

On meromorphic functions whose image has finite spherical area

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Abstract

In this paper, we study meromorphic functions on a domain $\Omega \subset \mathbb{C}$ whose image has finite spherical area, counted with multiplicity. The paper is composed of two parts. In the first part, we show that the limit of a sequence of meromorphic functions is naturally defined on Ω union a tree of spheres. In the second part, we show that a set $E \subset \Omega$ is removable if and only if it is negligible for extremal distance.

1 Introduction

Let $\Omega \subset \mathbb{C}$ be a domain in the complex plane. A holomorphic function on Ω belongs to the classical Dirichlet space $\mathcal{D} = \mathcal{D}(\Omega)$ if the Euclidean area of its image counted with multiplicity is finite:

$$\mathcal{A}_{\mathbb{C}}(F, \Omega) = \int_{\Omega} |F'(z)|^2 |dz|^2 < \infty. \quad (1.1)$$

In this paper, we present two mostly independent vignettes on the “spherical Dirichlet space” $\mathcal{F} = \mathcal{F}(\Omega)$ which consists of meromorphic functions on Ω whose images have finite spherical area, i.e.

$$\mathcal{A}_{\hat{\mathbb{C}}}(F, \Omega) = \int_{\Omega} \left(\frac{2|F'(z)|}{1 + |F(z)|^2} \right)^2 |dz|^2 < \infty. \quad (1.2)$$

1.1 Bubbling of meromorphic functions

Suppose that $\{F_n\}$ is a sequence of meromorphic functions on a domain $\Omega \subset \mathbb{C}$ for which

$$\mathcal{A}_{\hat{\mathbb{C}}}(F_n, \Omega) = \int_{\Omega} \left(\frac{2|F'_n(z)|}{1 + |F_n(z)|^2} \right)^2 |dz|^2 \leq C. \quad (1.3)$$

If $C < 4\pi$, then $\{F_n\}$ is a normal family since the image of each F_n misses a positive area subset of the sphere. In general, $\{F_n\}$ is only *quasinormal* in the sense that a subsequence converges locally uniformly in $\Omega \setminus S$, where S is a finite set.

We assume that no point $p \in S$ is redundant, i.e. that $\{F_n\}$ is not normal in any neighbourhood of p . This implies that for any $p \in S$ and $r > 0$,

$$\mathcal{A}_{\hat{\mathbb{C}}}(F_n, B(p, r)) \geq 4\pi,$$

for all n sufficiently large. In particular, the cardinality of S is at most $\lfloor C/4\pi \rfloor$. We pass to a further subsequence so that the measures

$$\mu_n = F_n^* dA_{\hat{\mathbb{C}}} = \left(\frac{2|F'_n(z)|}{1 + |F_n(z)|^2} \right)^2 |dz|^2$$

converge weakly to a measure μ . The limiting measure μ may have point masses at the points of S . In [GKR20], Grahl, Kraus and Roth observed that $\mu(\{p\}) \geq 4\pi$ for any $p \in S$, and suggested that mass is quantized:

$$\mu(\{p\}) = 4\pi D, \quad \text{for some integer } D \geq 1.$$

It is intuitively clear that a part of the structure is lost when one takes the naive pointwise limit $F : \Omega \rightarrow \hat{\mathbb{C}}$. After reading their manuscript, the author showed the following theorem which describes the “full limit” of the meromorphic functions $F_n : \Omega \rightarrow \hat{\mathbb{C}}$:

Theorem 1.1. *After passing to a subsequence, the full limit of the F_n is naturally a meromorphic function defined on a multi-nodal surface X obtained by gluing trees of spheres to Ω at points of S . The full limit consists of:*

- A meromorphic function $F : \Omega \rightarrow \hat{\mathbb{C}}$, obtained as the pointwise limit of the functions $F_n : \Omega \rightarrow \mathbb{C}$ on $\Omega \setminus S$.

- Each sphere $\Sigma_k \subset X$, $k = 1, 2, \dots, N$, comes equipped with a rational function $\mathcal{R}_k : \Sigma_k \rightarrow \hat{\mathbb{C}}$ of degree $D_k \geq 1$. These rational functions are obtained as appropriate rescaling limits of the F_n and are determined uniquely up to pre-composition with affine maps $z \rightarrow az + b$ in $\text{Aut } \mathbb{C}$.

The limiting mass is $\mu(\{p\}) = 4\pi D$, where $D = \sum D_k$ is the sum of the degrees of the rational maps associated to spheres contained in trees attached at p .

Remark. A multi-nodal surface X is a topological space obtained by gluing a finite or countable collection of surfaces $\{\mathcal{S}_k\}$ at a discrete set of points, so that each point in X is contained in at most finitely surfaces \mathcal{S}_k . A point in X is called a *regular point* if it is contained in exactly one surface \mathcal{S}_k and a *multi-node* if it is contained in more than one surface. (In the literature, a *node* refers to a multi-node that is contained in exactly two surfaces. A *nodal surface* is a multi-nodal surface, where each multi-node is at worst a node.)

Actually, Theorem 1.1 was already known in much greater generality: it was proved for pseudo-holomorphic mappings into almost Kähler manifolds by Parker and Wolfson [PW93] and for quasiregular mappings by Pankka and Souto [PS19]. The locally univalent case has also been obtained by Li and Shafrir [LS94] using PDE techniques. Nevertheless, we hope that our elementary argument will be accessible to a wider audience as it involves only elementary notions from complex analysis and topology.

One definition from topology that will feature prominently in our argument is that of an *orientation-preserving branched cover* $\hat{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$: a continuous self-mapping of the sphere, which is a local homeomorphism outside of a finite set of points, where it is topologically equivalent to $z \rightarrow z^n$ for some $n \geq 2$. If the topological degree of \hat{F} is d , then \hat{F} has $2d - 2$ critical points counted with multiplicity (as is the case for a rational function of degree d) and every point $w \in \hat{\mathbb{C}}$ that is not a critical value is covered exactly d times. In particular, the spherical area of the image (counted with multiplicity) $\mathcal{A}_{\hat{\mathbb{C}}}(F, \hat{\mathbb{C}}) = 4\pi d$.

1.2 Removable sets

A compact set $E \subset \Omega$ is called *removable* for Dirichlet functions if any holomorphic function in $\mathcal{D}(\Omega \setminus E)$ extends to a holomorphic function in $\mathcal{D}(\Omega)$. Similarly, we say that a set $E \subset \Omega$ is removable for spherical Dirichlet functions if any meromorphic function in $\mathcal{F}(\Omega \setminus E)$ extends to a meromorphic function in $\mathcal{F}(\Omega)$.

In a classical work [AB50], Ahlfors and Beurling showed that a set is removable for Dirichlet functions if and only if it is a NED (negligible for extremal distance) set. A set E is NED if for any rectangle \mathcal{R} , the modulus of curves connecting a pair of opposite sides that avoid E computes the modulus of \mathcal{R} . NED sets have 2-dimensional Lebesgue measure zero and are totally disconnected, but the converse is far from true.

NED sets have a number of other characterizations, for example, a set E is NED if any conformal embedding $\mathbb{C} \setminus E \rightarrow \mathbb{C}$ is linear. In the second half of the paper, we show that the NED condition also characterizes removable sets for spherical Dirichlet functions:

Theorem 1.2. *A compact set E is removable for spherical Dirichlet functions if and only if it is NED.*

One direction is easy:

Lemma 1.3. *Suppose a compact set $E \subset \Omega$ is removable for spherical Dirichlet functions. Then, it is also removable for usual Dirichlet functions and therefore NED.*

Proof. We first notice that any \mathcal{F} -removable set E has 2-dimensional Lebesgue measure zero. Otherwise, one can find a non-trivial quasiconformal homeomorphism $w^\mu : \mathbb{C} \rightarrow \mathbb{C}$ whose dilatation is supported on E . As w^μ is injective, $\mathcal{A}_{\hat{\mathbb{C}}}(w^\mu, \Omega \setminus E) \leq 4\pi$ and $w^\mu \in \mathcal{F}(\Omega \setminus E)$. Since w^μ is conformal on $\Omega \setminus E$ but not conformal on Ω , it cannot extend to a meromorphic function on Ω .

Suppose F is a holomorphic function in $\mathcal{D}(\Omega \setminus E)$. If E is removable for the class \mathcal{F} , then F extends to a meromorphic function on Ω which satisfies

$$\int_{\Omega} |F'(z)|^2 |dz|^2 = \int_{\Omega \setminus E} |F'(z)|^2 |dz|^2 < \infty.$$

As the last condition is incompatible with F having poles, F is holomorphic on Ω . Thus, $F \in \mathcal{D}(\Omega)$ as desired. \square

The original argument of Ahlfors and Beurling which showed that NED sets are Dirichlet removable used a special identity involving conformal maps to slit domains, while a newer proof by Hedberg [Hed74] utilized the connection between the Dirichlet energy and the condenser capacity. Since these miraculous connections are unavailable in the spherical setting, it is unlikely that one can extend these approaches to spherical Dirichlet functions. Our proof is a slight improvement of the beautiful isoperimetric argument of Parker and Wolfson [PW93] who observed that points are removable for spherical Dirichlet functions.

In the case when F is defined on the complex plane, one can show that the singularity at infinity is removable using Nevanlinna theory:

Lemma 1.4. *Suppose $F : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ is a meromorphic function whose image has finite spherical area, counted with multiplicity:*

$$\mathcal{A}_{\hat{\mathbb{C}}}(F, \mathbb{C}) = \int_{\mathbb{C}} \left(\frac{2|F'(z)|}{1 + |F(z)|^2} \right)^2 |dz|^2 < \infty.$$

Then F is a rational function. If $\deg F = d$ then $\mathcal{A}_{\hat{\mathbb{C}}}(F, \mathbb{C}) = 4\pi d$.

Proof. From the Ahlfors-Shimizu interpretation of Nevanlinna theory [Hay64, Section 1.5], it is easy to see that the function F can attain any value $a \in \hat{\mathbb{C}}$ only a finite number of times. One merely needs to use the fact that the Nevanlinna counting function is bounded above by the Ahlfors-Shimizu characteristic:

$$\sum_{\substack{F(z)=a \\ |z|<r}} \log \frac{r}{|z|} \leq \frac{1}{4\pi} \int_0^r \mathcal{A}_{\hat{\mathbb{C}}}(F, B(0, t)) \cdot \frac{dt}{t} + O(1), \quad F(0) \neq a.$$

By Picard's theorem, the singularity at infinity cannot be essential, and therefore, is at worst a pole. Hence, F is a rational function. \square

We also mention a third approach to removability due to Chen and Li [CL91] who used the method of moving planes to prove Lemma 1.4 for locally univalent functions. While their result is unable to handle holomorphic functions with critical points, it is applicable to other non-linear PDEs and works in higher dimensions.

1.3 Rescaling limits

Suppose $F : \Omega \rightarrow \hat{\mathbb{C}}$ is a meromorphic function. A *rescaling* of F is a map of the form $F \circ m$ where m is a Möbius transformation.

Corollary 1.5. *Suppose that $\{F_n\}$ is a sequence of meromorphic functions on a domain $\Omega \subset \hat{\mathbb{C}}$ satisfying (1.3) and $\tilde{F}_n = F_n \circ m_n$ is a sequence of rescalings of $F_n : \Omega \rightarrow \hat{\mathbb{C}}$ defined on balls $B(0, R_n)$ with $R_n \rightarrow \infty$. After passing to a subsequence, the maps \tilde{F}_n converge locally uniformly to a rational function \mathcal{R} on \mathbb{C} minus a finite set of points.*

Proof. Since the spherical areas

$$\mathcal{A}_{\hat{\mathbb{C}}}(\tilde{F}_n, B(0, R)) = \mathcal{A}_{\hat{\mathbb{C}}}(F_n, m_n(B(0, R_n))) \leq C,$$

are uniformly bounded, the sequence \tilde{F}_n is quasi-normal on any ball $B(0, R)$. After passing to a subsequence, the \tilde{F}_n converge to a meromorphic function $\tilde{F} : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ satisfying

$$\mathcal{A}_{\hat{\mathbb{C}}}(\tilde{F}, \mathbb{C}) = \lim_{R \rightarrow \infty} \mathcal{A}_{\hat{\mathbb{C}}}(\tilde{F}, B(0, R)) \leq \lim_{R \rightarrow \infty} \left\{ \liminf_{n \rightarrow \infty} \mathcal{A}_{\hat{\mathbb{C}}}(\tilde{F}_n, B(0, R)) \right\} \leq C,$$

locally uniformly outside a finite set of cardinality at most $\lfloor C/4\pi \rfloor$. From Lemma 1.4, it follows that $\tilde{F} = \mathcal{R}$ is a rational function. \square

We say that a sequence of rescalings \tilde{F}_n is *trivial* if the limit \mathcal{R} is a constant function. The rational functions \mathcal{R}_k , $k = 1, 2, \dots, N$ in Theorem 1.1 will be obtained as rescalings of the sequence $\{F_n\}$ near points of S . One difficulty in Theorem 1.1 is to figure out where to rescale. This is a somewhat delicate matter: for instance, in the proof of Zalcman's lemma [Mar19], one rescales at local maxima of certain functions associated with the F_n . This approach is guaranteed to provide at least one rescaling limit, but in general, is unable to give the full set of non-trivial rescaling limits. The approach of Li and Shafrir [LS94] uses a similar idea and therefore suffers from the same drawback.

2 Mass quantization

Since the issue of bubbling is a local matter, by shrinking the domain Ω , we may assume that the set $S = \{p\}$ consists of a single point. For simplicity of exposition, we assume that the sequence F_n satisfies the following three conditions:

1. The limit function F is not constant,
2. F is holomorphic near $p \in S$,
3. p is not a critical point of F , i.e. $F'(p) \neq 0$.

We call such sequences of meromorphic functions *elementary*. In Section 4, we will explain how to handle the case of general sequences.

Lemma 2.1 (Mass quantization). *Suppose $F_n : \Omega \rightarrow \hat{\mathbb{C}}$ is an elementary sequence of meromorphic functions with $\mathcal{A}_{\hat{\mathbb{C}}}(F_n, \Omega) \leq C$. After passing to a subsequence, we may assume that the measures $\mu_n = F_n^* dA_{\hat{\mathbb{C}}}$ converge weakly to a measure μ on Ω . Then, $\mu(\{p\}) = 4\pi D$ for some integer $D \geq 1$.*

Proof. Choose $r > 0$ sufficiently small so that F is univalent in a neighbourhood of $\overline{B(p, r)}$. In particular, $\overline{B(p, r)} \subset \Omega$ contains no critical points of F . By discarding countably many radii, we may assume that each map F_n has no critical points on $\partial B(p, r)$, although it may have plenty of critical points inside $B(p, r)$.

We orient the boundary $\partial B(p, r)$ counter-clockwise, so that $F(\partial B(p, r))$ is a Jordan curve which winds counter-clockwise around $F(p)$. Since $F_n \rightarrow F$ uniformly on a neighbourhood of $\partial B(p, r)$, for n sufficiently large, $F_n(\partial B(p, r))$ will also be a Jordan curve which winds counter-clockwise around $F(p)$.

We extend $F|_{B(p, r)}$ to an orientation-preserving diffeomorphism $\hat{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ by selecting an orientation-preserving diffeomorphism $\hat{\mathbb{C}} \setminus B(p, r) \rightarrow \hat{\mathbb{C}} \setminus F(B(p, r))$ that agrees with F on $\partial B(p, r)$.

In a similar way, we extend each $F_n|_{B(p, r)}$ to an orientation-preserving branched cover $\hat{F}_n : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ by selecting an orientation-preserving diffeomorphism $\hat{\mathbb{C}} \setminus B(p, r) \rightarrow \hat{\mathbb{C}} \setminus F_n(B(p, r))$ that agrees with F_n on $\partial B(p, r)$. Since the F_n converge uniformly to F on $\partial B(p, r)$, one can choose the extensions $\hat{F}_n|_{\hat{\mathbb{C}} \setminus B(p, r)}$ which tend to $\hat{F}|_{\hat{\mathbb{C}} \setminus B(p, r)}$ uniformly in the spherical metric.

Let d_n be the topological degree of \hat{F}_n . Since \hat{F}_n is a $d_n : 1$ mapping of the Riemann sphere to itself,

$$\mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}_n, B(p, r)) + \mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}_n, \hat{\mathbb{C}} \setminus B(p, r)) = 4\pi d_n.$$

As $\mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}_n, \hat{\mathbb{C}} \setminus B(p, r)) < 4\pi$, the degrees d_n are uniformly bounded above. We pass to a subsequence for which $\{d_n\}$ is constant.

Since $d = 1$ is the topological degree of \hat{F} ,

$$\mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}, B(p, r)) + \mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}, \hat{\mathbb{C}} \setminus B(p, r)) = 4\pi d.$$

It is clear that the degree of \hat{F}_n can only drop in the limit, i.e. $d \leq d_n$, and bubbling occurs when there is a strictly inequality: $d < d_n$.

As

$$\mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}_n, \hat{\mathbb{C}} \setminus B(p, r)) \rightarrow \mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}, \hat{\mathbb{C}} \setminus B(p, r)),$$

the spherical area of $F_n(B(p, r))$ can only drop by a multiple of 4π , i.e.

$$\lim_{n \rightarrow \infty} \mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}_n, B(p, r)) = \mathcal{A}_{\hat{\mathbb{C}}}(\hat{F}, B(p, r)) + 4\pi D,$$

where $D = d_n - d$. The $4\pi D$ drop in area must also be accompanied with a drop in $2D$ critical points. Since the convergence $F_n \rightarrow F$ is uniform away from p , by Hurwitz theorem, these critical points must tend to p as $n \rightarrow \infty$. \square

3 Tree structure

In this section, we prove Theorem 1.1 for elementary sequences of meromorphic functions. Continuing the discussion from the previous section, we pass to a further subsequence so that the critical values of \hat{F}_n converge in $\hat{\mathbb{C}}$ and denote the limiting critical value set by \mathcal{V} . Pick a round ball $\mathcal{B} = B(w, \rho)$ so that

$$B(w, 2\rho) \subset \mathbb{C} \setminus (\mathcal{V} \cup F(B(p, r))).$$

The pre-image $\hat{F}_n^{-1}(\mathcal{B})$ consists of d_n connected components. Since exactly one of these connected components is outside $B(p, r)$, the remaining $d_n - 1$ components are compactly contained in $B(p, r)$, which we label $\mathcal{B}_1^{(n)}, \mathcal{B}_2^{(n)}, \dots, \mathcal{B}_{d_n-1}^{(n)}$.

By construction, F_n maps each topological disk $\mathcal{B}_i^{(n)}$ conformally onto \mathcal{B} and each Jordan curve $\gamma_i^{(n)} := \partial\mathcal{B}_i^{(n)}$ homeomorphically onto $\partial\mathcal{B}$. Let $z_i^{(n)}$ be the pre-image of w contained in $\mathcal{B}_i^{(n)}$ and $r_i^{(n)} := \text{diam } \mathcal{B}_i^{(n)}$. By Koebe's distortion theorem, the domains $\mathcal{B}_i^{(n)}$ are uniformly round: there exists constants $C_1, C_2 > 0$, independent of n , such that

$$B(z_i^{(n)}, C_1 \cdot r_i^{(n)}) \subset \mathcal{B}_i^{(n)} \subset B(z_i^{(n)}, C_2 \cdot r_i^{(n)}).$$

Clusters. Passing to a subsequence if necessary, the curves $\gamma_i^{(n)} := \partial\mathcal{B}_i^{(n)}$ become organized into *clusters*, with $\gamma_i^{(n)}$ and $\gamma_j^{(n)}$ in the same cluster if

$$\text{diam } \gamma_i^{(n)} \asymp \text{dist}(\gamma_i^{(n)}, \gamma_j^{(n)}) \asymp \text{diam } \gamma_j^{(n)}, \quad \text{as } n \rightarrow \infty.$$

In other words, we ask that the ratios

$$\frac{\text{dist}(\gamma_i^{(n)}, \gamma_j^{(n)})}{\text{diam } \gamma_j^{(n)}} \quad \text{and} \quad \frac{\text{diam } \gamma_i^{(n)}}{\text{diam } \gamma_j^{(n)}}$$

are bounded from 0 and ∞ .

Spheres. We let N denote the number of clusters. By Corollary 1.5, each cluster \mathcal{C}_j has a non-constant rescaling limit:

$$F_n(z_j^{(n)} + r_j^{(n)} z) \rightarrow \mathcal{R}_j(z),$$

where the convergence is uniform on $\mathbb{C} \setminus S_j$, where S_j is a finite set. Since the degree of \mathcal{R}_j is just the size of the cluster \mathcal{C}_j , the degrees of the rational functions \mathcal{R}_j add up to $d_n - 1 = D$.

From the construction, it is clear that the rational functions \mathcal{R}_j are uniquely determined up to pre-composition with affine maps in $\text{Aut } \mathbb{C}$.

Lemma 3.1. *The degrees of the rescaling limits account for the drop in the degree:*

$$\deg \mathcal{R}_1 + \deg \mathcal{R}_2 + \cdots + \deg \mathcal{R}_N = D.$$

Tree structure. In turn, the clusters are naturally organized into a tree of spheres structure. We say that the sphere $(\Sigma_i, \mathcal{R}_i)$ is a *descendant* of $(\Sigma_j, \mathcal{R}_j)$ if for any $\gamma_i \in \mathcal{C}_i$ and $\gamma_j \in \mathcal{C}_j$,

$$\frac{\text{diam } \gamma_i^{(n)}}{\text{diam } \gamma_j^{(n)}} \rightarrow 0, \quad \text{dist}(\gamma_i^{(n)}, \gamma_j^{(n)}) \lesssim \text{diam } \gamma_j^{(n)}, \quad \text{as } n \rightarrow \infty,$$

and write $(\Sigma_i, \mathcal{R}_i) \prec (\Sigma_j, \mathcal{R}_j)$. We view every sphere $(\Sigma_i, \mathcal{R}_i)$ as a descendant of (Ω, F) . The relation \prec is clearly transitive. We say that $(\Sigma_i, \mathcal{R}_i)$ is a *direct descendant* of $(\Sigma_j, \mathcal{R}_j)$ if there is no index k for which $(\Sigma_i, \mathcal{R}_i) \prec (\Sigma_k, \mathcal{R}_k) \prec (\Sigma_j, \mathcal{R}_j)$.

Assembly. If $(\Sigma_i, \mathcal{R}_i)$ is a direct descendant of $(\Sigma_j, \mathcal{R}_j)$, then we attach $(\Sigma_i, \mathcal{R}_i)$ to $(\Sigma_j, \mathcal{R}_j)$ by gluing the point at infinity in Σ_i to

$$p_{i,j} = \lim_{n \rightarrow \infty} \frac{z_i^{(n)} - z_j^{(n)}}{r_j^{(n)}} \in \Sigma_j.$$

Similarly, if $(\Sigma_i, \mathcal{R}_i)$ is a direct descendant of (Ω, F) , then we attach $(\Sigma_i, \mathcal{R}_i)$ to (Ω, F) by gluing the point at infinity in Σ_i to $p \in \Omega$.

As $\mathcal{R}_i(\infty) = \mathcal{R}_j(p_{i,j})$, the full limit $(F, \mathcal{R}_1, \dots, \mathcal{R}_N)$ is continuous on the multi-nodal surface X obtained by gluing Ω and the spheres $\Sigma_1, \Sigma_2, \dots, \Sigma_N$.

4 General sequences

We now explain how to handle general sequences of meromorphic functions F_n .

Q. What happens if F has a pole at p instead of being holomorphic?

A. We can instead work with $1/F_n$. Under this transformation, the spherical derivative remains unchanged:

$$\frac{2|(1/F_n)'|}{1 + |1/F_n|^2} = \frac{2|F_n'/F_n^2|}{1 + |1/F_n|^2} = \frac{2|F_n'|}{1 + |F_n|^2}.$$

Q. What happens if F has a critical point at p of order $a \geq 1$?

A. The proof proceeds as before, but this time, we cannot extend $F_n|_{B(p,r)}$ and $F|_{B(p,r)}$ to be one-to-one on $\hat{\mathbb{C}} \setminus B(p,r)$. When defining the extensions $\hat{F}_n|_{B(p,r)}$ and $\hat{F}|_{B(p,r)}$, we add a critical point of order a at infinity. The extensions are facilitated by the following lemma:

Lemma 4.1. (i) *Suppose $F : \partial B(p,r) \rightarrow \mathbb{C}$ is a C^1 immersed curve in the plane such that $F(p) \notin F(\partial B(p,r))$ and*

$$\frac{d}{d\theta} \arg(F(p + re^{i\theta}) - F(p)) > 0.$$

If $F(\partial B(p,r))$ winds $k = a + 1$ times counter-clockwise around $F(p)$, then F extends to a C^1 orientation-preserving branched cover $\hat{F} : \hat{\mathbb{C}} \setminus \overline{B(p,r)} \rightarrow \hat{\mathbb{C}}$ which is $k : 1$ in a neighbourhood of infinity and has no other critical points.

(ii) *Suppose F is a C^1 mapping defined on the closed ball $\overline{B(p,r)}$ whose restriction to $\partial B(p,r)$ satisfies the assumptions of part (i). Then F extends to a C^1 branched cover \hat{F} of the Riemann sphere.*

Sketch of proof. We may choose \hat{F} so that it maps the ray $[p + re^{i\theta}, \infty)$ diffeomorphically onto the ray $[F(p + re^{i\theta}), \infty)$. In part (ii), a little care needs to be taken so that the extension \hat{F} is C^1 on $\partial B(p,r)$. \square

The assumptions of the above lemma hold when $r > 0$ is sufficiently small. We point out some minor differences in the computation:

- \hat{F} is a rational map of degree $d = a + 1$.
- The degree of \hat{F}_n drops by $D = d_n - d = d_n - (a + 1)$.
- $B(p,r)$ contains $2D$ extra critical points of F_n , which disappear in the limit.
- $a + 1$ connected components of $\hat{F}_n^{-1}(\mathcal{B})$ are outside $B(p,r)$.
- $d_n - (a + 1) = D$ connected components of $\hat{F}_n^{-1}(\mathcal{B})$ are compactly contained in $B(p,r)$, so Lemma 3.1 remains true in this context.

Q. What happens if the limit F is a constant function?

A. Since bubbling is a local matter, we may assume that the domain Ω is bounded. We can then work with the perturbed sequence $F_{n,1}(z) = F_n(z) + z$. Since

$$\frac{2|F'_n(z) + 1|}{1 + |F_n(z) + z|^2} \asymp \frac{2|F'_n(z)|}{1 + |F_n(z)|^2},$$

when either quantity is large, the spherical areas $\mathcal{A}_{\hat{\mathbb{C}}}(F_n(z) + z, \Omega)$ are uniformly bounded if and only if the $\mathcal{A}_{\hat{\mathbb{C}}}(F_n, \Omega)$ are. The construction in Section 3 produces a tree of spheres $(\Sigma_k, \mathcal{R}_{k,1})$ for the perturbed sequence of meromorphic functions. The rational functions for the original sequence are simply $\mathcal{R}_k(z) = \mathcal{R}_{k,1}(z) - p$, where p is the point in Ω to which the branch of the tree containing Σ_k is attached. (When we zoom in near $p \in S$, the function z looks like a constant.)

Further remark. In the locally univalent case, Li and Shafrir [LS94] observed that all bubbles are simple (correspond to rational functions of degree 1) and are attached directly to Ω . In this case, the limit function has to be constant. This can be seen from the construction of the tree of spheres as the formation of non-simple bubbles and higher-order bubbles involves critical points. W. Chen [Che99] has constructed sequences of meromorphic functions without critical points which feature an arbitrary number of simple bubbles.

5 Preliminaries for removability

We now turn to the second half of the paper, where we show that NED sets are removable for spherical Dirichlet functions.

5.1 Notation

Let $E \subset \Omega$ be a compact set and $F \in \mathcal{F}(\Omega \setminus E)$ be a spherical Dirichlet function. For a rectifiable curve $\gamma \subset \Omega$, we denote the spherical length of its image counted with multiplicity by

$$L(\gamma) = \int_{\gamma \setminus E} \frac{2|F'(z)|}{1 + |F(z)|^2} |dz|.$$

We will write Γ for the domain enclosed by γ . Similarly, for an open set $U \subset \Omega$, we write

$$\mathcal{E}(U) = \int_{U \setminus E} \frac{4|F'(z)|^2}{(1 + |F(z)|^2)^2} \cdot |dz|^2.$$

5.2 NED sets

In [VG77, Theorem 3.1], Vodopyanov and Goldshtein showed that NED sets are removable for continuous $W_{\text{loc}}^{1,2}$ functions: if $F \in W_{\text{loc}}^{1,2}(\Omega \setminus E, \mathbb{R})$ extends continuously to Ω , then $F \in W_{\text{loc}}^{1,2}(\Omega, \mathbb{R})$. Another proof is given in [Nta21]. It is easy to see that the Vodopyanov-Goldshtein characterization of NED sets is also valid when one considers Sobolev spaces with values in the complex numbers \mathbb{C} or the Riemann sphere $\hat{\mathbb{C}}$.

Remark. (i) Since we are dealing with continuous functions, there is no difficulty in defining Sobolev spaces with values in the Riemann sphere by passing to coordinate charts.

(ii) From the Vodopyanov-Goldshtein characterization of NED sets, it follows that in order to show that a compact set $E \subset \Omega$ is removable for $\mathcal{F}(\Omega)$, it is enough to show that any spherical Dirichlet function $F \in \mathcal{F}(\Omega \setminus E)$ extends to a continuous function from Ω to $\hat{\mathbb{C}}$. Indeed, as NED sets have Lebesgue measure zero, the membership $F \in \mathcal{F}(\Omega \setminus E) \cap W_{\text{loc}}^{1,2}(\Omega, \hat{\mathbb{C}})$ implies that F is weakly holomorphic (in the sense of distributions) on Ω , so that by Weyl's lemma, F is strongly holomorphic on Ω .

One of our main tools will be a lemma due to Ntalampekos which allows one to perturb curves off NED sets. Recall that a family of curves $\Gamma_0 \subset \Omega$ has modulus zero, if for any $\varepsilon > 0$, there exists a measurable function $\lambda : \Omega \rightarrow [0, \infty)$ such that

$$\int_{\Omega} \lambda(z)^2 |dz|^2 < \varepsilon \quad \text{and} \quad \int_{\gamma} \lambda(z) |dz| \geq 1, \quad \gamma \in \Gamma_0.$$

Lemma 5.1 (Ntalampekos perturbation lemma). *Let $\Omega \subset \mathbb{C}$ be a domain in the plane and γ be a rectifiable path in Ω that lies outside an exceptional curve family $\Gamma_0 = \Gamma_0(F)$ of modulus zero. There exists a path $\hat{\gamma}$ with the same endpoints as γ that avoids the set E (except possibly, at the endpoints) with*

$$d_{\text{Haus}}(\gamma, \hat{\gamma}) < \varepsilon, \quad |L(\hat{\gamma}) - L(\gamma)| < \varepsilon.$$

If γ is a Jordan curve, then one may choose the perturbed path $\hat{\gamma} \subset \Omega \setminus E$ to be a smooth Jordan curve.

The above statement is a consequence of [Nta21, Theorem 7.1, (I) \Leftrightarrow (VI)] with

$$\rho(z) = \frac{2|F'(z)|}{1 + |F(z)|^2} \cdot \chi_{\Omega \setminus E}(z).$$

Remark. (i) Since the metric ρ is continuous on $\Omega \setminus E$, once we perturb a Jordan curve γ off E , it is easy to perturb γ further to obtain a smooth curve.

(ii) Lemma 5.1 implies that NED sets are totally disconnected, however, the NED property is much more restrictive. For instance, by [AB50, Theorem 10], NED sets are *metrically removable*: any two points $x, y \in \mathbb{C}$ can be joined by a curve which avoids the set E (except possibly, at the endpoints) whose Euclidean length is arbitrarily close to $|x - y|$. One can also deduce the metric removability of NED sets from [Nta21, Theorem 7.1] with $\rho = \chi_{\mathbb{C}}$. For more information on metrically removable sets, we refer the reader to the work of Kalmykov, Kovalev and Rajala [KKR19].

5.3 Length-area estimates

We will use the following classical length-area estimate:

Lemma 5.2. *Suppose $A = A(p; s, r) = \{z \in \mathbb{C} : r < |z| < s\}$ is a round annulus contained in Ω of modulus $m = (1/2\pi) \log(r/s)$. There exists a simple closed curve $\gamma \notin \Gamma_0$ that separates the two boundary components such that*

$$L(\gamma)^2 \leq C_1(m) \cdot \mathcal{E}(A).$$

Similarly, there exists a simple curve $\delta \notin \Gamma_0$ that connects the two boundary components with

$$L(\delta)^2 \leq C_2(m) \cdot \mathcal{E}(A).$$

The constants $C_1(m)$ and $C_2(m)$ are uniform as m ranges over a compact subset of $(0, \infty)$. The curve γ can be taken to be a circle $\partial B(p, \rho)$ with $s < \rho < r$ while δ can be taken to be a radial line segment $[se^{i\theta}, re^{i\theta}]$.

The above lemma allows us to surround NED sets by short multicurves:

Lemma 5.3. *Let $E \subset \Omega$ be a compact NED set. For any $\varepsilon > 0$, one can find a finite collection of smooth Jordan curves $\gamma_1, \gamma_2, \dots, \gamma_n \subset \Omega \setminus E$ such that:*

- (i) *the domains Γ_i have disjoint closures and cover E ,*
- (ii) $\sum_i L(\gamma_i)^2 < \varepsilon.$
- (iii) $\sum_i \mathcal{E}(\Gamma_i) < \varepsilon.$

If U is an open set containing E , we can choose the curves γ_i to lie in U .

Proof. Since NED sets have measure zero, by shrinking U if necessary, we may assume that $\mathcal{E}(U) < \varepsilon$. We then choose $\delta > 0$ so that $\text{dist}(E, \partial U) \geq 4\delta$.

Let \mathcal{A} be the collection of round annuli $A(p; 2\delta, 3\delta) \subset U$ with $p \in \delta\mathbb{Z}^2$. For each annulus $A \in \mathcal{A}$, we can use Lemma 5.2 to select a simple curve $\eta_A \subset A$ that separates the two boundary components of A with $L(\eta_A)^2 \leq C_1 \cdot \mathcal{E}(A)$. Using Lemma 5.1 to perturb the curves η_A if necessary, we may assume that they don't pass through E .

Since a point is contained in a bounded number of annuli $A \in \mathcal{A}$,

$$\sum_{A \in \mathcal{A}} L(\eta_A)^2 \leq C_1 \sum_{A \in \mathcal{A}} \mathcal{E}(A) < C_2 \varepsilon.$$

Let $\{\Gamma_i\}$ be the bounded complementary components of $\mathbb{C} \setminus \bigcup_{A \in \mathcal{A}} \eta_A$ which contain a point of E . Since E is a compact set, the collection $\{\Gamma_i\}$ is finite. Let n be its cardinality.

Since each $\gamma_i = \partial\Gamma_i$, $i = 1, 2, \dots, n$, is composed of arcs from a bounded number of η_A ,

$$\sum_{i=1}^n L(\gamma_i)^2 < C_3 \varepsilon.$$

By shrinking the domains Γ_i slightly (and changing $L(\gamma_i)$ and $\mathcal{E}(\Gamma_i)$ by an arbitrarily small amount), we can make the γ_i smooth and $\overline{\Gamma_i}$ disjoint. For instance, one can form the Riemann maps $\varphi_i : \mathbb{D} \rightarrow \Gamma_i$ and replace γ_i with $\gamma_{i,r} = \varphi_i(\partial\mathbb{D}_r)$ for some r sufficiently close to 1. Indeed, as the curves $\{\gamma_i\}$ are rectifiable, the associated Riemann maps $\{\varphi_i\}$ have derivative in the Hardy space H^1 and the $L(\gamma_{i,r})$ converge to $L(\gamma_i)$.

To complete the proof, we replace ε by $\min(\varepsilon, \varepsilon/C_3)$ to get (ii). □

5.4 An isoperimetric estimate

Let $\omega = \frac{4dx \wedge dy}{(1+|x|^2+|y|^2)^2}$ denote the volume form on the sphere $\hat{\mathbb{C}}$. For a point $p \in \hat{\mathbb{C}}$, let

- U_p be the hemisphere centered at p ,
- p^* be its diametrically opposite point.

Since $\hat{\mathbb{C}} \setminus \{p^*\}$ is contractible, there is a 1-form β_p on $\hat{\mathbb{C}} \setminus \{p^*\}$ so that $\omega = d\beta_p$. By subtracting a 1-form with constant coefficients from β_p , we may assume that β_p vanishes at p , so that

$$\|\beta_p(x)\| \leq C_p \operatorname{dist}_{\hat{\mathbb{C}}}(p, x), \quad x \in U_p.$$

In particular, if η is a curve passing through p and is contained in U_p , then

$$\left| \int_{\eta} \beta_p \right| \leq C_p \cdot \ell_{\hat{\mathbb{C}}}(\eta)^2, \quad (5.1)$$

where $\ell_{\hat{\mathbb{C}}}(\eta)$ is the length of η as measured in the spherical metric $\rho_{\hat{\mathbb{C}}} = \frac{2|dz|}{1+|z|^2}$. If we select the 1-forms $\{\beta_p\}_{p \in \hat{\mathbb{C}}}$ in an $\mathrm{SO}(3)$ -invariant way, then the constant C_p in (5.1) will be independent of p .

Lemma 5.4. *Suppose $\Gamma \setminus \bigcup_{i=1}^n \bar{\Gamma}_i \subset \mathbb{C}$ is a domain with smooth Jordan boundary and F is a meromorphic function defined on a neighbourhood of $\Gamma \setminus \bigcup_{i=1}^n \bar{\Gamma}_i$. Then,*

$$\operatorname{dist}\left(\mathcal{E}\left(\Gamma \setminus \bigcup \bar{\Gamma}_i\right), 4\pi\mathbb{Z}\right) \leq C \left(\sum L(\gamma_i)^2 + L(\gamma)^2\right),$$

where $C > 0$ is a universal constant.

In the proof below, we think of $F(\gamma)$ and $F(\gamma_i)$ as parametrized curves and $F(\Gamma \setminus \bigcup \bar{\Gamma}_i)$ as a parametrized surface immersed in the sphere.

Proof. We may assume that $L(\gamma)$ and $L(\gamma_i)$, $i = 1, 2, \dots, n$, are less than $\pi/2$, otherwise there is nothing to prove. Pick arbitrary points p and p_i on $F(\gamma)$ and $F(\gamma_i)$ respectively. By construction, $F(\gamma)$ and $F(\gamma_i)$ are contained in hemispheres U and U_i centered at p and p_i respectively.

Perturbing the curves γ and γ_i if necessary, we may assume that they do not pass through the critical points of F , so that $F(\gamma)$ and $F(\gamma_i)$ are smooth immersed curves.

Choose smooth immersed disks $D \subset U$ with $\partial D = F(\gamma)$ and $D_i \subset U_i$ with $\partial D_i = F(\gamma_i)$, $i = 1, 2, \dots, n$. Then,

$$S = F\left(\Gamma \setminus \bigcup_{i=1}^n \overline{\Gamma}_i\right) \cup D \cup \bigcup_{i=1}^n D_i$$

is a closed immersed surface in the sphere. Since the homology class of S is an integral multiple of the homology class of the sphere,

$$\int_{F(\Gamma \setminus \bigcup_{i=1}^n \overline{\Gamma}_i)} \omega + \int_D \omega + \sum_{i=1}^n \int_{D_i} \omega = 4\pi k,$$

for some $k \in \mathbb{Z}$. By Stokes theorem,

$$4\pi k - \mathcal{E}\left(\Gamma \setminus \bigcup_{i=1}^n \overline{\Gamma}_i\right) = \int_{F(\gamma)} \beta + \sum_{i=1}^n \int_{F(\gamma_i)} \beta_i. \quad (5.2)$$

It remains to bound the terms on the right hand side of (5.2) using (5.1). □

6 An energy estimate

In this section, we show the following estimate on the decay of energy:

Lemma 6.1. *Let $F \in \mathcal{F}(\Omega \setminus E)$ be a spherical Dirichlet function. If $B(p, r) \subset \Omega$ and $\mathcal{E}(B(p, r)) \leq 2\pi$, then*

$$\mathcal{E}(B(p, s)) \leq (s/r)^\alpha \cdot \mathcal{E}(B(p, r)), \quad 0 < s < r, \quad (6.1)$$

where $\alpha > 0$ is a universal constant.

The proof of Lemma 6.1 uses the following isoperimetric bound:

Lemma 6.2. *Suppose $\gamma \subset \Omega$ is a Jordan curve with $\mathcal{E}(\Gamma) < 2\pi$. If $\gamma \notin \Gamma_0$ is not exceptional, then $\mathcal{E}(\Gamma) \lesssim L(\gamma)^2$.*

Proof. By the Ntalampekos perturbation lemma, we may assume that $\gamma \subset \Omega \setminus E$ is a smooth Jordan curve. For any $\varepsilon > 0$, we can use Lemma 5.3 to produce a collection of smooth Jordan domains $\{\Gamma_i\}$ with disjoint closures such that

$$E \cap \Gamma \subset \bigcup_{i=1}^n \Gamma_i \subset \Gamma, \quad \sum_{i=1}^n \mathcal{E}(\Gamma_i) < \varepsilon, \quad \sum_{i=1}^n L(\gamma_i)^2 < \varepsilon.$$

By Lemma 5.4, we have

$$\mathcal{E}\left(\Gamma \setminus \bigcup_{i=1}^n \overline{\Gamma_i}\right) \lesssim L(\gamma)^2 + \sum_{i=1}^n L(\gamma_i)^2 \leq L(\gamma)^2 + \varepsilon,$$

so that $\mathcal{E}(\Gamma) = \mathcal{E}\left(\Gamma \setminus \bigcup_{i=1}^n \overline{\Gamma_i}\right) + \sum_{i=1}^n \mathcal{E}(\Gamma_i) \lesssim L(\gamma)^2 + 2\varepsilon$. The lemma follows since $\varepsilon > 0$ was arbitrary. \square

Following [PW93], the proof of Lemma 6.1 runs as follows:

Proof of Lemma 6.1. Suppose $0 < \rho < r$. By Lemma 6.2 and the Cauchy-Schwarz inequality,

$$\begin{aligned} \mathcal{E}(B(p, \rho)) &\lesssim \left(\int_{\partial B(p, \rho) \setminus E} \frac{2|F'(z)|}{1 + |F(z)|^2} |dz| \right)^2 \\ &\leq 2\pi\rho \int_{\partial B(p, \rho) \setminus E} \frac{4|F'(z)|^2}{(1 + |F(z)|^2)^2} |dz| \\ &\lesssim \rho \cdot \frac{d\mathcal{E}(B(p, \rho))}{d\rho}, \end{aligned}$$

provided that $\partial B(p, \rho) \not\subset \Gamma_0$ is not exceptional. Rearranging, we get

$$\frac{d}{d\rho} \log \mathcal{E}(B(p, \rho)) \geq \frac{\alpha}{\rho},$$

for some $\alpha > 0$. Since almost every circle $\partial B(p, \rho)$ is not exceptional, we can integrate with respect to ρ to obtain (6.1). \square

7 Continuity

As explained in Section 5.2, in order to show that NED sets are removable for spherical Dirichlet functions (Theorem 1.2), it is enough to show:

Lemma 7.1. *Suppose $\Omega \subset \mathbb{C}$ is a domain in the plane and $E \subset \Omega$ is a compact NED set. Any spherical Dirichlet function $F \in \mathcal{F}(\Omega \setminus E)$ is continuous as a map from Ω to the Riemann sphere.*

In fact, the argument below will show that F is locally Hölder continuous with exponent $\alpha/2$, where α is the constant from Lemma 6.1.

Lemma 7.2. *Given a point $p \in \Omega$ such that $B(p, 2r) \subset \Omega$ and $\mathcal{E}(B(p, 2r)) \leq 2\pi$, one can find two families of curves $\{\hat{\gamma}_n\}_{n=0}^\infty$ and $\{\hat{\ell}_n\}_{n=0}^\infty$ in $B(p, r) \setminus E$ such that:*

1. $\hat{\gamma}_n$ is a Jordan curve contained in the annulus $A(p; 0.9r/2^n, r/2^n)$, and separates its two boundary components.
2. $\hat{\ell}_n$ is a Jordan arc which connects $\hat{\gamma}_n$ and $\hat{\gamma}_{n+1}$, and except for its endpoints, lies in the topological annulus bounded by these two curves.
3. $L(\hat{\gamma}_n) \lesssim 2^{-\alpha n/2}$ and $L(\hat{\ell}_n) \lesssim 2^{-\alpha n/2}$.

Proof. By Lemmas 5.2 and 6.1, there exists a sequence of nested concentric circles $\gamma_n = \partial B(p, r_n)$, $n = 0, 1, 2, \dots$, with $0.9r/2^n < r_n < r/2^n$ such that

$$L(\gamma_n) \lesssim \sqrt{\mathcal{E}(B(p, r/2^{n-1}))} \lesssim 2^{-\alpha n/2},$$

as well as a sequence of line segments

$$\ell_n = [p + (0.9r/2^{n+1})e^{i\theta_n}, p + (r/2^n)e^{i\theta_n}]$$

with

$$L(\ell_n) \lesssim \sqrt{\mathcal{E}(B(p, r/2^{n-1}))} \lesssim 2^{-\alpha n/2}.$$

We first apply the Ntalampekos perturbation lemma to the circles γ_n to produce the desired Jordan curves $\hat{\gamma}_n \in A(p; 0.9r/2^n, r/2^n) \setminus E$. We then apply the Ntalampekos perturbation lemma to the line segments ℓ_n to obtain Jordan arcs $\tilde{\ell}_n \in \Omega \setminus E$ with the same endpoints, and obeying the same estimate. Since the arc $\tilde{\ell}_n$ runs from $\partial B(p, r/2^n)$ to $\partial B(p, 0.9r/2^{n+1})$, a sub-arc $\hat{\ell}_n$ connects $\hat{\gamma}_n$ and $\hat{\gamma}_{n+1}$. \square

Proof of Lemma 7.1. Fix a ball $B(q, r/2) \subset \Omega$ with $\mathcal{E}(B(q, 5r/2)) \leq 2\pi$. Suppose p, p' are two points in $B(q, r/2) \setminus E$. In order to estimate the spherical distance between $F(p)$ and $F(p')$, we connect $F(p), F(p')$ by a curve. To that end, we use Lemma 7.2 to form four families of curves $\{\hat{\gamma}_n\}_{n=0}^\infty$, $\{\hat{\ell}_n\}_{n=0}^\infty$, $\{\hat{\gamma}'_n\}_{n=0}^\infty$ and $\{\hat{\ell}'_n\}_{n=0}^\infty$ associated to the pairs (p, r) and (p', r) .

Since the Euclidean distance $s = |p - p'|$ between p and p' is less than r , there is a unique integer $m \geq 0$ so that

$$\frac{r}{2^{m+1}} \leq |p - p'| < \frac{r}{2^m}.$$

It is easy to see the curves

$$\hat{\gamma}_m \subset A(p; 0.9r/2^m, r/2^m) \quad \text{and} \quad \hat{\gamma}'_m \subset A(p'; 0.9r/2^m, r/2^m)$$

intersect: they both enclose the midpoint of the line segment $[p, p']$ but neither curve encloses the other. Pick an arbitrary point $p'' \in \hat{\gamma}_m \cap \hat{\gamma}'_m$ in the intersection. Concatenating pieces of curves from these four families (with indices $\geq m$) produces a curve $\gamma_{p \leftrightarrow p'} \subset \Omega$ which joins p to p' and passes through p'' with $L(\gamma_{p \leftrightarrow p'}) \lesssim (s/r)^{\alpha/2}$. The proof is complete. \square

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