Approximating Jordan Curves by Julia Sets

Oleg Ivrii

November 25, 2013

Abstract

In this note, we show that a Jordan curve can be approximated by polynomial quasi-circle Julia sets in the Hausdorff topology.

1 Overview

In this note, we explain how to approximate compact sets by Julia sets. We show the following:

Theorem 1. *Any Jordan curve can be approximated by quasi-circle Julia sets of polynomials.*

We can replace a Jordan curve by any "compact connected set". Our argument is in two steps: first, we give an explicit construction of a certain polynomials whose Jordan curve lies near the circle but slightly inside the unit circle, the polynomial is approximately linear. Then, we modify this polynomial by quasi-conformal conjugacy to make its Julia set near any prescribed Jordan curve.

Remark. The paper [Li] gives a *constructive* proof that any compact connected set can be approximated by polynomial Julia sets but these may not necessarily be quasi-circles.

2 Some wonderful polynomials

Consider the polynomial $F(z) = z^n + 0.5z$ with n large. Slightly inside the unit circle, the dynamics are very close to $z \to 0.5z$ while slightly outside, the dynamics iterates to infinity very quickly. We show that the transition between the two regimes a quasi-circle:

Lemma 1. *If n is large, Julia set of $F(z)$ is a quasi-circle.*

Proof. Notice the polynomial F has a super attracting fixed point at infinity and an attracting fixed point at 0. The map F also has $n-1$ repelling fixed points In addition, F has $n-1$ critical points $c_1, c_2, \ldots, c_{n-1}$ which lie on the circle $\{z : |z| = (0.5/n)^{1/n-1}\}$. It is easy to see that for n large, the critical points iterate to the origin: e.g $|c_i|^n < 0.5/n$ and therefore, $F(c_i) \subset D(0, 0.51)$. From then on, it is evident that c_i tends to 0 at an exponential rate. As all the critical points of $F(z)$ converge to the attracting fixed point at the origin, the Julia set is a quasi-circle.

To see this, we first notice that the Fatou set consists of the basin of infinity and the components that make up the basin of the origin. There could be no other components because they require critical points. Next, we see that the immediate basin of the origin maps to itself in a $n : 1$ manner and hence cannot have any other pre-images. Therefore, the Julia set must be a Jordan curve that separates the basin of infinity and the basin of the origin. □

3 Radial Beltrami coefficients

In this section, we reduce the problem of approximating arbitrary Jordan curves to approximating quasi-circles $w_V(S^1)$ which are images of the unit circle by radial Beltrami coefficients. Call a Beltrami coefficient $V(z)$ on the unit disk *radial* if it is is constant on radial rays, i.e if it is of the form $V(z) = V(\theta) \cdot \chi_{\mathbb{D}} \cdot \frac{d\bar{z}}{dz}$.

Theorem 2. *A smooth Jordan curve γ can be represented as the image of the unit circle by a quasi-conformal map g with a radial Beltrami coefficient.*

Proof. Suppose the Jordan curve divides the Riemann sphere into two domains: the "interior" domain Ω_- and the "exterior" domain Ω_+. Let $R_\pm : \mathbb{D}_\pm \to \Omega_\pm$ be the Riemann maps. The welding function $\psi = (R_-)^{-1} \circ R_+$ on the unit circle is bi-Lipschitz. Set

$$g(z) = \begin{cases} R_+(z) & \text{for } z \in \mathbb{D}_+, \\ R_-(r \cdot \psi(e^{i\theta})) & \text{for } z \in \mathbb{D}_-, \end{cases} \tag{1}$$

It is easy to check that the map $r \cdot e^{i\theta} \to r \cdot \psi(e^{i\theta})$ is quasi-symmetric; hence $g(z)$ is quasi-conformal as desired. □

4 Some invariant Beltrami coefficients

We construct new polynomials by quasi-conformal conjugacy. For this purpose, we find invariant Beltrami coefficients μ_n that approximate a radial Beltrami coefficient $V(z)$ in the following sense: given any $\epsilon > 0$, for n sufficiently large, we construct a

Beltrami coefficient μ_n that is supported on the filled Julia set of F_n, such that outside the annulus $A_\epsilon := \{z : 1 - \epsilon < |z| < 1 + \epsilon\}$, we have $||\mu_n - V||_\infty < \epsilon$ while on the annulus A_ϵ, $||\mu_n||_\infty = ||V||_\infty$. (Since K-quasi-conformal maps form a normal family, quasi-conformal deformations supported on small sets or of small norm are close to the identity).

Near the origin, the linearizing coordinate $\varphi_n(z) := \lim_{k\to\infty}(0.5)^{-k} \cdot F_n^{\circ k}(z)$ conjugates F to multiplication by 0.5. This means that

$$\varphi_n(F_n(z)) = (0.5) \cdot \varphi_n(z). \tag{2}$$

The above equation determines φ_n uniquely up to the normalization $\varphi_n'(0) = 1$.

The Beltrami coefficient $V(\theta)\frac{d\bar{z}}{dz}$ in the plane is obviously invariant under multiplication by 0.5. Therefore, $\mu_n = \varphi_n^*(V(\theta)\frac{d\bar{z}}{dz})$ is invariant under F_n. For our polynomials, the linearizing maps φ_n are essentially identity on a large subset of the disk and hence $\mu_n \approx V(z)$. It follows that the polynomials $G_n := (w_{\mu_n})^{-1} \circ F_n \circ w_{\mu_n}$ have the desired property: their Julia sets $w_{\mu_n}(\mathcal{J}(F_n))$ approximate $w_V(S^1)$ in the Hausdorff topology.

References

[I] Ivrii, *Incompleteness of the Weil-Petersson metric on spaces of Blaschke products*, In preparation.

[Li] Lindsey, K. A. *Shapes of polynomial Julia sets*, `arxiv:1209.0143v2`, 2013.