Inner Functions, Composition Operators, Symbolic Dynamics and Thermodynamic Formalism

Oleg Ivrii and Mariusz Urbański

Abstract

In this paper, we use thermodynamic formalism to study the dynamics of inner functions F acting on the unit disk. If the Denjoy-Wolff point of F is in the open unit disk, then without loss of generality, we can assume that $F(0) = 0$ so that 0 is an attracting fixed point of F and the Lebesgue measure on the unit circle is invariant under F. Utilizing the connection between composition operators, Aleksandrov-Clark measures and Perron-Frobenius operators, we develop a rudimentary thermodynamic formalism which allows us to prove the Central Limit Theorem and the Law of the Iterated Logarithm for Sobolev multipliers and Hölder continuous observables.

Under the more restrictive, but natural hypothesis that F is a one component inner function, we develop a more complete thermodynamic formalism which is sufficient for orbit counting, assuming only the $(1 + \varepsilon)$-integrability of $\log |F'|$. As one component inner functions admit countable Markov partitions of the unit circle, we may work in the abstract symbolic setting of countable alphabet subshifts of finite type. Due to the very weak hypotheses on the potential, we need to pay close attention to the regularity of the complex Perron-Frobenius operators $\mathcal{L}_s$ with $\operatorname{Re} s > 1$ near the boundary.

Finally, we discuss inner functions with a Denjoy-Wolff point on the unit circle. We assume a parabolic type behavior of F around this point and we introduce the class of parabolic one component inner functions. By making use of the first return map, we deduce various stochastic laws and orbit counting results from the aforementioned abstract symbolic results.

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1 Introduction

A *finite Blaschke product* is a holomorphic self-map of the unit disk which extends to a continuous dynamical system on the unit circle. It is determined by the location of its zeros up to a rotation:

$$F(z) = e^{i\theta} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_i \in \mathbb{D}.$$

Loosely speaking, an *inner function* is a holomorphic self-map of the disk which extends to a measure-theoretic dynamical system on the unit circle. More precisely, we require that for a.e. $\theta \in [0, 2\pi)$, the radial boundary value $F(e^{i\theta}) := \lim_{r \to 1} F(re^{i\theta})$ exists and has absolute value 1.

One can classify inner functions according to the location of the Denjoy-Wolff fixed point. We mostly deal with the case when the Denjoy-Wolff fixed point is in the open unit disk. After a conjugation, we may assume that $F(0) = 0$. We call such inner functions *centered*. By the Schwarz Lemma, under iteration, all points in the unit disk tend to 0. We write $m = d\theta/2\pi$ for the normalized Lebesgue measure on the unit circle.

To set the stage, we recall a number of dynamical properties of m such as invariance, ergodicity and mixing due to Sullivan-Shub [SS85] and Pommerenke [Pom81]:

Lemma 1.1. *Suppose that $F(z)$ is an inner function such that $F(0) = 0$.*

(i) *Lebesgue measure on the unit circle is invariant under F, that is, $m(F^{-1}(E)) = m(E)$ for any measurable set $E \subset \partial \mathbb{D}$.*

(ii) *The Lebesgue measure on the unit circle is ergodic, that is, any invariant set $E \subset \mathbb{D}$ has measure 0 or 1.*

(iii) *The map $F : \partial\mathbb{D} \to \partial\mathbb{D}$ is mixing, that is, for any measurable sets $A, E \in \partial\mathbb{D}$,*

$$m(A \cap F^{-n}(E)) \to m(A) \cdot m(E), \qquad as \ n \to \infty.$$

The above properties may be proved by examining Poisson extensions of characteristic functions.

In this paper, we employ thermodynamic formalism to show stronger statistical properties of inner functions such as the Central Limit Theorem, the Law of the Iterated Logarithm and the Almost Sure Invariance Principle. We also obtain an Orbit Counting Theorem for one component inner functions satisfying a mild integrability condition. Later, we also discuss Orbit Counting for parabolic one component inner functions.

1.1 Thermodynamic Formalism and Stochastic Laws for Centered Inner Functions

Our first objective is to study stochastic properties of arbitrary inner functions F with $F(0) = 0$ acting on the unit circle. We begin by surveying a number of results on the spectral properties of the *composition operator*

$$C_F : h \to h \circ F$$

acting on various spaces of real and holomorphic functions on the unit circle. We then define the *Perron-Frobenius* or *transfer operator* L_F with potential $-\log|F'|$ as the dual of the composition operator with respect to the L^2 pairing on the unit circle.

In Section 3, we will see that the operator L_F admits a convenient representation in terms of Aleksandrov-Clark measures:

$$(L_F g)(\alpha) = \int_{\partial\mathbb{D}} g(z) d\mu_\alpha(z), \qquad \alpha \in \partial\mathbb{D},$$

which generalizes the classical definition

$$(L_F g)(x) = \sum_{F(y)=x} |F'(y)|^{-1} g(y),$$

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valid for finite Blaschke products. In fact, the two definitions agree precisely when the Alexandrov-Clark measures $\{\mu_\alpha\}$ are purely atomic. More generally, one can define Perron-Frobenius operators for other potentials by taking adjoints of weighted composition operators.

We use duality arguments to show that for a large class of potentials, the Perron-Frobenius operator has a spectral gap when acting on spaces of Sobolev and Hölder functions. From here, stochastic laws follow quite easily: the Central Limit Theorem is an immediate consequence of Gordin's results [Gor69] while the Law of the Iterated Logarithm and the Almost Sure Invariance Principle follow from the work of Gouëzel [Gou10].

We now describe the Central Limit Theorem (CLT) in greater detail. As is standard in ergodic theory and dynamical systems, we use the notation

$$S_n h(x) := h(x) + h(F(x)) + \cdots + h(F^{\circ(n-1)})(x)$$

to repesent the sum of a function h along the forward orbit of x. One says that the Central Limit Theorem holds for an observable $h : \partial\mathbb{D} \to \mathbb{R}$ and a measure μ on $\partial\mathbb{D}$ if the random variables $\frac{S_n h}{\sqrt{h}}$ on the probability space $(\partial\mathbb{D}, \mu)$ converge in distribution to the Gaussian distribution

$$\mathcal{N}(\overline{h}, \, \sigma^2(h - \overline{h})), \qquad \overline{h} = \int_{\partial\mathbb{D}} h(x) d\mu(x),$$

for some constant $\sigma^2 = \sigma^2(h) \geq 0$ depending on h.

Theorem 1.2. *If F is a centered inner function, then the Central Limit Theorem holds with respect to the Lebesgue measure on the unit circle for*

- *any Sobolev multiplier $h \in \mathcal{M}(W^{1/2,2}(\partial\mathbb{D}))$,*

- *any Hölder continuous function $h \in C^\alpha(\partial\mathbb{D})$ with $\alpha > 0$.*

Furthermore, if F is not a finite Blaschke product and h is non-constant, then $\sigma^2(h) \neq 0$.

The space of Hölder continuous functions $C^\alpha(\partial\mathbb{D})$ is an algebra in the sense that the product of two functions in $C^\alpha(\partial\mathbb{D})$ is again a function in $C^\alpha(\partial\mathbb{D})$.

Loosely speaking, for $\beta > 0$, the Sobolev space $W^{\beta,2}(\partial\mathbb{D})$ consists of $L^2(\partial\mathbb{D})$ functions which have β derivatives in $L^2(\partial\mathbb{D})$. In terms of Fourier coefficients,

this means that

$$\sum_{n=-\infty}^{\infty} |n|^{2\beta} |\hat{h}(n)|^2 < \infty.$$

For $0 < \beta \leq 1/2$, the space $W^{\beta,2}(\partial\mathbb{D})$ is not an algebra, so we work with the multiplier space $\mathcal{M}(W^{\beta,2}(\partial\mathbb{D}))$ instead. A function γ is called a $W^{\beta,2}(\partial\mathbb{D})$ *multiplier* if for any function $g \in W^{\beta,2}(\partial\mathbb{D})$, the product $g\gamma \in W^{\beta,2}(\partial\mathbb{D})$, in which case,

$$\|g\gamma\|_{W^{\beta,2}(\partial\mathbb{D})} \leq C\|g\|_{W^{\beta,2}(\partial\mathbb{D})}, \tag{1.1}$$

for some $C > 0$. The infimum of constants C one can put in the right hand side of (1.1) is called the *multiplier norm* and is denoted by $\|\gamma\|_{\mathcal{M}(W^{\beta,2}(\partial\mathbb{D}))}$.

For concreteness, we take $\beta = 1/2$. It is well known that any Sobolev multiplier is a bounded function, and for any $\varepsilon > 0$,

$$W^{1/2+\varepsilon,2}(\partial\mathbb{D}) \subset \mathcal{M}(W^{1/2,2}(\partial\mathbb{D})) \subset W^{1/2,2}(\partial\mathbb{D}).$$

The first inclusion shows that any C^1 function is a $W^{1/2,2}$ multiplier. For more on multiplication in Sobolev spaces, we refer the reader to the survey [BH22].

In additional to the Lebesgue measure, we also establish the Central Limit Theorem with respect to equilibrium measures associated to potentials of the form $-\log|F'| + \gamma$, where γ is a positive function in $C^\alpha(\partial\mathbb{D})$ of small norm.

1.2 Orbit Counting for Centered and Parabolic One Component Inner Functions

To delve deeper into understanding the dynamical properties of inner functions, we restrict to a natural class of inner functions called *one component inner functions* introduced by W. Cohn in [Coh82]. According to the original definition, an inner function $F : \mathbb{D} \to \mathbb{D}$ is a one component inner function if the set $\{z \in \mathbb{D} : |F(z)| < c\}$ is connected for some $c \in (0,1)$. From the point of view of dynamical systems, it is more useful to say that a one component inner function is an inner function whose set of singular values is compactly contained in the unit disk.

An equivalence of the two definitions, as well as a number of basic properties of one component inner functions will be given in Section 8. In particular, we will see that one component inner functions admit sufficiently nice

Markov partitions, which allows us to view them as conformal graph directed Markov systems introduced in [MU03] and more systematically treated in [URM22b, KU23]. We may therefore study the dynamics of one component inner functions on the unit circle by working in the abstract symbolic setting of countable alphabet subshifts of finite type.

Let $F : \mathbb{D} \to \mathbb{D}$ be a centered one-component inner function. For $x \in \partial\mathbb{D}$ and $T > 0$, consider the *counting function*

$$n(x, T) := \#\big\{(n \geq 0, y \in \partial\mathbb{D}) : F^{\circ n}(y) = x \text{ and } \log|(F^{\circ n})'(y)| < T\big\}.$$

More generally, for a Borel set $B \subset \partial\mathbb{D}$, one can form the function $n(x, T, B)$ which counts the number of iterated pre-images y that lie in B. We show:

Theorem 1.3. *Let F be a one component inner function with $F(0) = 0$. Assume that F has infinite degree or is a finite Blaschke product of degree $d \geq 2$ other than $z \to z^d$. Under the hypothesis*

$$\int_{\partial\mathbb{D}} \big(\log|F'(z)|\big)^{1+\varepsilon} dm < \infty, \qquad \text{for some } \varepsilon > 0, \tag{1.2}$$

we have

$$n(x, T, B) \sim \frac{e^T}{\int_{\partial\mathbb{D}} \log|F'(z)| dm} \cdot m(B) \qquad \text{as } T \to \infty,$$

where $B \subset \partial\mathbb{D}$ is a Borel set with $m(\partial B) = 0$.

Theorem 1.4. *Let F be a one component inner function such that $F(0) = 0$, which is not a rotation. If the Lebesgue measure of F has finite entropy, i.e.*

$$\int_{\partial\mathbb{D}} \log|F'(z)| dm < \infty, \tag{1.3}$$

then

$$\frac{1}{T}\left\{\int_0^T \frac{n(x, t, B)}{e^t}\, dt\right\} \sim \frac{1}{\int_{\partial\mathbb{D}} \log|F'(z)| dm} \cdot m(B) \qquad \text{as } T \to \infty.$$

The above theorems were obtained for finite Blaschke products in [Ivr15, Section 7] based on renewal-theoretic considerations of S. Lalley. In order to

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study orbit counting for inner functions, we use the approach pioneered in [PoU17] which involves examining the Poincaré series

$$\eta_x^B(s) = \sum_{n \geq 0} \sum_{\substack{F^{\circ n}(y) = x \\ y \in B}} |(F^{\circ n})'(y)|^{-s} = \int_0^\infty e^{-sT} \, dn(x, T, B).$$

Following the strategy outlined in [PoU17], we show:

Lemma 1.5. *Let F be a one component inner function with $F(0) = 0$, which satisfies the integrability condition (1.2) for some $0 < \varepsilon < 1$. Suppose $B \subset \partial \mathbb{D}$ is a Borel set with $m(\partial B) = 0$. For any $x \in \partial \mathbb{D}$, the function $\eta_x^B(s)$ is*

    (1) *holomorphic from $\overline{\mathbb{C}_1^+} \to \mathbb{C}$,*

    (2) *continuous from $\overline{\mathbb{C}_1^+} \setminus \{1\} \to \mathbb{C}$.*

    (3) *near $s = 1$,*

$$\eta_x^B(s) - \frac{1}{\int_{\partial \mathbb{D}} \log |F'(z)| dm} \cdot \frac{1}{s - 1} = O\left(\frac{1}{|s - 1|^{1 - \varepsilon}}\right).$$

If we merely assume that the inner function F has finite entropy, then the above conclusions hold if we replace (3) with

    (3$'$) *near $s = 1$,*

$$\eta_x^B(s) - \frac{1}{\int_{\partial \mathbb{D}} \log |F'(z)| dm} \cdot \frac{1}{s - 1} = o\left(\frac{1}{|s - 1|}\right).$$

Once we show the above lemma, Theorem 1.3 and Theorem 1.4 follow follow from the Tauberian theorems of Wiener-Ikehara and Hardy-Littlewood respectively. For the convenience of the reader, we recall these theorems below:

Theorem 1.6 (Wiener-Ikehara). *Suppose $\mu \geq 0$ is a locally finite measure on $(0, +\infty)$ and*

$$\eta(s) = \int_0^\infty e^{-sT} d\mu$$

converges in the right half-plane $\mathbb{C}_1^+ = \{\operatorname{Re} s > 1\}$. If

$$\eta(s) - \frac{c}{s - 1}$$

extends to an L^1_{loc} function on the vertical line $\{\operatorname{Re} s = 1\}$, then

$$\mu([0, T]) \sim c \, e^T.$$

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Theorem 1.7 (Hardy-Littlewood). *If*

$$\eta(s) \sim \frac{c}{s-1}$$

as $s \to 1^+$ along the real axis, then

$$\frac{1}{T} \int_0^T \frac{d\mu(t)}{e^t} \to c.$$

Note that the Hardy-Littlewood Tauberian Theorem has weaker hypotheses than Wiener-Ikehara Tauberian theorem but also a weaker conclusion. The Hardy-Littlewood theorem is usually stated in the half-plane $\{\operatorname{Re} s > 0\}$ rather than $\{\operatorname{Re} s > 1\}$, which explains the e^t in the denominator. For the proofs, we refer the reader to [MV07, Corollary 8.7] and [Har49, Theorem 98].

Remark. (i) In the treatise [PoU17], one asks that the Poincaré series admits a meromorphic continuation to a slightly larger half-plane $\mathbb{C}_{1-p}^+$ with a simple pole at $s = 1$, which translates to the more stringent requirement

$$\int_{\partial \mathbb{D}} |F'(z)|^p dm < \infty, \qquad \text{for some } p > 0.$$

Due to the weak integrability hypothesis on the inner function F, the Poincaré series usually does not have a meromorphic continuation beyond the line vertical line $\{\operatorname{Re} s = 1\}$. As such, we are forced to deal with the subtle issues of differentiability and holomorphy of complex Perron-Frobenius operators $\mathcal{L}_s$ with $\operatorname{Re} s \geq 1$ near the vertical line $\{\operatorname{Re} s = 1\}$.

(ii) Although, we will not pursue this here, one can also count periodic orbits. Using the methods of [PoU17], one can show that

$$n_{\mathrm{Per}}(T, B) \sim \frac{e^T}{\int_{\partial \mathbb{D}} \log |F'(z)| dm} \cdot m(B) \qquad \text{as } T \to \infty,$$

where

$$n_{\mathrm{Per}}(T) = \#\big\{ y \in \partial \mathbb{D} : F^{\circ n}(y) = y \text{ for some } n \geq 1 \text{ and } \log |(F^{\circ n})'(y)| < T \big\}.$$

(iii) For the finite Blaschke products $F(z) = z^d$, with $d \geq 2$, the counting function $n(x, T)$ is a step function, so the asymptotics in Theorem 1.3 do no hold. This exceptional behaviour is caused by the failure of D-genericity. Showing that other centered one component inner functions satisfy this mixing condition will be an important ingredient in the proof of Theorem 1.3.

In Section 9, we define the class of parabolic one component inner functions and prove an analogue of Theorem 1.3. It is convenient to view parabolic inner functions as holomorphic self-maps of the upper half-plane $\mathbb{H}$, in which case, the absolutely continuous invariant measure is just the Lebesgue measure ℓ on the real line $\mathbb{R}$. In this setting, the Orbit Counting Theorem reads as follows:

Theorem 1.8. *Let $F : \mathbb{H} \to \mathbb{H}$ be a doubly parabolic one component inner function, with a doubly parabolic fixed point at infinity. Suppose that*

$$\int_{\mathbb{R}} \left(\log |F'(z)| \right)^{1+\varepsilon} d\ell < \infty,$$

for some $\varepsilon > 0$. If $B \subset \mathbb{R}$ is a bounded Borel set with $\ell(B) < \infty$ and $\ell(\partial B) = 0$, then

$$n(x, T, B) \sim \frac{e^T}{\int_{\mathbb{R}} \log |F'(z)| d\ell} \cdot \ell(B) \qquad \text{as } T \to \infty.$$

1.3   Orbit Counting in Symbolic Dynamics

Since our approach to orbit counting is very general, it is applicable to a plethora of other dynamical systems, not just to inner functions. We will deduce Theorems 1.3, 1.4 and 1.8 from a general orbit counting theorem, valid in the setting of countable alphabet subshifts of finite type. A complete explanation of the terms will be given in Section 5.

Theorem 1.9. *Let E be a countable set, $A : E \times E \to \{0,1\}$ be a finitely irreducible incidence matrix and $\psi : E_A^\infty \to \mathbb{R}$ be a robust potential such that*

$$\int_{E_A^\infty} \psi^{1+\varepsilon} d\mu_\psi < +\infty,$$

for some $\varepsilon > 0$. Given $\xi \in E_A^\infty$ and a Borel set $B \subset E_A^\infty$ with $m_\psi(\partial B) = 0$, the counting function

$$N_\xi^B(T) := \#\left\{ \omega \in E_\xi^* : \ \omega\xi \in B \ \text{and} \ S_{|\omega|}(-\psi)(\omega\xi) \leq T \right\}$$

satisfies

$$\lim_{T \to \infty} \frac{N_\xi^B(T)}{e^T} = \frac{\rho_\psi(\xi)}{\int_{E_A^\infty}(-\psi) d\mu_\psi} \cdot m_\psi(B).$$

Theorem 1.10. *Let E be a countable set, $A : E \times E \to \{0, 1\}$ be a finitely irreducible incidence matrix and $\psi : E_A^\infty \to \mathbb{R}$ be a normal potential such that*

$$\int_{E_A^\infty} \psi d\mu_\psi < +\infty.$$

Given $\xi \in E_A^\infty$ and a Borel set $B \subset E_A^\infty$ with $m_\psi(\partial B) = 0$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{N_\xi^B(t)}{e^t} dt = \frac{\rho_\psi(\xi)}{\int_{E_A^\infty} (-\psi) d\mu_\psi} \cdot m_\psi(B).$$

1.4 Notes and references

Blaschke-lacunary series. In [NS22, NS23], A. Nicolau and O. Soler i Gibert proved a Central Limit Theorem for "Blaschke-lacunary series"

$$\sum_{n=1}^N a_n F^{\circ n}(z),$$

where $\{a_n\}_{n=0}^\infty$ is a sequence of complex numbers satisfying

$$a_n \notin \ell^2, \qquad \lim_{N \to \infty} \frac{|a_N|^2}{\lim_{n=1}^N |a_n|^2} = 0. \tag{1.4}$$

More precisely, they show that for an appropriate choice of scaling factors σ_N, the distribution of the functions

$$\frac{\sqrt{2}}{\sigma_N} \cdot \sum_{n=1}^N a_n F^{\circ n}(z),$$

on the unit circle converges to the standard complex Gaussian. As explained in [NS23], the condition (1.4) is sharp in the sense if (1.4) is not satisfied, then the Central Limit Theorem cannot hold. The proof leverages that iterates of the observable $h(z) = z$ possesses a certain "uniform indepedence property" which is made precise by BMO estimates. For further results in this direction, we refer the reader to [Nic20, DN23].

Part I

General Centered Inner Functions

2 Background on Composition Operators

In this section, we assume that F is a holomorphic self-map of the unit disk with $F(0) = 0$. We survey a number of results on the spectral properties of the composition operator $C_F : g \to g \circ F$ acting on various spaces of real and holomorphic functions. We first recall a classical lemma due to Koenigs from 1884:

Lemma 2.1. *Suppose F is a holomorphic self-map of the unit disk with $F(0) = 0$. If $F'(0) \neq 0$, then the eigenvalues of $C_F : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ are 1 and $F'(0)^k$, $k = 1, 2, \ldots$. All have multiplicity 1. On the other hand, if $F(0) = 0$, then C_F only has the constant eigenfunction $\mathbf{1}$.*

A proof can be found in [Sha93, Section 6.1]. The Koenigs eigenfunction

$$\varphi(z) = \lim_{n \to \infty} F'(0)^{-n} \cdot F^{\circ n}(z)$$

is well known in complex dynamics. It serves as a linearizing coordinate near an attracting fixed point with a non-zero multiplier:

$$\varphi(F(z)) = F'(0) \cdot \varphi(z), \qquad \varphi'(0) = 1.$$

The other eigenfunctions are powers of the Koenigs eigenfunction.

2.1 Hardy Spaces

The Hardy space H^p consists of holomorphic functions on the unit disk which satisfy

$$\|g\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta < \infty.$$

To study the action of C_F on the space H^2, one uses the following two facts, which one can find in [Sha00, Sections 2 and 3]:

Lemma 2.2 (Littlewood-Paley formula). *For any function $g \in H^2$,*

$$\|g\|_{H^2} = |g(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |g'(z)|^2 \log \frac{1}{|z|} |dz|^2. \qquad (2.1)$$

Lemma 2.3. *Let $F : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disk which vanishes at the origin and*

$$N_F(z) = \sum_{F(w)=z} \log \frac{1}{|z|}, \qquad z \in \mathbb{D},$$

be its Nevanlinna counting function. *We have:*

(i) *The inequality $N_F(z) \le \log \frac{1}{|z|}$ holds for any $z \in \mathbb{D}$.*

(ii) *If F is an inner function, then $N_F(z) = \log \frac{1}{|z|}$ outside a set of logarithmic capacity zero.*

(iii) *If the equality $N_F(z) = \log \frac{1}{|z|}$ holds at a single point, then F is an inner function.*

Using Lemmas 2.2 and 2.3, it is easy to see that C_F is bounded on H^2:

$$\begin{aligned}
\|C_F g\|_{H^2}^2 &= |g(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |(g \circ F)'(z)|^2 \log \frac{1}{|z|} |dz|^2 \\
&= |g(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |g'(z)|^2 N_F(z) |dz|^2 \\
&\le |g(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |g'(z)|^2 \log \frac{1}{|z|} |dz|^2 \\
&= \|g\|_{H^2}^2.
\end{aligned}$$

If F is an inner function and $g(0) = 0$, then the inequalities in the computation above are equalities and we have $\|C_F g\|_{H^2}^2 = \|g\|_{H^2}^2$. As explained in [Sha00, Theorem 5.1], if F is not inner, then $C_F g$ is a strict contraction on the orthogonal complement of the constant functions:

Lemma 2.4. *Let F be a holomorphic self-map of the unit disk with $F(0) = 0$. If F is an inner function, then C_F is an isometry on H^2. Otherwise, F is a strict contraction on $H_0^2 = \{g \in H^2 : g(0) = 0\}$, i.e. $\|C_F\|_{H_0^2 \to H_0^2} < 1$.*

The *spectrum* σ of an operator consists of all $\lambda \in \mathbb{C}$ so that $\lambda I - F$ does not have a bounded inverse. The *spectral radius* is the radius of the smallest closed disk centered at the origin which contains the spectrum. Since C_F is a bounded

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operator acting on a Banach space, its spectral radius may be computed by the formula

$$r_{H^2}(C_F) = \lim_{n \to \infty} \|C_F^n\|^{1/n}.$$

We are more interested in the *essential spectum* of C_F, which is the part of the spectrum that survives compact perturbations. More precisely,

$$\sigma_e(C_F) := \bigcap_K \sigma(C_F - K),$$

where K ranges over compact operators. The *essential spectral radius* is the radius of the smallest disk centered at the origin which contains the essential spectrum. Outside $\overline{B(0, r_e(C_F))}$, the spectrum $\sigma(C_F)$ can consist only of isolated eigenvalues of finite multiplicity.

In the literature, one often says that an operator has a *spectral gap* if its essential spectral radius r_e is strictly smaller than its spectral radius r. For our purposes, it is preferable to use a more restrictive definition: we say that an operator has a spectral gap if it has a simple eigenvalue λ with $|\lambda| = r$, and the rest of the spectrum is contained in a ball $B(0, \rho)$ for some $\rho < r$.

The *essential norm* of C_F is defined as

$$\|C_F\|_{e, H^2 \to H^2} := \inf_K \{\|C_F - K\|_{H^2 \to H^2}\}.$$

The spectral radius formula shows that $r_{e, H^2}(C_F) \leq \|C_F\|_{e, H^2 \to H^2}$. In an important work [Sha87], J. H. Shapiro computed the essential norm of C_F when acting on H^2:

Theorem 2.5. *If F is a holomorphic self-map of the unit disk, then the essential norm of the composition operator C_F acting on H^2 is*

$$\|C_F\|_{e, H^2 \to H^2} = \limsup_{|z| \to 1} \frac{N_F(z)}{\log(1/|z|)}.$$

As a consequence of Lemma 2.4, we see that the spectral properties of the composition operator C_F on H^2 can detect whether or not a holomorphic self-map of the unit disk is an inner function:

Corollary 2.6. *Let F be a holomorphic self-map of the unit disk with $F(0) = 0$. If F is an inner function, then C_F is an isometry on H^2. Otherwise,*

$$r_{e, H^2}(C_F) \leq \|C_F\|_{e, H^2 \to H^2} < 1$$

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and the eigenvalue 1 with the eigenfunction **1** *is isolated and simple.*

For a general exponent $1 \le p < \infty$, Bourdon and Shapiro [BS97] showed the following theorem which implies that Corollary 2.6 holds for all Hardy spaces H^p with $1 \le p < \infty$:

Theorem 2.7. *If F is a holomorphic self-map of the unit disk, then*

$$r_{e,H^p}(C_F)^p \le r_{e,H^2}(C_F)^2, \qquad 1 \le p < \infty,$$

with equality holding whenever $1 < p < \infty$.

2.2   Weighted Bergman Spaces

While the results of the previous section are quite elegant, for the purposes of studying the dynamics of inner functions, they are slightly disappointing since the composition operators C_F do not exhibit a spectral gap in this case. We therefore turn our attention to Bergman spaces.

For $\alpha > -1$, the weighted Bergman space A_α^p consists of holomorphic functions that satisfy

$$\|g\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |g(z)|^p \, dA_\alpha < \infty, \qquad dA_\alpha = \frac{1}{\pi}(\alpha+1)(1-|z|^2)^\alpha |dz|^2.$$

As in [HKZ00], we normalize the area form dA_α so that $\|\mathbf{1}\|_{A_\alpha^p}^2 = 1$. When the exponent $p = 2$, the above condition can be rephrased in terms of the coefficients of the power series expansion at 0:

$$\|g\|_{A_\alpha^2}^2 = \sum_{n=0}^\infty \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}|a_n|^2 \asymp \sum_{n=0}^\infty \frac{|a_n|^2}{n^{1+\alpha}} < \infty. \qquad (2.2)$$

For weighted Bergman spaces, the spectral properties of C_F are related to the behaviour of the weighted Nevanlinna counting function

$$N_{F,\gamma}(z) = \sum_{F(w)=z} \left(\log \frac{1}{|z|}\right)^\gamma, \qquad \text{with } \gamma = \alpha + 2.$$

As explained in [Sha87, Section 6], the proof of the upper bound in Theorem 2.5 can be easily adapted to weighted Bergman spaces. Several years later, P. Poggi-Corradini [PC98] showed that Shapiro's bound was sharp for $\alpha = 0, 1$:

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Theorem 2.8. *If F is a holomorphic self-map of the unit disk and $\alpha > -1$, then the essential spectral radius of the composition operator C_F acting on A_α^2 satisfies*

$$\|C_F\|_{e, A_\alpha^2 \to A_\alpha^2} \leq \limsup_{|z| \to 1} \frac{N_{F, \alpha+2}(z)}{\log(1/|z|)^{\alpha+2}}.$$

Equality holds for $\alpha = 0, 1$.

Using the above theorem, it is easy to see that if F is not a rotation, then C_F has a spectral gap:

Corollary 2.9. *Let F be a holomorphic self-map of the unit disk with $F(0) = 0$, which is not a rotation. For $-1 < \alpha < \infty$, the essential spectral radius of the composition operator C_F acting on A_α^2 is strictly less than 1, while the eigenvalue 1 with eigenfunction $\mathbf{1}$, is isolated and simple.*

Proof. The proof is based on the following elementary observation: if $\{a_i\}$ is a collection of positive real numbers with $\sum a_i \leq x$, then

$$\sum a_i^\gamma \leq x^\gamma, \qquad \gamma = \alpha + 2 > 1.$$

In addition, equality can hold if and only if one of the numbers a_i's is equal to x and all the remaining numbers a_i are equal to 0. If one can guarantee that $a_i \leq \theta x$ for some $0 < \theta < 1$ and all $i \in \mathbb{N}$, then one has the stronger estimate

$$\sum_{i \in \mathbb{N}} a_i^\gamma \leq \{\theta^\gamma + (1-\theta)^\gamma\} x^\gamma.$$

In light of this observation and Theorem 2.8, to show that $r_{e, A_\alpha^2}(C_F) < 1$, it suffices to prove that

$$\limsup_{|z| \to 1, \, F(w) = z} \frac{\log(1/|w|)}{\log(1/|z|)} < 1. \tag{2.3}$$

Since F is not a hyperbolic isometry,

$$\gamma := 1 - \sup_{w_1 \in \partial B_{\mathrm{hyp}}(0,1)} \{d_{\mathbb{D}}(0, F(w_1))\} > 0,$$

where $d_{\mathbb{D}}(\cdot, \cdot)$ denotes hyperbolic distance on $\mathbb{D}$. For a point $w \in \mathbb{D}$ with $d_{\mathbb{D}}(0, w) \geq 1$, let w_1 be the point on the hyperbolic geodesic $[0, w]$ which is

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located a hyperbolic distance 1 from the origin. By the triangle inequality and the Schwarz lemma,

$$\begin{aligned} d_{\mathbb{D}}(0, F(w)) &\leq d_{\mathbb{D}}(0, F(w_1)) + d_{\mathbb{D}}(F(w_1), F(w)) \\ &\leq (1 - \gamma) + (d_{\mathbb{D}}(0, w) - 1) \\ &= d_{\mathbb{D}}(0, w) - \gamma, \end{aligned}$$

which implies that

$$1 - |F(w)| < c(1 - |w|), \qquad w \notin B_{\mathrm{hyp}}(0, 1),$$

for some $0 < c < 1$. The inequality (2.3) follows since $\log(1/|w|) \sim 1 - |w|$ as $|w| \to 1$.

Since we have proved that the essential spectral radius of C_F is less than 1, the eigenvalue 1 is isolated and simple by Lemma 2.1. $\qquad\square$

A similar situation holds for the weighted Bergman spaces A_α^p with general exponents p. In [MS02, Theorem 5 and Proposition 7], MacCluer and Saxe proved an analogue of Theorem 2.7 for the unweighted Bergman spaces A^p. Their argument was generalized to the weighted case in [ACKS20, Theorem 2.8]:

Theorem 2.10. *Let F be a holomorphic self-map of the unit disk. The essential spectral radius of C_F acting on A_α^p satisfies*

$$r_{e, A_\alpha^p}(C_F)^p \leq r_{e, A_\alpha^p}(C_F)^2, \qquad 1 \leq p < \infty,$$

with equality for $p > 1$. In particular, unless F is a rotation, then

$$r_{e, A_\alpha^p}(C_F) < 1, \qquad 1 \leq p < \infty.$$

From the above theorem, it follows that Corollary 2.9 also holds for the weighted Bergman spaces A_α^p with $\alpha > -1$ and $1 < p < \infty$.

2.3   Sobolev Spaces

Below, we identify distributions on the unit circle with their harmonic extensions to the unit disk. Since the composition operator C_F is a *real* operator,

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it preserves the anti-holomorphic spaces $\overline{H^2}$ and $\overline{A_\alpha^2}$ in addition to their holomorphic counterparts. It follows that C_F also preserves $L^2 = H^2 + \overline{H^2}$ and the Sobolev spaces $W^{-\beta,2} = (W^{\beta,2})^* = A_\alpha^2 + \overline{A_\alpha^2}$ for any $\beta = (1+\alpha)/2 > 0$. The intersection $H^2 \cap \overline{H^2} = A_\alpha^2 \cap \overline{A_\alpha^2} = \mathbb{C} \cdot \mathbf{1}$ consists of the constant functions. One can check these decompositions by comparing power series expansions.

3   Transfer Operators

For a finite Blaschke product F with $F(0) = 0$, the *transfer operator* with potential $-\log|F'|$ acts on functions on the unit circle by

$$(L_F g)(x) = \sum_{F(y)=x} |F'(y)|^{-1} g(y). \qquad (3.1)$$

Since the Lebesgue measure on the unit circle is invariant under F,

$$L_F \mathbf{1} = \mathbf{1}.$$

The invariance of the Lebesgue measure also shows that the transfer operator is the adjoint of the the composition operator with respect to the L^2-pairing on the unit circle:

$$\int_{\partial \mathbb{D}} L_F g(x) \overline{h(x)} dm(x) = \int_{\partial \mathbb{D}} g(x) \overline{h(F(x))} dm(x). \qquad (3.2)$$

For a general inner function F, we simply *define* the transfer operator as the adjoint of the composition operator.

In this section, we describe a number of spaces on which L_F acts, represent L_F in terms of Aleksandrov-Clark measures and discuss a few simple consequences of the spectral gap such as the exponential decay of correlations.

3.1   Duals of some Function Spaces

We now describe the duals of some function spaces of harmonic and holomorphic functions with respect to the L^2-pairing on the unit circle. As in Section 2.3, we identify functions f, g in the unit disk with their boundary distributions

$\tilde{f}, \tilde{g}$ on the unit circle. The L^2-pairing can be interpreted in terms of Fourier series expansions:

$$\langle \tilde{f}, \tilde{g} \rangle = \left\langle \sum a_n e^{in\theta}, \sum b_m e^{im\theta} \right\rangle = \sum a_n \overline{b_n}.$$

Alternatively, one can dilate the functions f, g and take the limit as $r \to 1$:

$$\langle f, g \rangle = \lim_{r \to 1} \int_{|z|=1} f(rz) \overline{g(rz)} \, |dz|.$$

Hardy spaces. Recall from Theorem 2.7 that the composition operator preserves the Hardy space H^p for any $1 \le p < \infty$. Therefore, the transfer operator preserves $(H^p)^* \cong H^q$ where $1/p + 1/q = 1$. Since the transfer operator is a *real* operator, it must preserve $\overline{H^q}$ as well. In particular, the transfer operator preserves $L^q = H^q + \overline{H^q}$ for $1 < q \le \infty$.

Weighted Bergman spaces. We saw in Theorem 2.10 that the composition operator C_F has a spectral gap on the weighted Bergman space A_α^p for any $1 \le p < \infty$ and $\alpha > -1$. The dual of A_α^p with respect to the $\langle \cdot, \cdot \rangle_{A_\alpha^p}$-pairing is A_α^q where q is the conjugate exponent. To obtain the dual with respect to the L^2-pairing on the circle, one needs to take a fractional integral of order $1 + \alpha$, i.e. $(A_\alpha^p)^* \cong \mathbf{I}_{1+\alpha}(A_\alpha^q)$. Below, we consider the cases when $p = 1, 2$ in detail, where it is particularly simple to write down the dual space explicitly.

$p = 2$. Recall that in terms of power series expansions, the space A_α^2 consists of holomorphic functions on the unit disk such that

$$\sum_{n=0}^{\infty} n^{-1-\alpha} |a_n|^2 < \infty.$$

Its dual is therefore the Dirichlet-type space $\mathcal{D}_{1+\alpha}$ which consists of holomorphic functions on the unit disk such that

$$\sum_{n=0}^{\infty} n^{1+\alpha} |a_n|^2 < \infty.$$

Again, using the fact that the transfer operator is a real operator, we see that it preserves the Sobolev space $W^{(1+\alpha)/2,2} = \mathcal{D}_{1+\alpha} + \overline{\mathcal{D}_{1+\alpha}}$. Since adjoint operators have the same spectral and essential spectral radii, L_F has a spectral

gap on $\mathcal{D}_{1+\alpha}$ and $W^{(1+\alpha)/2,2}(\partial\mathbb{D})$. Furthermore, as the adjoint operator has the same isolated eigenvalues with the same multiplicity, 1 is a simple isolated eigenvalue with eigenfunction $\mathbf{1}$ in each of those spaces.

Remark. The standard Dirichlet space $\mathcal{D} = \mathcal{D}_1$ corresponds to the case when $\alpha = 0$ and consists of holomorphic functions such that

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

The associated Sobolev space is $W^{1/2,2} = \mathcal{D} + \overline{\mathcal{D}}$.

$p = 1$. By [Zhu05, Theorem 7.6], for $0 < \beta < 1$, the dual space

$$(A^1_{\beta-1})^* \cong \mathrm{Hol}(\mathbb{D}) \cap C^\beta(\partial\mathbb{D})$$

consists of holomorphic functions on the unit disk that have Hölder continuous extensions on the closed unit disk. The same theorem identifies

$$(A^1)^* \cong \mathrm{Hol}(\mathbb{D}) \cap Z(\partial\mathbb{D})$$

with the space of holomorphic functions on the unit disk that admit Zygmund extensions. More generally, if we take duals of A^1_α with $\alpha > 0$, we get the holomorphic functions in $C^{n,\beta}$ with some integer $n \geq 0$ and $0 < \beta < 1$, as well as in $C^{n,Z}$, the space of n-fold integrals of Zygmund functions. As above, the transfer operator has a spectral gap when acting on these spaces and their real-valued counterparts.

3.2   Aleksandrov-Clark Measures

Let F be a holomorphic self-map of the unit disk. For $\alpha \in \partial\mathbb{D}$, consider the function
$$h_\alpha(z) = \frac{\alpha + F(z)}{\alpha - F(z)}, \qquad z \in \mathbb{D}.$$

Since h_α is holomorphic and has positive real part, it is the Herglotz extension of a positive measure on the unit circle:

$$h_\alpha(z) = \frac{\alpha + F(z)}{\alpha - F(z)} = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_\alpha(\zeta). \tag{3.3}$$

The collection of measures $\{\mu_\alpha\}_{\alpha\in\partial\mathbb{D}}$ are known as the *Aleksandrov-Clark measures* of F. For basic properties of Aleksandrov-Clark measures, we refer the reader to [CMR06, PS06, Sak07].

- With the normalization $F(0) = 0$, each measure μ_α has unit mass. When F is an inner function, for each $\alpha \in \partial\mathbb{D}$, $\mathrm{Re}\, h_\alpha$ has radial limit zero a.e. on the unit circle, which means that the measure μ_α is singular.

- The Aleksandrov-Clark measure μ_α is concentrated on the set of pre-images of α. For instance, μ_α has a point mass at $\zeta \in \partial\mathbb{D}$ if and only if F has an angular derivative at ζ with $F(\zeta) = \alpha$, in which case,

$$\mu_\alpha(\zeta) = |F'(\zeta)|^{-1}.$$

  In general, μ_α may not be a discrete measure, i.e. be a countable convex sum of point masses. In fact, there are holomorphic self-maps of the unit disk, even inner functions, that do not possess an angular derivative at any point on the unit circle.

- The Aleksandrov-Clark measures $\{\mu_\alpha\}$ vary continuously in α in the weak-$*$ topology. More generally, given a sequence of holomorphic self-maps of the unit disk $F_n \to F$ converging uniformly on compact subsets of the unit disk and unimodular numbers $\alpha_n \to \alpha$, the Aleksandrov-Clark measures $\mu_{F_n,\alpha_n} \to \mu_{F,\alpha}$ converge in weak-$*$.

- Aleksandrov's disintegration theorem says that for any continuous function $f(\zeta)$ on the unit circle,

$$\int_{\partial\mathbb{D}} \left(\int_{\partial\mathbb{D}} f(\zeta) d\mu_\alpha(\zeta) \right) dm(\alpha) = \int_{\partial\mathbb{D}} f(\zeta) dm(\zeta).$$

  In other words, the Aleksandrov-Clark measures $\{\mu_\alpha\}$ are just the conditional measures of the Lebesgue measure on $F^{-1}(\epsilon)|\epsilon$, where ϵ is the partition of the unit circle $\partial\mathbb{D}$ into points.

With help of the Aleksandrov-Clark measures, one can write down an explicit formula for the adjoint of the composition operator:

$$(L_F g)(\alpha) = \int_{\partial\mathbb{D}} g(z) d\mu_\alpha(z). \tag{3.4}$$

Indeed, the above formula simplifies to (3.1) for finite Blaschke products, while the general case follows from approximation.

3.3 Comparing Definitions

In the following theorem, we characterize inner functions for which the *adjoint* definition of the transfer operator coincides with the *classical* definition of the transfer operator (3.1) a.e.

Theorem 3.1. *Suppose F is a centered inner function. The Alexandrov-Clark measures $\{\mu_\alpha\}$ are discrete for almost every $\alpha \in \partial \mathbb{D}$ if and only if F has an angular derivative at a.e. point on the unit circle.*

For the proof, we will need the following lemma due to Craizer [Cra91, Lemma 5.4]:

Lemma 3.2. *If F a centered inner function has an angular derivative at a.e. point on the unit circle, then there exists a sequence of finite Blaschke products $\{F_n\}_{n=1}^\infty$ converging to F uniformly on compact subsets of the unit disk so that*

$$|F_n'(x)| < 2|F'(x)|,$$

for a.e. $x \in \partial \mathbb{D}$ and all integers $n \geq 1$.

Proof of Theorem 3.1. Suppose F has an angular derivative at a.e. point on the unit circle. We want to show that for almost every $\alpha \in \partial \mathbb{D}$,

$$\sum_{\substack{\zeta \in F^{-1}(\alpha) \\ |F'(\zeta)| < \infty}} \frac{1}{|F'(\zeta)|} = 1.$$

Let $F_n \to F$ be a sequence of finite Blaschke products from Lemma 3.2. Fix an $\varepsilon > 0$. Since F has angular derivatives a.e., we can choose $N > 0$ sufficiently large so that

$$m\big(\{|F'| > N/2\}\big) \leq \varepsilon^2,$$

which in turn, implies that

$$m\big(\{|F_n'| > N\}\big) \leq \varepsilon^2, \tag{3.5}$$

for all $n \geq n_0(N)$ sufficiently large.

Call a point $\alpha \in \partial\mathbb{D}$ (ε, N)-*good* for F_n if most of the mass of $\mu_{F_n,\alpha}$ comes from atoms of size $\geq 1/N$:

$$\sum_{\substack{\zeta \in F_n^{-1}(\alpha) \\ |F_n'(\zeta)| \leq N}} \frac{1}{|F_n'(\zeta)|} \geq 1 - \varepsilon. \tag{3.6}$$

It follows from (3.5) that F_n is (ε, N)-good on a set $\mathcal{G}_n$ of Lebesgue measure $\geq 1 - \varepsilon$. Let $\mathcal{G}$ be the set of points on the unit circle that lie in infinitely many $\mathcal{G}_n$. As $\mu_{F_n,\alpha} \to \mu_{F,\alpha}$ weakly, $m(\mathcal{G}) \geq 1 - \varepsilon$ and almost every $\alpha \in \mathcal{G}$ is (ε, N)-good for F. For these $\alpha \in \partial\mathbb{D}$, the discrete part of $\mu_{F,\alpha}$ has mass at least $\geq 1 - \varepsilon$. The lemma follows since $\varepsilon > 0$ was arbitrary. $\qquad\square$

For another proof, see [CMR06, Theorem 9.6.1].

Corollary 3.3. *Suppose F is a centered inner function. If F has an angular derivative at a.e. point on the unit circle, then F is essentially countable-to-one, i.e. the set $F^{-1}(\alpha)$ is countable for a.e. $\alpha \in \partial\mathbb{D}$.*

3.4   Equidistribution

We keep assuming that F is a centered inner function. Let $\alpha \in \partial\mathbb{D}$ be a point on the unit circle. The following lemma says that as $n \to \infty$, the pre-images $F^{-n}(\alpha)$ become equidistributed on the unit circle with respect to Lebesgue measure. A similar statement holds for the periodic points $\{x \in \partial\mathbb{D} : F^{\circ n}(x) = x\}$ of period n.

Lemma 3.4. *For an inner function F with $F(0) = 0$, we have:*

(i) *For every $\alpha \in \partial\mathbb{D}$ the sequence $\{\mu_{F^{\circ n},\alpha}\}_{n=1}^{\infty}$ converges weakly to the Lebesgue measure m on the unit circle.*

(ii) *As $n \to \infty$, the measures $\{\mu_{F^{\circ n}(z)/z,1}\}_{n=1}^{\infty}$ converge weakly to the Lebesgue measure m on the unit circle.*

Proof. We only prove (i) as the proof of (ii) is similar. According to the definition of the Aleksandrov-Clark measures,

$$\frac{\alpha + F^{\circ n}(z)}{\alpha - F^{\circ n}(z)} = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \, d\mu_{F^{\circ n},\alpha}(\zeta).$$

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Since 0 is an attracting fixed point, as $n \to \infty$, the left hand side tends to 1 for any $z \in \mathbb{D}$. In turn, this implies that $\mu_{F^{\circ n}, \alpha}$ converge weakly to the Lebesgue measure. $\square$

Corollary 3.5. *If F is a centered inner function, then the set of its periodic points is dense in the unit circle.*

3.5 Consequences of the Spectral Gap

We have seen in Section 3.1 that if F is a centered inner function, then L_F has a spectral gap on the spaces $W^{\beta,2}$ with $\beta > 0$ and C^α with $\alpha > 0$.

Let R be the Riesz projection onto the one-dimensional eigenspace spanned by the eigenfunction $\mathbf{1}$ (corresponding to the eigenvalue 1) and $S = \mathrm{Id} - R$ be the complementary Riesz projection onto the rest of the spectrum. From the general properties of Riesz projections, we know that:

- The Riesz projections R and S commute with L_F.

- $RS = SR = 0$.

- $\sigma(L_F|_{\mathrm{Im}\,R}) = \{1\}$ and $\sigma(L_F|_{\mathrm{Im}\,S}) = \sigma(L_F) \setminus \{1\}$.

We write $L_F = L_F(R + S) = R + \Delta$ where $\Delta = L_F S$. As the spectral radius $r(\Delta) < 1$, we have
$$\|\Delta^n g\| \le C\theta^n,$$
for some constants $\theta \in (0, 1)$, $C \in (0, +\infty)$ and all integers $n \ge 1$. Integrating both sides of the equality
$$L_F^n g = Rg + \Delta^n g$$
over the unit circle, and then taking $n \to \infty$, we get
$$\lim_{n \to \infty} \int_{\partial \mathbb{D}} L_F^n g \, dm = Rg.$$

Since $L_F^* m = m$, the left hand side is simply $\int_{\partial \mathbb{D}} g \, dm$. Therefore, the Riesz projection is given by
$$Rg = \overline{g} \cdot \mathbf{1}, \qquad \text{where } \overline{g} = \int_{\partial \mathbb{D}} g \, dm.$$

We conclude that on the set of mean zero functions, $L_f = \Delta$ is a contraction.

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We state the following lemma for Sobolev spaces, but it is also true for Hölder continuous functions with the same proof:

Lemma 3.6. *If F is a centered inner function, then the following hold:*

(i) For any $g \in W^{\beta,2}$, the sequence of functions $\{L_F^n g\}_{n=1}^\infty$ converges to the constant function $\int_{\partial\mathbb{D}} g(x)dm$ in $W^{\beta,2}$.

(ii) There exist constants $\theta \in (0,1)$ and $C \in (0,+\infty)$ such that for any two functions $g, h \in W^{\beta,2}$ with $\int_{\partial\mathbb{D}} g \, dm = 0$ or $\int_{\partial\mathbb{D}} h \, dm = 0$,

$$\left| \int_{\partial\mathbb{D}} g(h \circ F^{\circ n}) dm \right| < C\theta^n.$$

Since $W^{\beta,2}$ is dense in $L^2(\partial\mathbb{D}, m)$, the above lemma implies that F is mixing with respect to the Lebesgue measure m:

Corollary 3.7. *If F is a centered inner function, then the following hold:*

(i) For any $g \in L^2(\partial\mathbb{D}, m)$, the sequence of functions $\{L_F^n g\}_{n=1}^\infty$ converges to the constant function $\int_{\partial\mathbb{D}} g(x)dm$ in $L^2(\partial\mathbb{D}, m)$.

(ii) For any $g, h \in L^2(\partial\mathbb{D}, m)$ with $\int_{\partial\mathbb{D}} g \, dm = 0$ or $\int_{\partial\mathbb{D}} h \, dm = 0$,

$$\left| \int_{\partial\mathbb{D}} g(h \circ F^{\circ n}) dm \right| \to 0, \qquad as \ n \to \infty.$$

3.6 Composition Operators Acting on Measures

We have seen that the composition operator C_F acts isometrically on the space $L^2(\partial\mathbb{D}, m)$. We can extend C_F to a continuous operator on the Borel probability measures on the unit circle equipped with the weak-$*$ topology by specifying the integrals of continuous functions:

$$\int_{\partial\mathbb{D}} \phi \, d(C_F\nu) = \int_{\partial\mathbb{D}} \left\{ \int_{F^{-1}(\alpha)} \phi \, d\mu_\alpha \right\} d\nu(\alpha) = \int_{\partial\mathbb{D}} L_F\phi \, d\nu. \qquad (3.7)$$

A monotone class argument similar to the one in [CMR06, Chapter 9.4] shows that the above formula is valid for any bounded Borel function ϕ.

Let $\mathcal{R}_F \subset \partial\mathbb{D}$ be the set of points where $\lim_{r\to 1} F(r\zeta)$ exists and is unimodular. By [CL66, Theorem 2.6], $\mathcal{R}_F$ is an $F_{\sigma\delta}$ set and therefore Borel. Plugging $\phi = \chi_{\mathcal{R}_F^c}$ into (3.7) and using the fact that Aleksandrov-Clark measures do not charge $\mathcal{R}_F^c$ shows the following lemma:

Lemma 3.8. *If F a centered inner function, then for any Borel probability measure ν on the unit circle, the measure $C_F\nu$ is supported on the set of points on the unit circle where $\lim_{r\to 1} F(r\zeta)$ exists and is unimodular.*

Lemma 3.9. *Let F be a centered inner function and ν be a probability measure on the unit circle such that $C_F\nu = \nu$. Then, ν is an F-invariant measure supported on the set of points where the forward orbit of F belongs to the unit circle.*

Proof. Plugging in $\phi = \chi_{F^{-1}(E)}$ where E is a Borel set into (3.7), we get

$$C_F\nu(F^{-1}(E)) = \nu(E),$$

which means that ν is F-invariant. An inductive argument shows ν cannot charge any of the sets $F^{-n}(\mathcal{R}_F^c)$ with $n \geq 0$. $\qquad\square$

4 Perturbative Thermodynamic Formalism and the Central Limit Theorem for Centered Inner Functions

In this section, we use weighted composition operators to establish the rudiments of thermodynamic formalism for inner functions:

1. For a $W^{1/2,2}$ or C^α potential of sufficiently small norm, we construct conformal and equilibrium measures on the unit circle.

2. We then discuss analytic families of weighted composition operators and show their eigenvalues vary analytically, from which we deduce the Central Limit Theorem.

3. We then give a sufficient condition for the asymptotic variance to not vanish, thereby completing the proof of Theorem 1.2.

For concreteness, we work with the Sobolev potentials and leave the space of Hölder potentials to the reader.

4.1   Weighted Composition Operators

Let F be a centered inner function. The *weighted composition operator $C_{F,\gamma}$* is given by

$$C_{F,\gamma}(g) = \gamma(z) \cdot (g \circ F(z)), \quad z \in \mathbb{D}.$$

We first examine the action of $C_{F,\gamma}$ on various spaces of holomorphic functions. When $\gamma \in H^\infty$ is a bounded analytic function, multiplication by γ preserves the Hardy space H^2 and the Bergman space A^2, so that $C_{F,\gamma}$ acts on H^2 and A^2. Taking the adjoint with respect to the L^2-pairing on the unit circle, we see that the *weighted transfer operator*

$$(L_{F,\gamma}g)(\alpha) := L_F(\gamma g)(\alpha) = \int_{\partial \mathbb{D}} \gamma \cdot g \, d\mu_\alpha \tag{4.1}$$

acts on the Hardy space H^2 and the Dirichlet space $\mathcal{D}$.

For applications to dynamical systems, we need to consider the action of weighted composition operators on real Banach spaces. If γ is an L^∞ function, then the operator $C_{F,\gamma}$ preserves L^2. In order for $C_{F,\gamma}$ to preserve $W^{-1/2,2}$, it is enough to take $\gamma \in \mathcal{M}(W^{-1/2,2})$, which is a more stringent condition.

Remark. (i) When $F(z) = z$ is the identity mapping, $C_{F,\gamma}$ is bounded on $W^{-1/2,2}$ if and only if $\gamma \in \mathcal{M}(W^{-1/2,2})$. However, for other inner functions F, the set of admissible γ may be strictly larger.

(ii) A simple duality argument shows that $\mathcal{M}(W^{-1/2,2}) = \mathcal{M}(W^{1/2,2})$. Any multiplier of $W^{-1/2,2}$ is a bounded function, but the converse is not true.

For future reference, we also make the following simple observations:

Lemma 4.1. *If F is a centered inner function and γ is a weight in $L^\infty(\partial \mathbb{D}, m)$, then for any functions $g \in L^\infty(\partial \mathbb{D}, m)$ and $h \in L^2(\partial \mathbb{D}, m)$, we have*

$$L_{F,\gamma}(g \circ F \cdot h) = g \cdot L_{F,\gamma}(h).$$

Proof. Since for any $\alpha \in \partial \mathbb{D}$, the Aleksandrov-Clark measure μ_α is supported on the set of points $\zeta \in \partial \mathbb{D}$ for which $\lim_{r \to 1} F(r\zeta) = \alpha$, we have:

$$L_{F,\gamma}(g \circ F \cdot h)(\alpha) = \int_{\partial \mathbb{D}} \gamma h \cdot g \circ F \, d\mu_\alpha$$

$$= g(\alpha) \int_{\partial \mathbb{D}} \gamma h \, d\mu_\alpha$$

$$= g(\alpha) \cdot L_{F,\gamma}(h)(\alpha),$$

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as desired. $\square$

Lemma 4.2. *If F is a centered inner function and γ is a weight in $L^\infty(\partial\mathbb{D})$, then for any function $h \in L^2(\partial\mathbb{D})$ and $\alpha \in \partial\mathbb{D}$, we have*

$$L_{F,\gamma}^n(h)(\alpha) = \int_{\partial\mathbb{D}} \Pi_n\gamma \cdot h \, d\mu_{F^{\circ n},\alpha},$$

where

$$\Pi_n\gamma(x) = \gamma(x)\gamma(F(x))\ldots\gamma(F^{\circ(n-1)}(x)).$$

Proof. For weighted composition operators, it is clear that $C_{F,\gamma}^n = C_{F^{\circ n},\Pi_n\gamma}$. Taking adjoints, we get

$$L_{F,\gamma}^n(h)(\alpha) = L_{F^{\circ n},\Pi_n\gamma}(h)(\alpha).$$

The lemma follows after expressing the weighted transfer operator on the right in terms of Aleksandrov-Clark measures as in (4.1). $\square$

In order to study stochastic laws via thermodynamic formalism, we need to work with families of weighted transfer operators. Suppose $s \to \gamma_s$ is a family of weights in $\mathcal{M}(W^{1/2,2})$, which depend analytically on a parameter $s \in U$, where $U \subset \mathbb{C}$ is an open set containing the origin and $\gamma_0 = \mathbf{1}$. From the definition of the weighted transfer operators (4.1), it is not difficult to see that $s \to L_{F,\gamma_s}$ defines an analytic function from U to the space of bounded linear operators on $W^{1/2,2}$ with derivative $(d/ds)\, L_{F,\gamma_s} = L_{F,\dot\gamma_s}$.

In Section 3.1, we saw that L_{F,γ_0} has a spectral gap on $W^{1/2,2}$: it has a simple isolated eigenvalue $\lambda_0 = 1$, while the rest of the spectrum is contained in a ball $B(0,r)$ with $r < 1$. From the Kato-Rellich Perturbation Theorem, it follows that for any complex number s sufficiently close to 0, L_{F,γ_s} also has a spectral gap: it possesses a simple isolated eigenvalue $\lambda_s \in \mathbb{C}$, while the rest of the spectrum is contained in $B(0,r_s)$ with $r_s < |\lambda_s|$. Moreover, perturbation theory tells us that in the presence of a spectral gap, both the eigenvalue λ_s and the Riesz projection R_s onto the one-dimensional eigenspace associated to the eigenvalue λ_s vary analytically in s.

From the defining property of the Riesz projection $L_{F,\gamma_s}R_s = R_sL_{F,\gamma_s}$, it is clear that $\rho_s = R_s\mathbf{1}$ generates the λ_s-eigenspace of L_{F,γ_s}. The number

$$P(s) = \log r_{W^{1/2,2}}(L_{F,\gamma_s}) = \log \lambda_s$$

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is called the *topological pressure* of the weight $\gamma_s \in \mathcal{M}(W^{1/2,2})$, where the branch of logarithm is chosen so that $P(0) = 0$.

4.2 Stochastic Laws with respect to the Lebesgue Measure

We now change the notation and write $\mathcal{L}_{-\log|F'|+g} := L_{F,e^g}$, as is standard in thermodynamic formalism. In other words, we specify the transfer operator by the potential $-\log|F'| + g$ rather than the weight e^g.

It is easy to see that if $g \in \mathcal{M}(W^{1/2,2})$ is a Sobolev multiplier, then for any complex number $s \in \mathbb{C}$,

$$e^{sg} = 1 + sg + \frac{s^2}{2!}g^2 + \frac{s^3}{3!}g^2 + \dots$$

is also a multiplier with

$$\|e^{sg}\|_{\mathcal{M}(W^{1/2,2})} \leq \exp\left(|s| \cdot \|g\|_{\mathcal{M}(W^{1/2,2})}\right).$$

In this case, the weighted transfer operator $\mathcal{L}_s := \mathcal{L}_{-\log|F'|+sg}$ is bounded on $W^{1/2,2}$ and varies analytically with s. From the discussion in the previous section, there exists a Riesz projection $R_s : W^{1/2,2} \to \mathbb{C} \cdot \rho_s$, defined for all complex numbers s sufficiently close to 0, which satisfies

$$\mathcal{L}_s R_s = R_s \mathcal{L}_s \qquad \text{and} \qquad \mathcal{L}_s \rho_s = e^{P(s)} \rho_s,$$

where $\rho_s = R_s \mathbf{1}$.

The following lemma [PP90, Propositions 4.10 and 4.11] describes the first two derivatives of the topological pressure:

Lemma 4.3. *Let $g \in \mathcal{M}(W^{1/2,2})$ be a Sobolev multiplier. Then,*

$$\dot{P}(0) = \int_{\partial\mathbb{D}} g \, dm.$$

If in addition g has mean 0, i.e. $\int_{\partial\mathbb{D}} g \, dm = 0$, then

$$\ddot{P}(0) = \sigma^2(g) = \lim_{n\to\infty} \frac{1}{n} \int_{\partial\mathbb{D}} (S_n g)^2 \, dm.$$

Proof. (i) Let f_s be the eigenfunction of the weighted composition operator $h \to e^{sg} h \circ F$ with eigenvalue $e^{P(s)}$. When $s = 0$, the eigenfunction $f_0 = \mathbf{1}$ and $P(0) = 0$. Differentiating

$$e^{sg(z)} f_s(F(z)) = e^{P(s)} f_s(z) \tag{4.2}$$

with respect to s at 0, gives

$$g(z) + \dot{f}_0(F(z)) = \dot{P} + \dot{f}_0(z).$$

Integrating with respect to m, we get

$$\int_{\partial\mathbb{D}} g(z) dm(z) = \dot{P} + \left\{ \int_{\partial\mathbb{D}} \dot{f}_0(z) dm(z) - \int_{\partial\mathbb{D}} \dot{f}_0(F(z)) dm(z) \right\}.$$

Since measure m is F-invariant, the two integrals cancel out.

(ii) Differentiating (4.2) twice at $s = 0$ gives

$$g(z)^2 + 2g(z)\dot{f}_0(F(z)) + \ddot{f}_0(F(z)) = \ddot{P}(0) + \ddot{f}_0(z).$$

Integrating with respect to m, we get

$$\int_{\partial\mathbb{D}} g^2 dm + 2 \int_{\partial\mathbb{D}} g \dot{f}_0 dm = \ddot{P}(0).$$

By instead differentiating the n-th iterate, we obtain

$$\int_{\partial\mathbb{D}} (S_n g)^2 dm + 2 \int_{\partial\mathbb{D}} S_n g \cdot \dot{f}_0 dm = n \ddot{P}(0).$$

The statement follows after dividing by n, taking $n \to \infty$ and applying the ergodic theorem. $\qquad\square$

From here, one can prove the Central Limit Theorem for real-valued observables in $\mathcal{M}(W^{1/2,2})$ as in [PP90, Proposition 4.13]. Alternatively, one can appeal to the work of S. Gouëzel [Gou10] which shows that the process $\{g \circ F^n : n \geq 1\}$ satisfies an *almost sure invariance principle* (ASIP). Loosely speaking, the ASIP says that one can redefine the process $\{g \circ F^n : n \geq 1\}$ on some probability space with a Brownian motion B_t, so that $S_n g$ is close to $B_{\sigma^2 n}$. With the help of the ASIP, one can obtain stochastic laws such as the Central Limit Theorem and the Law of the Iterated Logarithm for $g \circ F^n$ from the corresponding facts for Brownian motion, essentially for free.

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4.3   When is the Variance Non-Zero?

To complete the proof of Theorem 1.2, we need a mechanism for showing that the asymptotic variance of an observable $g \in \mathcal{M}(W^{1/2,2})$ with respect to the Lebesgue measure is not equal to zero:

Theorem 4.4. *Suppose F is a centered inner function that is not a finite Blaschke product. If $g \in \mathcal{M}(W^{1/2,2})$ is a real-valued non-constant function of mean zero, then $\sigma^2(g) \neq 0$.*

Before providing the proof of the above theorem, we make the following preliminary observation:

Lemma 4.5. *Suppose $g \in \mathcal{M}(W^{1/2,2})$ is a real-valued function on the unit circle. If $\|g\|_{\mathcal{M}(W^{1/2,2})}$ is sufficiently small, then for any $u \in W^{1/2,2}$, the sequence $\left\{ \mathcal{L}^n_{-\log|F'|+ig}(u) \right\}_{n=0}^{\infty}$ is bounded in $W^{1/2,2}$.*

Proof. When $\|g\|_{\mathcal{M}(W^{1/2,2})}$ is small, then $\sigma_{e,W^{1/2,2}}(\mathcal{L}_{-\log|F'|+ig}) < 1$. Outside the essential spectral radius, $\mathcal{L}_{-\log|F'|+ig}$ may only have countably many isolated eigenvalues of finite multiplicity. Therefore, to prove the lemma, it is enough to show that the operator $\mathcal{L}_{-\log|F'|+ig}$ acting on $W^{1/2,2}$ does not possess any eigenvalues with modulus greater than 1. However, this is easy: as $\mathcal{L}_{-\log|F'|+ig}$ is an isometry on $L^2(\partial\mathbb{D}, m)$, it cannot have any $L^2(\partial\mathbb{D}, m)$ eigenvalues outside the closed unit disk, let alone any $W^{1/2,2}$ eigenvalues. $\square$

For the remainder of this section, we assume that $\|g\|_{\mathcal{M}(W^{1/2,2})}$ is sufficiently small so that the assumptions of Lemma 4.5 are satisfied. The proof of Theorem 4.4 relies on [PP90, Propositions 4.12 and 4.2]:

Lemma 4.6. *If $\sigma^2(g) = 0$, then g is cohomologous to zero, that is, one can write*

$$g(x) = \zeta(F(x)) - \zeta(x), \tag{4.3}$$

for some real-valued function $\zeta \in L^2(\partial\mathbb{D}, m)$.

Lemma 4.7. *Any $L^2(\partial\mathbb{D}, m)$ eigenfunction of $e^{-ig(x)}C_F$ belongs to $W^{1/2,2}$.*

While the two lemmas above are stated for different function spaces in [PP90], the proofs are similar to the ones given in [PP90]. Below, we present

the proof of Lemma 4.7 as there are additional technicalities arising from working with Aleksandrov-Clark measures and allow the interested reader to consult [PP90] for the proof of Lemma 4.6.

Proof of Lemma 4.7. Suppose $w(x)$ is an L^2 eigenfunction of the weighted composition operator $e^{-ig(x)}C_F$:

$$e^{-ig(x)}w(F(x)) = \lambda w(x).$$

As the Lebesgue measure is ergodic with respect to F, $|\lambda| = 1$ and $|w|$ is constant a.e. Since w is not identically 0, by scaling it appropriately, we can make its absolute value 1 a.e. Multiplying the identity

$$w(F^{\circ n}(x)) \cdot \overline{w(x)} = \lambda^n e^{iS_n g(x)}$$

by a function $h \in W^{1/2,2}$, integrating with respect to the Aleksandrov-Clark measure $\mu_{F^{\circ n},\alpha}$ and using Lemma 4.2, we get

$$w(\alpha) \cdot \left[\mathcal{L}^n_{-\log|F'|}(h\overline{w})\right](\alpha) = \lambda^n \cdot \left[\mathcal{L}^n_{-\log|F'|+ig}h\right](\alpha). \tag{4.4}$$

As the sequence of functions $\{\mathcal{L}^n_{-\log|F'|+ig}h\}_{n=0}^{\infty}$ is bounded in $W^{1/2,2}$ by Lemma 4.5 and the inclusion $W^{1/2,2} \subset L^2(\partial\mathbb{D}, m)$ is compact, we can pass to a subsequence so that

$$\mathcal{L}^{n_k}_{-\log|F'|+ig}h \to h^* \text{ in } L^2, \qquad \text{for some } h^* \in W^{1/2,2}.$$

We pass to a further subsequence so that $\lambda^{n_k} \to \lambda^* \in \partial\mathbb{D}$. In view of Lemma 3.7, taking the L^2-limit as $n \to \infty$ in (4.4) gives

$$w \int_{\partial\mathbb{D}} \overline{w}h\,dm = \lambda^* h^*.$$

Since w is assumed to be non-zero, there exists an $h \in W^{1/2,2}$ so that

$$\int_{\partial\mathbb{D}} \overline{w}h\,dm \neq 0,$$

from which we conclude that w is a scalar multiple of h^* a.e. $\qquad\square$

By Lemma 4.6, if $\sigma^2(g) = 0$, then the weighted composition operator $e^{-ig}C_F$ possesses an $L^\infty(\partial\mathbb{D}, m)$ eigenfunction of with eigenvalue 1, namely $e^{i\zeta(x)}$. As $L^\infty(\partial\mathbb{D}, m) \subset L^2(\partial\mathbb{D}, m)$, Lemma 4.7 tells us that $e^{i\zeta(x)}$ is actually a $W^{1/2,2}$ eigenfunction. Therefore, to prove Theorem 4.4, we may show that $e^{-ig}C_F$ has no $W^{1/2,2}$ eigenfunctions.

Lemma 4.8. *Suppose $h \in \mathcal{D}$ is not constant and F is an inner function that is not a finite Blaschke product. Then $h \circ F \notin \mathcal{D}$.*

In the proof below, we will use the fact that if F is not a finite Blaschke product, then for a.e. point $w \in \mathbb{D}$, the pre-image $F^{-1}(w)$ is an infinite set. Actually, Frostman's theorem [Mas12, Theorem 2.5] says that the set of exceptional points has logarithmic capacity zero, but we will not need this stronger conclusion.

Proof. The Dirichlet seminorm

$$\int_{\mathbb{D}} |(h \circ F)'(z)|^2 \, |dz|^2$$

counts the weighted area of the image of $h \circ F$. Since F has infinite degree, $h \circ F$ covers a.e. point in the image of h infinitely many times. Therefore, if the image of h has positive area, then $h \circ F \notin \mathcal{D}$. Since non-constant holomorphic functions are open mappings, this can only happen if h is constant. $\square$

In particular, the above lemma implies that if an inner function belongs to the Dirichlet space, then it is a finite Blaschke product.

Proof of Theorem 4.4. As noted above, if $\sigma^2(g) = 0$, then the weighted composition operator $e^{-ig}C_F$ has a non-trivial eigenfunction in $W^{1/2,2}$:

$$e^{-ig(x)}w(F(x)) = \lambda w(x).$$

Clearly, the right hand side is in $W^{1/2,2}$. To obtain a contradiction, we show that the left hand side is not in $W^{1/2,2}$. Since $g \in \mathcal{M}(W^{1/2,2})$ implies that $e^{ig} \in \mathcal{M}(W^{1/2,2})$, this reduces our task to showing that $w(F(x)) \notin W^{1/2,2}$. By decomposing w into holomorphic and anti-holomorphic parts, its enough to consider the case when w is a holomorphic function in $W^{1/2,2}$, i.e. an element of the Dirichlet space $\mathcal{D}$. The existence of such a w is ruled out by Lemma 4.8. $\square$

4.4 Conformal and Equilibrium Measures

We now switch to Hölder potentials to avoid subtle issues arising from the fact that functions in the Sobolev space $W^{1/2,2}$ may be not be bounded. (One can resolve this technicality by working in the space $W^{\beta,2}$ for some $\beta > 1/2$, but by the Sobolev embedding theorem, these functions have some Hölder regularity, so we might as well work in C^α.)

Lemma 4.9. *Suppose $\gamma : \partial\mathbb{D} \to (0,\infty)$ is a positive function of class C^α for some $0 < \alpha < 1$. When $\|\gamma - 1\|_{C^\alpha}$ is sufficiently small, $L_{F,\gamma}$ has a spectral gap on C^α and the eigenspace associated to the dominant eigenvalue $\lambda > 0$ is spanned by a strictly positive function ρ_γ.*

Proof. In Section 3.1, we used a duality argument to show that L_F has a spectral gap on C^α. It is not difficult to see the spectral gap persists for $L_{F,\gamma}$ when $\|\gamma - 1\|_{C^\alpha}$ is small by applying the Kato-Rellich Perturbation Theorem to the analytic family of operators $s \to L_{F,1+s(\gamma-1)}$ and using the operator norm estimate $\|L_{F,\gamma_1} - L_{F,\gamma_2}\|_{\mathcal{B}(C^\alpha)} \lesssim \|\gamma_1 - \gamma_2\|_{C^\alpha}$.

Since $L_{F,\gamma}$ is a real operator, the one-dimensional eigenspace associated to the dominant eigenvalue λ is spanned by a real-valued function ρ_γ. (As $\overline{\rho_\gamma}$ is an eigenfunction with eigenvalue $\overline{\lambda}$, the spectral gap implies that $\overline{\rho_\gamma} = \rho_\gamma$ and $\overline{\lambda} = \lambda$.) Since the zero set of ρ_γ is F-invariant and ρ_γ is continuous, ρ_γ never vanishes by Lemma 3.4. Multiplying ρ_γ by -1 if necessary, we may assume that the eigenfunction ρ_γ is strictly positive. $\qquad\square$

Below, we construct conformal and equilibrium measures associated to a weight γ satisfying the hypotheses of the above lemma. We start with the conformal measure:

Lemma 4.10 (Conformal measure). *There exists a unique probability measure m_γ on $\partial\mathbb{D}$ such that $C_{F,\gamma} m_\gamma = \lambda m_\gamma$.*

We have seen in Section 3.6 that the composition operator C_F naturally extends to a positive operator on the space of Borel measures on the unit circle. Evidently, so does the weighted composition operator:

$$C_{F,\gamma}\nu := \gamma \, d(C_F\nu).$$

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In view of (3.7), for any bounded Borel function $\phi \in L^\infty(\partial\mathbb{D}, m)$, we have

$$\int_{\partial\mathbb{D}} \phi \, d(C_{F,\gamma}\nu) = \int_{\partial\mathbb{D}} \gamma\phi \, d(C_F\nu) = \int_{\partial\mathbb{D}} L_F(\gamma\phi)d\nu = \int_{\partial\mathbb{D}} L_{F,\gamma}\phi \, d\nu. \qquad (4.5)$$

Proof of Lemma 4.10. Step 1. Let $\mathcal{P}(\partial\mathbb{D})$ denote the collection of Borel probability measures on the unit circle endowed with topology of weak convergence. Consider the map $T : \mathcal{P}(\partial\mathbb{D}) \to \mathcal{P}(\partial\mathbb{D})$ given by

$$T\nu := \frac{C_{F,\gamma}\nu}{(C_{F,\gamma}\nu)(\partial\mathbb{D})}.$$

By the Schauder-Tychonoff Fixed Point Theorem, T has a fixed point. Thus, there exists a measure $m_\gamma \in \mathcal{P}(\partial\mathbb{D})$ such that

$$C_{F,\gamma}m_\gamma = \tilde{\lambda}m_\gamma, \qquad \text{for some } \tilde{\lambda} > 0.$$

We refer to the measure m_γ as a *conformal measure* associated to γ.

Step 2. The equation $C_{F,\gamma}^n m_\gamma = \tilde{\lambda}^n m_\gamma$ shows that

$$\tilde{\lambda}^n = \int_{\partial\mathbb{D}} L_{F,\gamma}^n \mathbf{1} \, dm_\gamma. \qquad (4.6)$$

As the eigenfunction ρ_γ is pinched between two positive constants, i.e.

$$C_1 \cdot \mathbf{1} \leq \rho_\gamma \leq C_2 \cdot \mathbf{1},$$

we have

$$C_1 \cdot L_{F,\gamma}^n \mathbf{1} \leq \lambda^n \rho_\gamma \leq C_2 \cdot L_{F,\gamma}^n \mathbf{1},$$

so the right hand side of (4.6) is comparable to λ^n. As a result, $\tilde{\lambda} = \lambda$.

Step 3. Let m_γ be a probability measure on the unit circle which satisfies $C_{F,\gamma}m_\gamma = \lambda m_\gamma$. Since $L_{F,\gamma}$ has a spectral gap on C^α, for any $g \in C^\alpha$, the sequence of functions $\{\lambda^{-n}L_{F,\gamma}^n g\}$ converges in C^α to $(\int_{\partial\mathbb{D}} gdm_\gamma)\rho_\gamma$. As C^α is dense in the continuous functions on the unit circle, this determines the measure m_γ uniquely. $\qquad\square$

Having constructed the conformal measure m_γ associated to the potential γ, we may normalize the eigenfunction ρ_γ so that $\int_{\partial\mathbb{D}} \rho_\gamma dm_\gamma = 1$.

Lemma 4.11 (Equilibrium measure). *There exists a unique F-invariant probability measure μ_γ on $\partial\mathbb{D}$ which is absolutely continuous with respect to m_γ. The Radon-Nikodym derivative is $d\mu_\gamma/dm_\gamma = \rho_\gamma$.*

Proof. Step 1. It is easy to see that the *equilibrium measure* $\mu_\gamma = \rho_\gamma dm_\gamma$ is invariant under F: if $g \in C(\partial\mathbb{D})$, then

$$\int_{\partial\mathbb{D}} g \circ F \, d\mu_\gamma = \int_{\partial\mathbb{D}} g \circ F \cdot \rho_\gamma \, dm_\gamma$$
$$= \lambda^{-1} \int_{\partial\mathbb{D}} L_{F,\gamma}(g \circ F \cdot \rho_\gamma) \, dm_\gamma$$
$$= \lambda^{-1} \int_{\partial\mathbb{D}} g L_{F,\gamma} \rho_\gamma \, dm_\gamma$$
$$= \int_{\partial\mathbb{D}} g \, d\mu_\gamma,$$

where we have used the duality relation (4.5) and Lemma 4.1.

Step 2. The duality relation (4.5) implies an exponential decay of correlations for C^α observables with respect to the measures m_γ and $\mu_\gamma = \rho_\gamma \, dm_\gamma$ as in Lemma 3.6. From the density of $C^\alpha(\partial\mathbb{D}) \subset L^2(\partial\mathbb{D})$, it follows F is mixing with respect to μ_γ.

Step 3. If there was another F-invariant measure $\tilde{\mu}$ absolutely continuous with respect to m_γ, then the Radon-Nikodym derivative

$$g = \frac{d\tilde{\mu}}{d\mu_\gamma}$$

would be an F-invariant function. Since F is ergodic with respect to μ_γ (being mixing), then g must be constant μ_γ-a.e. This proves uniqueness. $\qquad\square$

Remark. We use the term equilibrium measure to give μ_γ a name. We do not claim that the equilibrium measure maximizes $h_\mu(F) + \int_{\partial\mathbb{D}} \gamma \, d\mu$ over all F-invariant probability measures. Nor is it clear that the maximizer is unique as $h_{\mu_\gamma}(F) + \int_{\partial\mathbb{D}} \gamma \, d\mu_\gamma$ could be infinite.

Previously in Section 4.2, we have established the Central Limit Theorem and other stochastic laws with respect to the Lebesgue measure. Similar arguments show that the Central Limit Theorem also holds with respect to the equilibrium measures constructed above.

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Remark. When F is an inner function that is not a finite Blaschke product and γ is non-constant, it is not possible to normalize the transfer operator by setting

$$\widehat{L}_{F,\gamma}g = \frac{1}{\lambda_\gamma \rho_\gamma} \cdot L_{F,\gamma}(\rho_\gamma g),$$

so that $\widehat{L}_{F,\gamma}\mathbf{1} = \mathbf{1}$, since this amounts to replacing γ with

$$\widehat{\gamma}(x) = \gamma(x) + \rho_\gamma(F(x)) - \rho_\gamma(x) - \log \lambda_\gamma,$$

which is not in C^α. While this is not an essential obstruction, it does make computing the derivatives of pressure and proving the CLT slightly more cumbersome. We leave the details to the reader.

Part II

Life on the Shift Space

5 Refined Thermodynamic Formalism and Stochastic Laws for Countable Alphabet Subshifts of Finite Type

In this section, we recall the rudimentary notions of thermodynamic formalism on the shift space from [PP90, MU03, URM22b]. We then define the classes of *normal* and *robust* potentials that mimic the properties of the potential $-\log|F'|$ associated to a one component inner function F. One advantage of this somewhat abstract approach of working on the shift space, rather than directly on the unit circle, is that the arguments are applicable to a plethora of other dynamical systems, not just to inner functions.

A normal or a robust potential can be put into a holomorphic family of potentials indexed by the right half-plane $\mathbb{C}_1^+ = \{s \in \mathbb{C} : \operatorname{Re} s > 1\}$. In order to study the differentiability properties of the transfer operator $\mathcal{L}_{s\psi}$ near the line $s = 1$, we introduce the modified transfer operators $\mathcal{L}_{s,p}$ and develop some of their basic properties.

5.1 Shift Space

Let E be a countable (finite or infinite) alphabet and $A : E \times E \to \mathbb{R}$ be a matrix whose entries are either 0 or 1. The *shift space* $E_A^\infty \subset E^\infty$ consists of all infinite words $\xi = \xi_0 \xi_1 \xi_2 \ldots$ with $\xi_i \in E$ such that $A(\xi_i, \xi_{i+1}) = 1$. We call words in E_A^∞ *admissible*. The shift map $\sigma : E_A^\infty \to E_A^\infty$ takes a word and removes its first letter.

We let E_A^* denote the collection of all finite admissible words in the alphabet E. We denote the length of $\tau \in E_A^*$ by $|\tau|$. For $\tau \in E_A^*$, we let $[\tau]$ denote the *cylinder* of all infinite words that begin with τ.

Below, we will always assume that the incidence matrix A is *finitely primitive*, i.e. there exists a finite collection of words $\Lambda \subset E_A^*$ of the same length such that for any two letters a and b in E there exists $\tau \in \Lambda$ such that the word $a\tau b$ is admissible.

We endow the shift space E_A^∞ with the metric

$$d(\xi, \xi') = 2^{-|\xi \wedge \xi'|},$$

where $|\xi \wedge \xi'|$ denotes the length of the common prefix of ξ, ξ'. We denote the space of (complex-valued) continuous functions on E_A^∞ by $C(E_A^\infty)$ and the space of bounded continuous functions by $C_{\mathrm{b}}(E_A^\infty)$.

Remark. If the alphabet is infinite, then the two spaces are usually not the same, i.e. $C_{\mathrm{b}}(E_A^\infty) \neq C(E_A^\infty)$. For instance, if $E = \mathbb{N}$ and $A(i,j) = 1$ for all $i, j \in \mathbb{N}$, then the function $f(\xi) = \xi_1$ is continuous but not bounded.

For $\alpha > 0$, we endow the space of Hölder continuous functions $C^\alpha(E_A^\infty)$ with the norm

$$\|f\|_\alpha := \|f\|_\infty + v_\alpha(f),$$

where

$$v_\alpha(f) := \sup \left\{ \frac{|f(\xi') - f(\xi)|}{d(\xi, \xi')^\alpha} : \xi \neq \xi' \in E_A^\infty \right\}.$$

Lemma 5.1. *The space of Hölder continuous functions $C^\alpha(E_A^\infty)$ forms an algebra. In fact,*

$$\|fg\|_\alpha \leq 3\|f\|_\alpha \cdot \|g\|_\alpha.$$

We leave the proof as an exercise for the reader. In particular, the above lemma implies that the exponential of a Hölder continuous function is Hölder continuous.

5.2 Transfer Operators

To a potential $\psi : E_A^\infty \to \mathbb{C}$, one can associate the Perron-Frobenius or *transfer* operator

$$\mathcal{L}_\psi g(\omega) = \sum_{a \in E} e^{\psi(a\omega)} \cdot g(a\omega),$$

where we use the convention that a term is 0 if it features an inadmissible word. We say that a potential ψ is *summable* if

$$\sum_{a \in E} \sup_{[a]} |e^\psi|,$$

and *level 1 Hölder continuous*[1] with exponent $\alpha > 0$ if

$$\sup_{a \in E} v_\alpha(\psi|_{[a]}) < \infty.$$

It is explained in [MU03, Chapter 2.3] or [URM22b, Lemma 17.6.2], that if ψ is a real-valued summable potential, then $\mathcal{L}_\psi$ preserves the space $C_{\mathrm{b}}(E_A^\infty)$. Furthermore, according to [MU03, Chapter 2.4], comp. also [URM22b, Theorems 8.1.12 and 18.1.9], if one additionally assumes that ψ is level 1 Hölder continuous, then $\mathcal{L}_\psi$ also preserves $C^\alpha(E_A^\infty)$, where it has a spectral gap:

Theorem 5.2 (Ruelle-Perron-Frobenius)**.** *For a real-valued summable level 1 Hölder continuous potential ψ on E_A^∞, we have:*

(i) *The operator $\mathcal{L}_\psi : C^\alpha(E_A^\infty) \to C^\alpha(E_A^\infty)$ has a maximal positive eigenvalue λ_ψ.*

(ii) *The eigenvalue λ_ψ is simple and the rest of the spectrum of $\mathcal{L}_\psi$ is contained in a ball centered at the origin of radius strictly less than λ_ψ.*

(iii) *The dual operator $\mathcal{L}_\psi^* = C_{\sigma,\psi} : \mathcal{M}(E_A^\infty) \to \mathcal{M}(E_A^\infty)$ has a unique probability eigenmeasure m_ψ and the corresponding eigenvalue is also λ_ψ.*

(iv) *The eigenspace of the eigenvalue λ_ψ is generated by a strictly positive Hölder continuous eigenfunction ρ_ψ with $m_\psi(\rho_\psi) = 1$.*

(v) *The Borel probability measure $\mu_\psi = \rho_\psi \, dm_\psi$ is invariant under the shift map $\sigma : E_A^\infty \to E_A^\infty$.*

(vi) *For any $g \in L^1(E_A^\infty)$, we have*

$$\int_{E_A^\infty} \mathcal{L}g \, dm = \lambda_\psi \int_{E_A^\infty} g dm_\psi. \tag{5.1}$$

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[1]The term "Hölder continuous on cylinders" is also used.

(vii) *For any function $g \in C^\alpha(E_A^\infty)$,*

$$\lambda_\psi^{-n} \int_{E_A^\infty} \mathcal{L}^n g \, dm \longrightarrow \left(\int_{E_A^\infty} g \, dm_\psi \right) \cdot \rho_\psi. \qquad (5.2)$$

We refer to m_ψ as the *conformal measure* associated to ψ, while μ_ψ is known as the *invariant Gibbs state* or *equilibrium measure* for ψ. From property (iv), it follows that the spaces $L^p(E_A^\infty, m_\psi) = L^p(E_A^\infty, \mu_\psi)$ are equal for any $1 \le p \le \infty$. The *topological pressure* $P(\psi)$ of ψ is defined as $\log \lambda_\psi$.

The following theorem describes the spectrum of the transfer operator associated to a complex-valued potential:

Theorem 5.3. *Let $\psi = u + iv$ be a complex-valued summable level 1 Hölder continuous potential on E_A^∞ with $L_u \mathbf{1} = \mathbf{1}$. The spectrum of $\mathcal{L}_\psi$ acting on $C^\alpha(E_A^\infty)$ is contained in the closed unit disk. On the unit circle, $\mathcal{L}_\psi$ may have at most one simple eigenvalue. Apart from this possible eigenvalue, the rest of the spectrum is contained in a disk of strictly smaller radius. The following conditions are equivalent:*

(i) *$e^{-iv}C_\sigma$ does not have an $L^2(m_u)$ eigenvalue.*

(ii) *$e^{-iv}C_\sigma$ does not have a C^α eigenvalue.*

(iii) *the spectral radius of $\mathcal{L}_\psi$ is less than λ_u.*

(iv) *v is cohomologous to a potential v_1 which takes values in $2\pi\mathbb{Z}$, that is,*

$$v - v_1 = w - w \circ f, \qquad \textit{for some } w \in C^\alpha(E_A^\infty).$$

(v) *the* length spectrum *of v*

$$\left\{ S_n v(\xi) : \sigma^n(\xi) = \xi \textit{ for some } n \ge 1 \right\} \subset 2\pi\mathbb{Z}.$$

The proofs of the equivalences are scattered in Sections 4 and 5 of [PP90]. While [PP90] deals with finite alphabets, the proofs carry over to countable alphabets.

We say that a potential $\psi = u + iv$ is *D-generic* if $e^{-iav}C_\sigma$ fails to have a C^α eigenvalue for any $a \ne 0$, or alternatively, if the length spectrum of ψ is not contained in a discrete subgroup of $\mathbb{R}$.

5.3 Families of Potentials

We say that a potential $\psi : E_A^\infty \to \mathbb{R}$ is *normal* if it is

- level 1 Hölder continuous for some exponent $\alpha > 0$,

- summable,

- negative: $\sup \psi < 0$,

- centered: $P(\psi) = 0$.

Recall that $\mathbb{C}_1^+$ is the right half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$ and $\overline{\mathbb{C}_1^+}$ is its closure. Given a normal potential ψ, we embed $\mathcal{L}_\psi$ in a holomorphic family of operators $\mathcal{L}_{s\psi}$ indexed by $s \in \mathbb{C}_1^+$. When the potential is clear from context, we write $\mathcal{L}_s$ instead of $\mathcal{L}_{s\psi}$. The following result can be found in [MU03, Chapter 2.6] and [URM22b, Lemma 20.2.2, Theorems 20.2.3 and 20.1.12]:

Theorem 5.4. *If $\psi : E_A^\infty \to \mathbb{R}$ is a normal potential, then the function*

$$s \to \mathcal{L}_s, \qquad \mathbb{C}_1^+ \to \mathcal{B}(C^\alpha(E_A^\infty))$$

is holomorphic. In addition, in some open subset of $\mathbb{C}_1^+$ which contains $(1, +\infty)$, the operator $\mathcal{L}_s$ has a simple isolated eigenvalue λ_s whose modulus is equal to the spectral radius of $\mathcal{L}_s$, and λ_s varies holomorphically in this open set.

5.4 Summability conditions

Before continuing, we prove the following simple lemma:

Lemma 5.5. *Suppose ψ is a negative level 1 Hölder continuous potential. For $s, p \geq 0$, the quantities*

$$\sum_{a \in E} \inf_{[a]} |\psi^p e^{s\psi}|, \qquad \sum_{a \in E} \sup_{[a]} |\psi^p e^{s\psi}|, \qquad \int_{E_A^\infty} |\psi^p| dm_{s\psi}$$

are comparable. In particular, if one of the quantities is finite, then they all are.

Proof. As $\psi < -C_1$ is bounded above by a negative constant and $\sup_{[a]} \psi - \inf_{[a]} \psi \leq v_\alpha^{(1)}(\psi)$ for any $a \in E$,

$$\inf_{[a]} |\psi^p| \asymp \sup_{[a]} |\psi^p| \qquad \text{and} \qquad \inf_{[a]} |e^{s\psi}| \asymp \sup_{[a]} |e^{s\psi}|,$$

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which shows that the first two quantities are comparable. Since

$$m_{s\psi}([a]) = \int_{E_A^\infty} \chi_{[a]} dm_{s\psi} = \int_{E_A^\infty} \mathcal{L}_{s\psi} \chi_{[a]} dm_{s\psi} \asymp \sup_{[a]} |e^{s\psi}|,$$

the integral

$$\int_{E_A^\infty} |\psi^p| dm_{s\psi} \asymp \sum_{a \in E} \sup_{[a]} |\psi^p| \cdot m_{s\psi}([a]) \asymp \sum_{a \in E} \sup_{[a]} |\psi^p| \cdot \sup_{[a]} |e^{s\psi}|$$

as desired. □

If one (and hence all) of the quantities in the above lemma is finite, then we say that the potential $\psi : E_A^\infty \to \mathbb{R}$ is (s, p)-*summable*. We will often write $p = n + \varepsilon$ where $n = \lfloor p \rfloor$ and $0 \le \varepsilon < 1$.

5.5   Modified Transfer Operators

In order to study the regularity of the operators $\mathcal{L}_s$ near the vertical line $\{\mathrm{Re}\, s = 1\}$, we introduce the (modified) transfer operators

$$\mathcal{L}_{s,p} g = \mathcal{L}_s(\psi^p g), \qquad s \in \overline{\mathbb{C}_1^+}, \qquad p \ge 0. \tag{5.3}$$

Using Lemma 5.5, it is not difficult to show that for a normal (s, p)-summable potential, the operator $\mathcal{L}_{s,p}$ is bounded on $C_{\mathrm{b}}(E_A^\infty)$. The following theorem describes the behaviour of $\mathcal{L}_{s,p}$ on the space of Hölder continuous functions $C^\alpha(E_A^\infty)$:

Theorem 5.6. *Suppose $\psi : E_A^\infty \to \mathbb{R}$ is a normal $(1, n+\varepsilon)$-summable potential. Then, $s \to \mathcal{L}_s$ is*
   (i) *a $C_{\mathrm{loc}}^{n+\varepsilon}$ mapping from $\overline{\mathbb{C}_1^+}$ to $\mathcal{B}(C^\alpha(E_A^\infty))$.*
   (ii) *a holomorphic mapping from $\mathbb{C}_1^+$ to $\mathcal{B}(C^\alpha(E_A^\infty))$, with*

$$\frac{d^k}{ds^k} \mathcal{L}_s = \mathcal{L}_{s,k}, \qquad k \in \mathbb{N}. \tag{5.4}$$

Proof. The proof of (i) is quite cumbersome and is deferred to Appendix A. Fix a real number $q \ge 0$. Below, we show that the operator

$$\mathcal{L}_{s,q} : \mathbb{C}_1^+ \to \mathcal{B}(C^\alpha(E_A^\infty))$$

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is holomorphic and $\mathcal{L}'_{s,q} = \mathcal{L}_{s,q+1}$. As the series

$$\mathcal{L}_{s,q}g(\omega) = \sum_{a \in E}(\psi^q e^{s\psi})(a\omega) \cdot g(a\omega)$$

converges locally uniformly on the open half-plane $\mathbb{C}_1^+$, it can be differentiated term-by-term for any $s \in \mathbb{C}_1^+$ and $g \in C^\alpha(E_A^\infty)$, so the *pointwise derivative*

$$\frac{d}{ds}(\mathcal{L}_{s,q}g(\omega)) = \mathcal{L}_{s,q+1}g(\omega).$$

By Morera's theorem, to verify that the operator $\mathcal{L}_{s,q}$ depends holomorphically on $s \in \mathbb{C}_1^+$, it suffices to check that for every loop $\gamma \subset \mathbb{C}_1^+$, the integral $\int_\gamma \mathcal{L}_{s,q}ds$ is the zero operator[2]. This is easy: as $\mathcal{L}_{s,q}$ is linear and bounded,

$$\left[\left(\int_\gamma \mathcal{L}_{s,q}ds\right)(g)\right](\omega) = \sum_{a \in E}\left(\int_\gamma \psi^q e^{s\psi}ds\right)(a\omega) \cdot g(a\omega) = 0,$$

by Cauchy's theorem. Having satisfied ourselves that $s \to \mathcal{L}_{s,q}$ is holomorphic as a mapping valued in $\mathcal{B}(C^\alpha(E_A^\infty))$, it has a *Banach space derivative*, which must coincide with the pointwise derivative computed above. $\square$

Remark. In general, the existence of the pointwise derivative does not imply the existence of a Banach space derivative, which is a much more stringent condition.

5.6 Projections and Eigenvalues

If $s \in [1, +\infty)$, then $\mathcal{L}_s$ has a maximal positive eigenvalue λ_s. Since $\mathcal{L}_s$ is continuous in $s \in \overline{\mathbb{C}_1^+}$, isolated simple eigenvalues vary continuously in s. Therefore, we can follow λ_s in a closed half-ball $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$, for some $\delta > 0$. For these parameters s, we can define the Riesz projection R_s onto the corresponding 1-dimensional eigenspaces. We normalize the eigenfunction ρ_1 of $\mathcal{L}_1$ so that $\int_{E_A^\infty} \rho_1 dm = 1$ and define

$$\rho_s := R_s\rho_1.$$

---

[2]In Appendix A, we will see that $s \to \mathcal{L}_{s,q}$ is continuous on $\mathbb{C}_1^+$, so the operator-valued integral is well-defined.

Lemma 5.7. *If ψ is a normal $(1, n + \varepsilon)$-summable potential, then $s \to R_s$ is a $C^{n+\varepsilon}$ mapping from $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$ to $\mathcal{B}(C^\alpha(E_A^\infty))$.*

Proof. We explicitly show the lemma for $p = n + \varepsilon \in [1, 2]$, which suffices for Orbit Counting. The general case is similar but more computationally involved.

$(p = 1)$ Since ψ is centered, $\lambda_1 = 1$. After decreasing $\delta > 0$ if necessary, we can find an $\eta > 0$ so that for any $s \in \overline{\mathbb{C}_1^+} \cap B(1, \delta)$,

$$\sigma(\mathcal{L}_s) \cap \overline{B(1, \eta)} = \{\lambda_s\}$$

consists of a simple isolated eigenvalue. Set $\gamma = \partial B(1, \eta)$. For any $s \in \overline{\mathbb{C}_1^+} \cap B(1, \delta)$,

$$R_s = \frac{1}{2\pi i} \int_\gamma (z \operatorname{Id} - \mathcal{L}_s)^{-1} dz.$$

By the first resolvent formula,

$$R_t - R_s = \frac{1}{2\pi i} \int_\gamma (z \operatorname{Id} - \mathcal{L}_t)^{-1} (\mathcal{L}_t - \mathcal{L}_s)(z \operatorname{Id} - \mathcal{L}_s)^{-1} dz,$$

if $s, t \in \overline{\mathbb{C}_1^+} \cap B(1, \delta)$. By Theorem 5.6,

$$R_s' = \frac{1}{2\pi i} \int_\gamma (z \operatorname{Id} - \mathcal{L}_s)^{-1} \mathcal{L}_s' (z \operatorname{Id} - \mathcal{L}_s)^{-1} dz \tag{5.5}$$

defines a continuous mapping from $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$ to $\mathcal{B}(C^\alpha(E_A^\infty))$.

$(p = 2)$ A similar computation yields a formula for the second derivative of the resolvent. By (5.5),

$$
\begin{aligned}
R_t' - R_s' &= \frac{1}{2\pi i} \int_\gamma \{ R_t \mathcal{L}_t' R_t - R_s \mathcal{L}_s' R_s \} dz \\
&= \frac{1}{2\pi i} \int_\gamma \left\{ (R_t - R_s) \mathcal{L}_t' R_t + R_s (\mathcal{L}_t' - \mathcal{L}_s') R_t + R_s \mathcal{L}_s' (R_t - R_s) \right\} dz.
\end{aligned}
$$

Therefore,

$$\frac{R_t' - R_s'}{t - s} = \frac{1}{2\pi i} \int_\gamma \left\{ \frac{R_t - R_s}{t - s} \cdot \mathcal{L}_t' R_t + R_s \cdot \frac{\mathcal{L}_t' - \mathcal{L}_s'}{t - s} \cdot R_t + R_s \mathcal{L}_s' \cdot \frac{R_t - R_s}{t - s} \right\} dz.$$

After taking $t \to s$, we see that

$$R_s'' = \frac{1}{2\pi i} \int_\gamma \{ R_s' \mathcal{L}_s' R_s + R_s \mathcal{L}_s'' R_s + R_s \mathcal{L}_s' R_s' \} dz, \tag{5.6}$$

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defines a continuous mapping from $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$ to $\mathcal{B}(C^\alpha(E_A^\infty))$.

$(1 < p < 2)$ Using the formula for $R'_t - R'_s$ above, we can express

$$\frac{|R'_t - R'_s|}{|t - s|^\varepsilon}$$

as an integral over γ. Since the integrand is bounded, R' is a C^ε mapping from $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$ to $\mathcal{B}(C^\alpha(E_A^\infty))$. $\qquad \square$

Corollary 5.8. *If ψ is a normal $(1, n + \varepsilon)$-summable potential, then $s \to \lambda_s$ is a $C^{n+\varepsilon}$ function on $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$.*

Proof. We explicitly show the corollary for when $p = n$ is an integer, and when $p = n + \varepsilon \in (1, 2)$, which suffices for Orbit Counting. The general case follows the same pattern.

$(\varepsilon = 0)$ Let $F : C^\alpha(E_A^\infty) \to \mathbb{C}$ be a bounded linear functional for which $F(\rho_1) \neq 0$. The equality $\mathcal{L}_s R_s \rho_1 = \lambda_s R_s \rho_1$ tells us that

$$\lambda_s = \frac{F(\mathcal{L}_s R_s \rho_1)}{F(R_s \rho_1)}.$$

The corollary now follows from differentiability properties of $\mathcal{L}_s$ and R_s discussed in Theorem 5.6 and Lemma 5.7.

$(1 < n + \varepsilon < 2)$ To see that $s \to \lambda_s$ is $C^{1+\varepsilon}$ function on $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$, it suffices to show that the derivative is

$$\lambda'_s = \frac{F(\mathcal{L}'_s(R_s \rho_1) \cdot R'_s \rho_1) F(R_s \rho_1) - F(\mathcal{L}_s R_s \rho_1) F(R'_s \rho_1)}{F(R_s \rho_1)^2}.$$

a C^ε function. This follows from the fact that C^ε functions and operators form an algebra. $\qquad \square$

5.7 Stochastic Laws

We compute the first derivative of pressure at $s = 1$ by following the argument in [PP90, Proposition 4.10]:

Lemma 5.9. *If ψ is a normal $(1, 1)$-summable potential, then*

$$\lambda'(1) = P'(1) := \lim_{s \to 1,\, s \in \mathbb{C}_1^+} P'(s) = \int_{E_A^\infty} \psi d\mu_1. \tag{5.7}$$

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Proof. Since $\rho_s = R_s \rho_1$, we can use Lemma 5.7 to conclude the existence of the one-sided derivative

$$\rho_1'(\xi) := \lim_{s \to 1,\, s \in \mathbb{C}_1^+} \rho_s'(\xi).$$

Differentiating $\mathcal{L}_s \rho_s = e^{P(s)} \rho_s$ at $s \in \mathbb{C}_1^+$ and taking $s \to 1$, we get

$$\mathcal{L}_1' \rho_1 + \mathcal{L} \rho_1' = P'(1) \rho_1 + \rho_1'.$$

After integrating the above equation with respect to m_1 and using (5.1), we obtain

$$\int_{E_A^\infty} \mathcal{L}_1' \rho_1 dm_1 = P'(1) \int_{E_A^\infty} \rho_1 dm_1.$$

By (5.4) and the fact that $\mu_1 = \rho_1 dm_1$ was designed to be a probability measure, the above equality simplifies to

$$\int_{E_A^\infty} \mathcal{L}_1(\psi \rho_1) dm_1 = P'(1).$$

Applying the identity (5.1) one more time, we end up with

$$\int_{E_A^\infty} \psi d\mu_1 = P'(1),$$

which is what we wanted to show.

The equality $\lambda'(1) = P'(1)$ follows from the definition of pressure as $\log \lambda$ and the normalization $P(1) = 0$. $\qquad\square$

In a similar fashion, one can follow the proof of [PP90, Proposition 4.11] to compute the one-sided second derivative of pressure at $s = 1$:

Lemma 5.10. *Suppose ψ is a normal $(1, 2)$-summable potential on E_A^∞. Let $\overline{\psi} = \int_{E_A^\infty} \psi d\mu_1$ and $\phi = \psi - \overline{\psi}$. Consider the family of potentials $\psi_s = \psi + (s-1)\phi$ defined on $\overline{\mathbb{C}_1^+}$. The one-sided second derivative of pressure is*

$$P''(1) := \lim_{s \to 1,\, s \in \mathbb{C}_1^+} P''(s) = \sigma^2(\phi), \tag{5.8}$$

where

$$\sigma^2(\phi) = \lim_{n \to \infty} \frac{1}{n} \int_{\partial \mathbb{D}} (S_n \phi)^2 d\mu$$

is the asymptotic variance *of ϕ.*

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Proof. Arguing as in the proof of Lemma 5.9 shows that $P'(1) = \int_{E_A^\infty} \phi d\mu_1 = 0$. In view of the $(1,2)$-summability hypothesis, Lemma 5.7 implies the existence of the one-sided derivatives

$$\rho_1'(\xi) := \lim_{s \to 1,\, s \in \mathbb{C}_1^+} \rho_s'(\xi), \qquad \rho_1''(\xi) := \lim_{s \to 1,\, s \in \mathbb{C}_1^+} \rho_s''(\xi).$$

Taking the second derivative of $\mathcal{L}_s \rho_s = e^{P(s)} \rho_s$ and tending $s \to 1$ in $\mathbb{C}_1^+$ gives

$$\mathcal{L}_1'' \rho_1 + 2\mathcal{L}_1' \rho_1' + \mathcal{L}_1 \rho_1'' = P''(1)\rho_1 + 2P'(1)\rho_1' + \rho_1''$$
$$= P''(1)\rho_1 + \rho_1''.$$

Integrating with respect to m_1, we obtain

$$\int_{E_A^\infty} \{\mathcal{L}_1'' \rho_1 + 2\mathcal{L}_1' \rho_1'\} dm_1 = P''(1),$$

which simplifies to

$$\int_{E_A^\infty} \phi^2 d\mu_1 + 2\int_{E_A^\infty} \rho_1' \cdot \phi dm_1 = P''(1).$$

Replacing ϕ by $S_n \phi$ and dividing by n, we get

$$\frac{1}{n} \int_{E_A^\infty} (S_n \phi)^2 d\mu_1 + 2\int_{E_A^\infty} \frac{\rho_1'}{\rho_1} \cdot \frac{S_n \phi}{n} d\mu_1 = P''(1).$$

The result follows after taking $n \to \infty$ and applying the ergodic theorem. $\square$

Once we know the existence of the second derivative of the pressure along the imaginary axis, one can conclude the Central Limit Theorem for ϕ as in [URM22b, Theorem 13.9.15]. For a $(1,3)$-summable potential, one could follow the argument of [PP90, Theorem 4.13] to obtain the Central Limit Theorem for ϕ with an $O(1/\sqrt{n})$ error term.

6 Orbit Counting in Symbolic Dynamics

In this section, we prove our main results in symbolic dynamics, Theorems 1.9 and 1.10. We follow the approach in [PoU17], paying closer to attention to the behaviour of the Poincaré series near the line $\operatorname{Re} s = 1$.

6.1 Poincaré Series

For a normal potential $\psi : E_A^\infty \to (-\infty, 0)$ and a point $\xi \in E_A^\infty$, the *counting function with offset* $\varphi \in C^\alpha(E_A^\infty)$ is given by

$$N_\xi^{\varphi,\psi}(T) := \#\big\{\omega \in E_\xi^* \ : \ -S_{|\omega|}\psi(\omega\xi) - \varphi(\omega\xi) \le T\big\},$$

where $E_\xi^* \subset E_A^*$ is the set of finite words ω such that $\omega\xi$ is admissible. We will often suppress φ and ψ from the notation and abbreviate $f = f_1 = e^\varphi$ and $f_s = e^{s\varphi}$.

Example. (i) When the offset $\varphi = \mathbf{0}$, $N_\xi^{\varphi,\psi}(T)$ counts the pre-images of ξ under the shift map $\sigma : E_A^\infty \to E_A^\infty$:

$$N_\xi(T) := \#\big\{\omega \in E_\xi^* \ : \ S_{|\omega|}(-\psi)(\omega\xi) \le T\big\}.$$

(ii) Let $\tau \in E_A^*$ be a finite word. Taking $\varphi(\xi) = S_{|\tau|}\psi(\tau\xi)$ in the definition above, we get

$$N_\xi^{[\tau]}(T) := \#\big\{\omega \in E_\xi^* \text{ starting with } \tau \ : \ S_{|\omega|}(-\psi)(\omega\xi) \le T\big\}.$$

Up to bounded error, this counts the number of pre-images of ξ under the shift map $\sigma : E_A^\infty \to E_A^\infty$ that lie in $[\tau]$:

$$\#\big\{\omega \in E_\xi^* \ : \ \omega\xi \in [\tau] \text{ and } S_{|\omega|}(-\psi)(\omega\xi) \le T\big\}.$$

(Since ψ is negative and summable, only finitely many shorts words $\omega \in E_\xi^*$ of length less than $|\tau|$ satisfy $S_{|\omega|}(-\psi)(\omega\xi) \le T$.)

Following [PoU17], to study the asymptotics of $N_\xi(T)$ as $T \to \infty$, we examine the *Poincaré series*

$$\eta(s) := \sum_{n=0}^\infty \mathcal{L}_s^n(e^{s\varphi}) = \int_0^\infty e^{-sT} dN_\xi(T) = \sum_{\omega \in E_A^*} \exp\big(s(S_{|\omega|}\psi + \varphi)\big)(\omega\xi).$$

Lemma 6.1. *Suppose ψ is a robust $(1, 1+\varepsilon)$-summable potential, with $0 \le \varepsilon \le 1$. For any $\xi \in E_A^\infty$, the following statements hold:*

(1) The series defining $\eta(s)(\xi)$ converges absolutely uniformly on compact subsets of $\mathbb{C}_1^+$ and defines a holomorphic function.

(2) The function $s \to \eta(s)(\xi) \in \mathbb{C}$ has a continuous extension to $\overline{\mathbb{C}_1^+} \setminus \{1\}$.

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$(3, \varepsilon = 0)$ *As* $s \to 1$ *in* $\mathbb{C}_1^+$,

$$\eta(s)(\xi) - \frac{(-\lambda_1')^{-1} R_1 f(\xi)}{s - 1} = o(|s - 1|^{-1}).$$

$(3, 0 < \varepsilon < 1)$ *As* $s \to 1$ *in* $\mathbb{C}_1^+$,

$$\eta(s)(\xi) - \frac{(-\lambda_1')^{-1} R_1 f(\xi)}{s - 1} = O(|s - 1|^{\varepsilon - 1}),$$

$(3, \varepsilon = 1)$ *As* $s \to 1$ *in* $\mathbb{C}_1^+$,

$$\eta(s)(\xi) - \frac{(-\lambda_1')^{-1} R_1 f(\xi)}{s - 1}$$

tends to a finite limit.

Remark. As ψ is a negative potential, $\lambda_1' \neq 0$ by Lemma 5.9, so the expression in (3) is well-defined.

We temporarily assume Lemma 6.1 and prove Theorem 1.9:

Proof of Theorem 1.9 assuming Lemma 6.1. By Lemma 6.1 and the hypotheses of the theorem, for any $\xi \in E_A^\infty$,

$$\eta(s)(\xi) - \frac{(-\lambda_1')^{-1} R_1 f(\xi)}{s - 1}$$

extends to an L_{loc}^1 function on the vertical line $\{\operatorname{Re} s = 1\}$. By the Wiener-Ikehara Tauberian theorem (Theorem 1.6),

$$\lim_{T \to \infty} \frac{N_\xi(T)}{e^T} = -(\lambda_1')^{-1} R_1 f(\xi).$$

Inserting the expression for λ_1' from Lemma 5.9, we get

$$\lim_{T \to \infty} \frac{N_\xi(T)}{e^T} = \frac{\rho_\psi(\xi)}{\int_{E_A^\infty} (-\psi) d\mu_\psi} \cdot \int_{E_A^\infty} e^{\varphi(\xi)} dm_\psi(\xi). \tag{6.1}$$

Setting $\varphi = \mathbf{0}$ in (6.1) shows that the theorem holds for $B = E_A^\infty$:

$$\lim_{T \to \infty} \frac{N_\xi(T)}{e^T} = \frac{\rho_\psi(\xi)}{\int_{E_A^\infty} (-\psi) d\mu_\psi}.$$

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More generally, setting $\varphi(\xi) = S_{|\tau|}\psi(\tau\xi)$ into (6.1), shows that the theorem holds for cylinders $B = [\tau]$:

$$\lim_{T \to \infty} \frac{N_\xi^{[\tau]}(T)}{e^T} = \frac{\rho_\psi(\xi)}{\int_{E_A^\infty}(-\psi)d\mu_\psi} \cdot m_\psi([\tau]),$$

Clearly, the theorem is also valid when $B \subset E_A^\infty$ is a finite union of disjoint cylinders. We next consider the case when B is an open set with $m_\psi(\partial B) = 0$. For any $\varepsilon > 0$, we can find a set $B_1 \subset B$ which is a finite union of disjoint cylinders with $m_\psi(B_1) \geq m_\psi(B) - \varepsilon$. Clearly,

$$\lim_{T \to \infty} \frac{N_\xi^B(T)}{e^T} \geq \frac{\rho_\psi(\xi)}{\int_{E_A^\infty}(-\psi)d\mu_\psi} \cdot (m_\psi(B) - \varepsilon),$$

Taking $\varepsilon \to 0$ shows the lower bound

$$\lim_{T \to \infty} \frac{N_\xi^B(T)}{e^T} \geq \frac{\rho_\psi(\xi)}{\int_{E_A^\infty}(-\psi)d\mu_\psi} \cdot m_\psi(B).$$

The above formula for the open set $E_A^\infty \setminus \overline{B}$ gives the corresponding upper bound.

Since complements of open sets are closed sets, the theorem also holds for closed sets B with $m_\psi(\partial B) = 0$. From here, it is easy to see that the theorem holds for an arbitrary measurable set $B \subset E_A^\infty$ with $m_\psi(\partial B) = 0$ by squeezing B between its interior and closure. $\qquad\square$

The argument above also shows Theorem 1.10 for D-generic potentials, provided that one uses Hardy-Littlewood Tauberian theorem (Theorem 1.7) instead of the Wiener-Ikehara Tauberian theorem (Theorem 1.6). Note that the $\varepsilon = 0$ case of Lemma 6.1 implies that

$$\eta(s)(\xi) \sim \frac{(-\lambda_1')^{-1}R_1 f(\xi)}{s-1},$$

as the numerator is non-zero.

6.2 Analytic Behaviour near $s = 1$

To complete the proof of Theorem 1.9, it remains to prove Lemma 6.1.

Proof of Lemma 6.1. By Theorem 5.3 and D-genericity of ψ, the spectral radius of $\mathcal{L}_s$ is less than 1 for any $s \in \overline{\mathbb{C}_1^+} \setminus \{1\}$, from which parts (1) and (2) follow easily. We therefore focus our attention on part (3).

Step 1. Consider the operator

$$\Delta_s = \mathcal{L}_s - \lambda_s R_s.$$

Since $\mathcal{L}_s R_s = R_s \mathcal{L}_s$ and R_s is the Riesz projection onto $\ker(\lambda_s \operatorname{Id} - \mathcal{L}_s)$,

$$R_s \Delta_s = \Delta_s R_s = 0.$$

Consequently,

$$\mathcal{L}_s^n = \lambda_s^n R_s + \Delta_s^n, \qquad n \geq 0.$$

Since $\|\Delta_1\|_\alpha < 1$, after shrinking $\delta > 0$ if necessary, we may assume that $\|\Delta_s\|_\alpha < 1$ for all $s \in \overline{\mathbb{C}_1^+} \cap B(1, \delta)$. This guarantees that the series

$$s \to \Delta_\infty(s) := \sum_{n=0}^\infty \Delta_s^n(f_s)$$

converges uniformly in $\mathcal{B}(C^\alpha(E_A^\infty))$ and thus defines a continuous function from $\overline{\mathbb{C}_1^+} \cap B(1, \delta)$ to $\mathcal{B}(C^\alpha(E_A^\infty))$.

Step 2. Since $|\lambda_s| < 1$ for all $\mathbb{C}_1^+ \cap B(1, \delta)$, we can write

$$\eta(s) = \sum_{n=0}^\infty \lambda_s^n R_s f_s + \sum_{n=0}^\infty \Delta_s^n(f_s) = (1 - \lambda_s)^{-1} R_s f_s + \Delta_\infty(s).$$

A computation shows that

$$\begin{aligned}
\eta(s) - \frac{(-\lambda_1')^{-1} R_1 f_1}{s - 1} &= \frac{\eta(f, s)(s - 1) + (\lambda_1')^{-1} R_1 f_1}{s - 1} \\
&= \frac{(1 - \lambda_s)^{-1} R_s f_s (s - 1) + \Delta_\infty(s)(s - 1) + (\lambda_1')^{-1} R_1 f_1}{s - 1} \\
&= \frac{(1 - \lambda_s)^{-1} R_s f_s (s - 1) + (\lambda_1')^{-1} R_1 f_1}{s - 1} + \Delta_\infty(s).
\end{aligned}$$

We denote the first summand in this formula by $A(s)$. Since $\Delta_\infty(s)$ is continuous at $s = 1$, to prove the lemma, we only need to analyze $A(s)$.

Step 3. We compute:

$$A(s) = \frac{(R_s f_s - R_1 f_1)(s-1) + R_1 f_1\big((s-1) + (\lambda_1')^{-1}(1 - \lambda_s)\big)}{(s-1)(1 - \lambda_s)}$$

$$= \frac{R_s(f_s - f_1)}{s-1} \cdot \frac{s-1}{1 - \lambda_s} + \frac{(R_s - R_1)f_1}{s-1} \cdot \frac{s-1}{1 - \lambda_s}$$

$$\qquad + R_1 f_1 \cdot \frac{s - 1 + (\lambda_1')^{-1}(1 - \lambda_s)}{(s-1)(1 - \lambda_s)}$$

$$= A_1(s) + A_2(s) + A_3(s).$$

In view of Lemmas 5.7 and 5.8, we have

$$\lim_{\overline{\mathbb{C}_1^+} \cap B(1,\delta_1) \ni s \to 1} \frac{R_s(f_s - f_1)}{s-1} = R_1(f_1 \log f_1) \in C^\alpha(E_A^\infty),$$

$$\lim_{\overline{\mathbb{C}_1^+} \cap B(1,\delta_1) \ni s \to 1} \frac{(R_s - R_1)f_1}{s-1} = R_1' f_1 \in C^\alpha(E_A^\infty)$$

and

$$\lim_{\overline{\mathbb{C}_1^+} \cap B(1,\delta_1) \ni s \to 1} \frac{s-1}{1 - \lambda_s} = -\frac{1}{\lambda_1'},$$

so that $A_1(s)$ and $A_2(s)$ are continuous at $s = 1$. Hence,

$$\eta(s) - \frac{(-\lambda_1')^{-1} R_1 f_1}{s-1} = R_1 f_1 \cdot \frac{s - 1 + (\lambda_1')^{-1}(1 - \lambda_s)}{(s-1)(1 - \lambda_s)} + B(s),$$

for some function $B(s)$ defined on $\overline{\mathbb{C}_1^+} \cap B(1,\delta)$ which is continuous at $s = 1$.

Step 4. $(\varepsilon = 0)$ If ψ is $(1,1)$-summable, then

$$\lambda_s = \lambda_1 + \lambda_1'(s-1) + o(|s-1|), \qquad \text{as } s \to 1 \text{ in } \mathbb{C}_1^+,$$

and

$$\frac{s - 1 + (\lambda_1')^{-1}(1 - \lambda_s)}{(s-1)(1 - \lambda_s)} = o(|s-1|^{-1}).$$

$(0 < \varepsilon < 1)$ If ψ is $(1, 1 + \varepsilon)$-summable for some $0 < \varepsilon < 1$, then

$$\lambda_s = \lambda_1 + \lambda_1'(s-1) + O(|s-1|^{1+\varepsilon}), \qquad \text{as } s \to 1 \text{ in } \mathbb{C}_1^+,$$

and

$$\frac{s - 1 + (\lambda_1')^{-1}(1 - \lambda_s)}{(s-1)(1 - \lambda_s)} = O(|s-1|^{\varepsilon-1}).$$

($\varepsilon = 1$) If ψ is $(1, 2)$-summable, then

$$\lambda_s = \lambda_1 + \lambda_1'(s - 1) + \frac{1}{2}\big(\lambda_1'' + o(1)\big)(s - 1)^2, \qquad \text{as } s \to 1 \text{ in } \mathbb{C}_1^+,$$

and

$$\varlimsup_{\overline{\mathbb{C}_1^+} \cap B(1,\delta_1) \ni s \to 1} \frac{s - 1 + (\lambda_1')^{-1}(1 - \lambda_s)}{(s - 1)(1 - \lambda_s)} = \frac{1}{2} \cdot \frac{\lambda_1''}{(\lambda_1')^2}.$$

The proof is complete. □

6.3 Failure of D-Genericity

Suppose ψ is a normal $(1, 1)$-summable potential. If ψ is not D-generic, then the limit

$$\lim_{T \to \infty} \frac{N_\xi^B(T)}{e^T}$$

may not exist in the traditional sense. Below, we show that the limit exists up to taking a Cesàro average as in Theorem 1.10. (In Section 6.1, we have already established Theorem 1.10 for D-generic potentials.)

According to Theorem 5.3,

$$\psi = \phi + w - w \circ \sigma,$$

for some function ϕ which takes values in a discrete subgroup $a\mathbb{Z} \subset \mathbb{R}$. We assume that a has been chosen so that ϕ does not take values in any proper subgroup of $a\mathbb{Z}$. Note that $a \neq 0$ as ψ is not identically zero (by assumption, ψ is negative). Theorem 5.3 also tells us that the operator $\mathcal{L}_\psi$ has spectral radius less than 1 for any $s \in \overline{\mathbb{C}_1^+} \setminus \{1 + (2\pi i/a)\mathbb{Z}\}$.

Arguing as in the proof of Lemma 6.1 shows:

Lemma 6.2. *Suppose ψ is a normal $(1, 1)$-summable potential, which is cohomologous to a function ϕ which takes values in a discrete subgroup $a\mathbb{Z} \subset \mathbb{R}$ (and in no smaller discrete subgroup). For any $\xi \in E_A^\infty$, the following statements hold:*

(1) The series defining $\eta(s)(\xi)$ converges absolutely uniformly on compact subsets of $\mathbb{C}_1^+$ and defines a holomorphic function.

(2) The function $s \to \eta(s)(\xi) \in \mathbb{C}$ has a continuous extension to $\overline{\mathbb{C}_1^+} \setminus \{1 + (2\pi i/a)k : k \in \mathbb{Z}\}$.

(3) *As* $s \to 1$ *in* $\mathbb{C}_1^+$,

$$\eta(s)(\xi) - \frac{(-\lambda_1')^{-1} R_1 f(\xi)}{s - 1} = o(|s - 1|^{-1}).$$

From here, one can deduce Theorem 1.10 from the Hardy-Littlewood Tauberian theorem as in the D-generic case. We leave the details to the reader.

Part III

One Component Inner Functions

7 Fundamental Properties and Examples

According to the original definition of W. Cohn in [Coh82], an inner function F is a *one component inner function* if the set $\{z \in \mathbb{D} : |F(z)| < r\}$ is connected for some $0 < r < 1$.
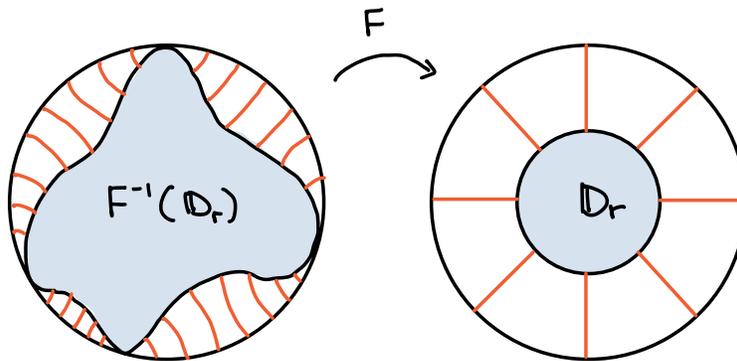

Figure 1: Anatomy of a one component inner function.

In his paper, Cohn gave a geometric description of one component inner functions, which is more suitable for applications in dynamical systems, see Fig. 1 above:

Lemma 7.1 (Cohn)**.** *Let F be a one component inner function and assume that $0 < r < 1$ is chosen so that $F^{-1}(\mathbb{D}_r)$ is connected. Then the region $F^{-1}(\mathbb{D}_r)$ is a Jordan domain. If $F^{-1}(\mathbb{D}_r)$ is compactly contained in the unit disk, then F is a finite Blaschke product. Otherwise, $\mathbb{D} \setminus F^{-1}(\mathbb{D}_r)$ consists of countably many simply-connected regions $\{\Omega_i\}$. On each region Ω_i, F acts as a universal covering map to the annulus $A(0; r, 1)$. The map F is also a universal covering map from $\widetilde{\Omega}_i$ to the annulus $A(0; r, 1/r)$, where $\widetilde{\Omega}_i$ is the double of Ω_i across the unit circle.*

We say that an inner function F is *singular* at a point $\zeta \in \partial\mathbb{D}$ if it does not admit an analytic extension to a neighbourhood of ζ. Let $\Sigma \subset \partial\mathbb{D}$ be the set of analytic singularities of F. It is clear from this definition that Σ is a closed set. While one usually thinks of inner functions as holomorphic self-maps of the unit disk, one may also view F as a meromorphic function on $\hat{\mathbb{C}} \setminus \Sigma$.

The following theorem provides a converse to Lemma 7.1, as well as a number of other characterizations of one component inner functions:

Theorem 7.2. *Let $F : \mathbb{D} \to \mathbb{D}$ be an inner function. The following are equivalent:*

(a) The set $F^{-1}(B(0, r))$ is connected for some $0 < r < 1$.

(b) There is an annulus $A = A(0; \rho, 1)$ such that $F : \mathbb{D} \to \mathbb{D}$ is a covering map over A.

(c) There is an annulus $\widetilde{A} = A(0; \rho, 1/\rho)$ such that $F : \hat{\mathbb{C}} \setminus \Sigma \to \hat{\mathbb{C}}$ is a covering map over $\widetilde{A}$.

(d) The singular set $\Sigma \subset \partial\mathbb{D}$ has Lebesgue measure 0, the derivative $F'(\zeta) \to \infty$ as $\zeta \in \partial\mathbb{D} \setminus \Sigma$ approaches Σ, and

$$\left| \frac{F''(\zeta)}{F'(\zeta)^2} \right| \leq C, \qquad \zeta \in \partial\mathbb{D} \setminus \Sigma, \tag{7.1}$$

for some constant $C > 0$.

If any of the above conditions hold, then Σ is equal to the set of points on the unit circle whose radial limit does not exist or is not unimodular.

Remark. (i) We say that $v \in \mathbb{D}$ is a *regular value* of an inner function $F : \mathbb{D} \to \mathbb{D}$ if F is a covering map over some open neighbourhood of v. A point in the unit disk which is not a regular value is called a *singular value*.

(ii) The equivalence (a) $\Leftrightarrow$ (d) is due to Aleksandrov [Ale02]. For additional characterizations of one component inner functions in terms of Carleson boxes and Aleksandrov-Clark measures, we refer the reader to [NR21] and [Bes14] respectively.

To prove Theorem 7.2, we will use several facts about inner functions and cluster sets. We begin by recalling [IK22, Lemma 1.3]:

Lemma 7.3. *Let F be an inner function. If V is a Jordan domain compactly contained in the unit disk, then any connected component of $F^{-1}(V)$ is a Jordan domain.*

For a function g on the unit disk and a point $\zeta \in \partial \mathbb{D}$, the *cluster set* $\mathrm{Cl}(g, \zeta)$ consists of all possible limits of $g(z_n)$, for sequences $z_n \in \mathbb{D}$ converging to ζ. The following lemma is well known, e.g. see [Mas12, Theorem 2.6]:

Lemma 7.4. *Let F be an inner function. A point $\zeta \in \partial \mathbb{D}$ belongs to Σ if and only if the cluster set $\mathrm{Cl}(F, \zeta)$ is the closed unit disk.*

We will also need the following elementary observation:

Lemma 7.5. *Suppose $g : \mathbb{D} \to \mathbb{D}$ is a inner function, which extends continuously to $\partial \mathbb{D} \setminus \{-1, 1\}$. If g preserves the upper and lower semicircles, then g is a Möbius transformation.*

Proof. We begin by observing that g is injective on $\partial \mathbb{D} \cap \mathbb{H}$ and $\partial \mathbb{D} \cap \overline{\mathbb{H}}$, being orientation-preserving but not surjective. Let α be a point on the unit circle, which is not the radial limit of g at -1 or $+1$, if they exist. As the Aleksandrov-Clark measure μ_α can be supported on at most one point, (3.3) tells us that g is either a constant or a Möbius transfomation. The assumption rules out the former case, so that g is a Möbius transformation. $\square$

Proof of Theorem 7.2. We show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) and (c) $\Leftrightarrow$ (d):

(a) $\Rightarrow$ (b). By Lemma 7.3, $F^{-1}(\mathbb{D}_r)$ is a Jordan domain. If $F^{-1}(\mathbb{D}_r)$ is compactly contained in the unit disk, then F is a finite Blaschke product. Otherwise, $\mathbb{D} \setminus F^{-1}(\mathbb{D}_r)$ consists of countably many simply-connected regions Ω_i, bounded by a curve γ_i in the unit disk and an arc I_i on the unit circle.

Usually γ_i and I_i share two distinct endpoints, however, when there is only one complementary region, then γ_i and I_i meet twice at a single point.

In view of Lemma 7.4, F extends continuously to each open arc I_i. Therefore, F is continuous on $\overline{\Omega_i}$, with possible exception of one or two points where γ_i meets the unit circle. To see that F acts as a universal covering map from Ω_i onto the annulus $A = A(0; r, 1)$, we lift F to the universal cover of $A = A(0; r, 1)$, compose with conformal maps as necessary and apply Lemma 7.5.

(b) $\Rightarrow$ (c). Suppose $F : \mathbb{D} \to \mathbb{D}$ is a covering map over an annulus $A(0; \rho, 1)$. Let $\Omega_1, \Omega_2, \ldots$ denote the connected components of $F^{-1}(A(0; \rho, 1))$. For each $i = 1, 2, \ldots$, let $\widetilde{\Omega}_i$ be the double of Ω_i across the unit circle. In view of Lemma 7.4, F extends holomorphically across $I_i = \widetilde{\Omega}_i \cap \partial\mathbb{D}$. From the Schwarz reflection principle, it follows that F is a covering map from $\widetilde{\Omega}_i$ to $A(0; \rho, 1/\rho)$.

(c) $\Rightarrow$ (a). Take a slightly smaller annulus $A(0; r, 1/r) \subset A(0; \rho, 1/\rho)$. Each connected component of $F^{-1}(A(0; r, 1/r))$ is a Jordan domain $\widetilde{\Omega}_i$ which is symmetric with respect to the unit circle. The boundaries of the connected components can only meet in the set Σ. As such, $\mathbb{D} \setminus F^{-1}(A(0; r, 1/r))$ is connected so that F is a one component inner function.

(c) $\Rightarrow$ (d). Let $\zeta \in \partial\mathbb{D}$ be a point at which the radial limit of F belongs to the unit circle. By applying Koebe's Quarter Theorem to the holomorphic branch of F^{-1} on the ball $B\big(F(\zeta), 1 - \rho\big)$ that maps $F(\zeta)$ to ζ, we get

$$\mathscr{B} = B\left(\zeta, \frac{1 - \rho}{4|F'(\zeta)|}\right) \subset \hat{\mathbb{C}} \setminus \Sigma.$$

In particular, Σ has Lebesgue measure 0 and $F'(\zeta) \to \infty$ as $\mathrm{dist}(\zeta, \Sigma) \to 0$.

Recall that the class $\mathcal{S}$ of conformal maps $\varphi : \mathbb{D} \to \mathbb{C}$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$ is compact in the topology of uniform convergence on compact subsets. In particular, there exists a constant $C > 0$ so that the second derivative $|\varphi''(0)| \leq C$ for every $\varphi \in \mathcal{S}$. After rescaling appropriately, we may apply this bound to F on $\mathscr{B}$ to get (7.1).

(d) $\Rightarrow$ (c). Let $\zeta \in \partial\mathbb{D} \setminus \Sigma$. Together with Gronwall's inequality, (7.1) implies that there exists $c > 0$ so that

$$|F'(\omega)| < 2|F'(\zeta)| \text{ on the arc } I = I\left(\zeta, \frac{c}{|F'(\zeta)|}\right)$$

of length $2c/|F'(\zeta)|$ centered at ζ. The assumption that $F'(\omega) \to \infty$ as $\mathrm{dist}(\omega, \Sigma) \to 0$ assures us that $I \subset \partial\mathbb{D} \setminus \Sigma$. By shrinking $c > 0$ if necessary, we can guarantee that $F|_I$ omits an arc of length π.

In particular, $F : \mathbb{D} \cup I \cup (\hat{\mathbb{C}} \setminus \mathbb{D}) \to \hat{\mathbb{C}}$ is contained in a compact family of meromorphic functions. An argument involving rescaling and normal families shows that F is injective on the ball

$$\mathscr{B} = B\left(\zeta, \frac{c_2}{|F'(\zeta)|}\right) \subset \hat{\mathbb{C}} \setminus \Sigma,$$

for some $0 < c_2 < c$. Koebe's Quarter Theorem then shows that $F(\mathscr{B})$ contains a ball centered at $F(\zeta)$ whose radius bounded is bounded below by $c_3 = c_2/4$. It remains to set $\rho = 1 - c_3$. $\qquad \square$

Remark. The proof of (c) $\Rightarrow$ (d) shows that for every integer $m \geq 2$ there exists a constant $C_m \in (0, +\infty)$ such that

$$\left|\frac{F^{(m)}(\zeta)}{F'(\zeta)^m}\right| \leq C_m, \qquad \zeta \in \partial\mathbb{D} \setminus \Sigma. \tag{7.2}$$

For the original proof, see [Ale02, Corollary 1].

We now describe two illuminating classes of examples of one component inner functions.

Character-Automorphic Functions. Let $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ be a Fuchsian group which possesses a fundamental polygon P with finitely many sides, including at least one ideal side. In other words, Γ is a geometrically finite Fuchsian group of co-infinite area. We can form an infinite Blaschke product whose zeros constitute an orbit of Γ:

$$g(z) = \prod_{\gamma \in \Gamma} -\frac{\gamma(0)}{|\gamma(0)|} \cdot \frac{z - \gamma(0)}{1 - \overline{\gamma(0)}z}.$$

The function g is related to the Green's function of the Riemann surface $\mathbb{D}/\Gamma$.

Contrary to what one may initially expect, the function g is not automorphic under Γ but only character-automorphic. A *character* v of a Fuchsian group $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ is a homomorphism from Γ to the unit circle. A function f on the unit disk is called *character automorphic* if

$$f(\gamma(z)) = v(\gamma) \cdot f(z), \qquad \gamma \in \Gamma.$$

It is easy to see that if $B(0, c)$ is a ball which meets all the finite sides of P, then the inverse image $F^{-1}(B(0, c))$ is connected. For further properties of the Blaschke product g, we refer the reader to [Pom76].

Stephenson's Cut and Paste Construction. Another natural way of building inner functions is Stephenson's cut and paste construction, see [Ste88, Bis93]. We confine ourselves to a particular example but the reader can easily modify it to construct a plethora of other examples.

Take an infinite collection of tiles T_j of the form $\mathbb{D} \setminus [1/2, 1)$, indexed by the integers. We form a simply-connected Riemann surface S by gluing the lower side of $[1/2, 1)$ in T_j to the upper side of $[1/2, 1)$ in T_{j+1}. The surface S comes equipped with a natural projection to the disk $\mathbb{D}$ which sends a point in a tile T_j to its representative in $\mathbb{D} \setminus [1/2, 1)$. We may uniformize $S \cong \mathbb{D}$ by taking 0 in the base tile T_0 to 0. In this uniformizing coordinate, the projection $F : S \to \mathbb{D} \setminus \{1/2\}$ becomes a holomorphic self-map of the disk. Since all the slits have been glued up, F is an inner function, and a little thought shows that it is the universal covering map of $\mathbb{D} \setminus \{1/2\}$.

It is clear from this construction, that if $B(0, c)$ is a ball with radius $c \in (0, 1/2)$, then the inverse image $F^{-1}(B(0, c))$ is connected.

8 Thermodynamic Formalism for Centered One Component Inner Functions

In this section, we discuss centered one component inner functions from a dynamical point of view. We show that a one component inner function admits a Markov partition, which allows us to code the unit circle by a countable alphabet shift space. For one component inner functions of finite entropy, we prove a variational principle and show that the potential $\psi(z) = -\log|F'|$ has a mixing property needed for Orbit Counting.

8.1 Markov Partitions

We first construct a *classical Markov partition* for a centered one component inner function and then modify it slightly in order to represent the unit circle

as the limit set of a Graph Directed Markov System (GDMS), the concept defined and explored in [MU03]. For more recent expositions on GDMS, we refer the reader to [URM22b, KU23].

Below, we will use the fact that a centered inner function is uniformly expanding on the unit circle:

$$\inf_{\zeta \in \partial \mathbb{D}} |F'(\zeta)| > 1, \tag{8.1}$$

where we use the convention that $|F'(\zeta)| = \infty$ if F does not have an angular derivative at ζ. For a proof, see [Mas12, Theorem 4.15]. For one component inner functions, this convention amounts to setting $|F'(\zeta)| = \infty$ on $\Sigma \subset \partial \mathbb{D}$.

Finite Blaschke Products. Suppose first that F is a finite Blaschke product of degree $d \geq 2$, for example $z \mapsto z^d$. Let p be one of the $d - 1$ repelling fixed points of F on the unit circle. The set $F^{-1}(p)$ divides the unit circle into d open arcs $I_1, I_2, \ldots, I_d$, which are mapped by F bijectively onto $\partial \mathbb{D} \setminus \{p\}$.

This Markov partition allows us to code points on the unit circle by infinite words in $\{1, 2, \ldots, d\}^{\mathbb{N}}$. More precisely, we say that a point $\zeta \in \partial \mathbb{D}$ is coded by

$$e_0 e_1 e_2 \ldots$$

if $e_n = i$ whenever $F^{\circ n}(\zeta) \in \overline{I_i}$. It is easy to see that any infinite word $\{1, 2, \ldots, d\}^{\mathbb{N}}$ arises as the code of some point on the unit circle, and all but countably many points are coded by unique words. The *exceptional set* $\mathscr{E}$ consists of the endpoints of the arcs I_i and their iterated pre-images under F. Points in $\mathscr{E}$ have two codings, while all other points have exactly one code.

Infinite-Degree Inner Functions. We now assume that F is a centered one component inner function that is not a finite Blaschke product. In this case, the singular set $\Sigma \subset \partial \mathbb{D}$ of F is not empty.

To construct a Markov partition of the unit circle for F, fix an arbitrary point $p \in \partial \mathbb{D}$ on the unit circle. Let $\{J_k\}_{k=1}^{\infty}$ be the collection complementary arcs in $\partial \mathbb{D} \setminus \Sigma$. Since for each $k \in \mathbb{N}$, the restriction $F|_{J_k}$ is an infinite degree covering map of $\partial \mathbb{D}$, the set $F^{-1}(p)$ partitions each J_k into infinitely many

arcs: $J_k = \bigcup_{l=-\infty}^{\infty} \overline{I_{k,l}}$. For convenience, we enumerate the arcs $\{I_i\} = \{I_{k,l}\}$ by a single index $i \in \mathbb{N}$.

As $F(I_i) = \partial \mathbb{D} \setminus \{p\}$, each closed arc $\overline{I_i}$ contains a fixed point of F. In particular, F has infinitely many fixed points. We may therefore assume that p is one of these fixed points.

We again have a near-bijection between infinite words in $\{1, 2, \dots\}^{\mathbb{N}}$ and points in $\partial \mathbb{D}$. This time, the exceptional set $\mathscr{E}$ of points without unique codings consists of iterated pre-images of endpoints of the arcs I_i, $i \in \mathbb{N}$, as well as points of Σ. The former have two codings, while the latter have no codings. Since the singular set Σ has Lebesgue measure zero, so does $\mathscr{E}$.

Graph Directed Markov Systems. From the perspective of GDMS, the Markov partitions constructed above are slightly inadequate. Arbitrarily choose a point $q \in \Sigma$. The points p and q divide the unit circle into two arcs X_1 and X_2, which we call *tiles* of the GDMS. For each tile X_j, $j = 1, 2$, choose an open set $X_j \subset U_j \subset A(0; r, 1/r)$, where $A(0; r, 1/r)$ is an annulus over which F is a covering map.

By construction, each arc I_i in the classical Markov partition is wholly contained in one of the two tiles. Recall that $F(I_i) = \partial \mathbb{D} \setminus \{p\}$. We can split I_i into two smaller arcs $I_{i,1}$ and $I_{i,2}$ with $F(I_{i,1}) = X_1$ and $F(I_{i,2}) = X_2$. We refer to the collection of intervals $\{I_n\} = \{I_{i,j}\}$ as the *Markov partition for the GDMS*.

We write $\phi_{i,j} : U_j \to \mathbb{C}$ for the holomorphic branch of F^{-1} which maps X_j onto $I_{i,j}$. We define the *alphabet* as the set of these as *contractions* $\{\phi_n\} = \{\phi_{i,j}\}$. Two contractions ϕ_m and ϕ_n can only be composed if the image of ϕ_n is contained in the tile on which ϕ_m is defined. We say that a word (finite or infinite)

$$e_0 e_1 e_2 \dots$$

is *admissible* if for any $n \geq 0$, the composition $\phi_{e_n} \circ \phi_{e_{n+1}}$ makes sense. It is easy to see that the incidence matrix which indicates admissible compositions is finitely primitive (see Section 5.1 for the definition).

The normalization $F(0) = 0$ guarantees that the inverse branches ϕ_{e_n} contract by a definite factor. In particular, admissible infinite words code single

points. The set of admissible infinite words is almost in bijection with points on the unit circle, as points in $\mathscr{E}$ do not have a unique coding.

Aleksandrov-Clark Measures. From the topological description of a one component inner function (Lemma 7.1), it follows that if F has a radial boundary value at a point $\zeta \in \Sigma$, then it cannot lie on the unit circle.

Lemma 8.1. *Let F be a one-component centered inner function. For any $\alpha \in \partial\mathbb{D}$, the Aleksandrov-Clark measure μ_α does not charge the set Σ. In particular, each measure μ_α is discrete and the adjoint definition of the transfer operator coincides with the classical one.*

As a result, little is lost when passing to the subset of points on the unit circle which have unique codes.

8.2 Some Properties of the Potential $-\log|F'|$

We now record some properties of the potential

$$\psi(z) = -\log|F'(z)|$$

associated to a one component centered inner function F which will be used in the sequel. In the language of Part II, one would say that ψ is a normal potential.

1. (Summable) The sum over the intervals I_n in the Markov partition (for the GDMS),

$$\sum_{n=1}^{\infty} \sup_{I_n} |e^\psi| < \infty.$$

   Indeed, by Koebe's Distortion Theorem, the above sum is essentially

$$\sum_{n=1}^{\infty} \sup_{I_n} |e^\psi| \asymp \sum_{n=1}^{\infty} |I_n| = 2\pi.$$

   More generally, for any integer $p \geq 0$, we say that a potential ψ is $(1, p)$-*summable* if

$$\sum_{n=1}^{\infty} \sup_{I_n} |\psi^p e^\psi| < \infty.$$

By Koebe's Distortion Theorem, the above condition is equivalent to

$$\int_{\partial \mathbb{D}} |\psi|^p dm < \infty.$$

2. (Negative) Since $F(0) = 0$, the derivative $|F'(z)| > c > 1$ on the unit circle $\partial \mathbb{D}$ by (8.1). As a result,

$$\sup_{z \in \partial \mathbb{D}} \psi(z) < 0.$$

3. (Centered) In Section 2, we saw that the spectral radius of the composition operator C_F was 1. As $\mathcal{L}_{-\log|F'|}$ is its adjoint, we have

$$P(-\log|F'|) = \log 1 = 0.$$

4. (Level 1 Lipschitz continuity)

Unless F is a finite Blaschke product, the potential $\psi(z) = -\log|F'(z)|$ is not Lipschitz on the unit circle, let alone bounded. Nevertheless, there is a constant $C \in [1, +\infty)$ such on each interval I_n from the Markov partition (for the GDMS), one has

$$|\psi(x) - \psi(y)| \leq C \cdot |F(x) - F(y)|, \tag{8.2}$$

which we call *level 1 Lipschitz continuity*. In other words, ψ is Lipschitz on each interval I_n from the Markov partition when it is rescaled to definite size, with a uniform Lipschitz constant.

To verify (8.2), we use the fundamental theorem of calculus:

$$|\psi(x) - \psi(y)| \leq \int_x^y \left| \frac{F''(z)}{F'(z)} \right| |dz|, \qquad x, y \in I_n.$$

From Koebe's Distortion Theorem, we know that

$$|F'(z)| \asymp \frac{|F(x) - F(y)|}{|x - y|}, \qquad x, y, z \in I_n.$$

Therefore,

$$|\psi(x) - \psi(y)| \leq \frac{|F(x) - F(y)|}{|x - y|} \int_x^y \left| \frac{F''(z)}{F'(z)^2} \right| |dz|.$$

Finally, by the analytic definition of one component inner functions (7.1), the integrand is bounded, which shows (8.2).

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8.3 Variational Principle

To proceed, we impose an additional assumption on our inner function F: we assume that its derivative lies in the Nevanlinna class:

$$\int_{\partial\mathbb{D}} \log|F'| dm < \infty.$$

In a beautiful work, M. Craizer [Cra91] showed that this condition is equivalent to the Lebesgue measure m having finite measure-theoretic entropy with respect to the action of F on the unit circle.

Let $\mathcal{P} = \{I_n\}_{n\in\mathbb{N}}$ be the classical Markov partition of the unit circle defined in Section 8.1. Since each arc $I_n \in \mathcal{P}$ is mapped onto $\partial\mathbb{D} \setminus \{p\}$, by Koebe's Distortion Theorem, $|F'(\zeta)| \asymp 1/|I_n|$ for $\zeta \in I_n$. Therefore,

$$\sum |I_n| \log \frac{1}{|I_n|} \asymp \int_{\partial\mathbb{D}} \log|F'| dm < \infty.$$

Notice that each element of the refined partition

$$\mathcal{P}^{(n)} = \mathcal{P} \vee \mathcal{F}^{-1}(\mathcal{P}) \vee \mathcal{F}^{-2}(\mathcal{P}) \vee \cdots \vee \mathcal{F}^{-(n-1)}(\mathcal{P})$$

is an arc and that the maximal size of arcs decays exponentially. As such, $\mathcal{P}$ is a one-sided generator in the sense that $\bigvee \mathcal{P}^{(n)}$ generates the σ-algebra of Borel subsets of the circle. Since $\mathcal{P}$ is a generating partition of finite entropy, by [PrU10, Theorem 1.9.7], the measure theoretic entropy is given by

$$h(F,m) = \int_{\partial\mathbb{D}} \log J_F \, dm,$$

where J_F is the measure-theoretic Jacobian of F. By definition, the measure-theoretic Jacobian is a non-negative function in $L^1(\partial\mathbb{D}, m)$ such that

$$m(F(E)) = \int_E J_F \, dm$$

for any measurable set E on which F is injective. In the setting of one component inner functions, $J_F(\zeta) = |F'(\zeta)|$ a.e. on the unit circle.

Theorem 8.2. *Suppose that F is a one component centered inner function whose derivative lies in the Nevanlinna class. If μ is an F-invariant probability measure on the unit circle, then its measure-theoretic entropy*

$$h(F,\mu) \leq \int_{\partial\mathbb{D}} \log|F'(\zeta)| d\mu.$$

Equality holds only for the normalized Lebesgue measure m.

The inequality in the variational principle above is neatly explained in [MU03, Section 2.1]. To see that the Lebesgue measure is the only measure for which equality holds, we refer the reader to Proof II of [PrU10, Theorem 4.6.2].

8.4 Spectral Gap on Smooth Functions

Below, we continue to abbreviate $\psi(x) = -\log|F'(x)|$ for simplicity. From the estimate (7.2), it is not difficult to see that for every integer $m \geq 1$ there exists a constant $\tilde{C}_m \in (0, +\infty)$ such that

$$\left| \frac{\psi^{(m)}(x)}{F'(x)^m} \right| \leq \tilde{C}_m, \qquad x \in \partial\mathbb{D} \setminus \Sigma, \tag{8.3}$$

where $\psi^{(m)}$ denotes the m-th derivative of ψ along the unit circle.

Lemma 8.3. *Suppose that F is a one component centered inner function, which is not a rotation. If $\operatorname{Re} s \geq 1$, then*

$$\mathcal{L}_s g(x) = \sum_{F(y)=x} e^{s\psi(y)} g(y)$$

satisfies the two-norm inequality of Ionescu-Tulcea and Marinescu: There exist constants $\theta \in (0,1)$ and $C \in (0, +\infty)$, which may depend on k and s such that

$$\|\mathcal{L}_s g(x)\|_{C^k} \leq \theta \cdot \|g\|_{C^k} + C \cdot \|g\|_{C^{k-1}}. \tag{8.4}$$

In particular,

$$\|\mathcal{L}_s^n g(x)\|_{C^k} \leq \theta^n \cdot \|g\|_{C^k} + \frac{C}{1-\theta} \cdot \|g\|_{C^{k-1}}. \tag{8.5}$$

is uniformly bounded for $n \geq 1$.

Proof. We treat the case when $k = 1$ as the general case is similar. Differentiating the definition of $\mathcal{L}_s g$, we get

$$\frac{d}{dx} \mathcal{L}_s g(x) = \sum_{F(y)=x} \frac{1}{|F'(y)|} \left\{ e^{s\psi(y)} g'(y) + s e^{s\psi(y)} \psi'(y) \cdot g(y) \right\}.$$

Thus,

$$\left| \frac{d}{dx} \mathcal{L}_s g(x) \right| \leq \sum_{F(y)=x} \left\{ |F'(y)|^{-1-\operatorname{Re}s} \|g\|_{C^1} + |s| \cdot |F'(y)|^{-\operatorname{Re}s} \left| \frac{F''(y)}{F'(y)^2} \right| \|g\|_{C^0} \right\},$$

which simplifies to (8.4). In the last step, we have used that

$$\sum_{F(y)=x} |F'(y)|^{-\operatorname{Re}s} \leq 1 \quad \text{and} \quad \sum_{F(y)=x} |F'(y)|^{-1-\operatorname{Re}s} \leq \theta < 1,$$

which can be proved in the same way as Corollary 2.9. $\qquad\square$

As in [PP90, Proposition 2.1 and Theorem 2.2], the above lemma implies that $\mathcal{L}_1$ has a spectral gap on $C^k(\partial \mathbb{D})$ for every integer $k \geq 1$:

Corollary 8.4. *If F is a centered one component inner function, then the operator $\mathcal{L}_1$ has a simple eigenvalue 1 with eigenfunction $\mathbf{1}$, while the rest of the spectrum is contained in a smaller ball centered at the origin. Moreover, the spectral radius $r_{C^k}(\mathcal{L}_{1+ia}) \leq 1$ for any $a \in \mathbb{R}$.*

8.5 Genericity and Orbit Counting

Suppose that F is a centered one component inner function with derivative in Nevanlinna class. Taking $s = 1+ia$ in Lemma 8.3, we see that for any function $u \in C^k$, the sequence $\{\mathcal{L}_{(-1+ia)\log|F'|}^n(u)\}_{n=0}^{\infty}$ is bounded in C^k, which is an analogue of Lemma 4.5. Adapting the proof of Lemma 4.7 to our current setting, we obtain:

Lemma 8.5. *Any L^2 eigenfunction of $e^{-ia\psi} C_F$ is C^∞.*

We now show:

Lemma 8.6. *Suppose F is an infinite degree one component centered inner function of finite entropy. For any $a \neq 0$, the operator $e^{-ia\psi} C_F$ does not possess any eigenfunctions of bounded variation, let alone any C^∞ eigenfunctions.*

Proof. We need to show that any BV solution of the eigenfunction equation

$$w(F(z)) = e^{ia \log |F'(z)|} \cdot w(z) \tag{8.6}$$

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is constant. Since the Lebesgue measure is ergodic, $|w(z)|$ is constant a.e. As w has bounded variation, after redefining w on a set of measure zero, we may assume that $|w(z)| = 1$ for all $z \in \partial\mathbb{D}$.

Let I_k be an interval from the classical Markov partition, described in Section 8.1. Recall that $F(I_k) = \partial\mathbb{D} \setminus \{p\}$. Taking the variation of both sides of (8.6) over I_k, we get

$$\|w\|_{\mathrm{BV}(\partial\mathbb{D}\setminus\{p\})} \leq |a| \int_{I_k} \log |F'(z)| dm(z) + \|w\|_{\mathrm{BV}(I_k)}.$$

As the Markov partition is infinite, we can choose I_k to make the right hand side as small as we want. This forces w to be constant on the unit circle. Since $a \neq 0$, (8.6) tells us that $\log |F'|$ is also constant, which can only happen if F is a Blaschke product of the form $z \mapsto z^d$. This contradicts the hypothesis that F has infinite degree. $\qquad\qquad\square$

In the language of Part II, Lemma 8.6 says that the potential $\psi(z) = -\log |F'(z)|$ is D-generic. Together with the discussion in Section 8.2, this completes the verification that ψ is a robust potential when F is an infinite degree one component inner function of finite entropy. For the case of finite Blaschke products, we refer the reader to [Ivr15, Section 7].

To deduce Theorems 1.3 and 1.4 as special cases of Theorems 1.9 and 1.10, we code the unit circle as in Section 8.1. In this case, the conformal and equilibrium measures for the potential $\psi \circ \pi^{-1} : E_A^\infty \to \mathbb{R}$ are equal to the pullback of the normalized Lebesgue measure on the unit circle under the coding map $\pi : E_A^\infty \to \partial\mathbb{D}$. Inserting these quantities into Theorems 1.9 and 1.10, one obtains Theorems 1.3 and 1.4 for points on the unit circle with unique codings, which are dense in the unit circle[3]. To obtain the full statements of Theorems 1.3 and 1.4, one can approximate a general point $x \in \partial\mathbb{D}$ by a sequence of points $x_n \in \partial\mathbb{D}$ in the image of π and apply Koebe's Distortion Theorem. We leave the details to the reader.

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[3]Recall from the construction in Section 8.1 that the exceptional set $\mathscr{E}$ of points on the unit circle which do not admit unique codings is forward invariant. In other words, if $x \notin \mathscr{E}$, then any repeated pre-image $y \notin \mathscr{E}$.

9 Thermodynamic Formalism for Parabolic One Component Inner Functions

An inner function $f : \mathbb{D} \to \mathbb{D}$ is called *parabolic* if its Denjoy-Wolff fixed point $p \in \partial\mathbb{D}$ and $f'(p) := \lim_{r\to 1} f'(rp) = 1$. It is convenient to think of parabolic inner functions as holomorphic self-maps of the upper half-plane with a Denjoy-Wolff point at infinity. For this purpose, take a Möbius transformation M which maps $\mathbb{D}$ to $\mathbb{H}$ and sends p to ∞. Then, $F = M \circ f \circ M^{-1}$ is a holomorphic self-map of the upper half-plane such that for Lebesgue almost every point $x \in \mathbb{R}$, the vertical boundary value $\lim_{y\to 0} F(x + iy)$ exists and is real.

We say that an inner function f is *doubly-parabolic* if f is holomorphic in a neighbourhood of p and

$$f(z) = p + c(z - p)^3 + \dots, \qquad c \neq 0,$$

as $z \to p$. Translating to the upper half plane, this means that F is doubly-parabolic if

$$F(z) = z - a/z + \dots, \qquad a > 0,$$

near infinity. By contrast, an expansion

$$f(z) = p + c(z - p)^2 + \dots, \qquad c \neq 0,$$

or

$$F(z) = z + T - a/z + \dots, \qquad T \neq 0, \qquad a > 0,$$

suggests that f and F are *singly-parabolic*.

We say that F is a *parabolic one component inner function* if one of the following equivalent conditions is satisfied:

Theorem 9.1. *Suppose F is a parabolic inner function, viewed as a map of the upper half-plane to itself, with a parabolic fixed point at infinity. The following are equivalent:*

(a) The set $F^{-1}(\{\operatorname{Im} z > y\})$ is connected for some $y > 0$.

(b) There is a strip $S = \{0 < \operatorname{Im} z < \tau\}$ such that $F : \mathbb{H} \to \mathbb{H}$ is a covering map over S.

(c) There is a strip $\tilde{S} = \{-\tau < \operatorname{Im} z < \tau\}$ such that $F : \mathbb{C} \setminus \Sigma_\infty \to \mathbb{C}$ is a covering map over $\tilde{S}$.

(d) *The singular set* $\Sigma_\infty \subset \mathbb{R}$ *has Lebesgue measure* 0, *the derivative* $F'(\zeta) \to \infty$ *as* $\zeta \in \mathbb{R}$ *approaches* Σ *and*

$$\left| \frac{F''(\zeta)}{F'(\zeta)^2} \right| \leq C, \qquad \zeta \in \mathbb{R} \setminus \Sigma_\infty, \tag{9.1}$$

for some constant $C > 0$.

In the theorem above, $\Sigma_\infty \subset \mathbb{R}$ is the set of points $x \in \mathbb{R}$ where F does not admit a finite analytic extension to a neighbourhood of x. By definition, Σ_∞ is a closed subset of the real line. The proof of Theorem 9.1 is essentially the same as that of Theorem 7.2. We leave the details to the reader.

In this section, we prove an orbit counting theorem for doubly-parabolic one component inner functions (dp1c inner functions). In order to apply the machinery developed in Section 6, we work with the first return map.

9.1 Expansion

The lemma below says that parabolic inner functions are expanding on the real line:

Lemma 9.2. *Suppose F is a parabolic inner function, viewed as a map of the upper half-plane to itself, with a parabolic fixed point at infinity. Then, $F'(x) > 1$ on $\mathbb{R}$.*

Proof. An inner function viewed as self-mapping of the upper half-plane has the form

$$F(z) = \alpha z + \beta + \int_{\mathbb{R}} \frac{1 + zw}{w - z} d\mu(w),$$

for some constants $\alpha > 0$, $\beta \in \mathbb{R}$ and finite positive singular measure μ on the real line, e.g. see [Tsu75]. Differentiating, we get

$$F'(z) = \alpha + \int_{\mathbb{R}} \frac{w^2 + 1}{(w - z)^2} d\mu(w). \tag{9.2}$$

From the above representation, it is readily seen that $\alpha = \lim_{t \to \infty} F'(it)$. For F to have a parabolic fixed point at infinity, we must have $\alpha = 1$. From the formula (9.2), it is clear that $F'(z) > 1$ for any $z \in \mathbb{R}$. $\qquad\square$

9.2   Markov Partitions

Let F be a dp1c inner function. The set $F^{-1}(\infty)$ partitions $(\mathbb{R} \cup \{\infty\}) \setminus \Sigma$ into countably many intervals $\{I_i\}_{i \in \Lambda}$ with $F(I_i) = \mathbb{R}$. The collection of intervals $\{I_i\}_{i \in \Lambda}$ is finite if and only if F is a finite Blaschke product. We refer to the collection $\{I_i\}_{i \in \Lambda}$ as the *basic Markov partition* for $F : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$.

Suppose $I_- = (-\infty, p_1^-)$ and $I_+ = (p_1^+, \infty)$ are the unbounded intervals in Λ. As the dynamics of F is repelling from ∞ in both directions, one can construct inverse orbits

$$\cdots < p_3^- < p_2^- < p_1^- \qquad \text{and} \qquad p_1^+ < p_2^+ < p_3^+ < \ldots$$

We refine the Markov partition Λ by subdividing I_+ and I_- into countably many intervals:

$$I_+ = \bigcup_{n=1}^{\infty} J_n^+, \qquad I_- = \bigcup_{n=1}^{\infty} J_n^-,$$

where $J_n^\pm$, $n \in \mathbb{N}$, is the segment between $p_n^\pm$ and $p_{n+1}^\pm$. By construction, F maps

- $J_{n+1}^\pm$ onto $J_n^\pm$ for any $n \geq 1$,

- J_1^+ onto $(-\infty, p_1)$,

- J_1^- onto (p_{-1}, ∞),

- each bounded basic interval $I_i \subset (p_1^-, p_1^+)$ onto $\mathbb{R}$.

Since, ∞ is a doubly-parabolic fixed point, diam $J_n^\pm \asymp 1/\sqrt{n}$ and is located $\asymp \sqrt{n}$ away from the origin.

For $N \geq 0$, define $X_N = [p_{N+1}^-, p_{N+1}^+]$. Given a bounded subset B of the real line, we may choose $N \geq 1$ sufficiently large so that $B \subset X_N$. From now on, we simply write X instead of X_N.

9.3   The First Return Map

We define the *first return map* $\hat{F} : X \to X$ as the first iterate of F which lies in X, that is,

$$\hat{F}(x) := F^{\circ N(x)}(x),$$

where $N(x) \geq 1$ is the smallest positive integer such that $F^{\circ N(x)}(x) \in X$. (Since any point in $\mathbb{R} \setminus X$ eventually maps to $J_1^\pm$ under the dynamics of F, the first return map is well defined outside the measure zero set $\Sigma_\infty \subset X$. In fact, any doubly parabolic Blaschke product is recurrent by [DM91, Theorem 4.2].)

We may view $\hat{F} : X \to X$ as a GDMS with the tiles $J_1^\pm, J_2^\pm, \ldots J_N^\pm$ and $[p_1^-, p_1^+]$ by splitting X into intervals that map univalently onto one of the tiles under $\hat{F}$. We leave it to the reader to check that the incidence matrix is finitely primitive.

Let ℓ_X be the normalized Lebesgue measure on X. As ℓ is invariant for $F : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$, ℓ_X is invariant for $\hat{F} : X \to X$. We now prove two lemmas which relate the properties of F and $\hat{F}$:

Lemma 9.3. *The Lyapunov exponent for the first return map $\hat{F}$ on X equals to the Lyapunov exponent of the original inner function F on $\mathbb{R}$:*

$$\int_X \log |\hat{F}'(x)| d\ell = \int_\mathbb{R} \log |F'(x)| d\ell.$$

The above lemma follows from a generalization of Kac's Lemma [URM22a, Proposition 10.2.5] and the identity $\log |(F^{\circ n})'(x)| = S_n \log |F'(x)|$.

Lemma 9.4. *For any $p \geq 1$, the first return map $\hat{F} : X \to X$ is $(1, p)$-integrable if and only if $F : \mathbb{R} \to \mathbb{R}$ is $(1, p)$-integrable, i.e.*

$$\int_X \big(\log |\hat{F}'(x)|\big)^p d\ell < \infty \quad \Longleftrightarrow \quad \int_\mathbb{R} \big(\log |F'(x)|\big)^p d\ell < \infty.$$

Proof. For $x \in \mathbb{R} \setminus X$, we continue to use the notation $N(x)$ for the smallest positive integer such that $\hat{F}(x) := F^{\circ N(x)}(x) \in X$.

Since diam $J_n^\pm \asymp n^{-1/2}$, the derivative $|(F^{\circ N(x)})'(x)| \asymp n^{1/2}$ for $x \in J_n^\pm$, whence

$$\int_{J_n^- \cup J_n^+} \big(\log |(F^{\circ n})'(x)|\big)^p d\ell(x) \lesssim n^{-1/2} \cdot (1 + \log n)^p.$$

Summing over $n = 1, 2, \ldots$, we obtain

$$\int_{\mathbb{R} \setminus X} \big(\log |(F^{\circ N(x)})'(x)|\big)^p d\ell(x) \lesssim \sum_{n=1}^{\infty} n^{-1/2} \cdot (1 + \log n)^p < +\infty.$$

From the above equation, the $(\Rightarrow)$ implication is immediate since

$$\log|\hat{F}'(x)| \geq \log|F'(x)|$$

for every $x \in \mathbb{R}$. By the elementary inequality $(A + B)^p \leq 2^p(A^p + B^p)$ and the invariance of the Lebesgue measure, we get

$$\int_X \big(\log|\hat{F}'(x)|\big)^p d\ell \lesssim \int_X \big(\log|F'(x)|\big)^p d\ell + \int_{\mathbb{R}\setminus X} \big(\log|(F^{\circ N(x)})'(x)|\big)^p d\ell,$$

which proves the $(\Leftarrow)$ implication. $\square$

9.4   Proof of the Orbit Counting Theorem 1.8

In order to apply the abstract symbolic results developed in Part II of this manuscript to the first return map $\hat{F} : X \to X$, we show that the potential

$$\psi(x) := -\log|\hat{F}'(x)|$$

is robust under the integrability hypothesis

$$\int_{\mathbb{R}} \big(\log|F'(z)|\big)^{1+\varepsilon} d\ell < \infty, \qquad \varepsilon > 0. \tag{9.3}$$

In view of Lemma 9.4, the assumption (9.3) is equivalent to the potential ψ being $(1, 1+\varepsilon)$-summable. From the formula (9.2), it is clear that ψ is negative as $X \subset \mathbb{R}$ is compact. An argument similar to the one in Section 8.2 shows that ψ is level 1 Lipschitz continuous (and therefore, it is level 1 α-Hölder continuous for any $0 < \alpha < 1$).

Since ψ is level 1 Hölder continuous and summable, the conformal and equilibrium measures for ψ are unique by Theorem 5.2. As ℓ_X functions as a conformal measure for ψ (with eigenvalue 1), $m_\psi = \ell_X$. Since the equilibrium measure μ_ψ is the unique $\hat{F}$-invariant measure on X that is absolutely continuous with respect to m_ψ, we must also have $\mu_\psi = \ell_X$. In particular, the eigenfunction $\rho_\psi = \mu_\psi/m_\psi = \mathbf{1}$ and the potential ψ is centered. Summarizing, we have proved that ψ is a normal $(1, 1 + \varepsilon)$-summable potential.

D-Genericity. According to [PoU17, Theorem 9.7], parabolic dynamical systems are automatically D-generic. For convenience of the reader, we give a brief sketch of the argument in the present setting: for any $n \geq 1$, there is a unique periodic point $q_n \in J_{n-1}^+$ of period n with itinerary

$$J_{n-1}^+ \to \cdots \to J_1^+ \to J_1^- \to J_n^+.$$

Inspection shows that the logarithms of the multipliers

$$L_n = \log |(F^{\circ n})'(q_n)|$$

satisfy

$$L_{n+1} - L_n \to 0, \qquad L_n \to \infty, \qquad \text{as } n \to \infty,$$

so they cannot be contained in a discrete subgroup of $\mathbb{R}$. It remains to notice that any periodic orbit of $F : \mathbb{R} \to \mathbb{R}$ which passes through X can be viewed as a periodic orbit of induced map $\hat{F} : X \to X$ with the same multiplier.

Conclusion. Applying Theorem 1.9 to the induced map $\hat{F} : X \to X$ and using Lemmas 9.3 and 9.4 shows that

$$\lim_{T \to \infty} \frac{n_{\hat{F}: X \to X}(x, T, B)}{e^T} = \frac{\ell_X(B)}{\int_X \log |\hat{F}'(z)| d\ell_X} = \frac{\ell(B)}{\int_{\mathbb{R}} \log |F'(z)| d\ell},$$

for $x \in X$ outside a Lebesgue measure zero set of points that do not have unique codings. Replacing $n_{\hat{F}: X \to X}(x, T, B)$ with $n_{\hat{F}: \mathbb{R} \to \mathbb{R}}(x, T, B)$, we see that Theorem 1.8 holds for a.e. $x \in X$.

Recalling that $X = X_N$ and taking $N \to \infty$ shows that Theorem 1.8 holds for a.e. $x \in \mathbb{R}$. Approximating a general point $x \in \mathbb{R} \cup \{\infty\}$ by a sequence of points x_n for which the conclusion of Theorem 1.8 is valid and using Koebe's Distortion Theorem as in Section 8.5 shows that Theorem 1.8 holds for all $x \in \mathbb{R} \cup \{\infty\}$.

A Continuity of Transfer Operators

In this appendix, $q \geq 0$ will a positive real number and $0 \leq \varepsilon < 1$. We show that if ψ is a normal $(1, q + \varepsilon)$-summable potential, then the modified

transfer operator $s \to \mathcal{L}_{s,q}$ given by (5.3) defines a $C^\varepsilon_{\text{loc}}$ function from $\overline{\mathbb{C}^+_1}$ to $\mathcal{B}(C^\alpha(E^\infty_A))$. We will often abbreviate $p = q + \varepsilon$.

The proof is not difficult but is somewhat tedious because one has to deal with the C^ε and C^α Hölder norms, both of which consist of two pieces (the L^∞ norm and the Hölder variation).

A.1 Some Properties of the Function $x \mapsto G(x) = x^p e^{-sx}$

Fix a real number $c > 0$. Loosely speaking, the following lemma says that for $x \in [c, \infty)$ and $s \in \overline{\mathbb{C}^+_1}$, the function $G(x) = x^p e^{-sx}$ behaves similarly to $G_0(x) = e^{-sx}$:

Lemma A.1. *The function $G(x, s) = x^p e^{-sx}$ enjoys the following properties on $[c, \infty) \times \overline{\mathbb{C}^+_1}$:*

1. *If $1 \leq \sigma_1 \leq \sigma_2$ and $x_1 \leq x_2$ then $G(x_2, \sigma_2) \lesssim G(x_1, \sigma_1)$.*

2. *$|G_x(x, s)| \lesssim |s| \cdot |G(x, s)|$.*

3. *$|G_s(x, s)| \lesssim x \cdot |G(x, s)|$.*

4. *$|G_{xs}(x, s)| \lesssim x \cdot |s| \cdot |G(x, s)|$.*

The implicit constants depend on the parameters $p \geq 0$ and $c > 0$.

Corollary A.2. *Suppose $(x_1, s_1), (x_2, s_2) \in (c, \infty) \times \overline{\mathbb{C}^+_1}$.*
  (i) If $x_{\min} = \min(x_1, x_2)$, then

$$\Delta_x = |G(x_2, s) - G(x_1, s)| \lesssim G(x_{\min}, s) \cdot |x_2 - x_1| \cdot |s|.$$

  (ii) If $\sigma_{\min} = \min(\operatorname{Re} s_1, \operatorname{Re} s_2)$, then

$$\Delta_s = |G(x, s_2) - G(x, s_1)| \lesssim G(x, \sigma_{\min}) \cdot x \cdot |s_2 - s_1|.$$

  (iii) With $x_{\min}$ and $\sigma_{\min}$ above, we have

$$\Delta_{xs} = |G(x_2, s_2) - G(x_1, s_2) - G(x_2, s_1) + G(x_1, s_1)|$$
$$\lesssim G(x_{\min}, \sigma_{\min}) \cdot |x_2 - x_1| \cdot \max(|s_1|, |s_2|) \cdot \min\big(1, |s_2 - s_1| \cdot \max(x_1, x_2)\big).$$

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Proof. (i, ii) By the fundamental theorem of calculus, $\Delta_x = \left| \int_{x_2}^{x_1} G_x(x, s) dx \right|$. Since G_x is a continuous function, it attains its average value at some point of $[x_1, x_2]$. Denoting it by x^*, we have

$$\Delta_x = |G(x_2, s) - G(x_1, s)| = |G_x(x^*, s)| \cdot |x_2 - x_1|,$$

for some $x^* \in [x_1, x_2]$. Part (i) follows from Properties 1 and 2. The proof of (ii) is similar.

(iii) To prove the "coarse" estimate, we write

$$\Delta_{xs} = \int_{x_1}^{x_2} \left\{ G_x(x, s_2) - G_x(x, s_1) \right\} dx.$$

and proceed as above. For the "fine" estimate, we use the representation

$$\Delta_{xs} = \left| \int_{s_2}^{s_1} \int_{x_2}^{x_1} G_{xs}(x, s) dx ds \right|,$$

which tells us that

$$\Delta_{xs} = |G_{xs}(x^*, s^*)| \cdot |x_2 - x_1| \cdot |s_2 - s_1|,$$

for some $x^* \in [x_1, x_2]$ and $s^* \in [s_1, s_2]$. The "fine" estimate now follows from Properties 1 and 4. $\qquad \square$

A.2 Proof of Boundedness

We first show that if ψ is a normal $(1, p)$-summable potential on E_A^∞, then for any $s \in \overline{\mathbb{C}_1^+}$, the linear operator $\mathcal{L}_{s,p}$ is bounded on $C^\alpha(E_A^\infty)$. For this purpose, we examine the behaviour of $\psi^p e^{s\psi}$ on a cylinder $[a] \subset E_A^\infty$:

Lemma A.3. *For any $p \geq 0$, $s \in \overline{\mathbb{C}_1^+}$ and normal $(1, p)$-summable potential ψ, we have*

$$\|\psi^p e^{s\psi} \circ a\|_\alpha \leq C(s, p, \psi) \cdot \sup_{[a]} |\psi^p e^{s\psi}|, \qquad a \in E. \tag{A.1}$$

Proof. The L^∞ norm

$$\|\psi^p e^{s\psi} \circ a\|_\infty = \sup_{[a]} |\psi^p e^{s\psi}|$$

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is trivially bounded by the right side of (A.1). To estimate the α-Hölder variation of $\psi^p e^{s\psi} \circ a$, we use Corollary A.2(i):

$$|\psi^p e^{s\psi}(a\omega) - \psi^p e^{s\psi}(a\tau)| \lesssim |\psi(a\omega) - \psi(a\tau)| \cdot |s| \cdot \sup_{[a]} |\psi^p e^{s\psi}|.$$

Dividing both sides by $d(\omega, \tau)^\alpha$ and taking the supremum over $\omega, \tau \in E_A^\infty$, we obtain

$$v_\alpha(\psi^p \exp(s\psi) \circ a) \lesssim v_\alpha^{(1)}(\psi) \cdot |s| \cdot \sup_{[a]} |\psi^p e^{s\psi}|.$$

Absorbing $v_\alpha^{(1)}(\psi) \cdot |s|$ into the implicit constant completes the proof. $\qquad\square$

Corollary A.4. *For any $p \geq 0$ and $s \in \overline{\mathbb{C}_1^+}$, the operator $\mathcal{L}_{s,p}$ is bounded on $C^\alpha(E_A^\infty)$.*

Proof. Suppose $g \in C^\alpha(E_A^\infty)$. We may write

$$\mathcal{L}_{s,p}g = \sum_{e \in E} (\psi^p e^{s\psi} \circ a) \cdot (g \circ a).$$

By Lemma 5.1,

$$\|\mathcal{L}_{s,p}g\|_\alpha \leq 3 \sum_{a \in E} \|\psi^p e^{s\psi} \circ a\|_\alpha \cdot \|g\|_\alpha \lesssim \|g\|_\alpha \cdot \sum_{a \in E} \sup_{[a]} |\psi^p e^{s\psi}|,$$

which is finite by Lemma 5.5. $\qquad\square$

A.3 Proof of Continuity (Part 1)

We first examine the case when $\varepsilon > 0$:

Lemma A.5. *If the potential ψ is $(1, q + \varepsilon)$ summable with $q \geq 0$ and $0 < \varepsilon < 1$, then $s \to \mathcal{L}_{s,q}$ is a $C_{\mathrm{loc}}^\varepsilon$ mapping from $\overline{\mathbb{C}_1^+}$ to $\mathcal{B}(C(E_A^\infty))$.*

Proof. We show that the quotient

$$\frac{\|\mathcal{L}_{s,q}g - \mathcal{L}_{t,q}g\|_{C(E_A^\infty)}}{|s - t|^\varepsilon}. \tag{A.2}$$

is uniformly bounded above over all $s, t \in \overline{\mathbb{C}_1^+}$. The numerator of (A.2) evaluated at a point $\omega \in E_A^\infty$ is

$$\left| \sum_{a \in E} (\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\omega) \cdot g(\omega) \right|,$$

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which by Corollary A.2(ii) is

$$\lesssim \|g\|_\infty \cdot \sum_{a \in E} \sup_{[a]} |\psi^q e^{\sigma_{\min}\psi}| \cdot \min\big(1, |(s-t)\psi(a\omega)|\big)$$

$$\lesssim \|g\|_\infty \cdot \sum_{a \in E} \sup_{[a]} |\psi^q e^{\sigma_{\min}\psi}| \cdot |(s-t)\psi(a\omega)|^\varepsilon$$

$$\lesssim \|g\|_\infty \cdot \sum_{a \in E} \sup_{[a]} |\psi^{q+\varepsilon} e^{\sigma_{\min}\psi}| \cdot |s-t|^\varepsilon,$$

where $\sigma_{\min} = \min(\operatorname{Re} s_1, \operatorname{Re} s_2)$. Therefore, (A.2) is

$$\lesssim \|g\|_\infty \cdot \sum_{a \in E} \sup_{[a]} |\psi^{q+\varepsilon} e^{\sigma_{\min}\psi}| < \infty,$$

as desired. $\qquad\square$

We now treat the case when $\varepsilon = 0$:

Lemma A.6. *If the potential ψ is $(1, q)$ summable for some $q \geq 0$, then the mapping $s \to \mathcal{L}_{s,q}$ is continuous from $\overline{\mathbb{C}_1^+}$ to $\mathcal{B}(C(E_A^\infty))$.*

Proof. Fix an $s \in \overline{\mathbb{C}_1^+}$. A simple estimate shows that for $t \in \overline{\mathbb{C}_1^+}$ with $|s-t| \leq 1$,

$$(\mathcal{L}_{s,q}g - \mathcal{L}_{t,q})(\omega) = \sum_{a \in E} (\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\omega) \cdot g(\omega)$$

$$\lesssim \|g\|_\infty \cdot \sum_{a \in E} \sup_{[a]} |\psi^q e^{\sigma_{\min}\psi}|$$

is dominated by a convergent series. Since the individual terms tend to 0 uniformly in ω as $t \to s$,

$$(\mathcal{L}_{s,q}g - \mathcal{L}_{t,q})(\omega) \to 0, \qquad \text{uniformly in } \omega \in E_A^\infty, \qquad \text{as } t \to s,$$

which is the desired continuity statement. $\qquad\square$

A.4 Proof of Continuity (Part 2)

As before, we first examine the case when $\varepsilon > 0$:

Lemma A.7. *If the potential ψ is $(1, q + \varepsilon)$ summable with $q \geq 0$ and $0 < \varepsilon < 1$, then $s \to \mathcal{L}_{s,q}$ is a $C_{\mathrm{loc}}^\varepsilon$ mapping from $\overline{\mathbb{C}_1^+}$ to $\mathcal{B}(C^\alpha(E_A^\infty))$.*

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Proof. Since we have already proved that $s \to \mathcal{L}_{s,q}$ is a $C^\varepsilon_{\mathrm{loc}}$ mapping from $\overline{\mathbb{C}^+_1}$ to $\mathcal{B}(C(E^\infty_A))$ in Lemma A.5, it remains to estimate the α-variation, which amounts to giving an upper bound for

$$\frac{\left|\sum_{a\in E}(\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\omega) \cdot g(\omega) - (\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\tau) \cdot g(\tau)\right|}{|s-t|^\varepsilon \cdot d(\omega,\tau)^\alpha},$$

for a fixed $s \in \overline{\mathbb{C}^+_1}$ and $|s-t| \leq 1$. We split the numerator of the above expression into two parts $A + B$, which we estimate separately.

For the first summand, the estimate is similar to the one in the proof of Lemma A.5:

$$A \leq \sum_{a\in E}\left|(\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\tau) \cdot (g(\tau) - g(\omega))\right|$$

$$\leq \|g\|_\alpha \cdot d(\omega,\tau)^\alpha \cdot \sum_{a\in E}\left|(\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\tau)\right|$$

$$\leq \|g\|_\alpha \cdot d(\omega,\tau)^\alpha \cdot \sum_{a\in E}|\psi^q e^{\sigma_{\min}\psi}| \cdot \min\left(1, |(s-t)\psi(a\tau)|\right)$$

$$\leq \|g\|_\alpha \cdot d(\omega,\tau)^\alpha \cdot \sum_{a\in E}\sup_{[a]}|\psi^q e^{\sigma_{\min}\psi}| \cdot |(s-t)\psi(a\tau)|^\varepsilon$$

$$\lesssim \|g\|_\alpha \cdot d(\omega,\tau)^\alpha \cdot |s-t|^\varepsilon \cdot \sum_{a\in E}\sup_{[a]}|\psi^{q+\varepsilon} e^{s_{\min}\psi}|,$$

where as usual, $\sigma_{\min} = \min(\mathrm{Re}\, s_1, \mathrm{Re}\, s_2)$.

For the other summand, we use Corollary A.2(iii):

$$B = \sum_{a\in E}\left|(\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\omega) \cdot g(\omega) - (\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\tau) \cdot g(\omega)\right|$$

$$\leq \|g\|_\infty \cdot \sum_{a\in E}\left|(\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\omega) - (\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\tau)\right|$$

$$\leq \|g\|_\infty \cdot \sum_{a\in E}\sup_{[a]}|\psi^q e^{\sigma_{\min}\psi}| \cdot |s| \cdot |\psi(a\omega) - \psi(a\tau)| \cdot \min\left(1, |s-t| \cdot \sup_{[a]}|\psi|\right)$$

$$\leq |s| \cdot \|g\|_\infty \cdot d(\omega,\tau)^\alpha \cdot v_\alpha^{(1)}(\psi) \cdot |s-t|^\varepsilon \cdot \sum_{a\in E}\sup_{[a]}|\psi^{q+\varepsilon} e^{s_{\min}\psi}|,$$

as desired. $\qquad\square$

We now treat the case when $\varepsilon = 0$:

Lemma A.8. *If the potential ψ is $(1, q)$ summable for some $q \geq 0$, then the mapping $s \to \mathcal{L}_{s,q}$ is continuous from $\overline{\mathbb{C}_1^+}$ to $\mathcal{B}(C^\alpha(E_A^\infty))$.*

Proof. Fix an $s \in \overline{\mathbb{C}_1^+}$. We need to show that

$$\frac{(\mathcal{L}_{s,q}g - \mathcal{L}_{t,q})(\omega) - (\mathcal{L}_{s,q}g - \mathcal{L}_{t,q})(\tau)}{d(\omega, \tau)^\alpha} \tag{A.3}$$

tends to 0 as $t \to s$. Expanding the definition of the modified transfer operator in (A.3), we get

$$\sum_{a \in E} \frac{(\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\omega)\, g(\omega) - (\psi^q e^{s\psi} \circ a - \psi^q e^{t\psi} \circ a)(\tau)\, g(\tau)}{d(\omega, \tau)^\alpha}.$$

As each individual term in the sum above tends to 0 as $t \to s$ and the sum is dominated by a convergent series

$$\lesssim \|g\|_\alpha \cdot \sum_{a \in E} |\psi^q e^{\sigma_{\min}\psi}|, \qquad \text{if } |t - s| \leq 1,$$

the entire sum tends to 0 as $t \to s$. $\qquad\qquad\square$

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