Entropy of universal covering maps

Oleg Ivrii

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Finite Blaschke products

A finite Blaschke product can be described as a holomorphic map from $\mathbb{D} \to \mathbb{D}$, which extends to a continuous dynamical system on the unit circle.

$$F(z) = e^{i\alpha} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_i \in \mathbb{D}.$$

Let $m = d\theta/2\pi$ be the Lebesgue measure on the unit circle. If $F(0) = 0$, then m is F-invariant, i.e.

$$m(E) = m\big(F^{-1}(E)\big), \qquad E \subset \mathbb{S}^1.$$

Entropy of finite Blaschke products

Assume $F'(0) \neq 0$ for simplicity.

The measure-theoretic entropy of m can be computed by Jensen's formula:

$$\begin{aligned}\int_{|z|=1} \log|F'(z)|dm &= \sum_{\text{crit}} \log\frac{1}{|c_i|} + \log|F'(0)| \\ &= \sum_{\text{crit}} \log\frac{1}{|c_i|} - \sum_{\text{zeros}} \log\frac{1}{|a_i|},\end{aligned}$$

where in the sum over the zeros of F, we omit the trivial zero at the origin.

Entropy of inner functions

An inner function is a holomorphic map from $\mathbb{D} \to \mathbb{D}$ such that for almost every $\theta \in [0, 2\pi)$, the radial limit

$$\lim_{r \to 1} F(re^{i\theta})$$

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Let $P \subset \mathbb{D} \setminus \{0\}$ be a finite set. Consider the universal covering map $\mathcal{U}_P : \mathbb{D} \to \mathbb{D} \setminus P$, normalized so that $\mathcal{U}_P(0) = 0$ and $\mathcal{U}_P'(0) > 0$.

Question. What is its entropy?

Can't we just apply Jensen's formula?

If we could apply Jensen's formula on the unit circle, the entropy

$$\int_{|z|=1} \log |\mathcal{U}_P'(z)| dm = -\sum_{\text{zeros}} \log \frac{1}{|z_i|}$$

would be negative.

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Theorem. (Pommerenke, 1976 & I, 2018)

$$\int_{|z|=1} \log |\mathcal{U}_P'(z)| dm = \sum_{\text{punctures}} \log \frac{1}{|p_i|} - \sum_{\text{zeros}} \log \frac{1}{|z_i|}.$$

BS decomposition

An inner function can be represented as a (possibly infinite) Blaschke product $\times$ singular inner function:

$$B(z) = e^{i\alpha} \prod_i -\frac{\overline{a_i}}{|a_i|} \cdot \frac{z - a_i}{1 - \overline{a_i} z}, \quad a_i \in \mathbb{D}, \quad \sum_i (1 - |a_i|) < \infty.$$

$$S(z) = \exp\left(- \int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z} d\sigma_\zeta \right), \quad \sigma \perp m, \quad \sigma \geq 0.$$

Here, B records the zero set, while S records the boundary zero structure.

Nevanlinna and Smirnov classes

The Nevanlinna class $\mathcal{N}$ consists of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ which satisfy

$$\lim_{r \to 1} \int_{|z|=r} \log^+ |f(z)| dm < \infty.$$

One has the following factorizations

$$\mathcal{N} = B(S_1/S_2)O, \qquad \mathcal{N}^+ = BSO,$$

where the outer function has the form

$$O(z) = \exp\left(\int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z} h(\zeta) dm_\zeta \right).$$

Inner functions of finite entropy

Theorem. (M. Craizer, 1991) Let F be an inner function with $F(0) = 0$. Then, $F' \in \mathcal{N}$ if and only if the Lebesgue measure m has finite entropy.

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In 1974, P. Ahern and D. Clark showed that F' admits a BSO decomposition, allowing us to define $\operatorname{Inn} F' := BS$, where B records the critical set of F and S records the boundary critical structure.

Can't we just apply Jensen's formula? (Part II)

If $f \in \mathcal{N}$ is in Nevanlinna class, it may happen that

$$\int_{|z|=1} \log|f(z)|dm \neq \lim_{r\to 1}\int_{|z|=r} \log|f(z)|dm.$$

In fact, the difference

$$\int_{|z|=1} \log|f(z)|dm - \lim_{r\to 1}\left\{\int_{|z|=r} \log|f(z)|dm\right\}$$

is just $\sigma(f)(\mathbb{S}^1)$.

Frostman Shifts

Let F be an inner function. The Frostman shift at $x \in \mathbb{D}$ is defined as

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Lemma. If $F \in \mathscr{J}$, then

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Theorem. $\sigma(F') \geq \sum_{x \in \mathbb{D}} \sigma(F_x)$.

Proof. For any $x \in \mathbb{D}$, $S(F_x) \,\big|\, \operatorname{Inn} F_x'$, but the singular measures $\sigma(F_x)$ are supported on different sets.

My favourite formula as a graduate student

Lemma. Let F be a finite Blaschke product with $F(0) = 0$. For any $x \in \mathbb{D} \setminus \{0\}$,

$$\log \frac{1}{|x|} = \sum_{F(y)=x} \log \frac{1}{|y|}.$$

More generally, if F is an inner function, then

$$\log \frac{1}{|x|} = \sum_{F(y)=x} \log \frac{1}{|y|} + \sigma(F_x)(\mathbb{S}^1).$$

Entropy of universal covering maps

Theorem. (Pommerenke, 1976 & I, 2018)

$$\int_{|z|=1} \log |\mathcal{U}_P'(z)| dm = \sum_{\text{punctures}} \log \frac{1}{|p_i|} - \sum_{\text{zeros}} \log \frac{1}{|z_i|}.$$

Intuitively, P is the set of **critical values** of $\mathcal{U}_P$.

To make this intuition rigorous, one approximates $\mathcal{U}_P$ by finite Blaschke products F_n with critical values $\subseteq P$.

[For simplicity, I will write $F = \mathcal{U}_P$.]

Upper bound

Since the entropy can only decrease after taking limits, we have

$$\begin{aligned}
\int_{|z|=1} \log|F'(z)|dm &\leq \liminf_{n\to\infty} \int_{|z|=1} \log|F_n'(z)|dm, \\
&= \liminf_{n\to\infty} \left\{ \log|F_n'(0)| + \sum_{\mathrm{crit}(F_n)} \log\frac{1}{|c_i|} \right\}, \\
&\leq \log|F'(0)| + \sum_{i=1}^{k} \log\frac{1}{|p_i|}.
\end{aligned}$$

Lower bound

The other direction is automatic:

$$\begin{aligned}\int_{|z|=1} \log|F'(z)|\,dm &= \lim_{r\to 1}\int_{|z|=r} \log|F'(z)|\,dm + \sigma(F')(\mathbb{S}^1),\\ &\geq \log|F'(0)| + \sum_{i=1}^{k} \sigma(F_{p_i})(\mathbb{S}^1),\\ &= \log|F'(0)| + \sum_{i=1}^{k} \log\frac{1}{|p_i|}.\end{aligned}$$

Construction of $\mathcal{U}_P : \mathbb{D} \to \mathbb{D} \setminus \{p_1, p_2, \ldots, p_k\}$.

Define a *tile* or *sheet* to be the shape

$$\mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i,$$

where γ_i are disjoint real-analytic arcs that join p_i to the unit circle.

An ∞-stack over γ_i is a countable collection of tiles $\{T_j\}_{j\in\mathbb{Z}}$, where the lower side of γ_i in T_j is identified with the upper side of γ_i in T_{j+1}.

Similarly, by an n-stack, we mean a set of n tiles with the above identifications made modulo n.

Construction of $\mathcal{U}_P : \mathbb{D} \to \mathbb{D} \setminus \{p_1, p_2, \ldots, p_k\}$.

0. Start with the base tile $T_e \cong \mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i$.
1. At each slit $\gamma_i \subset T_e$, we glue an ∞-stack.
2. To each of the $k-1$ unglued slits in each tile of generation 1, we glue a further ∞-stacks.

∞. Repeating this construction infinitely many times gives a Riemann surface S with a natural projection to $\mathbb{D}$.

Since S is simply-connected, $S \cong \mathbb{D}$. Since all slits get glued up, we get an inner function. It is easy to see that it is precisely $\mathcal{U}_P$.

For the finite approximations, glue in n-stacks and stop at generation n.

Dyakonov's question

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Theorem. (I, 2017) An inner function in $\mathscr{J}$ is uniquely determined by its critical structure up to post-composition with a Möbius transformation.

An inner function BS_σ can be represented as $\operatorname{Inn} F'$ for some $F \in \mathscr{J}$ if and only if σ is sufficiently concentrated (lives on a countable union of Beurling-Carleson sets).

Stable topology on inner functions

Definition. A Beurling-Carleson set E is a closed subset of the unit circle which has measure 0 such that $\sum |I_j| \cdot \log \frac{1}{|I_j|} < \infty$, where $\{I_j\}$ are the complementary intervals.

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We say that $F_n \to F$ converges in the stable topology if

- The convergence is uniform on compact subsets of the disk,
- $\operatorname{Inn} F_n' \to \operatorname{Inn} F' \quad \Longleftrightarrow \quad \operatorname{Out} F_n' \to \operatorname{Out} F'$.

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Theorem. (I, 2018) This happens if and only if the critical structures of the F_n are uniformly concentrated in Beurling-Carleson sense.

Thank you for your attention!