Asymptotic expansion of the Hausdorff dimension of the push-forward of Lebesgue measure

Oleg Ivrii

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In this note, we are interested in the asymptotic expansion of the Hausdorff dimension of the push-forward of the Lebesgue measure near $z \to z^2$. More precisely, let $h_\lambda$ be the conjugacy on the unit circle between $z^2$ and $z \cdot \frac{z^2 + \lambda}{1 + \lambda z}$. Suppose

$$ \text{H.dim } (h_\lambda)_* = 1 - \sum_{jk} a_{jk} \lambda^j \overline{\lambda}^k. \quad (1) $$

**Question.** Is the matrix $\{a_{jk}\}$ positive definite?

We explicitly compute the $2 \times 2$ minor of this matrix and verify that it is positive definite. Using a computer program, I computed a $7 \times 7$ minor and showed it to be positive definite.

We follow McMullen’s alternative calculation using DeMarco’s formula. For this purpose, it is simpler to perturb $z \to z^2$ through Blaschke products $f_k(z) = \frac{z^2 + k}{1 + k z^2}$.

Let $P(z) = z^2 + k, Q(z) = 1 + k z^2$, so that $f_k = P/Q$. Also, let $p(A, B) = A^2 + kB^2$ and let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be the cover of $f$, given by $F(A, B) = (p(A, B), p(B, A))$.

The critical points of $f$ are $c_1 = 0$ and $1/c_1 = \infty$. We will use the lifts $C_1 = (0, \mu)$ and $(\mu, 0)$ with $\mu^2 = 1 - |k|^2$ so that

$$ 4z_1 z_2 (1 - |k|^2) = |\det DF(z_1, z_2)| = 4 \prod |(z_1, z_2) \wedge C_i| = 4z_1 z_2 \mu^2. \quad (2) $$

Then (5.3) of McMullen’s paper says:

$$ L(f_k, (h_k)_*, m) = \log 2 + \sum G(C_i) - \log |\text{Res}(F)| \quad (3) $$

1
The resultant is the determinant of
\[
\begin{pmatrix}
1 & 0 & k & 0 \\
0 & 1 & 0 & k \\
k & 0 & 1 & 0 \\
0 & k & 0 & 1 \\
\end{pmatrix}
\]
which is \(1 - 2|k|^2 + |k|^4\).

Now let us examine the escape-rate functions \(\sum G(C_i)\). For this we purpose, consider the forward orbit of \((0, 1)\):

<table>
<thead>
<tr>
<th>Step</th>
<th>(A) up to weight 3</th>
<th>(B) up to weight 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(k)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(k + k^2)</td>
<td>(1 + k^2 \cdot \bar{k})</td>
</tr>
<tr>
<td>3</td>
<td>(k + k^2 + 2k^3 + \ldots)</td>
<td>(1 + 3k^2 \cdot \bar{k} + 2k^3 \cdot \bar{k} + \ldots)</td>
</tr>
<tr>
<td>4</td>
<td>(k + k^2 + 2k^3 + \ldots)</td>
<td>(1 + 7k^2 \cdot \bar{k} + 6k^3 \cdot \bar{k} + \ldots)</td>
</tr>
<tr>
<td>5</td>
<td>(k + k^2 + 2k^3 + \ldots)</td>
<td>(1 + 15k^2 \cdot \bar{k} + 14k^3 \cdot \bar{k} + \ldots)</td>
</tr>
</tbody>
</table>

Let us evaluate
\[
\frac{1}{2^n} \log \|F^n(0, 1)\|.
\]
Here the norm is the maximum norm over the two coordinates. It picks out the \(B\) column.

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Inspection reveals that
\[
a = k + k^2 + 2k^3 + O(|k|^4).
\]
The multiplier \(f'(a) \approx 2k\). By the Schwarz lemma, we see that \(|A_n/B_n - \gamma| = O(|k|^d)\) holds for \(n \geq d\).

Since \(B_{n+1} = B_n^2(1 + \bar{k} \cdot (A_n/B_n)^2)\) for \(n \geq 3\), we see that
\[
B_{n+1} = B_n^2 \left(1 + k^2 \bar{k} + 2k^3 \bar{k} + O(|k|^5)\right).
\]
This tells us that
\[
\frac{\log B_{n+1}}{2^{n+1}} - \frac{\log B_n}{2^n} = \frac{k^2 \bar{k} + 2k^3 \bar{k}}{2^{n+1}} + O(|k|^5)
\]
\[
\frac{\log B_3}{8} = \frac{3k^2 \bar{k} + 2k^3 \bar{k}}{8}
\]
Telescopic summation yields
\[
\lim_{n \to \infty} \frac{1}{2^n} \log B_n = \frac{k^2 \bar{k}}{2} + \frac{k^3 \bar{k}}{2}.
\]
Taking real parts, we see that
\[
G(1, 0) = \frac{1}{4} (k^2 \cdot \bar{k} + k \cdot \bar{k}^2) + \frac{1}{4} (k^3 \cdot \bar{k} + k \cdot \bar{k}^3) + \ldots
\]
Notice that \(G(\mu, 0) = G(1, 0) + \log \mu\).
Thus following McMullen, we see that DeMarco’s formula yields
\[
L(f, m) = \log 2 - \log(1 - 2|k|^2 + |k|^4) + \log(1 - |k|^2) + 2 \sum G(1, 0)
\]
\[
= \log 2 - \log(1 - |k|^2) + 2 \sum G(1, 0)
\]
\[
= \log 2 + |k|^2 + \frac{k^2 \cdot \bar{k} + k \cdot \bar{k}^2}{2} + \frac{|k|^4}{2} + \frac{k^3 \cdot \bar{k} + k \cdot \bar{k}^3}{2} + \ldots
\]
Hence,
\[
\delta = \frac{\log 2}{\log 2 + |k|^2 + \frac{1}{2} (k^2 \bar{k} + k \bar{k}^2) + \frac{1}{2} |k|^4 + \frac{1}{2} (k^3 \bar{k} + k \bar{k}^3) + \ldots}
\]
Now set
\[
k = \lambda / 2 - \lambda^2 / 4 + \lambda^3 / 8 + \ldots
\]
Thus
\[
\delta = \frac{1}{1 + \frac{1}{4 \log 2} |\lambda|^2 - \frac{1}{16 \log 2} \cdot (\lambda^2 \bar{\lambda} + \lambda \bar{\lambda}^2) + \frac{5}{32 \log 2} \cdot |\lambda|^4 - \frac{1}{32 \log 2} (\lambda^3 \bar{\lambda} + \lambda \bar{\lambda}^3) + \ldots}
\]
from which we see that
\[
\delta = 1 - \frac{|\lambda|^2}{4 \log 2} + \frac{\lambda^2 \bar{\lambda} + \lambda \bar{\lambda}^2}{16 \log 2} + \left( \frac{1}{(4 \log 2)^2} - \frac{5}{32 \log 2} \right) |\lambda|^4 + \frac{1}{32 \log 2} (\lambda^3 \bar{\lambda} + \lambda \bar{\lambda}^3) + \ldots
\]
Thus in matrix form, \(1 - \delta\) is
\[
\begin{pmatrix}
\frac{1}{4 \log 2} & -\frac{1}{16 \log 2} \\
\frac{5}{32 \log 2} & -\frac{1}{(4 \log 2)^2}
\end{pmatrix}
\]
which has positive determinant.