Asymptotic expansion of the Hausdorff dimension of the push-forward of Lebesgue measure

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In this note, we are interested in the asymptotic expansion of the Hausdorff dimension of the push-forward of the Lebesgue measure near $z \to z^2$. More precisely, let h_λ be the conjugacy on the unit circle between z^2 and $z \cdot \frac{z+\lambda}{1+\bar{\lambda}z}$. Suppose

$$\text{H.dim } (h_\lambda)_* = 1 - \sum_{jk} a_{jk} \lambda^j \overline{\lambda}^k. \tag{1}$$

Question. Is the matrix $\{a_{jk}\}$ positive definite?

We explicitly compute the 2×2 minor of this matrix and verify that it is positive definite. Using a computer program, I computed a 7×7 minor and showed it to be positive definite.

We follow McMullen's alternative calculation using DeMarco's formula. For this purpose, it is simpler to perturb $z \to z^2$ through Blaschke products $f_k(z) = \frac{z^2+k}{1+\bar{k}z^2}$.

Let $P(z) = z^2 + k, Q(z) = 1 + kz^2$, so that $f_k = P/Q$. Also, let $p(A, B) = A^2 + kB^2$ and let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be the cover of f, given by $F(A, B) = (p(A, B), p(B, A))$.

The critical points of f are $c_1 = 0$ and $1/c_1 = \infty$. We will use the lifts $C_1 = (0, \mu)$ and $(\mu, 0)$ with $\mu^2 = 1 - |k|^2$ so that

$$4z_1 z_2 (1 - |k|^2) = |\det DF(z_1, z_2)| = 4 \prod |(z_1, z_2) \wedge C_i| = 4z_1 z_2 \mu^2. \tag{2}$$

Then (5.3) of McMullen's paper says:

$$L(f_k, (h_k)_* m) = \log 2 + \sum G(C_i) - \log |\text{Res}(F)| \tag{3}$$

The resultant is the determinant of

$$\begin{pmatrix} 1 & 0 & k & 0 \\ 0 & 1 & 0 & k \\ \overline{k} & 0 & 1 & 0 \\ 0 & \overline{k} & 0 & 1 \end{pmatrix}$$

which is $1 - 2|k|^2 + |k|^4$.

Now let us examine the escape-rate functions $\sum G(C_i)$. For this we purpose, consider the forward orbit of $(0, 1)$:

Step	A up to weight 3	B up to weight 4
0	0	1
1	k	1
2	$k + k^2$	$1 + k^2 \cdot \overline{k}$
3	$k + k^2 + 2k^3 + \dots$	$1 + 3k^2 \cdot \overline{k} + 2k^3 \cdot \overline{k} + \dots$
4	$k + k^2 + 2k^3 + \dots$	$1 + 7k^2 \cdot \overline{k} + 6k^3 \cdot \overline{k} + \dots$
5	$k + k^2 + 2k^3 + \dots$	$1 + 15k^2 \cdot \overline{k} + 14k^3 \cdot \overline{k} + \dots$

Let us evaluate

$$\frac{1}{2^n} \log ||F^n(0, 1)||.$$

Here the norm is the maximum norm over the two coordinates. It picks out the B column.

Denote the attracting fixed point of f near 0 by a. The critical point 0 converges to p. Inspection reveals that

$$a = k + k^2 + 2k^3 + O(|k|^4).$$

The multiplier $f'(a) \approx 2k$. By the Schwarz lemma, we see that $|A_n/B_n - \gamma| = O(|k|^d)$ holds for $n \geq d$.

Since $B_{n+1} = B_n^2(1 + \overline{k} \cdot (A_n/B_n)^2)$ for $n \geq 3$, we see that

$$B_{n+1} = B_n^2 \Big(1 + k^2\overline{k} + 2k^3\overline{k} + O(|k|^5)\Big).$$

This tells us that

$$\begin{aligned} \frac{\log B_{n+1}}{2^{n+1}} - \frac{\log B_n}{2^n} &= \frac{k^2\overline{k} + 2k^3\overline{k}}{2^{n+1}} + O(|k|^5) \\ \frac{\log B_3}{8} &= \frac{3k^2\overline{k} + 2k^3\overline{k}}{8} \end{aligned}$$

Telescopic summation yields

$$\lim_{n\to\infty} \frac{1}{2^n} \log B_n = \frac{k^2\overline{k}}{2} + \frac{k^3\overline{k}}{2}.$$

Taking real parts, we see that

$$G(1,0) = \frac{1}{4}(k^2 \cdot \overline{k} + k \cdot \overline{k}^2) + \frac{1}{4}(k^3 \cdot \overline{k} + k \cdot \overline{k}^3) + \dots$$

Notice that $G(\mu, 0) = G(1,0) + \log \mu$.

Thus following McMullen, we see that DeMarco's formula yields

$$\begin{aligned} L(f,m) &= \log 2 - \log(1 - 2|k|^2 + |k|^4) + \log(1 - |k|^2) + 2 \sum G(1,0) \\ &= \log 2 - \log(1 - |k|^2) + 2 \sum G(1,0) \\ &= \log 2 + |k|^2 + \frac{k^2 \cdot \overline{k} + k \cdot \overline{k}^2}{2} + \frac{|k|^4}{2} + \frac{k^3 \cdot \overline{k} + k \cdot \overline{k}^3}{2} + \dots. \end{aligned}$$

Hence,

$$\delta = \frac{\log 2}{\log 2 + |k|^2 + \frac{1}{2}(k^2\overline{k} + k\overline{k}^2) + \frac{1}{2} \cdot |k|^4 + \frac{1}{2}(k^3\overline{k} + k\overline{k}^3) + \dots}$$

Now set

$$k = \lambda/2 - \lambda^2/4 + \overline{\lambda} \cdot \lambda^2/8 + \dots$$

Thus

$$\delta = \frac{1}{1 + \frac{1}{4\log 2}|\lambda|^2 - \frac{1}{16\log 2} \cdot (\lambda^2\overline{\lambda} + \lambda\overline{\lambda}^2) + \frac{5}{32\log 2} \cdot |\lambda|^4 - \frac{1}{32\log 2}(\lambda^3\overline{\lambda} + \lambda\overline{\lambda}^3) + \dots}$$

from which we see that

$$\delta = 1 - \frac{|\lambda|^2}{4\log 2} + \frac{\lambda^2\overline{\lambda} + \lambda\overline{\lambda}^2}{16\log 2} + \Big(\frac{1}{(4\log 2)^2} - \frac{5}{32\log 2}\Big)|\lambda|^4 + \frac{1}{32\log 2}(\lambda^3\overline{\lambda} + \lambda\overline{\lambda}^3) + \dots.$$

Thus in matrix form, $1 - \delta$ is

$$\begin{pmatrix} \frac{1}{4\log 2} & -\frac{1}{16\log 2} \\ -\frac{1}{16\log 2} & \frac{5}{32\log 2} - \frac{1}{(4\log 2)^2} \end{pmatrix}$$

which has *positive* determinant.