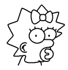
Differential Equations: Practice Exam

Solve any 5 problems. All problems are worth equal marks.
Time permitted: 3 hours. No aids allowed.



Problem 1

(a) Find the general real-valued solution of the homogenous second-order linear ODE with constant coefficients:

$$y'' + 2y' + y = 0.$$

(b) Find the general real-valued solution of the inhomogenous equation

$$y'' + 2y' + y = e^{-x}.$$

Hint. Search for a solution of the form $y_{\text{spec}} = Cx^n e^{-x}$.

Problem 2

(a) Determine all the equilibrium points of the system

$$\begin{cases} \dot{x} &= (2+x)(y-x), \\ \dot{y} &= (4-x)(y+x). \end{cases}$$

(b) Find the eigenvalues of the linearized systems near each equilibrium point.

(c) Use this information to classify the equilibrium points as nodal or spiral sources, sinks and saddles. (The problem has been cooked up so only the robust cases occur.)

(d) Draw the nullclines $\{\dot{x} = 0\}$ and $\{\dot{y} = 0\}$. In each complementary region, determine the general direction of the vector field, i.e. decide whether $\dot{x}$ and $\dot{y}$ are positive or negative.

(e) Sketch the phase portrait as best you can using the above information.

Problem 3

(a) Let $y(x)$ be the solution to the initial value problem

$$\begin{cases} y'(x) = x - y^2, \\ y(1) = 1. \end{cases}$$

Show that

$$\sqrt{x-1} \leq y(x) \leq \sqrt{x}, \qquad x \geq 1.$$

(b) Show that any non-zero solution the Hermite differential equation

$$u'' - 2xu' + 2\lambda u = 0$$

has at most finitely many zeros on the real line.

Hint. The substitution $v = e^{-x^2/2}u(x)$ transforms the DE into

$$v'' + (1 + 2\lambda - x^2)v = 0.$$

Problem 4

(a) Construct a homogenous second-order linear ODE with variable coefficients,

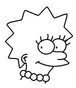
$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

with solutions $u(x) = x$ and $v(x) = \sin x$.

(b) Use the method of variation of parameters to solve the inhomogenous linear ODE

$$y'' + y = \frac{1}{\cos^3 x}.$$

Warning. No marks will be given for simply applying Lagrange's formula. You are supposed to derive it in the specific example.



Problem 5

Show that for any $a, b \in \mathbb{R}$, the boundary value problem

$$y'' - y \sin x = 0, \qquad x \in (0,1), \quad y(0) = a, \quad y(\pi) = b$$

has a unique solution. You need to prove both existence and uniqueness.

Problem 6

Solve the wave equation in $x \in [0,1]$, $t \in [0, \infty]$ with Neumann boundary conditions:

$$\begin{cases} u_{tt} = u_{xx}, \\ u_x(t,0) = u_x(t,1) = 0, \\ u(0,x) = f(x), \\ u_t(0,x) = 0. \end{cases}$$

with $f \in C^\infty([0,1])$. The answer should be an infinite orthonormal series.

Hint. First search for solutions of the form $u(t) = X(x)T(t)$.