_n\}$ covering γ to (g_m, D_m) and (h_n, E_n) respectively, we need to show that $g_i = h_j$ in $D_i \cap E_j$ whenever D_i and E_j intersect.

Suppose not, consider the indices (i, j) with the smallest $i + j$ for which this fails. Without loss of generality, we can assume that $s_i \geq t_j$.

The functions g_{i-1} and g_i agree in $D_{i-1} \cap D_i$ while g_{i-1} and h_j agree on $D_{i-1} \cap E_j$. It follows that $g_i = h_j$ in $D_{i-1} \cap D_i \cap E_j$ which is not empty, so then $g_i = h_j$ continues to hold in $D_i \cap E_j$.

More generally, suppose $U \subset \mathbb{C}$ is a domain and $f \in H(U)$. We wish to find an unramified Riemann surface (X, ℓ) over $\mathbb{C}$ and a biholomorphing mapping σ which takes U onto an open subset of X such that:

(1) $\ell \circ \sigma$ is the inclusion map of U into $\mathbb{C}$,

(2) f extends to a holomorphic function g on X (i.e $g \circ \sigma = f$),

(3) (X, ℓ) is the "largest" Riemann surface satisfying (1) and (2).

We explain condition (3). If (X', ℓ', σ') also satisfy (1) and (2), we require that there is a unique holomorphic map $h : X' \to X$ such that $\ell' = \ell \circ h$ and $\sigma' = \sigma \circ h$. In this case, $g' = g \circ h$ as they coincide on $\sigma(U) = \sigma'(U)$ $(g' \circ \sigma' = f = g \circ h \circ \sigma)$.

We shall not address this more general question.

5.2 Monodromy Theorem

Theorem (Monodromy)**.** *Suppose (f, D) can be analytically continued along any curve in Ω starting at a (the center of D). Then (1) If γ_0, γ_1 are homotopic curves in Ω from a to b, then continuations of (f, D) along γ_0 and γ_1 coincide. (2) If Ω is simply-connected, then there is a holomorphic function g on Ω such that $g = f$ on D.*

Proof of (1). Let γ_s, $s \in [0, 1]$ be a homotopy with endpoints a, b. Then (f, D) may analytically continued along each γ_s to (γ_s, D_s). Need to show that $g_0 = g_1$ (in $D_0 \cap D_1$).

Fix s. Let $\mathcal{C} = \{D = E_0, E_1, \dots E_n\}$ be a chain of disks covering γ_s. This means that there is a partition $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ with $\gamma([t_i, t_{i+1}]) \subset E_i$ for $i = 0, 1, \dots n$. We will find a $\delta > 0$ such that if $|s' - s| < \delta$, then $\mathcal{C}$ covers $\gamma_{s'}$ as well.

By the uniqueness of analytic continuation, it would follow that $g_s' = g_s$ for $|s - s'| < \delta$. By compactness of $[0, 1]$, we would have $g_0 = g_1$.

Choose ϵ to be less than the distance from $\gamma_s[0, 1]$ to $\mathbb{C} \setminus E_i$. Since γ is uniformly continuous, we have a $\delta > 0$ such that $|\gamma_s(t) - \gamma_{s'}(t)| < \epsilon$ for all $t \in [0, 1]$ whenever $|s - s'| < \delta$.

Proof of (2). To define g near $b \in \Omega$, choose any chain of disks $\mathcal{C} = \{D = E_0, E_1, \dots E_n\}$ with E_n containing b and analytically continue (f, D) to obtain the function element (g_b, E_n). By part (1), the construction of the function element (g_b, E_n) does not depend on the choice of $\mathcal{C}$.

From the construction, it is clear various g_b's patch together: we have to check that if $b' \in E_n$ is some other point, then the analytic continuation of (f, D) near b' matches up with (g_b, E_n). But we can just use precisely the same chain of disks!

5.3 Modular Function

We will introduce the modular function to show the following results:

Theorem (Little Picard)**.** *A non-constant entire function $f(z)$ omits at most one value (it is possible that one value is omitted, for instance $f(z) = e^z$ omits 0).*

Theorem (Big Picard)**.** *If f has an essential singularity at $z = a$, then in any neighbourhood of a, f takes every complex value, with one possible exception, infinitely many times.*

The Little Picard theorem is a special case of the Big Picard theorem: if f is entire, at infinity, either f has a pole (in which case, it is a polynomial) or it has an essential singularity (so $f(\frac{1}{z})$ has an essential singularity at the origin).

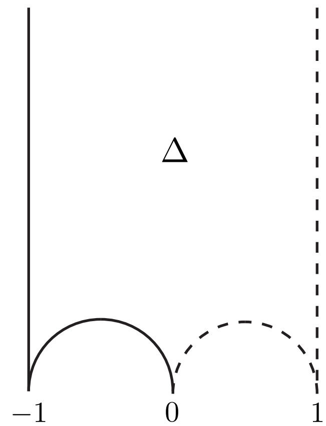
The *modular group* $G = \mathrm{SL}(2, \mathbb{Z})$ acts on the upper half-plane $\mathbb{H}^+$: if $\varphi \in G$ given by the matrix $(a, b; c, d)$ with $ad - bc = 1$ then $\varphi(z) = \frac{az+b}{cz+d}$.

Exercise: G is generated by $z + 1$, $-\frac{1}{z}$.

Losely speaking, a modular function is a holomorphic or a meromorphic function defined on $\mathbb{H}^+$ which is invariant under the action of G (or a subgroup of G).

To prove the Little Picard Theorem, we will construct a modular function invariant under the subgroup $\Gamma \subset G$ generated by $\sigma(z) = \frac{z}{2z+1}$ and $\tau(z) = z + 2$.

Suppose $z = x + iy$. Let $\Delta \subset \mathbb{C}$ be the region: $y > 0$, $-1 \leq x \leq 1$, $|2z + 1| \geq 1$, $|2z - 1| > 1$ (draw this region carefully, note that some boundary components are contained and some are not).



Theorem. *(1) If $\varphi_1, \varphi_2 \in \Gamma$ with $\varphi_1 \neq \varphi_2$, then the images $\varphi_1(\Delta)$ and $\varphi_2(\Delta)$ are disjoint.*
(2) The union $\bigcup_{\varphi \in \Gamma} \varphi(\Delta) = \mathbb{H}^+$ (Δ is the "fundamental domain" for the action of Γ).
(3) Let Γ_1 be the subgroup of G of elements $\varphi(z) = \frac{az+b}{cz+d}$ with a, d odd and b, c even. Then $\Gamma = \Gamma_1$.

Clearly, Γ is a subgroup of Γ_1 as σ, τ are in Γ_1. Let $(1')$ be the statement (1) with Γ_1 in place of Γ. It thus suffices to prove $(1')$ and (2).

Proof of $(1')$

If $\varphi(z) = \frac{az+b}{cz+d} \in G$ and $z \in \mathbb{H}^+$ then $\operatorname{Im} \varphi(z) = \operatorname{Im} \frac{az+b}{cz+d} \cdot \frac{c\bar{z}+d}{c\bar{z}+d} = \operatorname{Im} \frac{z(ad-bc)}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2}$.

To show $(1')$, it is suffices to show that Δ and $\varphi(\Delta)$ are disjoint for $\mathrm{id} \neq \varphi \in \Gamma_1$. If $c = 0$, $ad = 1$, so $a = d = \pm 1$ and thus $\varphi(z) = z + 2n$ in which case the conclusion is obvious.

Suppose instead $c \neq 0$. We claim that $|cz + d| \geq 1$ for $z \in \Delta$. If not, Δ intersects the interior of the circle $D = \{w : |cw + d| \leq 1\}$ centered at $-\frac{d}{c}$ with radius $\frac{1}{|c|}$.

If a circle with center on the real axis is to intersect Δ, it must contain at least one of the points $-1, 0, 1$. So, at least one of the points $-1, 0, 1$ lies in the interior of D. But if $w = -1, 0, 1$, $cw + d$ is an odd integer, so $|cw + d|$ cannot be less than 1 (i.e these three points are possibly on the boundary, but definitely are not in the interior of D).

So $\operatorname{Im} \varphi(z) \leq \operatorname{Im} z$ for $z \in \Delta$. Suppose both z_0 and $\varphi(z_0)$ are in Δ (this is where we need to find a contradiction). Applying the above to φ^{-1}, we obtain the reverse inequality $\operatorname{Im} z_0 \leq \operatorname{Im} \varphi(z_0)$, so $|cz_0 + d| = 1$, i.e z_0 lies on the boundary of D.

Then D must be the circle centered at $-\frac{1}{2}$ with radius $\frac{1}{2}$, i.e $c = 2, d = 1$, so

$$\varphi(z) = \frac{az+b}{cz+d} = \frac{(1+2b)z+b}{2z+1} = \frac{z}{2z+1} + b = \tau^n\sigma(z).$$

But then $\varphi(z_0)$ cannot possibly be in D (σ takes the semicircle $\{w \in \mathbb{H}^+ : |2w+1| = 1\}$ to the semicircle $\{w \in \mathbb{H}^+ : |2w-1| = 1\}$; more precisely $\sigma(-\frac{1}{2} + \frac{1}{2}e^{i\theta}) \to \frac{1}{2} + \frac{1}{2}e^{i(\pi-\theta)}$).

Proof of (2)

Let Σ be the union $\bigcup_{\varphi\in\Gamma} \varphi(\Delta) \subset \mathbb{H}^+$. Claim: Σ includes all $z \in \mathbb{H}^+$ such that $|2z - (2m+1)| \geq 1$ for all $m \in \mathbb{Z}$. This is almost trivial, up to some boundary components, this region is just $\bigcup_{n\in\mathbb{Z}} \tau^n\Delta$ – but these components belong to $\bigcup_{n\in\mathbb{Z}} \tau^n\sigma\Delta$.

Fix $w \in \mathbb{H}^+$. Choose a $\varphi_0 \in \Gamma$ which minimizes $|cw+d|$. Such a φ_0 exists: the imaginary part of $cw+d$ comes from cw, so c cannot be too big. Then d cannot be too big either for otherwise the real part $cw+d$ is large; so we really take minimum over finite amount of terms.

Let $z = \varphi_0(w)$. Thus z has the largest imaginary part out of all $\{\Gamma w\}$. Noting that $\sigma^{-1}(z) = \frac{z}{-2z+1}$, we consider:

$$(\sigma\tau^{-n})z = \sigma(z-2n) = \frac{z-2n}{2z-4n+1}, \qquad (\sigma^{-1}\tau^{-n})z = \sigma^{-1}(z-2n) = \frac{z-2n}{-2z+4n+1}$$

Hence $|2z-4n+1| \geq 1$, $|2z-4n-1| \geq 1$ for all integers n, i.e $|2z-(2m+1)| \geq 1$ for integral m. So $z \in \Sigma$ implies $w \in \Sigma$.

5.4 Modular Function II

Last time, we considered a group Γ acting on $\mathbb{H}^+$ generated by $\sigma(z) = \frac{z}{2z+1}$ and $\tau(z) = z+2$.

Theorem. *The exists a holomorphic function $\lambda(z)$ defined on the upper half-plane such that (1) $\lambda(z)$ is invariant under the action of Γ, (2) λ is injective on the fundamental domain Δ, (3) $\lambda(\Delta) = \mathbb{C}\setminus\{0,1\}$ and (4) $\mathbb{R}$ is the natural boundary of λ.*

Condition (4) means that λ cannot be extended to a holomorphic function beyond the real axis.

Let $\Delta_0 = \Delta \cap \{y > 0\}$. The Riemann mapping theorem tells us that there exists a biholomorphism $h : \Delta_0 \to \mathbb{H}^+$ which *extends continuously to a homeomorphism of closed domains* such that $h(0) = 0, h(1) = 1, h(\infty) = \infty$.

We could try to use methods of Section 3.4 to argue that h extends continuously to $\overline{\Delta_0}$ (although inventiveness is needed to argue continuously of h near $0, 1, \infty$). Its best just to note that Δ_0 is simply-connected and has Jordan curve boundary and appeal to Carathéodory's theorem.

The Schwartz reflection principle allows us to continue h to all of Δ by $h(-x+iy) = \overline{h(x+iy)}$. Then h extends continuously to the $\partial\Delta$ and takes all possible values exactly once on Δ.

Suppose $z \in \mathbb{H}^+$ and $\lambda(z)$, let $\varphi \in \Gamma$ be the automorphism for which $\varphi(z) \in \Delta$. Define $\lambda(z) = h(\varphi(z))$.

As boundary values of h on $\partial\Delta$ are real, we have $h(-1+iy) = h(1+iy) = h(\tau(-1+iy))$ and $h(-\frac{1}{2} + \frac{1}{2}e^{i\theta}) = h(\frac{1}{2} + \frac{1}{2}e^{i(\pi-\theta)}) = h(\sigma(h(-\frac{1}{2} + \frac{1}{2}e^{i\theta}))$. This tells us λ is continuous on $\mathbb{H}^+$ and holomorphic possibly except on $\bigcup_{\varphi\in\Gamma} \varphi(\partial G)$. A corollary to Morera's theorem tells us that actually λ is holomorphic on $\bigcup_{\varphi\in\Gamma} \varphi(\partial G)$ and thus holomorphic on $\mathbb{H}^+$.

To show (4), recall $h(\varphi(0)) = h(0) = 0$ for any $\varphi \in \Gamma$. If $\varphi(z) = \frac{az+b}{cz+d}$, we see that $\lambda(\frac{a}{c}) = h(0) = 0$ for all even a and odd b. Hence zeros of λ are dense on the real line and thus no holomorphic extension past the real line can be made.

5.5 Little Picard Theorem

We finally have all the tools to prove Picard's Little Theorem. Suppose f omits the values $a \neq b$. We may assume that $a = 0$ and $b = 1$ by replacing f by $\frac{f-a}{f-b}$ if necessary.

Last time we have constructed the modular function $\lambda : \mathbb{H}^+ \to \Omega = \mathbb{C} \setminus \{0, 1\}$ and showed that it is a covering map.

Suppose $D_0 \subset \Omega$ is a disk with center $f(0)$. We can choose a disk $E_0 \subset \mathbb{C}$ such that $f(E_0) \subseteq D_0$. Next choose $V_0 \in \mathbb{H}^+$ so that $\lambda|_{V_0}$ is a biholomorphism onto D_0.

Let $g_0 = (\lambda|_{V_0})^{-1} \circ f|_{E_0}$. We will show that (g_0, E_0) has unrestricted analytic continuation to $\mathbb{C}$. Indeed if $\gamma : [0, 1]$ is any curve starting at 0, cover γ by a chain of disks $E_0, E_1, \dots E_n$ so that $f(E_j)$ lies within some disk $D_j \subset \Omega$.

For each disk D_j, choose a V_j so that that $\lambda|_{V_j}$ is a biholomorphism onto D_j and V_j intersects V_{j-1}. Then $(\lambda|_{V_j})^{-1} \circ f|_{E_j} : D_j \to \mathbb{H}^+$ is the desired analytic continuation.

The Monodromy theorem gives us a holomorphic function $g : \mathbb{C} \to \mathbb{H}^+$. But such a function must be constant (composing with $\frac{z-i}{z+i}$, we get a map from the complex plane to the unit disk). Hence f itself must be constant.

Corollary. *If $f(z)$ is an entire function, then $f \circ f$ has a fixed point unless f is a translation $f(z) = z + a$ (f need not have a fixed point itself, e.g $f(z) = z + e^z$).*

Suppose $f \circ f$ has no fixed points. Then f has no fixed points either. Define $g(z) = \frac{f(f(z))-z}{f(z)-z}$. It is entire and omits values 0, 1 and hence constant, so $g(z) = c$ with $c \neq 0, 1$, i.e $f(f(z)) = c(f(z) - z)$.

Differentiating, we see that $f'(f(z))f'(z) - 1 = c(f'(z) - 1)$ so $f'(z)(f'(f(z) - c) = 1 - c$. As $c \neq 1$, $f'(z) \neq 0$ hence $f'(f(z))$ omits the values 0, c and hence is constant, i.e $f'(z)$ is constant, so $f(z) = az + b$. For f not to have fixed points, a must equal 1.

Corollary. *Any entire periodic function has a fixed point.*

Suppose f has period p, i.e $f(z) = f(z + p)$. If f has no fixed points then $f(z) - z$ omits the value 0. But it also omits the value p, otherwise $f(z + p) = f(z) = z + p$. By Picard's theorem, $f(z) = z + c$ but such functions are not periodic.

Lemma. *Let f, g be entire functions such that $f^2 + g^2 = 1$. There exists an entire function $h(z)$ so that $f = \cos(h(z))$ and $g = \sin(h(z))$.*

The equation factors $(f + ig)(f - ig) = 1$. The function $f + ig, f - ig$ are entire without zeros, so $f + ig = e^{\varphi(z)}$ and $f - ig = e^{-\varphi(z)}$, i.e $f = \frac{e^{\varphi}+e^{-\varphi}}{2} = \cos \circ h$ with $h = \frac{\varphi}{i}$ and similarly, $g = \sin \circ h$.

Chapter 6

Problems

1. Show the identities

   (a)
   $$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z-n)^2} = \frac{\pi^2}{(\sin \pi z)(\tan \pi z)}.$$

   (b)
   $$\frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2} = \frac{\pi}{\sin \pi z}.$$

   (c)
   $$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2 + a^2} = \frac{\pi}{a} \cdot \frac{\sinh 2\pi a}{\cosh 2\pi a - \cos 2\pi z}.$$

2. Prove the identity by taking logarithmic derivatives of both sides:
   $$\pi x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \sinh(\pi x).$$

3. Let $\mathcal{F}$ be the family of holomorphic functions in the unit disk satisfying $|f^{(n)}| \leq n!$ for all $n \geq 0$. Show that $\mathcal{F}$ is a normal family.

4. Suppose f is an entire function. Show that f has an n-th root if and only if all zeros of f have multiplicity divisible by n.

5. Suppose $f(z)$ is a holomorphic function defined on the unit disk with $|f(z)| \leq M$. Let the zeros of f are $a_1, a_2, \dots a_n$ counted with their multiplicities. Show that

$$|f(z)| \leq M \Big| \prod_{k=1}^{n} \frac{a_k - z}{1 - \overline{a_k} z} \Big|.$$

In particular, $|f(0)| \leq M \prod_{k=1}^{n} |a_k|$ and if $f(0) = 0$ then $|f(z)| \leq M|z|$.

6. (Blaschke product) Suppose $f(z)$ is a holomorphic function defined on the unit disk with a zero at 0 of order s and the other zeros $\{a_k\}$ satisfying $\sum_k (1 - |a_k|) < \infty$ (or equivalently $\sum_k \log |a_k| > -\infty$). It then admits a nice factorization $f = BG$ where B is a product of

$$B(z) = z^s \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \overline{a_k} z}$$

and $G(z)$ is a holomorphic functions without zeros. Show that $B(z)$ is holomorphic.

7. Show that a bounded holomorphic function admits a Blaschke product, i.e that the sum $\sum_k \log |a_k|$ converges (Hint: use the first corollary of Problem 5).

8. (Jensen's formula) Suppose $f(z)$ is a holomorphic function in the unit disk without zeros. As $\log |f(z)|$ is harmonic, we have $\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta$ for $0 < r < 1$. But what if $f(z)$ has zeros in the unit disk? Suppose that $f(0) \neq 0$ and denote its zeros by $\{a_k\}$. In this case,

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \sum_{|a_k| < r} \log \Big(\frac{r}{|a_k|} \Big)$$

Here $\log |f(0)|$ is not equal to, but is actually *less* than the mean value. Such functions are called *subharmonic*. Prove the above formula (Hint: first consider r for which no $\{a_k\}$ lie on $|z| = r$, also show that RHS is continuous).

9. (Canonical Product) Look back at the proof of Weierstrass' theorem (Section 2.3) and observe the following: Suppose $f(z)$ is an entire function with zeros $\{a_k\}$ satisfying $\sum_k \frac{1}{|a_k|^{m+1}} < \infty$ for some integer m. Then it is possible to choose all $m_k = m$.

10. Prove that

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

Hint: The Weierstrass theorem implies that

$$\sin \pi z = z e^{g(z)} \prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}$$

To find $g(z)$, take the logarithmic derivative and use Example 2 from Section 1.2.

11. (Wedderburn's lemma) Suppose f, g are entire functions without common zeros. Show that there exists entire functions a, b such that $af + bg = 1$.

12. Find the residues at the poles of the Γ function (Hint: use the functional equation).

13. (Bohr-Mollerup theorem) Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a logarithmically convex function satisfying $f(x+1) = f(x)$ and $f(1) = 1$. Then necessarily, $f(x) = \Gamma(x)$ for all $x > 0$.

    Hint: Show that for all natural $n \geq 2$ and positive x

$$(n-1)^x(n-1)! \leq f(x+n) \leq n^x(n-1)!$$

    and

$$\frac{n^x n!}{x(x+1)\cdots(x+n)} \leq f(x) \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \cdot \frac{x+n}{n}.$$

14. Show that the automorphisms of the upper half-plane which preserve i are given by

$$w(z) = \frac{z + \tan(\theta/2)}{1 - z\tan(\theta/2)}.$$

    Write the formula for $\theta = \pi$.

15. Find the automorphism group of $\mathbb{C}\setminus\{0\}$.

16. Show that two annuli are biholomorphic if and only if the ratio of their radii are the same (complete the proofs below).

    (a) Use the previous problem.

    (b) Use the uniqueness of solution of the Dirichlet problem: suppose that $f : A(0;1,r) \to A(0;1,R)$ is the biholomorphism and that f takes $\{z : |z| = 1\}$ to itself. Then $\log|f(z)| = \frac{\log R}{\log r} \cdot \log|z|$. Treating it as an equation in variables $z, \overline{z}$, differentiate with respect to z.

(c) Given an annulus, $A_r = A(0;1,r)$, we can consider its universal covering space $f_r : \mathbb{H}^+ \to A_r$ with f_r given by

$$f(z) = \exp\Big(-2\pi i \cdot \frac{\log z}{\log \lambda_r}\Big)$$

where $\lambda_r > 0$ depends on r. The fundamental group of the annulus (which is $\mathbb{Z}$) acts on $\mathbb{H}^+$, with generator acting by multiplication by λ_r. If two annuli A_r and A_R are biholomorphic, then these actions must be conjugate.

17. Show that two tori $\mathbb{C}/\Gamma_1$ (with Γ_1 generated by e_1, e_2) and $\mathbb{C}/\Gamma_2$ (with Γ_2 generated by f_1, f_2) are biholomorphic if and only if there is a fractional linear transformation with integer coefficients and determinant 1 which takes (e_1, e_2) to (f_1, f_2).

18. (Schwarz-Christoffel formula) Show that the mapping $F(w) : \mathbb{H}^+ \to \Omega$ (where Ω is a polygon) given by the formula

$$F(w) = C\int_0^w \prod_{k=1}^{n-1} (w - w_k)^{-\beta_k} dw + C'$$

is conformal for some distinct real numbers w_k and $\sum \beta_k = 2$.

19. Show that $F(w) = \int_0^w (1-w^n)^{-2/n} dw$ maps the unit disk onto the interior of a regular polygon with n sides.

20. Find the image of the unit disk under the mapping $F(z) = \frac{1}{z}\prod_{k=1}^n (z - a_k)^{\lambda_k}$ where λ_k are positive with $\sum_{k=1}^n \lambda_k = 2$.

21. Prove the *addition theorem*:

$$\mathfrak{p}(z_1 + z_2) = -\mathfrak{p}(z_1) - \mathfrak{p}(z_2) + \frac{1}{4}\left(\frac{\mathfrak{p}'(z_1) - \mathfrak{p}'(z_2)}{\mathfrak{p}(z_1) - \mathfrak{p}(z_2)}\right)^2.$$

22. Another form of the addition theorem: if $u + v + w = 0$ in $\mathbb{C}/\Gamma$ then

$$\det \begin{vmatrix} 1 & 1 & 1 \\ \mathfrak{p}(u) & \mathfrak{p}(v) & \mathfrak{p}(w) \\ \mathfrak{p}'(u) & \mathfrak{p}'(v) & \mathfrak{p}'(w) \end{vmatrix} = 0.$$

23. Suppose that $f(z)$ is a even doubly periodic function (let the group of periods be Γ). There exists points $a_1, a_2, \dots a_n; b_1, b_2, \dots b_n \in \mathbb{C}/\Gamma$ such that

$$f(z) = c \prod_{k=1}^{n} \frac{\mathfrak{p}(z) - \mathfrak{p}(a_k)}{\mathfrak{p}(z) - \mathfrak{p}(b_k)}.$$

24. Show that a doubly periodic function is a rational function of the $\mathfrak{p}$ and $\mathfrak{p}'$ (Hint: use the previous problem).

25. Show that while the function $f(z) = \sum_n z^{2^n}$ is holomorphic in the unit disk, it does not extend holomorphically to any larger open set (Hint: $f(z^2) = f(z) - z$).

    Remark. *This problem is a special case of the* Hadamard Gap theorem*.*

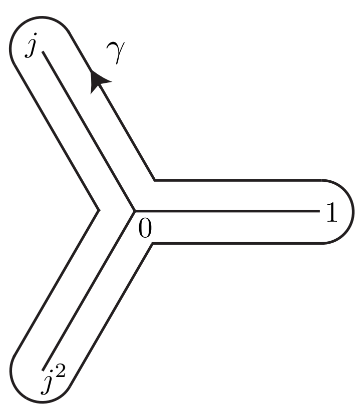
26. Find the radius of convergence of $f(z) = \sum_{k \geq 1} \frac{z^{2k}}{k+1}$. Find a maximal domain of existence (a maximal open set in $\mathbb{C}$ to which $f(z)$ may be analytically continued).

27. The set of solutions (z, w) of $w^2 - 2wz + 1 = 0$ can be completed to a compact Riemann surface over $P^1(\mathbb{C})$. Find the residues of the differential form $\frac{dz}{\sqrt{z^2-1}}$ points at infinity.

28. Prove

$$\int_0^1 \frac{dx}{\sqrt[3]{1 - x^3}} = \frac{2\pi}{3\sqrt{3}}.$$

    by integrating over a lift of γ in the Riemann Surface $X \subseteq P^2(\mathbb{C})$ given by the equation $x^3 + y^3 = z^3$ (show that the form being integrated is meromorphic and apply the Residue theorem).



29. Prove *Picard's theorem for meromorphic functions*: if a meromorphic function defined on the entire complex plane omits three values, it is necessarily constant.

30. Show that a non-constant holomorphic function defined on $\mathbb{C} \setminus \{0\}$ omits at most 1 value.

31. Show that the function $f(z) = ze^z$ attains every value from $\mathbb{C}$.

32. Suppose f, g are meromorphic functions such that $f^3 + g^3 = 1$. Show that actually f and g are constant functions. Is result still true if 3 is replaced by a larger positive integer?

33. Suppose f, g are entire functions satisfying $e^f + e^g = 1$. Show that f, g are actually constant functions.

Chapter 7

Further Results

7.1 Big Picard Theorem

To prove Big Picard Theorem, we need the concept of normal families of *meromorphic* functions. A family of meromorphic functions $\mathcal{S} \subset M(\Omega)$ is *normal* if it is every sequence in $\mathcal{S}$ has a subsequence that converges uniformly in the *spherical metric* on compact subsets of Ω.

Instead of using geodesic spherical distance, it is easier to use the equivalent metric $|z - z'|_{S^2} = \min(|z - z'|, |\frac{1}{z} - \frac{1}{z'}|)$. *Notation:* We will always write S^2 as a subscript when dealing with spherical distance.

Theorem. *Suppose $\{f_n\} \subset M(\Omega)$ which converges uniformly in the spherical metric to f. (1) Then either f is meromorphic or identically infinite. (2) Furthermore, if each f_n is holomorphic, the limit function f is also holomorphic or identically infinite.*

Remark. *The functions $f_n(z) = n$ converge to the identically infinite function, so the holomorphic functions do not form a closed subspace of continuous functions $C(\Omega, S^2)$ taken with the compact-open topology. But if we do add the identically infinite function, the above theorem implies that we really do get a closed subspace.*

Proof of (1): Case I. Suppose $a \in \Omega$ is such that $f(a) \neq \infty$. There is an ϵ for which $|f(a) - f_n(a)| < \epsilon$ when $n > N$ is sufficiently large. In particular, $|f_n(a)| < |f(a)| + \epsilon = M$ for some constant M.

As $\{f_n\}_{n>N} \cup \{f\}$ is compact, by the Arzela-Ascoli theorem, we see that it is uniformly S^2-equicontinuous. Let $\delta < d_{S^2}(M, \infty)$. There exists an r such that for $z \in D(a, r)$,

$|f_n(z) - f_n(a)|_{S^2} < \delta$ simultaneously for all $n > N$. It follows that $\{f_n(z)\}$ is bounded uniformly in $D(a, r)$. This means that near a, $f(z)$ is a holomorphic function.

Case II. Now consider the case $a \in \Omega$ where $f(a) = \infty$. Let $g_n(z) = f_n(\frac{1}{z})$ and $g(z) = f(\frac{1}{z})$. Then $g(a) = 0$ and g_n converges in $C(\Omega, S^2)$ to g. By Case I, we see that there exists a ball $D(a, r)$ on which g_n are holomorphic (for n sufficiently large) and converge uniformly on compact subsets to g.

We have two cases. Either g is identically 0, in which case f is identically infinite or the zeros of g are isolated and so $\frac{1}{g} = f$ is a meromorphic function with a pole at a.

Proof of (2): Now suppose that $\{f_n\}$ are actually holomorphic functions. To see that the limit f is holomorphic, we must show that Case II implies that f is identically infinite. Suppose $a \in \Omega$ is such that $f_n(a)$ tends to infinity. In this case, $g_n(z)$ never vanishes but $g(a) = 0$, so by Hurwitz's theorem, $g(z)$ must vanish identically, i.e f is identically infinite.

Theorem (Montel-Carathéodory)**.** *If a family of functions $\mathcal{S} \subset H(\Omega)$ avoids two values (say 0 and 1), it is normal.*

Notation: D will denote the unit disk and $\lambda : D \to \mathbb{C} \setminus \{0, 1\}$ will be a holomorphic covering map (compose the modular function constructed in Sections 5.3 and 5.4 with a fractional linear transformation).

As normality is a local property, it suffices to treat the case when Ω is a disk (let the center be a). Suppose $\{f_n\} \subset \mathcal{S}$. Passing to a subsequence, we may assume $f_n(a) \to \alpha$. First suppose that $\alpha \neq 1, 0, \infty$. This allows us to consider U, a small ball centered at α within $\mathbb{C} \setminus \{0, 1\}$.

As λ is a holomorphic covering map, we can lift each $f_n : \Omega \to \mathbb{C} \setminus \{0, 1\}$ to $g_n : \Omega \to D$ so that $g_n(a) \in U$. Since $|g_n(z)| \leq 1$, we can extract a subsequence g_{n_k} which converges uniformly on compact sets to a function $g : \Omega \to \overline{D}$.

But actually g is a function from Ω to D – otherwise, the maximum-modulus principle implies that $g(z) = c$ is constant (of absolute value 1). But this would imply that $(\lambda|_U)^{-1}(\alpha) = \lim_n (\lambda|_U)^{-1}(f_n(a)) = \lim_n g_n(a) = c$ which is absurd.

Suppose $K \subset \Omega$ is compact. On K, $|g(z)|$ is bounded by some constant $M < 1$. As g_{n_k} converges to g uniformly, there is a constant $M' < 1$ such that $|g_{n_k}(z)| < M'$ on K. But the set $\lambda(D(0, M'))$ is bounded, so $\{f_{n_k}\}$ bounded uniformly on K. Since this is true of all K, by Montel's theorem, $\{f_{n_k}\}$ is normal.

Now we deal with the three special values of α:

Suppose $\alpha = 1$. As f_n don't vanish, we choose $g_n = \sqrt{f_n} : \Omega \to \mathbb{C}$ so that $g_n(a) \to -1$. But then $g_n(\Omega) \subset \mathbb{C} \setminus \{0, 1\}$ and normality of f_n is equivalent to normality of g_n; so we have reduced the case $\alpha = 1$ to the general case.

If $\alpha = 0$, let $g_n = 1 - f_n$; this reduces the case when $\alpha = 1$.

If $\alpha = \infty$, let $g_n = \frac{1}{f_n}$. Again g_n are analytic and functions into $\mathbb{C} \setminus \{0, 1\}$; so have a subsequence g_{n_k} which converges uniformly on compact sets of Ω to some function g. But g_{n_k} have no zeros, so by Hurwitz's theorem, g is identically 0; and thus f is identically infinite.

Proof of the Big Picard Theorem

Without loss of generality, we may assume that f has an essential singularity at the origin. Suppose in some disk $D(0, R)$, $f(z)$ omits values 0, 1. Let $\Omega = D(0, R) \setminus \{0\}$. Define $f_n(z) = f(z/n)$. The Montel-Carathéodory theorem implies that f_n is a normal family of meromorphic functions.

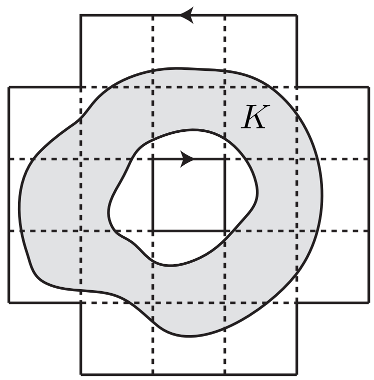
This means that some subsequence $\{f_{n_k}\}$ converges to g uniformly on $|z| = \frac{R}{2}$. If $\{f_{n_k}\}$ genuinely converges uniformly, then $f(z/n) \leq M$ for all integers n_k. Now consider the annulus $A(0; \frac{1}{n_{k+1}}, \frac{1}{n_k})$. On its boundary, $f \leq M$, by maximum-modulus, f must be bounded by M in its interior. Hence f is bounded near 0, so we actually have a removable singularity.

The other possibility is that g is identically infinite. But this means that $\lim_{z \to 0} f(z) = \infty$ meaning that f really has a pole at 0.

7.2 Runge's Theorem

Proposition. *Suppose $K \subset \Omega$ is compact. Then there exists line segments $\sigma_1, \sigma_2, \ldots \sigma_n$ lying in $\Omega \setminus K$ such that $f(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\sigma_k} \frac{f(\zeta)}{\zeta - z} d\zeta$ for all $z \in K$. Furthemore, the σ_k may taken of the same length and parallel to the coordinate axis.*

For this purpose, we can cover K by a grid of squares of side length less than $d(K, \partial\Omega)$ – see diagram. Let $\{Q_j\}$ be the squares which intersect K (then by hypothesis, the $\{Q_j\}$ lie inside Ω).



Consider the sum

$$S(z) = \frac{1}{2\pi i} \sum_j \int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta$$

If $z \in K$, we are tempted to evaluate $S(z) = \frac{1}{2\pi i} \sum_j f(z)\chi_{Q_j}(z) = f(z)$. This calculation has the slight problem that z may lie on a boundary of one of the Q_j's.

However, the sum in $S(z)$ has a great many cancellations: if two squares in $\{Q_j\}$ are adjacent then their common side will cancel out (due to opposite orientation). In fact, the only segments of ∂Q_j that will remain will lie *outside* of K. These are our σ_k.

Exercise: One can arrange the σ_k into finitely many disjoint cycles.

Lemma. *Suppose $K \subset \mathbb{C}$ is compact. We can approximate any function f holomorphic on a neighbourhood K uniformly by rational functions with poles lying outside K.*

Suppose f is a holomorphic function defined on $\Omega \supset K$. By the above proposition, we can choose segments $\{\sigma_k\}$ such that $f(z) = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\sigma_k} \frac{f(\zeta)}{\zeta - z} d\zeta$ holds for all $z \in K$.

Each integral $\int_\sigma \frac{f(\zeta)}{\zeta - z} d\zeta$ may be approximated by its Riemann sum $\frac{1}{n} \sum_{j=1}^n \frac{f(\zeta_j)}{\zeta_j - z}$ (here ζ_k lie on σ). By compactness of K and σ, this approximation is in fact uniform on K.

More precisely, as σ is compact, for a fixed $z \in K$ and any $\epsilon > 0$, if the partition in the Riemann sum is *fine enough*, then the integral differs from the Riemann sum by less than ϵ. But the integrand is *continuous* in z (at least while $z \in K$) so by compactness of K, we can make the partition so fine to make Riemann sum ϵ-approximate the integral simultaneously for all $z \in K$.

Theorem (Runge). *Suppose $K \subset \mathbb{C}$ is compact and E is a set containing a point from every bounded component of $\mathbb{C} \setminus K$. Then any function f holomorphic on a neighbourhood of K can be uniformly approximated by rational functions having poles in E.*

To obtain the theorem from the lemma, it suffices to prove the following: if a, b are in the same component of $\mathbb{C} \setminus K$ then $\frac{1}{z-a}$ can be approximated uniformly by polynomials of $\frac{1}{z-b}$. If we prove this claim we can "push" poles from bounded components of $\mathbb{C} \setminus K$ to E and poles lying in the unbounded component to infinity.

By transitivity of uniform approximation by polynomials, it suffices to show if $b \in D(a, r)$ then $\frac{1}{z-b}$ can be uniformly approximated by polynomials in $\frac{1}{z-a}$ on $\mathbb{C} \setminus \overline{D(a, r)}$. Proof is left as an exercise for the reader.

7.3 Prescribing Zeros in Arbitrary Domains

Recall that to construct an entire function with zeros $\{a_k\}$ we took the product of $G_{m_k}\Big(\frac{z}{a_k}\Big)$ where $G_m(w) = (1-w) \cdot \exp\Big(w + \frac{1}{2}w^2 + \cdots + \frac{1}{m}z^m\Big)$. Let $\Omega \subset \mathbb{C}$ be a domain and let $\{a_k\}$ be points accumulating only at the boundary of Ω.

Suppose $\{b_k\}$ is a sequence of points outside Ω such that $|b_k - a_k| \to 0$ (a *satellite sequence*). Exercise: Such a sequence exists if $\{a_k\}$ is bounded.

We propose the product $f(z) = \prod G_{m_k}\Big(\frac{a_k - b_k}{z_k - b_k}\Big)$. Suppose $r > 0$, for we work in $\Omega_{1/r} = \{z \in \Omega : d(z, \partial\Omega) > 1/r$ and consider terms with $r|a_k - b_k| < 1$:

$$g_k(z) = -\frac{1}{m_k + 1}\Big(\frac{a_k - b_k}{z_k - b_k}\Big)^{m_k+1} - \frac{1}{m_k + 2}\Big(\frac{a_k - b_k}{z_k - b_k}\Big)^{m_k+2} - \ldots$$

Thus,

$$|g_k(z)| \leq \frac{1}{m_k + 1}\Big(r|a_k - b_k|\Big)^{m_k+1}\Big(1 - r|a_k - b_k|\Big)^{-1}$$

Just like in the proof of the regular Weierstrass theorem, we need to choose m_k so that $\sum(r|a_k - b_k|)^{m_k+1}$ converges for *every* $r > 0$. Again, $m_k = k$ suffices.

Remark. *Unfortunately, if $\{a_k\}$ is unbounded, it is not always possible to choose a satellite sequence. For instance, let $\Omega = \mathbb{H}^+$ and $a_k = ik$. For this $\{a_k\}$ a satellite sequence does not exist.*

We continue the construction when $\{a_k\}$ happens to be unbounded. Pick a point $a \in \mathbb{C}$ which happens not to be one of the designated a_k, nor is one of their accumulation points. The mapping $h(z) = \frac{1}{z-a}$ will ensure that $\{h(a_k)\}$ is bounded.

Let $\tilde{f}(z)$ be a holomorphic function with $\{h(a_k)\}$ as zeros. By construction $f(z) = \tilde{f}(h(a))$ has the desired zeros; trouble is that it may not be a holomorphic at a (and it may not be possible to choose a outside of Ω). *But actually, it is holomorphic at a!*

But $\lim_{z\to\infty} G_m\left(\frac{a_k-b_k}{z_k-b_k}\right) = 1$ and as product converges absolutely and uniformly near infinity, we see that actually $\tilde{f}(\infty) = 1$, i.e $f(a) = 1$.

Corollary. *For any domain $\Omega \subseteq \mathbb{C}$, there exists a holomorphic function $f(z) \in H(\Omega)$ which cannot be continued analytically to any larger domain.*

Let $f(z)$ be a function whose zeros accumulate to every boundary point of Ω (actually, this is easier said than done).

It is even true that any domain $\Omega \subseteq \mathbb{C}$ is a *domain of holomorphy* which means that there is a holomorphic function $f(z) \in H(\Omega)$ such that for any point z_0, the power series expansion of $f(z)$ converges only inside Ω.

For instance, the principal branch of $\log z$ (which is defined on $\mathbb{C}$ without the negative real ray) cannot be extended anywhere on this ray, but power series expansions do converge beyond this ray.

7.4 Prescribing Singularities in Arbitrary Domains

We wish to construct a function $f(z)$ meromorphic in a domain Ω with poles $\{b_k\}$ (accumulating only at the boundary of Ω) and principal parts $P_k(\frac{1}{z-b_k})$.

Plan: write $\Omega = \bigcup_i K_i$ as the increasing union of compact sets. Let $L_i = K_{i+1} \setminus K_i$ and B_i be the b_k's which lie inside L_i. By Runge's theorem, exists a rational function $p_k(z)$ with poles lying outside K_{i+1} such that $|P_k(\frac{1}{z-b_k}) - p_k(z)| < \epsilon/2^i$ on K_i. We want to take

$$f(z) = \sum_k \frac{1}{\#B_i}\left\{P_k\left(\frac{1}{z-b_k}\right) - p_k(z)\right\}.$$

The contribution from the b_k's on L_j to K_i with $j > i$ is less than $\epsilon/2^j$, hence the series converges uniformly and absolutely implying that $f(z)$ is meromorphic.

We have to insure that application of Runge's theorem is *valid*; for this we need to take K_1 "large enough" so that each bounded component of $\mathbb{C} \setminus K_1$ contains a bounded component of $\mathbb{C} \setminus \Omega$ (then this will also be true of K_j, $j \geq 2$).

Lemma. *Given a compact subset $K \subset \Omega$, it is possible to choose a compact $K_1 \subset \Omega$ containing K such that every hole (compact component) of K_1 contains a hole of Ω.*

If $\Omega = \mathbb{C}$, we can let K_1 be giant ball, so in what follows we consider $\Omega \neq \mathbb{C}$.

Let $\Omega_r = \{z \in \Omega : d(z, \Omega) \leq r\}$ where r is some number less than $d(K, \Omega)$. Alternatively, we may describe the complement of Ω_r:

$$\mathbb{C} \setminus \Omega_r = \bigcup_{z \in \mathbb{C} \setminus \Omega} D(z, r).$$

In particular, Ω_r is closed. We would take $K_1 = \Omega_r$, but Ω_r may not be compact; so we take be a giant ball $\overline{B}$ containing K and set $K_1 = \Omega_r \cap \overline{B}$.

Let Z be a bounded component of $\mathbb{C} \setminus K_1$. As $\mathbb{C} \setminus \overline{B}$ is a connected and unbounded chunk of $\mathbb{C} \setminus K_1$, it cannot intersect Z; for if Z has common points with $\mathbb{C} \setminus \overline{B}$, Z would have to contain $\mathbb{C} \setminus \overline{B}$ which is absurd. Hence $Z \subset \Big(\mathbb{C} \setminus K_1\Big) \setminus \Big(\mathbb{C} \setminus \overline{B}\Big) = \mathbb{C} \setminus \Omega_r$.

Each $D(z, r)$ which makes up $\mathbb{C} \setminus \Omega_r$ is connected; so is either contained or disjoint from Z (that is why we have shown $Z \subset \mathbb{C} \setminus \Omega_r$), this tells us that Z is the union of disks $D(z, r)$ which have center in Z, i.e

$$Z = \bigcup_{z \in Z \cap (\mathbb{C} \setminus \Omega)} D(z, r) = \bigcup_{z \in Z \setminus \Omega} D(z, r).$$

As Z is non-empty, the above representation tells us that Z intersects $\mathbb{C} \setminus \Omega$, but then we are done: suppose Z intersects $\mathbb{C} \setminus \Omega$ at a point p and if S is the connected component of p in $\mathbb{C} \setminus \Omega$, then as $\mathbb{C} \setminus K_1 \supset \mathbb{C} \setminus \Omega$, then Z would have to contain all of S as desired.