

Critical structures of inner functions

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Abstract

A celebrated theorem of M. Heins says that up to post-composition with a Möbius transformation, a finite Blaschke product is uniquely determined by its critical points. K. Dyakonov suggested that it may be interesting to extend this result to infinite degree, however, one needs to be careful since different inner functions may have identical critical sets. In this work, we try parametrizing inner functions by 1-generated invariant subspaces of the weighted Bergman space A_1^2 . Our technique is based on the Liouville correspondence which provides a bridge between complex analysis and non-linear elliptic PDE.

1 Introduction

A finite Blaschke product $F(z)$ is a holomorphic self-map of the unit disk \mathbb{D} which extends to a continuous dynamical system on the unit circle $\mathbb{S}^1 = \partial\mathbb{D}$. The most common way to study Blaschke products is by examining their zero sets. It is not difficult to show that a finite Blaschke product $F(z)$ is uniquely determined by its zero set up to a rotation:

$$F(z) = e^{i\theta} \prod_{i=1}^d \frac{z - a_i}{1 - \overline{a_i}z}, \quad a_1, a_2, \dots, a_d \in \mathbb{D},$$

where $d \geq 1$ is the degree of F . This approach allows one to factor zeros of bounded analytic functions and leads to Beurling's invariant subspace theorem, which is one of the cornerstones of modern complex analysis and function theory.

In this work, we follow a less traveled path of examining the *critical sets* of Blaschke products, initiated by M. Heins [4] in 1962.

Theorem 1.1 (Heins). *A finite Blaschke product is uniquely determined by the set of its critical points up to post-composition with a Möbius transformation $m \in \text{Aut}(\mathbb{D})$, and furthermore, any set of $d - 1$ points in the unit disk arises as the critical set of some Blaschke product of degree d .*

Loosely speaking, an *inner function* is a holomorphic self-map of the unit disk which extends to a measure-theoretic dynamical system of the unit circle. More precisely, we want the radial boundary values to exist almost everywhere and have absolute value one.

If one wants to generalize Heins' result to the set Inn of inner functions, one is confronted with the following obstacle: different inner functions can have the same critical set. For example,

$$F_1(z) = z, \quad F_2(z) = \exp\left(\frac{z+1}{z-1}\right)$$

have no critical points. In order to distinguish F_1 and F_2 , one must record some additional information. In [5], the author explained how to parametrize inner functions of finite entropy (with derivative in the Nevanlinna class), answering a question posed by K. Dyakonov [1, 2]:

Theorem 1.2. *Let \mathcal{J} be the set of inner functions whose derivative lies in the Nevanlinna class. The natural map*

$$F \rightarrow \text{Inn}(F') \quad : \quad \mathcal{J} / \text{Aut}(\mathbb{D}) \rightarrow \text{Inn} / \mathbb{S}^1$$

is injective. The image consists of all inner functions of the form BS_μ where B is a Blaschke product and S_μ is the singular factor associated to a measure μ whose support is contained in a countable union of Beurling-Carleson sets.

By definition, a *Beurling-Carleson set* $E \subset \partial\mathbb{D}$ is a closed subset of the unit circle of zero Lebesgue measure whose complement is a union of arcs $\bigcup_k I_k$ with

$$\kappa(E) = \sum |I_k| \log \frac{1}{|I_k|} < \infty.$$

1.1 Beurling's theorem

It would be desirable to parametrize all inner functions by their critical structure, not just the relatively small subset \mathcal{J} . To understand how this might be done, let us see how one can parametrize *zero structures* of inner functions. The following statement expresses the fact that zero sets of functions in the Hardy space H^2 are Blaschke sequences:

$$\text{BP} / \mathbb{S}^1 = \{\text{zero-based subspaces of } H^2\}.$$

By definition, a (closed) subspace $X \subset H^2$ is *zero-based* if it is defined as the collection of functions in H^2 that vanish at a prescribed set of points. Taking the “closure” of the above statement, we arrive at the famous theorem of Beurling:

Theorem 1.3 (A. Beurling, 1949).

$$\begin{aligned} \text{Inn} / \mathbb{S}^1 &= \overline{\{\text{zero-based subspaces of } H^2\}} \\ &= \{\text{invariant subspaces of } H^2\}. \end{aligned}$$

The collection of closed subspaces of a Banach space carries a natural topology where $X_n \rightarrow X$ if for any convergent sequence $x_n \rightarrow x$ with $x_n \in X_n$, the limit $x \in X$, and conversely, any $x \in X$ can be approximated by a convergent sequence $x_n \rightarrow x$ with $x_n \in X_n$. The process of taking closure has been given the beautiful name *approximate spectral synthesis* by N. K. Nikol'skii [10, p. 34].

1.2 Critical structures of inner functions

Let $(H^2)'$ denote the space of derivatives of H^2 functions. According to the Littlewood-Paley identity,

$$\|f\|_{H^2}^2 = |f(0)|^2 + \frac{1}{2\pi} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} |dz|^2, \quad (1.1)$$

one can identify $(H^2)'$ with the *weighted Bergman space* A_1^2 which is the collection of all holomorphic function on the unit disk that satisfy

$$\|f\|_{A_1^2} = \left(\frac{2}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2) |dz|^2 \right)^{1/2} < \infty.$$

The starting point for our discussion is a beautiful result of Kraus [6] which says that critical sets of Blaschke products coincide with critical sets of H^2 functions:

$$\text{MBP} / \text{Aut}(\mathbb{D}) = \{\text{zero-based subspaces of } A_1^2\}.$$

The exact definition of the class MBP of *maximal Blaschke products* will be given later. At this moment, we only need to know that maximal Blaschke products enjoy the following three properties:

1. If C is a critical set of some Blaschke product, then it is the critical set of some maximal Blaschke product.
2. A maximal Blaschke product is uniquely determined by its critical set up to post-composition with a Möbius transformation in $\text{Aut}(\mathbb{D})$.
3. Maximal Blaschke products are indestructible: if $F \in \text{MBP}$, then $m \circ F \in \text{MBP}$ for any $m \in \text{Aut}(\mathbb{D})$.

For a function $H \in A_1^2$, the *subspace generated by H* is defined as the minimal closed subspace of A_1^2 which contains H and is invariant under multiplication by z :

$$[H] = \overline{\{Hp : p \text{ polynomial}\}}.$$

In the papers [12, 13], S. Shimorin showed that the closure of the zero-based subspaces in A_1^2 consists of subspaces that can be generated by a single function. In light of Shimorin's result, we can try to write down the "closure" of Kraus' theorem:

Conjecture 1.4. *If an invariant subspace of A_1^2 can be generated by a single function, it can be generated by the derivative of an essentially unique inner function.*

While we are not able to fully resolve this conjecture, we can show that any 1-generated invariant subspace of A_1^2 can be generated by the derivative of a bounded function. Previously, it was known that the generator could be chosen to be the derivative of a BMO function, see [3, Theorem 3.3].

1.3 Liouville's theorem

Given a conformal pseudometric $\lambda(z)|dz|$ on the unit disk with an upper semicontinuous density, its *Gaussian curvature* is given by

$$k_\lambda = -\frac{\Delta \log \lambda}{\lambda^2},$$

where the Laplacian is taken in the sense of distributions. It is well known that the Poincaré metric $\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$ has constant curvature -1 , the Euclidean metric $\lambda_{\mathbb{C}}(z) = 1$ is flat, while the spherical metric $\lambda_{\widehat{\mathbb{C}}}(z) = \frac{2}{1+|z|^2}$ has constant curvature $+1$. For a holomorphic self-map of the unit disk $F \in \text{Hol}(\mathbb{D}, \mathbb{D})$, consider the pullback

$$\lambda_F := F^* \lambda_{\mathbb{D}} = \frac{2|F'|}{1-|F|^2}.$$

Since curvature is a conformal invariant, $k_{\lambda_F} = -1$ on $\mathbb{D} \setminus \text{crit}(F)$ where $\text{crit}(F)$ denotes the critical set of F . However, on the critical set, $\lambda_F = 0$ while its curvature has δ -masses. Introducing the change of variables $u_F = \log \lambda_F$, we naturally arrive at the PDE

$$\Delta u = e^{2u} + 2\pi\nu, \quad \nu \geq 0, \tag{1.2}$$

where $\nu = \sum_{c \in \text{crit}(F)} \delta_c$ is an integral sum of point masses.

A theorem of Liouville [7, Theorem 5.1] states that the correspondence $F \rightarrow u_F$ is a bijection between

$$\text{Hol}(\mathbb{D}, \mathbb{D}) / \text{Aut}(\mathbb{D}) \iff \{\text{solutions of (1.2) with } \nu \text{ integral}\}.$$

Liouville's theorem allows one to interpret properties of holomorphic self-maps of the disk as properties of solutions of the Gauss curvature equation. For example, Heins theorem states that a “finite Blaschke product” is a holomorphic self-map of the disk for which “ $u_F \rightarrow \infty$ as $|z| \rightarrow 1$ ” while Theorem 1.2 states that an “inner function of finite entropy” corresponds to a “*nearly-maximal solution* of the Gauss curvature equation” for which

$$\limsup_{r \rightarrow 1} \int_{|z|=r} (u_{\mathbb{D}} - u) d\theta < \infty. \tag{1.3}$$

1.4 Weighted Gauss Curvature Equation

In the present work, we will be mostly working with the weighted Gauss curvature equation

$$\Delta u = |H|^2 e^{2u} \quad (\text{GCE}_H)$$

where $H \in A_1^2(\mathbb{D})$, introduced by Kraus [6]. At first glance, it seems that little is gained by working with GCE_H since it is equivalent to the usual Gauss curvature equation: u is a solution of $\Delta u = |H|^2 e^{2u}$ if and only if $v = u + \log |H|$ is a solution of

$$\Delta v = e^{2v} + 2\pi \sum_{c \in \mathcal{Z}(H)} \delta_c, \quad (1.4)$$

where $\mathcal{Z}(H)$ denotes the zero set of H counted with multiplicity. Nevertheless, it turns out that GCE_H is easier to work with than (1.4) since it features a holomorphic function instead of a measure.

We now give a brief summary of the argument in [6] that the zero set of a function $H \in A_1^2(\mathbb{D})$ is a critical set of some Blaschke product (the other direction is trivial). Kraus first noticed that GCE_H has at least one solution, see Theorem 2.1 below. In this case, GCE_H has a maximal solution $u_{H,\max}$ which dominates all other solutions pointwise. Liouville's theorem implies that

$$u_{H,\max}(z) = \log \frac{1}{|H(z)|} \cdot \frac{2|F'(z)|}{1 - |F(z)|^2}$$

for some holomorphic function $F : \mathbb{D} \rightarrow \mathbb{D}$, whose critical set contains the zero set of H . We can decompose $F = BSO$ into a product of a Blaschke factor, a singular inner factor and an outer factor. In [6], Kraus gave a very elegant argument that uses the maximality of $u_{H,\max}$ to rule out the existence of non-trivial singular and outer factors. Incidentally, F is the maximal Blaschke product with critical set C alluded to earlier.

In this paper, we initiate the study of *canonical solutions* of GCE_H . The advantage of the canonical solution over the maximal solution is that its Liouville map I_H records more information about H than its zero set. As the reader may guess, I_H is an inner function whose derivative has the same zero set as H and $I'_H \in [H]$. In fact, I_H only depends on the invariant subspace generated by H .

2 Background in PDE

In this section, we study the weighted Gaussian curvature equation (GCE_H)

$$\begin{cases} \Delta u = |H|^2 e^{2u} & \text{in } \mathbb{D}, \\ u = h, & \text{on } \partial\mathbb{D}, \end{cases} \quad (2.1)$$

for $H \in A_1^2(\mathbb{D}) \setminus \{0\}$. For a function u to be considered a *solution*, we require:

1. For any $\phi \in C_c^\infty(\mathbb{D})$,

$$\int_{\mathbb{D}} u \Delta \phi |dz|^2 = \int_{\mathbb{D}} |H|^2 e^{2u} \phi |dz|^2. \quad (2.2)$$

2. As $r \rightarrow 1$, the measures $u(re^{i\theta})d\theta \rightarrow h d\theta$ converge in the weak-* topology. (Unless otherwise specified, we interpret boundary values in this way.)

By analogy with subharmonic functions, we say that u is a *subsolution* if $\Delta u \geq |H|^2 e^{2u}$ in the sense of distributions and a *supersolution* if $\Delta u \leq |H|^2 e^{2u}$.

Theorem 2.1. *For $H \in A_1^2(\mathbb{D})$, (2.1) admits a unique solution for any $h \in L^\infty(\partial\mathbb{D})$. The solution is increasing in h : if $h_1 \leq h_2$, then $u_1 \leq u_2$.*

Uniqueness and monotonicity follow from Kato's inequality [11, Proposition 6.9] which states that if $u \in L_{\text{loc}}^1$ and $\Delta u \geq f$ in the sense of distributions with $f \in L_{\text{loc}}^1$, then $\Delta u^+ \geq f \cdot \chi_{u>0}$. As usual, $u^+ = \max(u, 0)$ denotes the positive part of u .

Proof of Theorem 2.1: uniqueness and monotonicity. By Kato's inequality,

$$\Delta(u_1 - u_2)^+ \geq |H|^2 (e^{2u_1} - e^{2u_2}) \cdot \chi_{\{u_1 > u_2\}} \geq 0$$

is a subharmonic function. The inequality $h_1 \leq h_2$ implies that $(u_1 - u_2)^+$ has zero boundary values. The maximum principle shows that $(u_1 - u_2)^+ \leq 0$ or $u_1 \leq u_2$. The same argument also proves uniqueness. \square

Existence is a standard application of Schauder's fixed point theorem, although for the convenience of the reader, we spell out the details. As usual, $G(z, \zeta) = \log \left| \frac{1-z\bar{\zeta}}{z-\zeta} \right|$ denotes the Green's function of the unit disk. We say that a measure μ on the unit disk is a *Blaschke measure* if $(1-|z|)d\mu(z)$ is a finite measure. The following lemma is well-known:

Lemma 2.2. *If μ is a Blaschke measure on the unit disk, then*

$$G_\mu(z) = \frac{1}{2\pi} \int_{\mathbb{D}} G(z, \zeta) d\mu(\zeta)$$

satisfies

$$\|G_\mu(z)\|_{W_0^{1,1}(\mathbb{D})} \leq C \int_{\mathbb{D}} (1 - |z|) d\mu(z),$$

and solves the linear Dirichlet problem

$$\begin{cases} \Delta u = -\mu, & \text{in } \mathbb{D}, \\ u = 0, & \text{on } \partial\mathbb{D}, \end{cases} \quad (2.3)$$

where the boundary condition is understood in terms of weak limits.

For a proof using Stampacchia's truncation method, see [11, Chapter 5].

Proof of Theorem 2.1: existence. Let P_h denote the harmonic extension of h to the unit disk. Since $h : \partial\mathbb{D} \rightarrow \mathbb{R}$ is bounded above by assumption, P_h is bounded above on the unit disk. Consider the closed convex set

$$\mathcal{K}_h = \left\{ v \in L^1(\mathbb{D}, |dz|^2), v \leq P_h \right\} \subset L^1(\mathbb{D}, |dz|^2)$$

and the operator

$$(Tv)(z) = P_h(z) - \frac{1}{2\pi} \int_{\mathbb{D}} e^{2v(\zeta)} |H(\zeta)|^2 G(z, \zeta) |d\zeta|^2. \quad (2.4)$$

The condition $H \in A_1^2(\mathbb{D})$ tells us that $d\mu_v(\zeta) = e^{2v(\zeta)} |H(\zeta)|^2 |d\zeta|^2$ is a Blaschke measure. Lemma 2.2 tells us that the integral in (2.4) lies in $W_0^{1,1}(\mathbb{D})$. Since $W_0^{1,1}(\mathbb{D})$ sits compactly inside $L^1(\mathbb{D})$, the operator T is compact and maps \mathcal{K}_h into itself. By Schauder's fixed point theorem, T has a fixed point. Applying Lemma 2.2 again, we see that any fixed point of T solves GCE_H . The proof is complete. \square

In the course of the previous proof, we saw:

Lemma 2.3. *If u is a solution of (2.1) with $h \in L^\infty(\partial\mathbb{D})$, then*

$$u(z) = Tu(z) = P_h(z) - \frac{1}{2\pi} \int_{\mathbb{D}} e^{2u(\zeta)} |H(\zeta)|^2 G(z, \zeta) |d\zeta|^2.$$

Conversely, if u satisfies the above formula, then u solves (2.1).

The following lemma says that the solution of GCE_H varies continuously in H :

Lemma 2.4. *Let $h \in L^\infty(\partial\mathbb{D})$ be a bounded function on the unit circle. Suppose $H_k \rightarrow H$ in $A_1^2(\mathbb{D})$. For each $k = 1, 2, \dots$, let u_k be the solution of GCE_{H_k} with boundary data h . Then, $u_k \rightarrow u$ pointwise a.e., where u is the solution of GCE_H with boundary data h .*

Proof. According to Lemma 2.3,

$$u_k(z) - P_h(z) = -\frac{1}{2\pi} \int_{\mathbb{D}} e^{2u_k(\zeta)} |H_k(\zeta)|^2 G(z, \zeta) |d\zeta|^2.$$

By Lemma 2.2, the functions $u_k(z) - P_h(z)$ are uniformly bounded in $W_0^{1,1}$. By Sobolev compactness, one can pass to a subsequence which converges in L^1 , and then to another subsequence which converges pointwise a.e. to some function $u(z)$. As the functions u_k are bounded above by $\|h\|_{L^\infty(\partial\mathbb{D})}$, the dominated convergence theorem tells us that

$$u(z) - P_h(z) = -\frac{1}{2\pi} \int_{\mathbb{D}} e^{2u(\zeta)} |H(\zeta)|^2 G(z, \zeta) |d\zeta|^2.$$

In other words, any subsequential limit of the functions u_k is a solution of GCE_H with boundary data h . Since the solution of GCE_H with boundary data h is unique, $u_k \rightarrow u$. \square

For future reference, we also record:

Lemma 2.5 (Comparison principle). *Suppose u is a subsolution and v is a supersolution. If $u \leq v$ on the boundary, then $u \leq v$ in the interior.*

3 Canonical solutions

Given a non-zero function $H \in A_1^2(\mathbb{D})$, let $u_{H,n}$ be the solution of the boundary value problem

$$\begin{cases} \Delta u = |H|^2 e^{2u}, & \text{in } \mathbb{D}, \\ u = n, & \text{on } \partial\mathbb{D}. \end{cases} \quad (3.1)$$

By Liouville's theorem,

$$u_{H,n} = \log \frac{1}{|H|} \frac{2|I'_n|}{1 - |I_n|^2}$$

for some holomorphic self-map I_n of the unit disk. From the monotonicity of solutions, we know that the functions $u_{H,n}$ are increasing in n and are clearly bounded above. Taking $n \rightarrow \infty$, we obtain the *canonical solution*

$$u_{H,\infty} := \lim_{n \rightarrow \infty} u_{H,n} = \log \frac{1}{|H|} \frac{2|I'|}{1 - |I|^2}. \quad (3.2)$$

From the above construction, it is clear that the canonical solution is the minimal solution which dominates all solutions with finite boundary data.

Our main theorem states:

Theorem 3.1. *Let $H \in A_1^2(\mathbb{D})$. The function $I = I_H$ is an inner function. The invariant subspace $[H]$ is generated by I'_n for any $n \in \mathbb{R}$ and contains I' .*

Remark. It is well-known that GCE_H has a maximal solution, which dominates any other solution pointwise. It may come as a surprise to the reader that the canonical and maximal solutions may be different. For instance, if H has no zeros in the disk but generates a non-trivial invariant subspace in $A_1^2(\mathbb{D})$, then the maximal solution $u_{H,\max} = \log \frac{1}{|H|} \frac{2}{1 - |z|^2}$ has the Liouville function $I_{H,\max}(z) = z$ but $I'_{H,\max} = 1 \notin [H]$.

For the convenience of the reader, we have split up the proof of Theorem 3.1 into a long sequence of lemmas. In order to verify Conjecture 1.4, one would also need to show that I' generates $[H]$. In Section 3.4, we will reduce Conjecture 1.4 to a statement in PDE, which we hope is more tractable.

3.1 Why is $I' \in [H]$?

Lemma 3.2. *If a solution $u = \log \frac{1}{|H|} \frac{2|F'|}{1 - |F|^2}$ is bounded above, then $F' \in [H]$.*

Proof. Since u is subharmonic,

$$u_{H,n} \leq \sup u \text{ on } \mathbb{D} \implies |F'/H| \leq e^{\sup u/2} \implies F' \in [H],$$

where we used that invariant subspaces of $A_1^2(\mathbb{D})$ are closed under multiplication by bounded holomorphic functions. \square

Lemma 3.3. *Suppose $\{F_n\} \subset \text{Hol}(\mathbb{D}, \mathbb{D})$ converges uniformly on compact subsets to $F \in \text{Hol}(\mathbb{D}, \mathbb{D})$. If each $F'_n \in [H]$, then $F' \in [H]$.*

Proof. While the F_n may not converge to F in $A_1^2(\mathbb{D})$, the Littlewood-Paley identity (1.1) tells us that the norms

$$\|F'_n\|_{A_1^2} \lesssim \|F_n\|_{H^2} \leq 1$$

are uniformly bounded. This allows us to pass to a subsequence that converges weakly. Since the F_n converge uniformly on compact subsets to F , the weak limit must also be F . It remains to use the following fact from functional analysis: *in a Banach space, a subspace is closed if and only if it is weakly closed.* \square

Corollary 3.4. *The Liouville map I_H associated to the canonical solution $u_{H,\infty}$ satisfies $I'_H \in [H]$.*

3.2 Why is I an inner function?

Lemma 3.5. *When $H = I'$ is the derivative of an inner function, we can write down the canonical solution explicitly:*

$$u_{I',n} = \log \frac{1}{|I'|} \frac{2|rI'|}{1-|rI|^2} = \log \frac{2r}{1-|rI|^2}, \quad \frac{2r}{1-r^2} = e^n.$$

$$u_{I',\infty} = \log \frac{1}{|I'|} \frac{2|I'|}{1-|I|^2} = \log \frac{2}{1-|I|^2}.$$

Lemma 3.6. *The Liouville map I_H associated to a canonical solution $u_{H,\infty}$ is an inner function.*

Proof. Since $I' \in [H]$, there exists a sequence of polynomials p_k such that $Hp_k \rightarrow I'$ in $A_1^2(\mathbb{D})$. Replacing $p_k(z)$ by $p_k(rz)$ if necessary, we may assume that the p_k have no zeros on the unit circle. By the comparison principle,

$$u_{Hp_k,n} \leq \log \frac{1}{|Hp_k|} \cdot \frac{2|I'|}{1-|I|^2}, \tag{3.3}$$

for any $n \in \mathbb{R}$. Note that if p_k has zeros inside the disk, then the RHS will be a supersolution of GCE_{Hp_k} rather than a solution, which makes it easier for (3.3) to hold. Taking $k \rightarrow \infty$ in (3.3), we get

$$u_{I',n} \leq \log \frac{2}{1 - |I|^2}.$$

Since this is true for any $n \in \mathbb{R}$, $u_{I',\infty} \leq \log \frac{2}{1 - |I|^2}$, which forces I to be inner. \square

Lemma 3.7. *The canonical solution $u_{H,\infty}$ depends only on the invariant subspace generated by H in $A_1^2(\mathbb{D})$.*

Proof. The proof of Lemma 3.6 shows that if $[H_1] \subset [H_2]$, then

$$\frac{|I'_{H_1}|}{1 - |I_{H_1}|^2} \leq \frac{|I'_{H_2}|}{1 - |I_{H_2}|^2}.$$

In particular, if H_1 and H_2 generate the same invariant subspace, then

$$\frac{|I'_{H_1}|}{1 - |I_{H_1}|^2} = \frac{|I'_{H_2}|}{1 - |I_{H_2}|^2}.$$

By Liouville's theorem, $I_{H_1} = I_{H_2}$ up to post-composition with an element of $\text{Aut}(\mathbb{D})$. The proof is complete. \square

3.3 Why does I'_n generate $[H]$?

In the proof below, we will use the following fact: the minimal harmonic majorant of a subharmonic function $u : \mathbb{D} \rightarrow [-\infty, \infty)$ is zero if and only if u has zero boundary data in the sense of weak limits of measures. This follows from the description of the minimal harmonic majorant of u as the limit of the Poisson extensions $P_{u|_{\partial\mathbb{D}_r}}$ as $r \rightarrow 1^-$.

Proof of Theorem 3.1. For concreteness, we will show that I'_0 generates $[H]$ as the general case is similar. Since the subharmonic function $u = \log \frac{1}{|H|} \frac{2|I'_0|}{1 - |I_0|^2}$ has zero boundary data, its minimal harmonic majorant is 0. For $0 < r < 1$, define

$$u_r = \log \frac{1}{|H_r|} \frac{2|rI'_0|}{1 - |rI_0|^2},$$

where

$$H_r := \frac{2r}{\phi_r} \cdot I'_0 \in [I'_0], \quad \phi_r(z) := \text{Out}_{1-|rI_0|^2}(z).$$

Here, ϕ_r is the outer function with absolute value $1 - |rI_0|^2$ on the unit circle. As ϕ_r is bounded away from zero, H_r is a product of I'_0 with a bounded holomorphic function, and hence it belongs to the invariant subspace generated by I'_0 . By construction, the minimal harmonic majorant of u_r is also 0.

Since the $|H_r|$ increase to $|H|$, after passing to a subsequence, the functions $H_r \rightarrow e^{i\theta} H$ converge in $A_1^2(\mathbb{D})$ for some $\theta \in [0, 2\pi)$. It follows that $H \in [I'_0]$ and therefore, $[H] = [I'_0]$. \square

3.4 Does I' generate $[H]$?

We now show that Conjecture 1.4 would follow from the following statement in PDE:

Conjecture 3.8. *Any solution of $\Delta u = |I'|^2 e^{2u}$ that is $\leq u_{I',\infty}$ can be approximated uniformly on compact subsets by solutions u_k that are bounded above.*

Proof that Conjecture 3.8 implies Conjecture 1.4. Let $H \in A_1^2(\mathbb{D})$ and I_0 and I be the Liouville functions of $u_{H,0}$ and $u_{H,\infty}$ respectively. Taking Conjecture 3.8 for granted, we can approximate

$$u_k = \log \frac{1}{|I'|} \frac{2|F'_k|}{1 - |F'_k|^2} \rightarrow \log \frac{1}{|I'|} \frac{2|I'_0|}{1 - |I_0|^2}.$$

Since the solutions u_k are bounded above, each $F'_k \in [I']$ by Lemma 3.2. By Lemma 3.3, $I'_0 \in [I']$. Therefore, $[I'] = [H]$ as desired. \square

Remark. Conjecture 3.8 is easy to believe since one can approximate a harmonic function by bounded harmonic functions. However, it turns out to be quite difficult and we can only verify it in special cases, for instance, when I' lies in the Nevanlinna class.

3.5 When are canonical solutions maximal?

Since solutions of GCE_H are in bijection with solutions of (1.4),

$$u_{H,\max} = \log \frac{1}{|H|} \frac{2|F'_{\mathcal{Z}(H)}|}{1 - |F_{\mathcal{Z}(H)}|^2},$$

where $F_{\mathcal{Z}(H)}$ is a maximal Blaschke product whose critical set is the zero set of H .

Lemma 3.9. *Let $H \in A_1^2(\mathbb{D})$ and I_H be the Liouville map of the canonical solution associated to H . The canonical and maximal solutions of GCE_H coincide if and only if $I_H \in \text{MBP}$.*

Proof. The canonical solution $u_{H,\infty} = \log \frac{1}{|H|} \frac{2|I'_H|}{1 - |I_H|^2}$ is equal to the maximal solution if and only if

$$\frac{|I'_H|}{1 - |I_H|^2} = \frac{|F'_{\mathcal{Z}(H)}|}{1 - |F_{\mathcal{Z}(H)}|^2}.$$

By Liouville's theorem, this happens precisely when I_H and $F_{\mathcal{Z}(H)}$ are related by an element of $\text{Aut}(\mathbb{D})$. The theorem follows from the fact that the class of maximal Blaschke products is invariant under post-compositions by automorphisms of the disk. \square

3.6 Boundary behaviour of canonical solutions

In [14], C. Sundberg showed that the Bergman canonical divisor with zero set $C \subset \mathbb{D}$ extends analytically past any open arc $J \subset \partial\mathbb{D}$ that does not meet the closure of C . In a similar spirit, D. Kraus and O. Roth [8] showed that a maximal Blaschke product with critical set C also extends analytically past any open arc $J \subset \partial\mathbb{D}$ that does not meet the closure of C . We establish an analogous result for Liouville maps of canonical solutions:

Theorem 3.10. *If $H \in A_1^2(\mathbb{D})$ extends analytically past an open arc $J \subset \partial\mathbb{D}$, then the Liouville map $I = I_H$ extends analytically through J .*

Proof. We prove the theorem under the additional assumption that H does not vanish on J , this assumption can be removed by Lemma 3.11 below. By Lemma 2.3,

$$u_n(z) \geq n - \frac{e^{2n}}{2\pi} \int_{\mathbb{D}} |H(\zeta)|^2 G(z, \zeta) |d\zeta|^2.$$

From this representation, it follows that for any $\zeta \in J$,

$$\liminf_{z \rightarrow \zeta} u_{H,n}(z) \geq n.$$

Since the $u_{H,n}$ are increasing in n ,

$$\lim_{z \rightarrow \zeta} u_{H,\infty}(z) = \infty.$$

Since H is bounded away from zero near ζ ,

$$\log \frac{|I'(z)|}{1 - |I(z)|^2}$$

tends to $+\infty$ as $z \rightarrow \zeta$. According to [9, Theorem 1.1], I extends analytically past the arc J . The proof is complete. \square

Lemma 3.11. *Suppose $H \in A_1^2(\mathbb{D})$ and $\zeta \in \partial\mathbb{D}$. Then, $[H(z)] = [H(z)(z - \zeta)]$.*

Proof. The inclusion $[H(z)] \supseteq [H(z)(z - \zeta)]$ is obvious. To prove the other inclusion, we notice that

$$H(z)(z^n - \zeta^n) \in [H(z)(z - \zeta)].$$

Taking $n \rightarrow \infty$, we get $H(z)\zeta^n \in [H(z)(z - \zeta)]$. Multiplying by ζ^{-n} , we see that $H(z) \in [H(z)(z - \zeta)]$ as desired. \square

4 Further remarks and open problems

We conclude with some remarks and open problems:

1. In the theory of Bergman spaces, one learns that any 1-generated invariant subspace $X \subset A_1^2$ can be generated by a Bergman inner function $\varphi = \varphi_X$, which solves a certain extremal problem. According to [3, Theorem 3.3], φ is the derivative of a BMO function. Since one expects φ to be the smoothest function in X , it is natural to wonder if φ is actually the derivative of a bounded function.

2. Let $X \subset A_1^2$ be a 1-generated invariant subspace and $X_n \rightarrow X$ be an approximate spectral synthesis by finite zero-based subspaces. Is it true that the canonical inner functions I_{X_n} tend to I_X ?
3. Can one give an alternative proof of Shimorin's result on the approximate spectral synthesis [12] in A_1^2 using the methods of this paper? That is, to show that one can approximate any 1-generated invariant subspace $X \subset A_1^2$ by zero-based ones.

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