

Prescribing inner parts of derivatives of inner functions

Oleg Ivrii

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Abstract

Let \mathcal{J} be the set of inner functions whose derivative lies in the Nevanlinna class. We show that up to a post-composition with a Möbius transformation, an inner function $F \in \mathcal{J}$ is uniquely determined by the inner part of its derivative. We also characterize inner functions which can be represented as $\text{Inn } F'$ for some $F \in \mathcal{J}$ in terms of the associated singular measure, namely, it must live on a countable union of Beurling-Carleson sets. This answers a question raised by K. Dyakonov.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. It is possible to define a finite Blaschke product as a proper holomorphic self-map of the unit disk. It is well known that a finite Blaschke product $F(z)$ is uniquely determined by its zero set up to a rotation:

$$F(z) = e^{i\psi} \prod_{i=1}^d \frac{z - a_i}{1 - \overline{a}_i z}, \quad a_1, a_2, \dots, a_d \in \mathbb{D}.$$

The number $d \geq 1$ is called the *degree* of F . Topological considerations show that F has $d - 1$ critical points (that is, zeros of F') lying in the unit disk. It is a

classical result of M. Heins [11, Section 29] that a finite Blaschke product is uniquely determined by the set of its critical points up to post-composition with a Möbius transformation $m \in \text{Aut}(\mathbb{D})$, and furthermore, any set of $d - 1$ points in the unit disk arises as the critical set of some Blaschke product of degree d .

We can give an alternative interpretation of Heins' result in terms of the following curious differentiation procedure: to a Blaschke product F of degree $d \geq 2$, we assign a Blaschke product B of degree $d - 1$ whose zeros are located at the critical points of F lying in the unit disk. Heins' theorem says that this correspondence is a bijection (between Blaschke products of degree d and Blaschke products of $d - 1$), provided one considers F modulo post-composition with Möbius transformations (as not to change its critical set) and B up to rotations (which preserve the zero set).

In this paper, we discuss an analogue of the above differentiation procedure in infinite degree considered by K. Dyakonov [8, 9]. We need some definitions. An *inner function* is a holomorphic self-map F of the unit disk such that for almost every $\theta \in [0, 2\pi)$, the radial limit $\lim_{r \rightarrow 1} F(re^{i\theta})$ exists and has absolute value 1. Let Inn denote the space of all inner functions. Consider the subspace \mathcal{J} of inner functions whose derivative lies in the Nevanlinna class \mathcal{N} , i.e. which satisfy

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F'(re^{i\theta})| d\theta < \infty. \quad (1.1)$$

According to the work of Ahern and Clark [1], F' admits an “inner-outer” decomposition $F' = \text{Inn } F' \cdot \text{Out } F'$, see Lemma 3.2 below. Intuitively, $\text{Inn } F' = BS$ describes the “critical structure” of the map F – the Blaschke factor records the locations of the critical points of F in the unit disk, while the singular inner factor describes the “boundary critical structure.”

The mapping $F \rightarrow \text{Inn } F'$ from $\mathcal{J}/\text{Aut}(\mathbb{D})$ to $\text{Inn} / \mathbb{S}^1$ clearly generalizes the construction outlined for finite Blaschke products above. The presence of the singular factor allows us to distinguish different inner functions with the same critical set, for instance, it differentiates the universal covering map $z \rightarrow \exp(\frac{z+1}{z-1})$ of the punctured disk $\mathbb{D} \setminus \{0\}$ from the identity mapping.

We show that up to post-composition with a Möbius transformation, an inner function in \mathcal{J} is uniquely determined by its critical structure and describe all possible critical structures of inner functions:

Theorem 1.1. *Let \mathcal{J} be the set of inner functions whose derivative lies in the Nevanlinna class. The natural map*

$$F \rightarrow \text{Inn}(F') : \mathcal{J} / \text{Aut}(\mathbb{D}) \rightarrow \text{Inn} / \mathbb{S}^1$$

is injective. The image consists of all inner functions of the form BS_μ where B is a Blaschke product and S_μ is the singular factor associated to a measure μ whose support is contained in a countable union of Beurling-Carleson sets.

Here, a *Beurling-Carleson set* is a closed subset of the unit circle of zero Lebesgue measure whose complement is a union of arcs $\bigcup_k I_k$ with $\sum |I_k| \log \frac{1}{|I_k|} < \infty$.

In [8], Dyakonov showed that $\text{Inn } F'$ is trivial if and only if F is a Möbius transformation. After reading Dyakonov's work, the author realized that a theorem of D. Kraus can be alternatively formulated as " $F \rightarrow \text{Inn } F'$ is a bijection from Maximal Blaschke Products in \mathcal{J} to the space of all Blaschke Products." The main focus of this paper will be to understand the role of singular factors.

We spend a moment to check that the map in Theorem 1.1 is well-defined:

Lemma 1.2. *If $F \in \mathcal{J}$ is an inner function, then for any Möbius transformation $T \in \text{Aut}(\mathbb{D})$, the Frostman shift $T \circ F \in \mathcal{J}$ and $\text{Inn}(T \circ F)' = \text{Inn } F'$.*

Proof. From the chain rule, we have $(T \circ F)'(z) = T'(F(z)) \cdot F'(z)$. Since $\log |T'|$ is bounded, $T \circ F \in \mathcal{J}$. The equality also tells us that the inner part $\text{Inn}(T \circ F)'$ is divisible by $\text{Inn } F'$. Using T^{-1} in place of T , we see that $\text{Inn } F'$ is divisible by $\text{Inn}(T \circ F)'$. Hence, $\text{Inn}(T \circ F)' = \text{Inn } F'$ agree (up to a unimodular constant). \square

1.1 Gauss curvature equation

Even though our problem originates in complex analysis, the techniques of this paper are essentially that of non-linear elliptic PDE. The connection between the two fields comes from Liouville's theorem which says that *up to post-composition with Möbius transformations in $\text{Aut}(\mathbb{D})$, holomorphic self-maps of the unit disk without critical points are in bijection with conformal metrics of constant curvature -4 .*

Let $\lambda_{\mathbb{D}} = \frac{1}{1-|z|^2}$ denote the Poincaré metric on the unit disk. It is not difficult to check that $\lambda_{\mathbb{D}}$ is a solution of the *Gauss curvature equation*

$$k_\lambda := -\frac{\Delta \log \lambda}{\lambda^2} = -4, \quad (1.2)$$

which may be alternatively written as

$$\Delta u = 4e^{2u}, \quad u : \mathbb{D} \rightarrow \mathbb{R}, \quad (1.3)$$

after the change of variables $u = \log \lambda$. Given a holomorphic self-map of the disk $F : \mathbb{D} \rightarrow \mathbb{D}$, set

$$\lambda_F := F^* \lambda_{\mathbb{D}} = \frac{|F'|}{1 - |F|^2}, \quad u_F := \log \lambda_F.$$

Since the Gaussian curvature is a conformal invariant, λ_F also has curvature -4 . Liouville's theorem says that all solutions of (1.2) arise in this way. We also note that by the Schwarz lemma, $\lambda_{\mathbb{D}}$ is the maximal solution of (1.2), in the sense that if λ is any other solution, then $\lambda < \lambda_{\mathbb{D}}$ pointwise.

We have the following correspondence:

Theorem 1.3. *The mapping $F \rightarrow u_F$ is a bijection between locally univalent inner functions in $\mathcal{J}/\text{Aut}(\mathbb{D})$ and nearly-maximal solutions of (1.3) satisfying*

$$\limsup_{r \rightarrow 1} \int_{|z|=r} (u_{\mathbb{D}} - u) d\theta < \infty. \quad (1.4)$$

For each $0 < r < 1$, we may view $(u_{\mathbb{D}} - u) d\theta$ as a positive measure on the circle of radius r . Subharmonicity guarantees the existence of a weak limit as $r \rightarrow 1$, which we denote $\mu[u]$. (The reader may take the boundary measure of the least harmonic majorant of $u_{\mathbb{D}} - u$.)

Theorem 1.4. *The mapping $u \rightarrow \mu[u]$ is injective. Its image consists of all finite measures whose support is contained in a countable union of Beurling-Carleson sets. If $u = u_F$, the singular measure $\sigma(F') = \mu(u_F)$.*

The reader interested in exploring connections with nonlinear elliptic PDEs with measure boundary values may examine the beautiful papers [10, 19].

1.2 Strategy

We now state several propositions which will be used to show Theorem 1.1. These will be proved in Sections 5 and 6 after we develop the necessary tools.

Lemma 1.5 (Decomposition rule). *An inner function $B_C S_\mu$ lies in the image of $F \rightarrow \text{Inn } F'$ if and only if its singular part S_μ does.*

Therefore, to describe the image of our mapping, it suffices to determine which singular inner functions S_μ can be represented as $S_\mu = \text{Inn } F'_\mu$ with $F_\mu \in \mathcal{J}$. If such an F_μ can be found (which is necessarily unique), we say that the measure μ is *constructible*.

Lemma 1.6 (Product rule). *Suppose measures μ_j , $j = 1, 2, \dots$ are constructible. If their sum $\mu = \sum_{j=1}^{\infty} \mu_j$ is finite, then μ is also constructible.*

Lemma 1.7 (Division rule). *If a measure μ is constructible, then any $\nu \leq \mu$ is also constructible.*

To obtain a large supply of constructible measures, we use the following result of Cullen [4]:

Lemma 1.8. *Suppose that the support of μ is contained in a Beurling-Carleson set. Then $S'_\mu \in \mathcal{N}$.*

Since S_μ divides S'_μ , the division rule implies that any measure μ supported on a Beurling-Carleson set is constructible. By the product rule, any measure supported on a countable union of Beurling-Carleson sets is also constructible. According to Theorem 1.1, any constructible measure is of this form. As a consequence, we see that Cullen's theorem is essentially sharp:

Corollary 1.9. *Suppose μ is a measure on the unit circle with $S'_\mu \in \mathcal{N}$. Then, the support of μ is contained in a countable union of Beurling-Carleson sets.*

On the other side of the spectrum, we have invisible measures. We say that a finite positive singular measure μ is *invisible* if for any measure $0 < \nu \leq \mu$, there does not exist a function $F_\nu \in \mathcal{J}$ with $\text{Inn } F'_\nu = S_\nu$. In Section 5, we will show

that any singular measure on the unit circle μ can be uniquely decomposed into a constructible part and an invisible part: $\mu = \mu_{\text{con}} + \mu_{\text{inv}}$. To complete the proof of Theorem 1.1, we give a criterion for a measure to be invisible:

Theorem 1.10. *Suppose μ is a measure on the unit circle which does not charge Beurling-Carleson sets. Then, it is invisible.*

In the late 1970s, Korenblum [12] and Roberts [22] independently showed that $\{S_\mu \mid \mu \text{ does not charge Beurling-Carleson sets}\}$ is the collection of inner functions which are cyclic in Bergman space. To prove Theorem 1.10, we first show that any measure μ with modulus of continuity $\omega_\mu(t) \leq Ct \log(1/t)$ is invisible. To obtain the full result, we adapt the argument from Roberts [22] to our setting. This involves an iterative procedure based on the decomposition of a measure that does not charge Beurling-Carleson sets into “ $t \log 1/t$ ”-pieces. We do not know if there is a direct connection between the two problems even if the solutions share the same motif.

1.3 Notation

Let m denote the Lebesgue measure on \mathbb{S}^1 , normalized to have unit mass. Given a Blaschke sequence C in the unit disk, let B_C be the Blaschke product with zero set C . In order for B_C to be uniquely defined, we may use the normalization $B_C(1) = 1$. The symbol F_C will be reserved for the maximal Blaschke product with critical set C . For a singular measure μ on the unit circle, we let S_μ be the associated singular inner function.

2 Background on conformal metrics

Given an at most countable set C in the unit disk (counted with multiplicity), the machinery of Kraus and Roth [13]–[17] seeks to construct a Blaschke product with critical set C . If such a Blaschke product does not exist, then the machinery does not produce anything. If there are Blaschke products with critical set C , the machinery produces the optimal or *maximal* Blaschke product F_C . Instead of constructing F_C directly, Kraus and Roth construct the conformal pseudometric $\lambda_{F_C} = F_C^* \lambda_{\mathbb{D}}$, the

pullback of the Poincaré metric on the disk. To explain their construction, we need several concepts:

(i) *SK-metrics.* We will say that a conformal pseudometric λ on a domain U has “constant curvature -4 ” if it vanishes on a discrete set of points $C \subset U$ (which may be empty) and satisfies

$$k_\lambda = -\frac{\Delta \log \lambda}{\lambda^2} = -4, \quad \lambda \in C^2, \quad \text{on } U \setminus C.$$

More generally, if $k_\lambda \leq -4$ on $U \setminus C$, then following Heins, we say that $\lambda(z)$ is a (regular) *SK-metric*. In reality, this is a slight abuse of notation since the distributional Laplacian may have point masses at points of C .

(ii) *Perron families.* According to [11, Section 13] or [16, Definition 4.11], a collection Φ of SK-metrics is a *Perron family* if it is closed under *modifications* and *taking maxima*. The first condition says that given a round disk $D \subset U$ and a metric $\lambda \in \Phi$, the unique SK-metric $M_D \lambda$ which agrees with λ on $U \setminus D$, is non-vanishing and has constant curvature -4 in D lies in Φ ; while the second condition says that for any $\lambda_1, \lambda_2 \in \Phi$, their pointwise maximum $\max(\lambda_1, \lambda_2)$ is also in Φ . Heins proved that if a Perron family is non-empty, then the supremum of all metrics in Φ is a non-vanishing conformal metric of curvature -4 .

(iii) *Liouville’s theorem.* Suppose λ is a conformal pseudometric defined on a simply-connected domain U . We say that λ vanishes at $c_i \in C$ with *multiplicity* m_i if

$$\lim_{z \rightarrow c_i} \frac{\lambda(z)}{|z - c_i|^{m_i}} = L_i, \quad \text{for some } 0 < L_i < \infty.$$

Liouville’s theorem [15, Theorem C] says that if λ has constant curvature -4 and all its zeros have *integral* multiplicities, then $\lambda_F = F^* \lambda_{\mathbb{D}}$ for some holomorphic function $F : U \rightarrow \mathbb{D}$. Furthermore, the function F is unique up to post-composition with a Möbius transformation.

For a set C in the unit disk, let Φ_C be the collection of all SK-metrics on \mathbb{D} which vanish on C . It clearly verifies the two axioms of being a Perron family on the domain $\mathbb{D} \setminus C$. Provided Φ_C is non-empty, one obtains a metric of constant curvature -4 and a holomorphic function $F_C : \mathbb{D} \rightarrow \mathbb{D}$ which vanishes on C to the correct order.

Leveraging the maximality of the metric λ_{F_C} , Kraus [13] proved that the outer and singular inner factors of F_C are trivial. In other words, F_C is a Blaschke product.

In the case when the critical set C is a Blaschke sequence, Kraus made the fundamental observation that $|B_C|\lambda_{\mathbb{D}}$ is an SK-metric which guarantees that the Perron family Φ_C is non-empty. (More generally, given a holomorphic function H with $\|H\|_\infty \leq 1$ and a metric λ of curvature -4 , $|H| \cdot \lambda$ is an SK-metric.)

Further exploiting the lower bound $\lambda_{F_C} \geq |B_C|\lambda_{\mathbb{D}}$, Kraus obtained the following remarkable result [13, Theorem 4.4]:

Theorem 2.1 (Kraus). *Suppose C is a Blaschke sequence in the disk and λ is a metric of constant curvature -4 which vanishes precisely at C with the correct multiplicity. Then $\lambda = \lambda_{F_C}$ if and only if*

$$\lim_{r \rightarrow 1} \int_{|z|=r} \log \frac{\lambda}{\lambda_{\mathbb{D}}} d\theta = 0. \quad (2.1)$$

In Section 3, we will use ideas of Ahern and Clark to show that the above theorem can be alternatively phrased as:

Corollary 2.2. *Suppose C is a Blaschke sequence in the disk. An infinite Blaschke product $F \in \mathcal{J}$ is the maximal Blaschke product associated to C if and only if the singular factor of $\text{Inn } F'$ is trivial, i.e. if $\text{Inn } F' = B_C$.*

In order to generalize the arguments of Kraus and Roth to allow for singular factors, we will need:

Lemma 2.3 (Fundamental Lemma). *For any inner function $F \in \mathcal{J}$,*

$$\lambda_F \geq |\text{Inn } F'| \lambda_{\mathbb{D}}. \quad (2.2)$$

In fact, λ_F is the smallest metric of constant curvature -4 with this property.

Note that the minimality of the metric λ_F implies that the map $F \rightarrow \text{Inn } F'$ from Theorem 1.1 is injective. Using the factorization $F' = \text{Inn } F' \cdot \text{Out } F'$, one can rewrite (2.2) as

$$\frac{1 - |F(z)|^2}{1 - |z|^2} \leq |\text{Out } F'(z)|, \quad (2.3)$$

which was proved by Dyakonov in [6, Theorem 2.1] using Julia’s lemma. The reader may also consult [7, Corollary 2.1] for additional remarks. We may therefore view the fundamental lemma as a refinement of Dyakonov’s theorem.

The proof of minimality will be given in Section 3.3. In Section 4, we will give an alternative proof of (2.2) by carefully approximating F by finite Blaschke products (finite Blaschke products are maximal by [11, Section 29]).

2.1 Hulls and wedges

We conclude this section with two natural constructions of conformal metrics which will play an important role in this work:

Wedge of two metrics. Given two inner functions $F, G \in \mathcal{J}$, consider the family $\Phi_{F,G}$ of SK-metrics that are pointwise less than $\min(\lambda_F, \lambda_G)$. This family is not empty: the metric $|\text{Inn } F'| \cdot |\text{Inn } G'| \cdot \lambda_{\mathbb{D}}$ is in it, as Lemma 2.3 shows. Taking the supremum of conformal metrics in $\Phi_{F,G}$, we get a regular conformal metric of constant curvature -4 . By Liouville’s theorem, it is the pullback of $\lambda_{\mathbb{D}}$ by a holomorphic function which we denote $F \wedge G$. By Lemma 3.6 below, $F \wedge G \in \mathcal{J}$.

Hull of a conformal metric. For an SK-metric κ , let Ψ_κ be the collection of all metrics of constant curvature -4 which are greater than κ and Φ_κ be the collection of all SK-metrics that are less than all metrics in Ψ_κ . Since Φ_κ is a Perron family, its supremum is a metric $\hat{\kappa}$ of curvature -4 . We call $\hat{\kappa}$ the *hull* of κ . From the definition, it is clear that $\hat{\kappa}$ is the smallest metric of curvature -4 which exceeds κ . In this terminology, Lemma 2.3 says that λ_F is the hull of $|\text{Inn } F'| \lambda_{\mathbb{D}}$.

3 Gap of a Nevanlinna function

By definition, the Nevanlinna class \mathcal{N} consists of holomorphic functions on the unit disk for which

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_{|z|=r} \log^+ |f(z)| d\theta < \infty, \quad z = re^{i\theta}. \quad (3.1)$$

It is well known that (unless f is identically zero) this condition is equivalent to the boundedness of

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_{|z|=r} |\log |f(z)|| d\theta.$$

Since $\log |f(z)|$ is a subharmonic function, $\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{|z|=r} \log |f(z)| d\theta$ exists and is finite. However, unlike the Hardy norms, it need not be the case that

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{|z|=r} \log |f(z)| d\theta = \frac{1}{2\pi} \int_{|z|=1} \log |f(z)| d\theta,$$

where in the integral in the right hand side, we consider the radial boundary values of f which are known to exist a.e. To understand the cause of the discrepancy, we consider the canonical decomposition of $f = B(S/S_1)O$ into a Blaschke product, a quotient of singular inner functions and an outer function:

$$\begin{aligned} B(z) &= \prod_i -\frac{\bar{a}_i}{|a_i|} \cdot \frac{z - a_i}{1 - \bar{a}_i z}, \\ (S/S_1)(z) &= \exp\left(-\int_{\mathbb{S}^1} \frac{\zeta + z}{\zeta - z} d\sigma_\zeta\right), \quad \sigma \perp m, \\ O(z) &= \exp\left(\int_{\mathbb{S}^1} \frac{\zeta + z}{\zeta - z} \log |f(\zeta)| dm_\zeta\right), \end{aligned}$$

see for instance [5]. Given an interval I on the unit circle, let rI denote its radial projection onto the circle $S_r = \{z : |z| = r\}$.

Lemma 3.1.

$$\text{gap}(f) := \frac{1}{2\pi} \int_{|z|=1} \log |f(z)| d\theta - \lim_{r \rightarrow 1} \left\{ \frac{1}{2\pi} \int_{|z|=r} \log |f(z)| d\theta \right\} = \sigma(\mathbb{S}^1).$$

More generally, if I is an interval on the unit circle,

$$\text{gap}_I(f) := \frac{1}{2\pi} \int_I \log |f(z)| d\theta - \lim_{r \rightarrow 1} \left\{ \frac{1}{2\pi} \int_{rI} \log |f(z)| d\theta \right\} = \sigma(I),$$

provided the endpoints of I do not charge σ .

Proof. As above, we decompose $f = B(S/S_1)O$. It suffices to analyze the three components separately. We begin with the Blaschke factor. We claim that

$$\frac{1}{2\pi} \int_{|z|=r} \log |B(z)| d\theta \rightarrow 0, \quad \text{as } r \rightarrow 1.$$

Factoring out z^m , we may assume that $B(0) \neq 0$. Let a_1, a_2, \dots be an enumeration of the zeros of B in the unit disk. By Jensen's formula and the fact that $|B(0)| = \prod |a_i|$,

$$\frac{1}{2\pi} \int_{|z|=r} \log |B(re^{i\theta})| d\theta = \sum_{|a_i| < r} \log \frac{r}{|a_i|} - \sum_{|a_i| < 1} \log \frac{1}{|a_i|},$$

which tends to zero as $r \rightarrow 1$. This proves the claim. Since $\log |B(z)| < 0$ is negative, we a fortiori have $\frac{1}{2\pi} \int_{rI} \log |B(z)| d\theta \rightarrow 0$ for any interval $I \subset \mathbb{S}^1$. Clearly, $\frac{1}{2\pi} \int_I \log |B(z)| d\theta = 0$ as well.

Since $\log |O(z)|$ is a harmonic function on the unit disk which is the Poisson extension of its radial boundary values,

$$\frac{1}{2\pi} \int_{rI} \log |O(z)| d\theta \rightarrow \frac{1}{2\pi} \int_I \log |O(z)| d\theta.$$

In other words, the outer factor also behaves as expected.

The singular factor exhibits the most interesting behaviour. Since $\log |(S/S_1)(z)|$ is the Poisson extension of the singular measure σ ,

$$\frac{1}{2\pi} \int_{rI} \log |(S/S_1)(z)| d\theta \rightarrow \sigma(I),$$

if the endpoints of I do not charge σ . On the other hand,

$$\frac{1}{2\pi} \int_I \log |(S/S_1)(z)| d\theta = 0$$

as the radial boundary values of $\log |(S/S_1)(z)|$ are zero a.e. on the unit circle. Putting the Blaschke, singular and outer parts together gives the statement of the lemma. \square

3.1 Applications to inner functions

We now apply Lemma 3.1 to study inner functions with derivative in the Nevanlinna class. We first give a slightly different perspective on a classical theorem due to Ahern and Clark:

Lemma 3.2 (Ahern-Clark). *For an inner function $F \in \mathcal{J}$, its derivative admits a BSO decomposition. In other words, the singular measure $\sigma(F') \geq 0$.*

Proof. By Lemma 1.2, it suffices to consider the case when $F(0) = 0$. Then $|F'(x)| \geq 1$ on the unit circle, e.g. see [20, Theorem 4.15]. In view of the fundamental inequality

$$|F'(rx)| \leq 4|F'(x)|, \quad x \in \mathbb{S}^1, \quad 0 < r < 1,$$

of Ahern and Clark [1], the dominated convergence theorem shows

$$\int_I \log^+ |F'(z)| dm - \lim_{r \rightarrow 1} \int_{rI} \log^+ |F'(z)| dm = 0,$$

for any interval $I \subset \mathbb{S}^1$. However, by Fatou's lemma, the negative part of the logarithm can only dissipate and therefore

$$\text{gap}_I(F') = \int_I \log |F'(z)| dm - \lim_{r \rightarrow 1} \int_{rI} \log |F'(z)| dm \geq 0.$$

This completes the proof. \square

The following lemma says that as $r \rightarrow 1$, the measures $\log \frac{\lambda_{\mathbb{D}}}{\lambda_F}(re^{i\theta}) dm$ converge weakly to $\sigma(F')$:

Lemma 3.3. *Let $I \subset \mathbb{S}^1$ be an interval. If $F \in \mathcal{J}$ then*

$$\frac{1}{2\pi} \int_I \log |F'(z)| d\theta = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{rI} \log \frac{1 - |F(z)|^2}{1 - |z|^2} d\theta. \quad (3.2)$$

Proof. From the contraction of the hyperbolic distance $d_{\mathbb{D}}(F(0), F(z)) \leq d_{\mathbb{D}}(0, z)$, it follows that the quotient $\frac{1 - |F(z)|^2}{1 - |z|^2} \geq c_{F(0)}$ is bounded below by a positive constant. By the Schwarz lemma,

$$\frac{1}{2\pi} \int_{rI} \max(\log |F'(z)|, \log c_{F(0)}) d\theta \leq \frac{1}{2\pi} \int_{rI} \log \frac{1 - |F(z)|^2}{1 - |z|^2} d\theta.$$

Applying the dominated convergence theorem like in the proof of Lemma 3.2 gives the \leq inequality in (3.2). For the \geq direction, we average Dyakonov's inequality (2.3) over $z \in rI$:

$$\frac{1}{2\pi} \int_{rI} \log |\text{Out } F'(z)| d\theta \geq \frac{1}{2\pi} \int_{rI} \log \frac{1 - |F(z)|^2}{1 - |z|^2} d\theta.$$

The lemma follows after taking $r \rightarrow 1$ since $\log |\text{Out } F'(z)|$ is the harmonic extension of $\log |F'(z)|$ considered as a function on the unit circle. \square

The reader may compare the above lemma with [2, Theorem 3]. If F is a locally univalent inner function (without critical points), then

$$\begin{aligned} F' \in \mathcal{N} &\iff \lim_{r \rightarrow 1} \int_{|z|=r} \log \frac{1 - |F(z)|^2}{1 - |z|^2} d\theta < \infty, \\ &\iff \lim_{r \rightarrow 1} \int_{|z|=r} \log \frac{\lambda_{\mathbb{D}}}{\lambda_F} d\theta < \infty, \end{aligned}$$

which is the statement of Theorem 1.3. For the second equivalence, one uses that

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{|z|=r} \log |F'(z)| d\theta = \log |F'(0)|$$

is necessarily finite.

3.2 Applications to conformal metrics

Lemma 3.4. *Suppose $F \in \mathcal{J}$ is an inner function for which*

$$\lambda_F \geq |B_C S_\mu| \cdot \lambda_{\mathbb{D}}. \quad (3.3)$$

Then, the singular measure $\sigma(F') \leq \mu$.

Proof. Let $I \subset \mathbb{S}^1$ be an interval. From the definition of λ_F ,

$$\int_{rI} \log \frac{\lambda_F}{|B_C S_\mu| \lambda_{\mathbb{D}}} dm = \int_{rI} \log \left(\frac{|F'| (1 - |z|^2)}{1 - |F|^2} \right) dm - \int_{rI} \log |B_C S_\mu| dm.$$

By Lemma 3.1 and the *easy* part of Lemma 3.3, as $r \rightarrow 1$, this tends to

$$0 \leq -\sigma(F')(I) + \sigma(S_\mu)(I),$$

at least if I is generic (there are extra terms if the endpoints of I charge any of these singular measures). \square

Remark. The same conclusion holds under the seemingly weaker assumption $\lambda_F \geq |B_C S_\mu O_h|$ where

$$O_h(z) = \exp \left(\int_{\mathbb{S}^1} \frac{\zeta + z}{\zeta - z} h(\zeta) dm_\zeta \right), \quad h : \mathbb{S}^1 \rightarrow \mathbb{R},$$

is an arbitrary outer function: the above computation results in $\sigma(F') \leq \mu - h dm$. Since $\sigma(F') \perp h dm$ are mutually singular, we have $\sigma(F') \leq \mu$ and $h \leq 0$.

Similar considerations show:

Lemma 3.5. *For any $F, G \in \mathcal{J}$ and interval $I \subset \mathbb{S}^1$,*

$$\lim_{r \rightarrow 1} \int_{|z|=r} \log(\lambda_F/\lambda_G) dm = -\sigma(F')(I) + \sigma(G')(I). \quad (3.4)$$

In particular, if $\lambda_F \geq \lambda_G$ then $\sigma(F') \leq \sigma(G')$.

Combining the above lemma with Theorem 2.1 gives Corollary 2.2.

Lemma 3.6. *If λ_G is a metric of curvature -4 such that $\lambda_G \geq |H|\lambda_{\mathbb{D}}$ for some bounded holomorphic function $H \not\equiv 0$, then $G \in \mathcal{J}$.*

Proof. Since H is a bounded holomorphic function, $\gamma_1 = \lim_{r \rightarrow 1} \int_{r\mathbb{S}^1} \log |H| dm$ is finite. The condition $\lambda_G \geq |H|\lambda_{\mathbb{D}}$ implies that the zeros of G' form a Blaschke sequence, which in turn implies that the integral $\gamma_2 = \lim_{r \rightarrow 1} \int_{r\mathbb{S}^1} \log |G'| dm$ is also finite. An inspection of the inequality

$$0 \leq \liminf_{r \rightarrow 1} \int_{r\mathbb{S}^1} \log \frac{\lambda_G}{|H|\lambda_{\mathbb{D}}} dm \leq -\gamma_1 + \gamma_2 - \limsup_{r \rightarrow 1} \int_{r\mathbb{S}^1} \log^+ |G'| dm$$

then shows that G' satisfies the Nevanlinna condition (3.1). It remains to prove that the outer part of G is trivial, so that G is an inner function. If this were not the case, then for a positive measure set of directions $\theta \in [0, 2\pi)$, $\limsup_{r \rightarrow 1} \lambda_G(re^{i\theta})$ would be finite. However, this contradicts the assumption $\lambda_G \geq |H|\lambda_{\mathbb{D}}$, since by the Lusin-Privalov theorem, the radial limit of $H(re^{i\theta})$ is non-zero almost everywhere. \square

3.3 Injectivity and Minimality

With the above preparations, we can show the injectivity statement of Theorem 1.1. If there were two functions $F, G \in \mathcal{J}$ with $\text{Inn } F' = \text{Inn } G' = B_C S_\mu$, then

$$\lambda_F \geq \lambda_{F \wedge G} \geq |B_C S_\mu| \cdot \lambda_{\mathbb{D}}. \quad (3.5)$$

Lemmas 3.4 and 3.5 imply that $(F \wedge G)'$ has the same inner part as F' . From the definition of curvature, $\Delta \log(\lambda_F/\lambda_{F \wedge G}) = 4(\lambda_F^2 - \lambda_{F \wedge G}^2)$. Hence $\log(\lambda_F/\lambda_{F \wedge G})$ is subharmonic and non-negative, yet

$$\lim_{r \rightarrow 1} \int_{|z|=r} \log(\lambda_F/\lambda_{F \wedge G}) d\theta \rightarrow 0, \quad (3.6)$$

which forces $\log(\lambda_F/\lambda_{F \wedge G}) = 0$. We deduce that $\lambda_F = \lambda_{F \wedge G} = \lambda_G$ and therefore $F = G$ up to post-composition with a Möbius transformation by Liouville's theorem.

Given an inner function $F \in \mathcal{J}$, we now show that λ_F is the smallest metric of constant curvature -4 that exceeds $|\text{Inn } F'| \lambda_{\mathbb{D}}$. Following Section 2.1, we consider the hull λ of the metric $|\text{Inn } F'| \lambda_{\mathbb{D}}$. The inequalities

$$\lambda_F \geq \lambda \geq |\text{Inn } F'| \lambda_{\mathbb{D}} \quad (3.7)$$

reveal that λ has exactly the same zero set as λ_F (counted with multiplicity). In particular, all the zeros of λ have integral multiplicities. Proceeding like in the proof of injectivity, we obtain $\lim_{r \rightarrow 1} \int_{|z|=r} \log(\lambda_F/\lambda) d\theta \rightarrow 0$ and $\lambda = \lambda_F$ as desired.

4 Stable approximations

In this section, we study convergent sequences of inner functions. We are particularly interested in *stable* approximations where the inner-outer decomposition is preserved in the limit:

Definition. Suppose $\{F_n\} \subset \mathcal{J}$ is a sequence of inner functions which converge uniformly on compact subsets of the disk to an inner function F . We say that F_n is a (*Nevanlinna*) *stable approximation* of F if

$$\text{Inn } F' = \lim_{n \rightarrow \infty} \text{Inn } F'_n, \quad \text{Out } F' = \lim_{n \rightarrow \infty} \text{Out } F'_n. \quad (4.1)$$

In general, we have inequalities in one direction:

Theorem 4.1. Suppose $\{F_n\} \subset \mathcal{J}$ is a sequence of inner functions which converge uniformly on compact subsets of the disk to a holomorphic function $F : \mathbb{D} \rightarrow \mathbb{D}$. Also assume that the $I_n = \text{Inn } F'_n$ converge to an inner function I . Then $F \in \mathcal{J}$ and the following inequalities hold:

$$\sigma(F') \leq \sigma(I), \quad (4.2)$$

$$|\text{Inn } F'| \geq |I|, \quad (4.3)$$

$$\int_{\mathbb{S}^1} \log |F'| dm \leq \lim_{n \rightarrow \infty} \int_{\mathbb{S}^1} \log |F'_n| dm. \quad (4.4)$$

Furthermore, either all of the above inequalities are equalities or none of them are.

Proof. Taking $n \rightarrow \infty$ in $\lambda_{F_n} \geq |I_n|\lambda_{\mathbb{D}}$ gives $\lambda_F \geq |I|\lambda_{\mathbb{D}}$. Lemma 3.4 then proves the first inequality (4.2).

Clearly, $\text{Inn } F'$ and $I = \lim(\text{Inn } F'_n)$ have the same zeros in the unit disk but may have different singular factors. However, it is easy to see that for singular inner functions, one has the inequality $S_1 \leq S_2$ if and only if $\sigma(S_1) \geq \sigma(S_2)$. The “if” direction is obvious, while the “only if” direction follows from the identity

$$0 \leq \lim_{r \rightarrow 1} \int_{rE} \log |S_2/S_1| dm = -\sigma(S_2)(E) + \sigma(S_1)(E),$$

valid for any generic interval $E \subset \mathbb{S}^1$ whose endpoints do not charge the measures $\sigma(S_1)$ and $\sigma(S_2)$. This proves (4.3) and shows that the equality cases in (4.2) and (4.3) coincide.

Since $F'_n \rightarrow F'$ uniformly on compact subsets of the disk, (4.3) is equivalent to the inequality $|\text{Out } F'(z)| \leq |\lim_{n \rightarrow \infty} \text{Out } F'_n(z)|$. Setting $z = 0$ and taking logarithms gives (4.4). However, if (4.4) is an equality, then by the maximum modulus principle applied to $\text{Out } F'(z)/(\lim_{n \rightarrow \infty} \text{Out } F'_n(z))$, we must have $|\text{Out } F'(z)| = |\lim_{n \rightarrow \infty} \text{Out } F'_n(z)|$ for all $z \in \mathbb{D}$. This completes the proof. \square

For some applications, we need a slightly more general version of the above theorem:

Theorem 4.2. *In the context of Theorem 4.1, suppose instead that the I_n converge to a non-zero holomorphic function $H : \mathbb{D} \rightarrow \mathbb{D}$ with the inner-outer decomposition $H = I \cdot O$. Then, the inequalities (4.2)–(4.4) still hold. The equality case in (4.4) implies that $\{F_n\}$ is a stable sequence, in particular, the outer factor $O = 1$ is trivial and the B_n converge to an inner function.*

The proof of Theorem 4.2 is nearly identical to that of Theorem 4.1, so we only sketch the details. Since $\|H\|_\infty \leq 1$, we have $|O(z)| \leq 1$ and $|I(z)| \geq |H(z)|$ for $z \in \mathbb{D}$. Following the proof of Theorem 4.1, we obtain the inequality $\lambda_F \geq |I \cdot O|\lambda_{\mathbb{D}}$. We may still use Lemma 3.6 to conclude that $F \in \mathcal{J}$. The remark after Lemma 3.4 allows us to conclude (4.2) and (4.3) in this more general case as well. We may weaken (4.3) to $|\text{Inn } F'| \geq |H|$, which is equivalent to (4.4). This time, the equality case in (4.4) forces $I = H$ and $O = 1$.

Remark. In view of Lemma 3.6, if a sequence of inner functions F_n converges to a function F with $F' \notin \mathcal{N}$, then $H = \lim(\text{Inn } F'_n)$ must be 0.

Craizer's argument from [3, Lemma 5.4] shows:

Lemma 4.3. *Any inner function $F \in \mathcal{J}$ admits a stable approximation by finite Blaschke products.*

Proof. According to a theorem of Frostman, e.g. see [20, Theorem 2.5], if $\xi \in \mathbb{D}$ avoids a set of zero logarithmic capacity, then the Frostman shift $T_\xi \circ F$ is a Blaschke product, where $T_\xi = \frac{z-\xi}{1-\bar{\xi}z}$. If ξ is not an exceptional point, we may choose a sequence $F_{n,\xi}$ of finite Blaschke products converging to F so that $T_\xi \circ F_{n,\xi}$ is a sequence of partial products of $T_\xi \circ F$. By [20, Corollary 4.13], we have

$$|(T_\xi \circ F_{n,\xi})'(x)| \leq |(T_\xi \circ F)'(x)|, \quad x \in \mathbb{S}^1.$$

It follows that

$$|(F_{n,\xi})'(x)| \leq \left[\frac{1+|\xi|}{1-|\xi|} \right]^2 |F'(x)|,$$

which leads to the estimate

$$\int_{\mathbb{S}^1} \log |F'_{n,\xi}(x)| dm \leq 2 \log \frac{1+|\xi|}{1-|\xi|} + \int_{\mathbb{S}^1} \log |F'(x)| dm.$$

Since we can choose ξ arbitrarily close to 0, we can diagonalize to find a sequence F_n of finite Blaschke products converging to F for which

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{S}^1} \log |F'_n(x)| dm \leq \int_{\mathbb{S}^1} \log |F'(x)| dm.$$

However, by Theorem 4.2, the lower bound is automatic and the sequence $\{F_n\}$ is stable. \square

Suppose $F \in \mathcal{J}$ is an inner function. The above lemma provides a Nevanlinna stable approximation $F_n \rightarrow F$ by finite Blaschke products. Since finite Blaschke products are maximal, we have $\lambda_{F_n} \geq |\text{Inn } F'_n| \lambda_{\mathbb{D}}$ for any $n \geq 1$. Taking $n \rightarrow \infty$ gives $\lambda_F \geq |\text{Inn } F'| \lambda_{\mathbb{D}}$. Note that there is no circular reasoning since the proof of Theorem 4.2 only relied on the easy part of Lemma 3.3.

Remark. We can endow the space of analytic functions $\mathcal{E} = \{f : f' \in \mathcal{N}\}$ with the *stable topology* by specifying that $f_n \rightarrow f$ if the f_n converge uniformly on compact sets to f and $\log |f'_n| dm \rightarrow \log |f'| dm$ weakly. Lemma 4.3 shows that finite Blaschke products are dense in \mathcal{J} while Theorem 4.2 implies that the subset $\mathcal{J} \subset \mathcal{E}$ is closed.

4.1 Example of an unstable approximation

We now give an example of a sequence of finite Blaschke products which is not Nevanlinna stable. Let F_n be the Blaschke product of degree $n+1$ with zeros at the origin and at $z_j = e^{j(2\pi i/n)} \cdot (1 - 1/n^2)$, $j = 1, 2, \dots, n$. With the normalization $F'_n(0) > 0$, the maps F_n converge to the identity since $\sum_{j=1}^n (1 - |z_j|) \rightarrow 0$ as $n \rightarrow \infty$. Recall that for $x \in \mathbb{S}^1$, one has the formula $|F'_n(x)| = 1 + \sum_{j=1}^n P_{z_j}(x)$, where P_z is the Poisson kernel as viewed from $z \in \mathbb{D}$, e.g. see [20, Theorem 4.15]. Computations show

$$\int_{I_j} \log |F'_n| dm \geq \int_{I_j} \log |1 + P_{z_j}| dm \gtrsim 1/n$$

where I_j consists of the points on the unit circle for which the closest zero is z_j . Hence, $|\text{Out } F'_n(0)| = \exp \int_{\mathbb{S}^1} \log |F'_n| dm > c > 1$ for some constant c independent of $n \geq 1$. Since the outer parts $\text{Out } F'_n$ do not converge to the constant function 1, neither can the inner parts $\text{Inn } F'_n$.

A modification of this construction can be used to show the existence of a sequence of finite Blaschke products $F_n \rightarrow z$ for which $\text{Inn } F'_n \rightarrow S_{\delta_1}$ and $\text{Out } F'_n \rightarrow 1/S_{\delta_1}$.

5 Understanding the image

In this section, we discuss the image of the map $F \rightarrow \text{Inn } F'$ and prove the decomposition, product and division rules from the introduction. We also show that the map $F \rightarrow \text{Inn } F'$ is not surjective by exhibiting a large class of invisible measures. A complete description of the image will be given in the next section.

5.1 Wedging F_μ with F_C

Theorem 5.1. (i) Suppose $F_\mu \in \mathcal{J}$ is an inner function with $\text{Inn } F'_\mu = S_\mu$. Let $F_{\mu,C} = F_\mu \wedge F_C$ where C is a Blaschke sequence. Then, $\text{Inn } F'_{\mu,C} = B_C S_\mu$.

(ii) Conversely, if $F_{\mu,C} \in \mathcal{J}$ is an inner function with $\text{Inn } F'_{\mu,C} = B_C S_\mu$, then there exists an inner function F_μ with $\text{Inn } F'_\mu = S_\mu$.

Proof. (i) Since $\lambda_{F_C} \geq \lambda_{F_{\mu,C}} \geq |B_C| \lambda_{F_\mu}$, the critical set of $F_{\mu,C}$ is precisely C with the correct multiplicity; while the inequalities $\lambda_{F_\mu} \geq \lambda_{F_{\mu,C}} \geq |B_C| \lambda_{F_\mu}$ show that $\sigma(F'_{\mu,C}) = \mu$ (one divides by λ_{F_μ} , integrates over $\{z : |z| = r\}$, tends $r \rightarrow 1$ and applies Lemma 3.5). Hence $\text{Inn } F'_{\mu,C} = B_C S_\mu$ as desired.

(ii) Suppose $F_{\mu,C} \in \mathcal{J}$ is an inner function with $\text{Inn } F'_{\mu,C} = B_C S_\mu$. Let F_n be a sequence of finite Blaschke products which converges to $F_{\mu,C}$ (stability is not required in this proof). For any $0 < r < 1$, we can form the sequence of finite Blaschke products $F_{n,r}$ by removing the critical points from F_n that lie in the ball $\{z : |z| < r\}$, and considering the maximal Blaschke product with the remaining critical points (with the normalization $F_{n,r}(0) = 0$ and $F'_{n,r}(0) > 0$). For each r , we pick a subsequential limit F_r of $F_{n,r}$. We may then extract a further subsequential limit F by taking $r \rightarrow 1$. By construction, we have

$$|B_C| \lambda_F \leq \lambda_{F_{\mu,C}} \leq \lambda_F.$$

Since the limit F cannot be constant, by Hurwitz' theorem, F has no critical points. The above inequalities imply $\sigma(F') = S_\mu$ and therefore $\text{Inn } F' = S_\mu$. \square

5.2 Division and product rules

In the next lemma, we show that any sequence “dominated” by a stable sequence is also stable:

Lemma 5.2. Suppose that $F_{C_n} \rightarrow F_{\mu_1 + \mu_2}$ is a stable sequence, and $C_{1,n} \subset C_n$ is a subset such that $B_{C_{1,n}}$ converges to S_{μ_1} . Then, $F_{C_{1,n}} \rightarrow F_{\mu_1}$.

Proof. Write $C_n = C_{1,n} \cup C_{2,n}$. From the assumptions, $B_{C_{1,n}} \rightarrow S_{\mu_1}$ and $B_{C_{2,n}} \rightarrow S_{\mu_2}$. After passing to a subsequence, we can ensure convergence:

$$F_{C_{1,n}} \rightarrow F_{\nu_1}, \quad \nu_1 \leq \mu_1,$$

$$F_{C_{2,n}} \rightarrow F_{\nu_2}, \quad \nu_2 \leq \mu_2.$$

The monotonicity of measures follows from Theorem 4.1. For each n , we have $\lambda_{F_{C_n}} \geq |B_{1,n}| \lambda_{F_{C_{2,n}}}$ and therefore, after taking $n \rightarrow \infty$, we see that

$$\lambda_{F_{\mu_1+\mu_2}} \geq |S_{\mu_1}| \lambda_{F_{\nu_2}}.$$

As is now standard, we may deduce

$$\mu_1 + \mu_2 \leq \mu_1 + \nu_2$$

by examining the inequality

$$0 \leq \lim_{r \rightarrow 1} \int_{rI} \log \frac{\lambda_{F_{\mu_1+\mu_2}}}{|S_{\mu_1}| \lambda_{F_{\nu_2}}} dm, \quad I \subset \mathbb{S}^1.$$

Hence $\nu_2 = \mu_2$ (and similarly $\nu_1 = \mu_1$) as desired. \square

The above lemma has a number of consequences:

Corollary 5.3. *If a measure μ is constructible, i.e. if F_μ exists, then all $\nu \leq \mu$ are also constructible. In particular, the image of the mapping $F \rightarrow \text{Inn } F'$ is closed under taking divisors.*

Indeed, given a stable approximation F_{C_n} to F_μ , it is not difficult to select $C_{1,n} \subset C_n$ so that $B_{C_{1,n}} \rightarrow S_\nu$.

Corollary 5.4. *If F_{μ_1} and F_{μ_2} are constructible, then $F_{\mu_1+\mu_2}$ is also constructible.*

The proof relies on the Solynin-type estimate

$$\lambda_{F_{C_1}} \lambda_{F_{C_2}} \geq \lambda_{F_{C_1 \cup C_2}} \lambda_{F_{C_1 \cap C_2}}, \quad (5.1)$$

valid when C_1 and C_2 are finite subsets of the disk counted with multiplicity. The proof of (5.1) is essentially that of [18, Lemma 2.8], so we only sketch the details. Consider the function

$$u = \log^+ \left(\frac{\lambda_{F_{C_1 \cup C_2}} \lambda_{F_{C_1 \cap C_2}}}{\lambda_{F_{C_1}} \lambda_{F_{C_2}}} \right).$$

We claim that it is subharmonic and non-negative in \mathbb{D} yet tends to 0 as $|z| \rightarrow 1$. This will show that it is equal to 0 identically. It is clearly non-negative by definition. To show that $u(z)$ is subharmonic, one can check that $\Delta u \geq 0$. We refer the reader to [18, Lemma 2.8] for the computation. For the last statement, note that by Lemma 2.3, for a finite Blaschke product F , the quotient $\lambda_F/\lambda_{\mathbb{D}} \rightarrow 1$ uniformly as $|z| \rightarrow 1$.

Proof of Corollary 5.4. Choose approximations $F_{C_{1,n}} \rightarrow F_{\mu_1}$ and $F_{C_{2,n}} \rightarrow F_{\mu_2}$ by finite Blaschke products. Making a small perturbation if necessary, we can assume that the sets $C_{1,n}$ and $C_{2,n}$ are disjoint. Let $C_n = C_{1,n} \cup C_{2,n}$ be their union. Passing to a subsequence, we may assume that $F_{C_n} \rightarrow F_\mu$ for some measure μ on the unit circle. By Solynin's estimate (5.1), we have

$$\log \frac{\lambda_{\mathbb{D}}}{\lambda_{F_{C_{1,n}}}} + \log \frac{\lambda_{\mathbb{D}}}{\lambda_{F_{C_{2,n}}}} \leq \log \frac{\lambda_{\mathbb{D}}}{\lambda_{F_{C_n}}}. \quad (5.2)$$

Taking $n \rightarrow \infty$ gives

$$\log \frac{\lambda_{\mathbb{D}}}{\lambda_{F_{\mu_1}}} + \log \frac{\lambda_{\mathbb{D}}}{\lambda_{F_{\mu_2}}} \leq \log \frac{\lambda_{\mathbb{D}}}{\lambda_{F_\mu}}. \quad (5.3)$$

By examining averages over rI and taking $r \rightarrow 1$, we discover that $\mu \geq \mu_1 + \mu_2$. Applying Corollary 5.3 shows that the measure $\mu_1 + \mu_2$ is constructible. \square

Remark. Solynin's original estimate from [23, 24] compares hyperbolic metrics on two domains in the plane with the hyperbolic metrics on their union and intersection:

$$\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z) \geq \lambda_{\Omega_1 \cup \Omega_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z), \quad z \in \Omega_1 \cap \Omega_2.$$

Corollary 5.5. *If $S'_\mu \in \mathcal{N}$ then μ is constructible.*

This follows from the division rule (Corollary 5.3) and the fact that S_μ divides S'_μ . As noted in the introduction, M. Cullen [4] verified the hypothesis of Corollary 5.5 when the support of μ is a Beurling–Carleson set.

5.3 Invisible measures

Let μ be a finite positive measure on the unit circle, which is singular with respect to the Lebesgue measure. We say μ is *invisible* if for any measure $0 < \nu \leq \mu$, there does not exist a function $F_\nu \in \mathcal{J}$ with $\text{Inn } F'_\nu = S_\nu$.

Lemma 5.6. *Either the map $F \rightarrow \text{Inn } F'$ is surjective or there exists an invisible measure.*

Proof. Suppose F_μ is not constructible. Since the hull of the metric $|S_\mu| \cdot \lambda_{\mathbb{D}}$ defined in Section 2.1 cannot vanish anywhere, it must be of the form λ_{F_ν} for some measure ν . (Lemma 3.6 explains why F_ν must be an inner function.) Applying Lemma 3.4, we see that $\nu < \mu$ since equality cannot hold. From the product rule (Corollary 5.4), it follows that the measure $\mu - \nu$ is invisible. More precisely, if $\sigma \leq \mu - \nu$ was constructible, then $\lambda_{F_\nu} > \lambda_{F_{\nu+\sigma/2}} > |S_\mu| \cdot \lambda_{\mathbb{D}}$ would contradict the definition of ν . \square

Actually, the above argument shows a little more:

Theorem 5.7. *A measure μ is invisible if and only if the hull of $|S_\mu| \cdot \lambda_{\mathbb{D}}$ is the Poincaré metric. More generally, any measure μ can be uniquely decomposed into a constructible part and an invisible part: $\mu = \mu_{\text{con}} + \mu_{\text{inv}}$, in which case, the hull of $|S_\mu| \cdot \lambda_{\mathbb{D}}$ is $\lambda_{F_{\mu_{\text{con}}}}$.*

We are now in a position to prove the countable version of the product rule (Lemma 1.6). Suppose we are given countably many constructible measures μ_j , $j = 1, 2, \dots$ such that their sum $\mu = \sum_{j=1}^{\infty} \mu_j$ is a finite measure. We claim that μ is constructible. According to Theorem 5.7, the hull of $|S_\mu| \cdot \lambda_{\mathbb{D}}$ is of the form λ_{F_ν} for some measure $\nu \leq \mu$. However, from Corollary 5.4, we know that $\tilde{\mu}_j = \mu_1 + \mu_2 + \dots + \mu_j$ is constructible. This shows that $\nu \geq \tilde{\mu}_j$ for any j , which forces $\nu = \mu$.

5.4 A criterion for invisibility

In this section, we only consider conformal metrics with strictly positive densities, that is, genuine metrics instead of pseudometrics. Given a positive continuous function u on $\mathbb{S}_r = \{z : |z| = r\}$, $0 < r < 1$, let $\Lambda_r[u]$ denote the unique conformal metric of curvature -4 on $\mathbb{D}_r = \{z : |z| < r\}$ which agrees with u on \mathbb{S}_r . For the existence and uniqueness of $\Lambda_r[u]$, we refer the reader to [11, Section 12] or [16, Appendix]. For a non-vanishing SK-metric λ , we will write $\Lambda[\lambda] = \hat{\lambda}$ for the minimal conformal metric of curvature -4 that exceeds λ .

Lemma 5.8. *The operation $u \rightarrow \Lambda_r[u]$ is monotone in u , that is, if $u \geq v$ then $\Lambda_r[u] \geq \Lambda_r[v]$.*

To see this, note that the function $h = \log^+(\Lambda_r[v]/\Lambda_r[u])$ is non-negative, subharmonic and identically zero on \mathbb{S}_r . To check that h is subharmonic, we use the definition of curvature and Kato's inequality (e.g. see [21, Proposition 6.6]):

$$\Delta h \geq (4\Lambda_r[v]^2 - 4\Lambda_r[u]^2) \cdot \chi_{v>u} \geq 0.$$

A similar argument shows:

Lemma 5.9. *Let λ be a non-vanishing conformal metric on the unit disk of curvature at most -4 . For $0 < r < 1$, the metric $\Lambda_r[\lambda(re^{i\theta})]$ is the minimal conformal metric of curvature -4 that exceeds λ on \mathbb{D}_r . The family of conformal metrics $\Lambda_r[\lambda(re^{i\theta})]$ is non-decreasing in r , and the limit*

$$\hat{\lambda} = \Lambda[\lambda] = \lim_{r \rightarrow 1} \Lambda_r[\lambda(re^{i\theta})] \quad (5.4)$$

is the minimal conformal metric of curvature -4 that exceeds λ on \mathbb{D} .

In general, it is difficult to evaluate $\Lambda_r[u]$ explicitly. In the next lemma, we do so when u is a constant function.

Lemma 5.10. *Given any $0 < c \leq 1$, there exists a unique $0 < r' \leq r$ so that $\Lambda_r[c \cdot \lambda_{\mathbb{D}}] = L^* \lambda_{\mathbb{D}}$ where $L(z) = \frac{r'}{r} \cdot z$ is the linear map $\mathbb{D}_r \rightarrow \mathbb{D}_{r'}$.*

The lemma follows by observing that since the metrics $(L_{r'})^* \lambda_{\mathbb{D}}$ are increasing in r' , there is a unique value of r' for which $c \cdot \lambda_{\mathbb{D}} = (L_{r'})^* \lambda_{\mathbb{D}}$ on \mathbb{S}_r .

Corollary 5.11. *We have*

$$\lim_{C \rightarrow \infty} \left[\lim_{r \rightarrow 1} \frac{\Lambda_r[C]}{\lambda_{\mathbb{D}}} \right] \rightarrow 1,$$

uniformly on compact subsets of the unit disk.

We can now prove:

Theorem 5.12. *Suppose μ is a singular measure on the unit circle which satisfies $\mu(I) \leq C|I| \log |1/I|$ for any interval $I \subset \mathbb{S}^1$ and some constant $C > 0$. Then, μ is invisible.*

Proof. From the product rule (Corollary 5.4), it is easy to see that a measure μ is invisible if and only if $\varepsilon \cdot \mu$ is for any $\varepsilon > 0$. This allows us to assume that $\mu(I) \leq \varepsilon|I| \log |1/I|$ which implies that the Poisson extension $P_\mu(z) \leq \varepsilon(A \log \frac{1}{1-|z|} + B)$ for some constants A and B . Hence, $|S_\mu| \lambda_{\mathbb{D}} \rightarrow \infty$ as $|z| \rightarrow 1$. The theorem now follows from the monotonicity principle (Lemma 5.8) and Corollary 5.11. \square

6 Roberts decompositions

In this section, we show that if μ does not charge Beurling-Carleson sets, then it is invisible, that is, any measure $0 < \nu \leq \mu$ cannot be in the image of the map $F \rightarrow \sigma(F')$. To upgrade the argument of Section 5.4, we will use the following theorem which is implicit in the work of Roberts [22]:

Theorem 6.1. *Suppose μ is a measure on the unit circle which does not charge Beurling-Carleson sets. Given a real number $c > 0$ and an integer $j_0 \geq 1$, μ can be expressed as a countable sum*

$$\mu = \sum_{j=1}^{\infty} \mu_j, \quad (6.1)$$

where each μ_j enjoys an estimate on the modulus of continuity:

$$\omega_{\mu_j}(1/n_j) \leq \frac{c}{n_j} \cdot \log n_j, \quad n_j := 2^{2^{j+j_0}}. \quad (6.2)$$

Here, $\omega_\mu(t) = \sup_{I \subset \mathbb{S}^1} \mu(I)$, with the supremum taken over all intervals of length t .

It will be important for us that the measure μ admits infinitely many decompositions with different parameters c and j_0 , where c can be made arbitrarily small and j_0 arbitrarily large. For convenience of the reader, we recall the proof.

Proof. For each $j = 1, 2, \dots$, we can define a partition P_j of the unit circle into n_j arcs of equal length (we consider half-open arcs which contain only one of the

endpoints, for example, the left endpoint). Since n_j divides n_{j+1} , each next partition can be chosen to be a refinement of the previous one. Given a measure μ on the unit circle, Roberts defines the *grating* of μ with respect to the sequence of partitions $\{P_j\}$. This procedure decomposes $\mu = \sum_{j=1}^{\infty} \mu_j + \nu$ so that (6.2) holds for each j , with the residual measure ν supported on a Beurling-Carleson set.

To define μ_1 , consider the intervals in the partition P_1 . Call an interval $I \in P_1$ *light* if $\mu(I) \leq (c/n_1) \cdot \log n_1$ and *heavy* otherwise. On a light interval, take $\mu_1 = \mu$, while on a heavy interval, let μ_1 be a multiple of μ so that the mass $\mu_1(I) = (c/n_1) \cdot \log n_1$. Clearly, $\mu_1 \leq \mu$. Consider the difference $\mu - \mu_1$ and grate it with respect to the partition P_2 to form the measure μ_2 , then consider $\mu - \mu_1 - \mu_2$ and grate it with respect to P_3 to form μ_3 , and so on. Continuing in this way, we obtain a sequence of measures μ_1, μ_2, \dots where each next measure is supported on the heavy intervals of the previous generation.

By construction, the bound (6.2) holds for all j . Since the residual measure ν is supported on the set of points which lie in heavy intervals at every stage, $\text{supp } \nu \subset \mathbb{S}^1 \setminus \mathcal{L}$, where \mathcal{L} is the union of interiors of light intervals of any generation. The relation $\log n_{j+1} = 2 \log n_j$ implies that $\mathbb{S}^1 \setminus \mathcal{L}$ is a Beurling-Carleson set:

$$\sum_{\text{light}} |I| \log \frac{1}{|I|} \lesssim 2^{j_0} + \sum_{\text{heavy}} |J| \log \frac{1}{|J|} \lesssim 2^{j_0} + \sum_{j=0}^{\infty} \sum_{J \in P_j \text{ heavy}} \mu_j(J) \leq 2^{j_0} + \mu(\mathbb{S}^1),$$

since any maximal light interval of generation $j \geq 2$ is contained in a heavy interval of the previous generation. \square

The estimate (6.2) on the modulus of continuity is easily seen to be equivalent to an estimate on the Poisson extension:

$$|P_{\mu_j}| \leq c' \cdot \log \frac{1}{1 - |z|^2}, \quad z \in B(0, 1 - 1/n_j). \quad (6.3)$$

Here, the constant c' can be taken to be cc_1 for some $c_1 > 0$. This is stated in [22, Lemma 2.2].

We will also need a simple lemma on conformal metrics:

Lemma 6.2. (i) *For any two singular measures μ_1 and μ_2 on the unit circle,*

$$\Lambda \left[|S_{\mu_1}| \cdot \Lambda [|S_{\mu_2}| \lambda_{\mathbb{D}}] \right] = \Lambda [|S_{\mu_1}| |S_{\mu_2}| \cdot \lambda_{\mathbb{D}}].$$

(ii) More generally,

$$\Lambda\left[|S_{\mu_1}| \cdot \dots \cdot \Lambda\left[|S_{\mu_{j-1}}| \cdot \Lambda\left[|S_{\mu_j}| \lambda_{\mathbb{D}}\right]\right] \dots\right] = \Lambda\left[|S_{\mu_1}||S_{\mu_2}| \cdots |S_{\mu_j}| \cdot \lambda_{\mathbb{D}}\right].$$

(iii) For $\mu = \sum_{j=1}^{\infty} \mu_j$, we have

$$\lim_{n \rightarrow \infty} \Lambda\left[|S_{\mu_1}| \cdot \dots \cdot \Lambda\left[|S_{\mu_{j-1}}| \cdot \Lambda\left[|S_{\mu_j}| \lambda_{\mathbb{D}}\right]\right] \dots\right] = \Lambda\left[|S_{\mu}| \lambda_{\mathbb{D}}\right].$$

Proof. (i) The \geq direction follows from the monotonicity of Λ . For the \leq direction, it suffices to show that

$$|S_{\mu_1}| \cdot \Lambda\left[|S_{\mu_2}| \lambda_{\mathbb{D}}\right] \leq \Lambda\left[|S_{\mu_1}||S_{\mu_2}| \cdot \lambda_{\mathbb{D}}\right]$$

or

$$|S_{\mu_1}| \cdot \Lambda_r\left[|S_{\mu_2}| \lambda_{\mathbb{D}}\right] \leq \Lambda_r\left[|S_{\mu_1}||S_{\mu_2}| \cdot \lambda_{\mathbb{D}}\right]$$

for any $0 < r < 1$, cf. Lemma 5.9. To this end, we form the function

$$u_r = \log^+ \left(\frac{|S_{\mu_1}| \cdot \Lambda_r\left[|S_{\mu_2}| \lambda_{\mathbb{D}}\right]}{\Lambda_r\left[|S_{\mu_1}||S_{\mu_2}| \cdot \lambda_{\mathbb{D}}\right]} \right)$$

defined on $\mathbb{D}_r = \{z : |z| < r\}$. Since it is subharmonic and vanishes on $\mathbb{S}_r = \partial\mathbb{D}_r$, it must be identically 0. This proves the \leq direction.

(ii) follows after applying (i) $j - 1$ times.

(iii) Let $\tilde{\mu}_j = \mu_1 + \mu_2 + \cdots + \mu_j$. By part (i), we have

$$|S_{\mu-\tilde{\mu}_j}| \cdot \Lambda\left[|S_{\tilde{\mu}_j}| \lambda_{\mathbb{D}}\right] \leq \Lambda\left[|S_{\mu}| \lambda_{\mathbb{D}}\right] \leq \Lambda\left[|S_{\tilde{\mu}_j}| \lambda_{\mathbb{D}}\right].$$

Since $|S_{\mu-\tilde{\mu}_j}| \rightarrow 1$, $\Lambda\left[|S_{\tilde{\mu}_j}| \lambda_{\mathbb{D}}\right]$ are decreasing and converge to $\Lambda\left[|S_{\mu}| \lambda_{\mathbb{D}}\right]$. \square

We can now prove Theorem 1.10:

Proof of Theorem 1.10. Step 1. Let $\mu = \mu_j$ be the Roberts decomposition (6.1) with parameters c and j_0 to be chosen later. In view of the invisibility criterion (Theorem 5.7), it suffices to show that

$$\lambda_j := \Lambda_{1-1/n_1} \left[|S_{\mu_1}| \cdot \dots \cdot \Lambda_{1-1/n_{j-1}} \left[|S_{\mu_{j-1}}| \cdot \Lambda_{1-1/n_j} \left[|S_{\mu_j}| \cdot \lambda_{\mathbb{D}} \right] \right] \dots \right] \quad (6.4)$$

is close to the hyperbolic metric at the origin, uniform in $j \geq 1$. Indeed, by the monotonicity properties of Λ , we have

$$\lambda_j \leq \Lambda \left[|S_{\mu_1}| \cdot \dots \Lambda \left[|S_{\mu_{j-1}}| \cdot \Lambda [|S_{\mu_j}| \cdot \lambda_{\mathbb{D}}] \right] \dots \right], \quad (6.5)$$

so that if λ_j is close to $\lambda_{\mathbb{D}}$, then so must

$$\Lambda [|S_{\mu_1}| |S_{\mu_2}| \cdots |S_{\mu_n}| \lambda_{\mathbb{D}}].$$

Step 2. The estimate on the modulus of continuity of μ_j implies that $|S_{\mu_j}| \lambda_{\mathbb{D}} \geq \lambda_{\mathbb{D}}^{4/5}$ on the circle \mathbb{S}_{1-1/n_j} . Here, we use the fact that we can choose $c' < 1/10$ in (6.3). We claim that

$$\Lambda_{1-1/n_j} [|S_{\mu_j}| \lambda_{\mathbb{D}}] \geq (1/2) \lambda_{\mathbb{D}}, \quad \text{on } \mathbb{S}_{1-1/n_{j-1}}. \quad (6.6)$$

Assuming (6.6), we have

$$|S_{\mu_{j-1}}| \cdot \Lambda_{1-1/n_j} [|S_{\mu_j}| \lambda_{\mathbb{D}}] \geq \lambda_{\mathbb{D}}^{4/5}, \quad \text{on } \mathbb{S}_{1-1/n_{j-1}}.$$

We could then inductively show that $\lambda_j \geq (1/2) \lambda_{\mathbb{D}}$ on \mathbb{S}_{1-1/n_1} . By Corollary 5.11, this would mean that λ_j is very close to $\lambda_{\mathbb{D}}$ at the origin, provided n_1 is large (this is where we use that j_0 can be made arbitrarily large.)

Step 3. Thus, we need to show that $\Lambda_{1-1/n_j} [|S_{\mu_j}| \lambda_{\mathbb{D}}] \geq (1/2) \cdot \lambda_{\mathbb{D}}$ on $\mathbb{S}_{1-1/n_{j-1}}$. Define $\varepsilon > 0$ by $1 - 1/n_j = 1 - \varepsilon$ so that $1 - 1/n_{j-1} = 1 - \varepsilon^{1/2}$. There exists a unique $0 < \ell < 1$ so that $\Lambda_{1-1/n_j} [\lambda_{\mathbb{D}}^{4/5}] = L^* \lambda_{\mathbb{D}}$ where $L(z) = \ell z$. Inspection shows that $1 - \ell \asymp \varepsilon^{4/5}$. Therefore,

$$\Lambda_{1-1/n_j} [|S_{\mu_j}| \lambda_{\mathbb{D}}] \geq \Lambda_{1-1/n_j} [\lambda_{\mathbb{D}}^{4/5}] = \frac{\ell}{1 - |\ell z|^2} \geq (1/2) \cdot \lambda_{\mathbb{D}}, \quad \text{on } \mathbb{S}_{1-1/n_{j-1}}$$

as desired. \square

Remark. We invite the reader to compare the above argument with the one given by Roberts [22] which uses the corona theorem to estimate the distance from an invariant subspace to the identity function in Bergman space. Even though the two settings are very different, the proofs share a similar theme.

The above theorem completes the strategy for proving Theorem 1.1 outlined in the introduction.

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