Critical values of inner functions

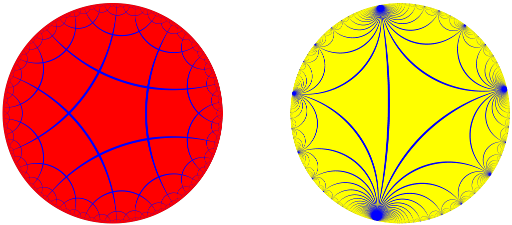
Oleg Ivrii and Uri Kreitner

June 20, 2023

Hyperbolic metric

In complex analysis, it is customary to equip the unit disk $\mathbb{D}$ with the hyperbolic metric

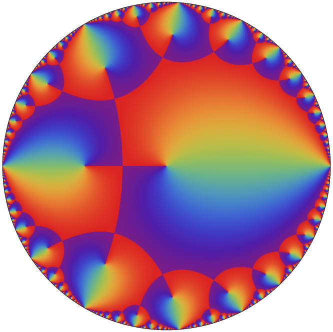
$$\lambda_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}.$$

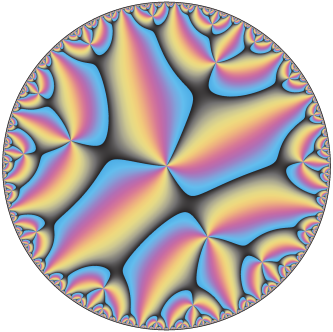


Then $\mathrm{Aut}(\mathbb{D})$ acts isometrically on $(\mathbb{D}, \lambda_{\mathbb{D}})$, while holomorphic mappings $F : \mathbb{D} \to \mathbb{D}$ are contractions.

Finite Blaschke products

A finite Blaschke product is a holomorphic self-map of the unit disk which is asymptotically a hyperbolic isometry.





Heins theorem

A finite Blaschke product is product of automorphisms of the disk:

$$F(z) = e^{i\alpha} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_i \in \mathbb{D}.$$

Heins theorem

A finite Blaschke product is product of automorphisms of the disk:

$$F(z) = e^{i\alpha} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_i \in \mathbb{D}.$$

Theorem. (M. Heins, 1962) Given a set C of $d-1$ points in the unit disk, there exists a unique Blaschke product of degree d with critical set C.

Heins theorem

A finite Blaschke product is product of automorphisms of the disk:

$$F(z) = e^{i\alpha} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_i \in \mathbb{D}.$$

Theorem. (M. Heins, 1962) Given a set C of $d - 1$ points in the unit disk, there exists a unique Blaschke product of degree d with critical set C.

We have uniqueness up to post-composition with Möbius transformations. If $m \in \mathrm{Aut}(\mathbb{D})$, we have $\mathrm{crit}\, m \circ F = \mathrm{crit}\, F$.

The quasigeodesic property

The hyperbolic metric $\lambda_{\mathbb{D}}$ has constant negative curvature

$$-\frac{\Delta \log \lambda_{\mathbb{D}}}{\lambda_{\mathbb{D}}^2} = -1.$$

Lemma. Let γ be a curve in $(\mathbb{D}, \lambda_{\mathbb{D}})$ and k_g be its geodesic curvature.

- If $k_g \leq 1$, then γ cannot intersect itself.
- If $k_g < c < 1$, then γ lies within a bounded distance of a hyperbolic geodesic.

Liouville correspondence

The Liouville correspondence provides a bridge between complex analysis and non-linear elliptic PDEs:

$$\left\{ \begin{array}{c} \text{hol. maps} \\ F : \mathbb{D} \to \mathbb{D} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{solutions of the} \\ \text{Gauss curvature equation} \end{array} \right\}$$

Liouville correspondence

The Liouville correspondence provides a bridge between complex analysis and non-linear elliptic PDEs:

$$\left\{ \begin{array}{c} \text{hol. maps} \\ F : \mathbb{D} \to \mathbb{D} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{solutions of the} \\ \text{Gauss curvature equation} \end{array} \right\}$$

A holomorphic self-map F of the unit disk defines the conformal pseudometric

$$\lambda_F = \frac{2|F'|}{1-|F|^2},$$

Liouville correspondence

The Liouville correspondence provides a bridge between complex analysis and non-linear elliptic PDEs:

$$\left\{ \begin{array}{c} \text{hol. maps} \\ F : \mathbb{D} \to \mathbb{D} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{solutions of the} \\ \text{Gauss curvature equation} \end{array} \right\}$$

A holomorphic self-map F of the unit disk defines the conformal pseudometric

$$\lambda_F = \frac{2|F'|}{1 - |F|^2},$$

which has constant negative curvature -1, away from the critical points of F,

Liouville correspondence

The Liouville correspondence provides a bridge between complex analysis and non-linear elliptic PDEs:

$$\left\{ \begin{array}{l} \text{hol. maps} \\ F : \mathbb{D} \to \mathbb{D} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{solutions of the} \\ \text{Gauss curvature equation} \end{array} \right\}$$

A holomorphic self-map F of the unit disk defines the conformal pseudometric

$$\lambda_F = \frac{2|F'|}{1-|F|^2},$$

which has constant negative curvature -1, away from the critical points of F, where it has concentrated negative curvature.

Liouville correspondence

The Liouville correspondence provides a bridge between complex analysis and non-linear elliptic PDEs:

$$\left\{ \begin{array}{c} \text{hol. maps} \\ F : \mathbb{D} \to \mathbb{D} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{solutions of the} \\ \text{Gauss curvature equation} \end{array} \right\}$$

The function $u_F = \log \lambda_F$ satisfies the Gauss curvature equation:

$$\Delta u = e^{2u} + 2\pi \sum_{c \in \operatorname{crit} F} \delta_c .$$

Liouville correspondence

The Liouville correspondence provides a bridge between complex analysis and non-linear elliptic PDEs:

$$\left\{ \begin{array}{c} \text{hol. maps} \\ F : \mathbb{D} \to \mathbb{D} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{solutions of the} \\ \text{Gauss curvature equation} \end{array} \right\}$$

The function $u_F = \log \lambda_F$ satisfies the Gauss curvature equation:

$$\Delta u = e^{2u} + 2\pi \sum_{c \in \operatorname{crit} F} \delta_c.$$

Any solution of the above equation arises from a holomorphic self-map of the unit disk, which is uniquely determined up to post-composition with an element of $\operatorname{Aut}(\mathbb{D})$.

Heins theorem (proof)

Construction of a finite Blaschke product F_C with critical set C:

1. Let u_C be the pointwise maximal solution of

$$\Delta u = e^{2u} + 2\pi \sum_{c \in C} \delta_c .$$

Heins theorem (proof)

Construction of a finite Blaschke product F_C with critical set C:

1. Let u_C be the pointwise maximal solution of

$$\Delta u = e^{2u} + 2\pi \sum_{c \in C} \delta_c.$$

2. Define F_C as the Liouville map of u_C.

Heins theorem (proof)

Construction of a finite Blaschke product F_C with critical set C:

1. Let u_C be the pointwise maximal solution of

$$\Delta u = e^{2u} + 2\pi \sum_{c \in C} \delta_c .$$

2. Define F_C as the Liouville map of u_C.
3. One uses the maximality of the solution u_C to conclude that F_C is a finite Blaschke product.

Inner functions

An inner function is a holomorphic self-map of $\mathbb{D}$ such that for almost every $\theta \in [0, 2\pi)$, the radial boundary value

$$\lim_{r \to 1} F(re^{i\theta})$$

exists and has absolute value 1.

Inner functions

An inner function is a holomorphic self-map of $\mathbb{D}$ such that for almost every $\theta \in [0, 2\pi)$, the radial boundary value

$$\lim_{r \to 1} F(re^{i\theta})$$

exists and has absolute value 1.

Different inner functions can have the same critical set. For example, $F_1(z) = z$ and $F_2(z) = \exp(\frac{z+1}{z-1})$ have no critical points.

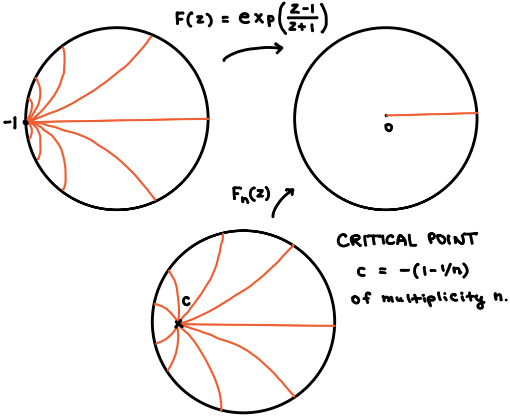


Figure: The universal covering map of the punctured disk.

BS decomposition

An inner function can be represented as a (possibly infinite) Blaschke product $\times$ singular inner function:

$$B(z) = e^{i\alpha} \prod_i -\frac{\overline{a_i}}{|a_i|} \cdot \frac{z - a_i}{1 - \overline{a_i} z}, \quad a_i \in \mathbb{D}, \quad \sum_i (1 - |a_i|) < \infty.$$

$$S(z) = \exp\left(- \int_{\mathbb{S}^1} \frac{\zeta + z}{\zeta - z} \, d\mu_\zeta \right), \quad \mu \perp m, \quad \mu \geq 0.$$

Here, B records the zero set, while S records the boundary zero structure.

Inner functions of finite entropy / Stable topology

Let $\mathscr{J}$ be the space of inner functions with derivative in the Nevanlinna class:

$$\int_{\partial\mathbb{D}} \log|F'(e^{i\theta})|dm < \infty,$$

where $F_n \to F$ if the convergence is uniform on compact sets and

$$\int_{\partial\mathbb{D}} \log|F_n'(z)|dm \to \int_{\partial\mathbb{D}} \log|F'(z)|dm.$$

In 1974, P. Ahern and D. Clark showed that F' admits a BSO decomposition, allowing us to define $\text{Inn}\, F' := BS$, where B records the critical set of F and S records the boundary critical structure.

Dyakonov's question

Theorem. (Kraus 2013, I. 2017) An inner function $F \in \mathscr{J}$ is uniquely determined by $\operatorname{Inn} F'$ up to post-composition with a Möbius transformation.

Dyakonov's question

Theorem. (Kraus 2013, I. 2017) An inner function $F \in \mathscr{J}$ is uniquely determined by Inn F' up to post-composition with a Möbius transformation.

An inner function BS_μ is a critical structure if and only if μ lives on a countable union of Beurling-Carleson sets.

Dyakonov's question

Theorem. (Kraus 2013, I. 2017) An inner function $F \in \mathscr{J}$ is uniquely determined by $\operatorname{Inn} F'$ up to post-composition with a Möbius transformation.

An inner function BS_μ is a critical structure if and only if μ lives on a countable union of Beurling-Carleson sets.

Definition. A Beurling-Carleson set E is a closed subset of the unit circle which has measure 0 such that

$$\|E\|_{\mathcal{BC}} := \sum |I_j| \cdot \log \frac{1}{|I_j|} < \infty,$$

where $\{I_j\}$ are the complementary arcs.

What this talk is about

For a finite Blaschke product F, we define:

- Critical point measure:

$$\mu_F = \sum_{c \in \text{crit}\, F} (1 - |c|) \cdot \delta_c$$

- Critical value measure:

$$\nu_F = \sum_{c \in \text{crit}\, F} (1 - |c|) \cdot \delta_{F(c)}.$$

Question. Can we extend these definitions to inner functions in a meaningful way?

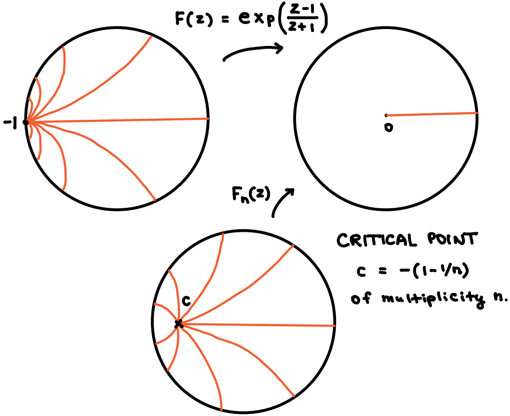


Figure: For the universal covering map of the punctured disk, the critical point measure is $\mu_F = \delta_{-1}$ and the critical value measure is $\nu_F = \delta_0$.

Critical value measures

Theorem 1. Suppose $F \in \mathscr{J}$ and $F_n \to F$ is a stable approximation by finite Blaschke products.

The critical value measures

$$\nu_{F_n} = \sum_{c \in \text{crit}\, F_n} (1 - |c|) \cdot \delta_{F_n(c)}$$

converge in the weak-$*$ topology to a measure ν_F.

Critical value measures

Theorem 1. Suppose $F \in \mathscr{J}$ and $F_n \to F$ is a stable approximation by finite Blaschke products.

The critical value measures

$$\nu_{F_n} = \sum_{c \in \text{crit}\, F_n} (1 - |c|) \cdot \delta_{F_n(c)}$$

converge in the weak-$*$ topology to a measure ν_F.

1. ν_F does not depend on the approximating sequence F_n.

Critical value measures

Theorem 1. Suppose $F \in \mathscr{J}$ and $F_n \to F$ is a stable approximation by finite Blaschke products.

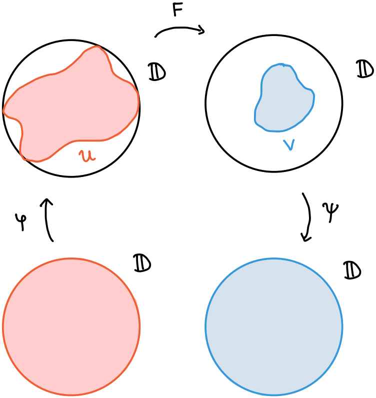
The critical value measures

$$\nu_{F_n} = \sum_{c \in \text{crit}\, F_n} (1 - |c|) \cdot \delta_{F_n(c)}$$

converge in the weak-$*$ topology to a measure ν_F.

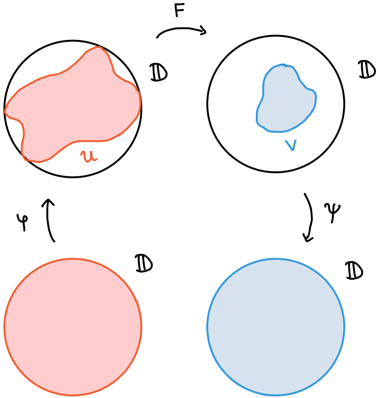
1. ν_F does not depend on the approximating sequence F_n.
2. The measure ν_F is supported on the open unit disk.

Components of inner functions



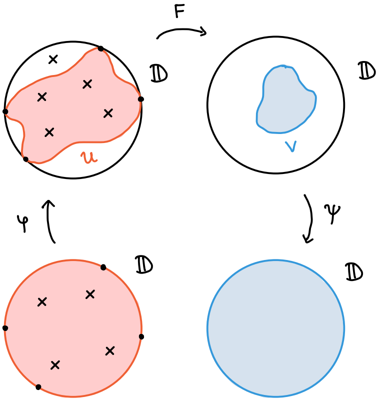
Suppose $V \subset \mathbb{D}$ is a Jordan domain and U is a connected component of the pre-image $F^{-1}(V)$.

Components of inner functions



Form the component inner function $F_U = \psi^{-1} \circ F \circ \varphi$, where φ, ψ are Riemann maps from $\mathbb{D}$ to U and V respectively.

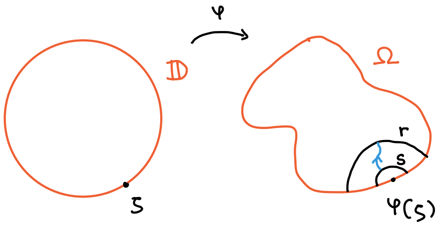
Components of inner functions



Theorem 2.

$$\operatorname{crit} F_U = \varphi^{-1}(\operatorname{crit} F), \qquad \varphi_* \, \sigma(F_U') = |(\varphi^{-1})'(\zeta)| \, d\sigma(F')|_{(\partial U \cap \partial \mathbb{D})}.$$

Angular derivatives



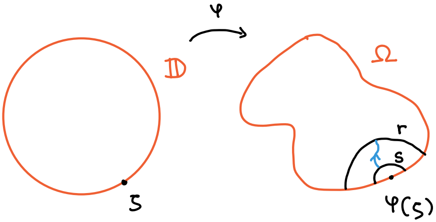
Suppose $\varphi : \mathbb{D} \to \Omega$ is a Riemann map onto a Jordan domain.

We say φ has an angular derivative at $\zeta \in \partial\mathbb{D}$ if

$$\lim_{r \to 1} \varphi'(r\zeta)$$

exists and is finite.

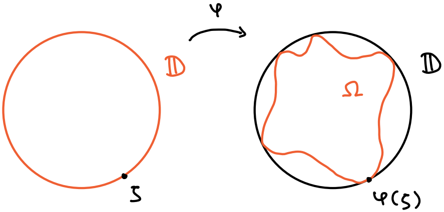
Angular derivatives



According to the Rodin-Warschawski theorem, φ has a non-zero angular derivative if and only if

$$\lim_{\substack{r,s\to 0\\ r>s}} \left\{ \frac{1}{\pi} \cdot \log \frac{r}{s} - \operatorname{Mod} \Gamma_{r,s} \right\} = 0.$$

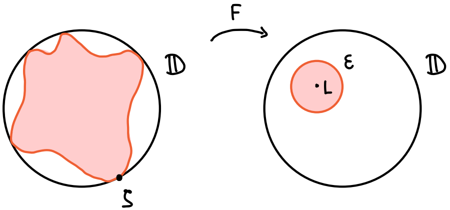
Angular derivatives



We are interested in the case when $\Omega \subset \mathbb{D}$.

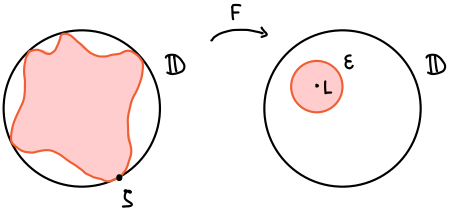
The Rodin-Warschawski theorem says that φ has a non-zero angular derivative at $\zeta \in \partial\mathbb{D}$ iff Ω is sufficiently thick at $\varphi(\zeta) \in \partial\mathbb{D}$.

Thick limits



We say that a holomorphic function $F : \mathbb{D} \to \mathbb{D}$ has thick limit L at $\zeta \in \partial\mathbb{D}$ if $\forall \varepsilon > 0$, some connected component of $F^{-1}(B(L, \varepsilon))$ is thick at ζ.

Thick limits



We say that a holomorphic function $F : \mathbb{D} \to \mathbb{D}$ has thick limit L at $\zeta \in \partial\mathbb{D}$ if $\forall \varepsilon > 0$, some connected component of $F^{-1}(B(L, \varepsilon))$ is thick at ζ.

By a result of K. Burdzy from 1986, this is the same as F having a minimal fine limit L at ζ.

Abundance of thick limits

Theorem 3. Suppose $F \in \mathscr{J}$ is an inner function with $\operatorname{Inn} F' = BS_\mu$. For almost every $\zeta \in \partial\mathbb{D}$ with respect to μ,

$$\operatorname*{thick-lim}_{z \to \zeta} F(z)$$

exists and lies in the open unit disk $\mathbb{D}$.

Abundance of thick limits

Theorem 3. Suppose $F \in \mathscr{J}$ is an inner function with $\operatorname{Inn} F' = BS_\mu$. For almost every $\zeta \in \partial\mathbb{D}$ with respect to μ,

$$\operatorname*{thick-lim}_{z \to \zeta} F(z)$$

exists and lies in the open unit disk $\mathbb{D}$.

Special case. Suppose μ is a singular measure on the unit circle supported on a Beurling-Carleson set E. Then,

$$\int_0^1 \mu(B(\zeta, \varepsilon))^{-1} d\varepsilon < \infty, \qquad \mu \text{ a.e. } \zeta \in \partial\mathbb{D}.$$

Fundamental lemma

Lemma (Dyakonov 1992, Kraus 2013, I. 2017)

For any inner function $F \in \mathscr{J}$, λ_F *is the minimal solution of* $\mathrm{GCE}(C)$ *which satisfies*

$$\lambda_F = \frac{2|F'|}{1-|F|^2} \geq |\operatorname{Inn} F'|\lambda_{\mathbb{D}}.$$

Corollary

Suppose $F_1, F_2 \in \mathscr{J}$ *with* $\operatorname{Inn} F_1' = S_{\mu_1}$ *and* $\operatorname{Inn} F_2' = S_{\mu_2}$. *If* $\mu_1 \leq \mu_2$ *then*

$$\lambda_{F_1} \geq \lambda_{F_2}.$$

Estimates for F'_μ with μ supported on E

Lemma (Coarse estimate, I. 2021)

For $\zeta \in \partial\mathbb{D} \setminus E$, we have

$$|F'_\mu(\zeta)| \le C\big(\mu(\partial\mathbb{D})\big) \cdot \mathrm{dist}(\zeta, E)^{-4}.$$

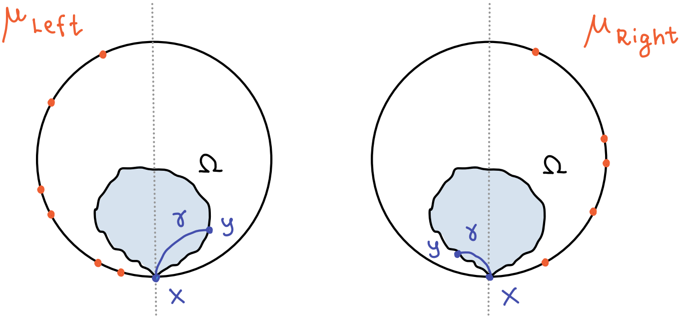
Lemma (Fine estimate)

Suppose $\zeta \in \partial\mathbb{D} \setminus E$. Write $z = (1-\delta)\zeta$ where $\delta = \mathrm{dist}(\zeta, E)$. If $P_\mu(z) \ge 1$ then

$$|F'_\mu(\zeta)| \le C \cdot \frac{P_\mu(z)}{\delta}, \qquad \zeta \in \partial\mathbb{D} \setminus E,$$

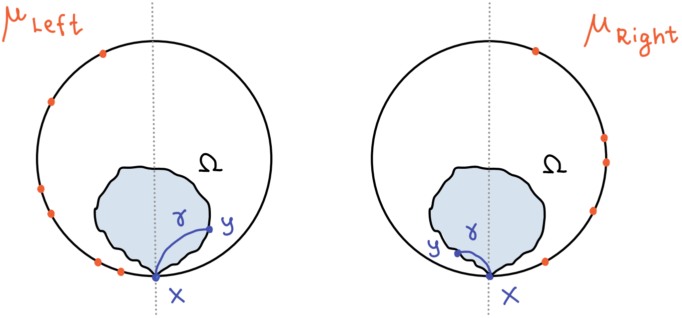
for some universal constant $C > 0$.

Abundance of thick limits (proof)



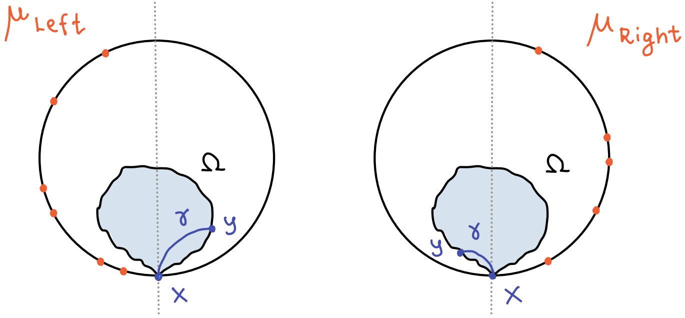
We want to show that $|F_\mu(x) - F_\mu(y)| \le \int_\gamma |F'_\mu(z)| \cdot |dz|$ is small.

Abundance of thick limits (proof)



To estimate $|F_\mu'(z)|$, we estimate either $\lambda_{F_{\mu_{\text{Left}}}}(z)$ or $\lambda_{F_{\mu_{\text{Right}}}}(z)$.

Abundance of thick limits (proof)



For μ-a.e. x, both measures $\mu_{\text{Left}(x)}$ and $\mu_{\text{Right}(x)}$ are substantial.

Thank you for your attention!