Critical values
of inner functions

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Hyperbolic metric

In complex analysis, it is customary to equip the unit disk $\mathbb{D}$ with the hyperbolic metric

$$\lambda_{\mathbb{D}} = \frac{2|dz|}{1 - |z|^2}.$$ 

Then $\text{Aut}(\mathbb{D})$ acts isometrically on $(\mathbb{D}, \lambda_{\mathbb{D}})$, while holomorphic mappings $F : \mathbb{D} \to \mathbb{D}$ are contractions.
Finite Blaschke products

A finite Blaschke product is a holomorphic self-map of the unit disk which is asymptotically a hyperbolic isometry.
A finite Blaschke product is product of automorphisms of the disk:

\[ F(z) = e^{i\alpha} \prod_{i=1}^{d} \frac{z - a_i}{1 - \bar{a}_i z}, \quad a_i \in \mathbb{D}. \]
Heins theorem

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We have uniqueness up to post-composition with Möbius transformations. If \( m \in \text{Aut}(\mathbb{D}) \), we have \( \text{crit } m \circ F = \text{crit } F \).
The quasigeodesic property

The hyperbolic metric $\lambda_D$ has constant negative curvature

$$- \frac{\Delta \log \lambda_D}{\lambda_D^2} = -1.$$  

Lemma. Let $\gamma$ be a curve in $(\mathbb{D}, \lambda_D)$ and $k_g$ be its geodesic curvature.

- If $k_g \leq 1$, then $\gamma$ cannot intersect itself.
- If $k_g < c < 1$, then $\gamma$ lies within a bounded distance of a hyperbolic geodesic.
Liouville correspondence

The Liouville correspondence provides a bridge between complex analysis and non-linear elliptic PDEs:

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\begin{align*}
\{ \text{hol. maps} & \} \quad \longleftrightarrow \quad \{ \text{solutions of the Gauss curvature equation} & \}
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The function \( u_F = \log \lambda_F \) satisfies the Gauss curvature equation:

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\Delta u = e^{2u} + 2\pi \sum_{c \in \text{crit } F} \delta_c.
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Any solution of the above equation arises from a holomorphic self-map of the unit disk, which is uniquely determined up to post-composition with an element of \( \text{Aut}(\mathbb{D}) \).
Heins theorem (proof)

Construction of a finite Blaschke product $F_C$ with critical set $C$:

1. Let $u_C$ be the pointwise maximal solution of

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3. One uses the maximality of the solution $u_C$ to conclude that $F_C$ is a finite Blaschke product.
Inner functions

An inner function is a holomorphic self-map of $\mathbb{D}$ such that for almost every $\theta \in [0, 2\pi)$, the radial boundary value

$$\lim_{r \to 1} F(re^{i\theta})$$

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Different inner functions can have the same critical set. For example, $F_1(z) = z$ and $F_2(z) = \exp\left(\frac{z+1}{z-1}\right)$ have no critical points.
Figure: The universal covering map of the punctured disk.

$$F(z) = \exp\left(\frac{z-1}{z+1}\right)$$

Critical point $c = -(1-\sqrt{n})$ of multiplicity $n$. 

Figure: The universal covering map of the punctured disk.
An **inner function** can be represented as a (possibly infinite) Blaschke product $\times$ singular inner function:

$$B(z) = e^{i\alpha} \prod_{i} -\frac{\overline{a_i}}{|a_i|} \cdot \frac{z - a_i}{1 - \overline{a_i}z}, \quad a_i \in \mathbb{D}, \quad \sum_{i}(1 - |a_i|) < \infty.$$  

$$S(z) = \exp\left(-\int_{S^1} \frac{\zeta + z}{\zeta - z} d\mu_\zeta\right), \quad \mu \perp m, \quad \mu \geq 0.$$  

Here, $B$ records the zero set, while $S$ records the boundary zero structure.
Let $\mathcal{I}$ be the space of inner functions with derivative in the Nevanlinna class:

$$\int_{\partial \mathbb{D}} \log |F'(e^{i\theta})| \, dm < \infty,$$

where $F_n \to F$ if the convergence is uniform on compact sets and

$$\int_{\partial \mathbb{D}} \log |F'_n(z)| \, dm \to \int_{\partial \mathbb{D}} \log |F'(z)| \, dm.$$

In 1974, P. Ahern and D. Clark showed that $F'$ admits a BSO decomposition, allowing us to define $\text{Inn} F' := BS$, where $B$ records the critical set of $F$ and $S$ records the boundary critical structure.
Dyakonov’s question

**Theorem.** (Kraus 2013, I. 2017) An inner function $F \in \mathcal{I}$ is uniquely determined by $\text{Inn } F'$ up to post-composition with a Möbius transformation.
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An inner function $BS_\mu$ is a critical structure if and only if $\mu$ lives on a countable union of Beurling-Carleson sets.
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**Theorem.** (Kraus 2013, I. 2017) An inner function $F \in \mathcal{J}$ is uniquely determined by $\text{Inn } F'$ up to post-composition with a Möbius transformation.

An inner function $BS_\mu$ is a critical structure if and only if $\mu$ lives on a countable union of Beurling-Carleson sets.

**Definition.** A Beurling-Carleson set $E$ is a closed subset of the unit circle which has measure 0 such that

$$\|E\|_{BC} := \sum |l_j| \cdot \log \frac{1}{|l_j|} < \infty,$$

where $\{l_j\}$ are the complementary arcs.
What this talk is about

For a finite Blaschke product $F$, we define:

- Critical point measure:
  \[
  \mu_F = \sum_{c \in \text{crit} F} (1 - |c|) \cdot \delta_c
  \]

- Critical value measure:
  \[
  \nu_F = \sum_{c \in \text{crit} F} (1 - |c|) \cdot \delta_{F(c)}.
  \]

**Question.** Can we extend these definitions to inner functions in a meaningful way?
Figure: For the universal covering map of the punctured disk, the critical point measure is $\mu_F = \delta_{-1}$ and the critical value measure is $\nu_F = \delta_0$. 
Critical value measures

**Theorem 1.** Suppose $F \in \mathcal{J}$ and $F_n \to F$ is a stable approximation by finite Blaschke products.

The critical value measures

$$\nu_{F_n} = \sum_{c \in \text{crit } F_n} (1 - |c|) \cdot \delta_{F_n}(c)$$

converge in the weak-\* topology to a measure $\nu_F$. 

1. $\nu_F$ does not depend on the approximating sequence $F_n$.
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Suppose $V \subset \mathbb{D}$ is a Jordan domain and $U$ is a connected component of the pre-image $F^{-1}(V)$. 
Form the **component inner function** $F_U = \psi^{-1} \circ F \circ \varphi$, where $\varphi, \psi$ are Riemann maps from $\mathbb{D}$ to $U$ and $V$ respectively.
Components of inner functions

Theorem 2.
\[ \text{crit } F_U = \varphi^{-1} (\text{crit } F), \quad \varphi_* \sigma(F'_U) = |(\varphi^{-1})'(\zeta)| \, d\sigma(F')|_{(\partial U \cap \partial D)}. \]
Angular derivatives

Suppose \( \varphi : \mathbb{D} \to \Omega \) is a Riemann map onto a Jordan domain. We say \( \varphi \) has an \textbf{angular derivative} at \( \zeta \in \partial \mathbb{D} \) if

\[
\lim_{r \to 1} \varphi'(r \zeta)
\]

exists and is finite.
Angular derivatives

According to the Rodin-Warschawski theorem, \( \varphi \) has a non-zero angular derivative if and only if

\[
\lim_{r,s \to 0, \ r > s} \left\{ \frac{1}{\pi} \cdot \log \frac{r}{s} - \text{Mod} \Gamma_{r,s} \right\} = 0.
\]
Angular derivatives

We are interested in the case when $\Omega \subset \mathbb{D}$.

The Rodin-Warschawski theorem says that $\varphi$ has a non-zero angular derivative at $\zeta \in \partial \mathbb{D}$ iff $\Omega$ is sufficiently thick at $\varphi(\zeta) \in \partial \mathbb{D}$. 
We say that a holomorphic function $F : \mathbb{D} \to \mathbb{D}$ has **thick limit** $L$ at $\zeta \in \partial \mathbb{D}$ if $\forall \varepsilon > 0$, some connected component of $F^{-1}(B(L, \varepsilon))$ is thick at $\zeta$. 

By a result of K. Burdzy from 1986, this is the same as $F$ having a **minimal fine limit** $L$ at $\zeta$. 

**Thick limits**
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By a result of K. Burdzy from 1986, this is the same as $F$ having a **minimal fine limit** $L$ at $\zeta$. 
Theorem 3. Suppose $F \in \mathcal{I}$ is an inner function with $\text{Inn } F' = BS_\mu$. For almost every $\zeta \in \partial \mathbb{D}$ with respect to $\mu$,

$$\text{thick-lim } z \to \zeta F(z)$$

exists and lies in the open unit disk $\mathbb{D}$. 

Abundance of thick limits

Theorem 3. Suppose $F \in \mathcal{J}$ is an inner function with \( \text{Inn } F' = BS_\mu \). For almost every $\zeta \in \partial \mathbb{D}$ with respect to $\mu$, \[
\text{thick-lim}_{z \to \zeta} F(z) \]
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Special case. Suppose $\mu$ is a singular measure on the unit circle supported on a Beurling-Carleson set $E$. Then,

\[
\int_0^1 \mu(B(\zeta, \epsilon))^{-1} d\epsilon < \infty, \quad \mu \text{ a.e. } \zeta \in \partial \mathbb{D}.
\]
Fundamental lemma

Lemma (Dyakonov 1992, Kraus 2013, I. 2017)

For any inner function $F \in \mathcal{J}$, $\lambda_F$ is the \textit{minimal} solution of
\[GCE(C)\]which satisfies

$$\lambda_F = \frac{2|F'|}{1 - |F|^2} \geq |\text{Inn} F'| \lambda_{\mathbb{D}}.$$ 

Corollary

Suppose $F_1, F_2 \in \mathcal{J}$ with $\text{Inn} F_1' = S_{\mu_1}$ and $\text{Inn} F_2' = S_{\mu_2}$. If $\mu_1 \leq \mu_2$ then

$$\lambda_{F_1} \geq \lambda_{F_2}.$$
Estimates for $F'_\mu$ with $\mu$ supported on $E$

Lemma (Coarse estimate, I. 2021)

For $\zeta \in \partial \mathbb{D} \setminus E$, we have

$$|F'_\mu(\zeta)| \leq C(\mu(\partial \mathbb{D})) \cdot \text{dist}(\zeta, E)^{-4}.$$ 

Lemma (Fine estimate)

Suppose $\zeta \in \partial \mathbb{D} \setminus E$. Write $z = (1 - \delta)\zeta$ where $\delta = \text{dist}(\zeta, E)$. If $P_\mu(z) \geq 1$ then

$$|F'_\mu(\zeta)| \leq C \cdot \frac{P_\mu(z)}{\delta}, \quad \zeta \in \partial \mathbb{D} \setminus E,$$

for some universal constant $C > 0$. 
We want to show that \(|F_\mu(x) - F_\mu(y)| \leq \int_\gamma |F'_\mu(z)| \cdot |dz|\) is small.
To estimate $|F'_{\mu}(z)|$, we estimate either $\lambda_{F_{\mu}^{\text{Left}}}(z)$ or $\lambda_{F_{\mu}^{\text{Right}}}(z)$. 
Abundance of thick limits (proof)

For $\mu$-a.e. $x$, both measures $\mu_{\text{Left}}(x)$ and $\mu_{\text{Right}}(x)$ are substantial.
Thank you for your attention!