Describing Blaschke products by their critical points

Oleg Ivrii

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Polynomials and finite Blaschke products

A polynomial (or a linear product) can be described as a **proper** holomorphic map from $\mathbb{C} \to \mathbb{C}$.

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A finite Blaschke product (or a hyperbolic polynomial) can be described as a **proper** holomorphic map from $\mathbb{D} \to \mathbb{D}$.

$$F(z) = e^{i\alpha} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_i \in \mathbb{D}.$$

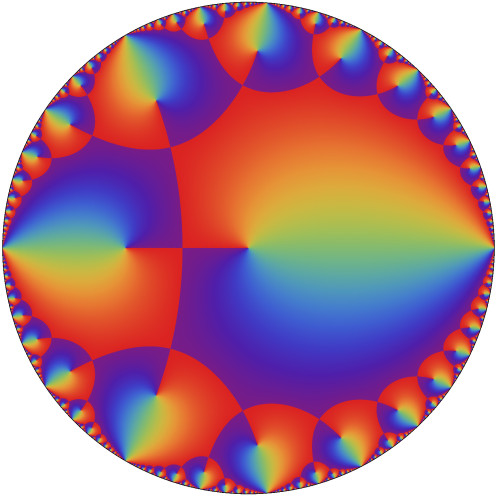

Figure: A dynamical portrait of $z \to z \cdot \frac{z+0.5}{1+0.5z}$.

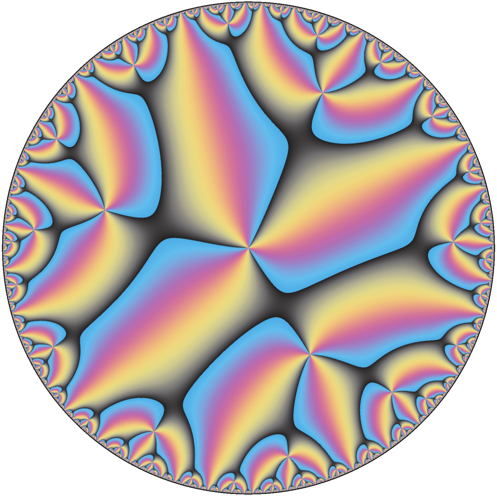

Figure: A dynamical portrait of $z \to z \cdot \frac{z+a}{1+\bar{a}z}$, $a = 0.5\, e^{2\pi i/3}$.

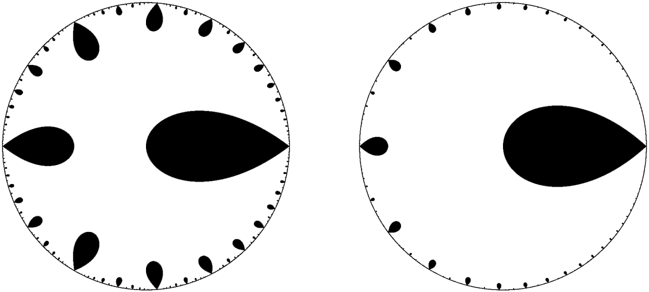

Figure: Invariant sets of $z \to z \cdot \frac{z+a}{1+\bar{a}z}$ as $a \to 1^-$ along the real axis.

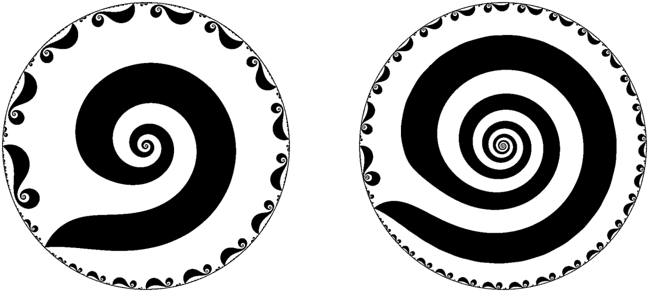

Figure: Invariant sets of $z \to z \cdot \frac{z+a}{1+\bar{a}z}$ as $a \to 1$ along a horocycle.

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In complex analysis, it is customary to equip the unit disk $\mathbb{D}$ with the hyperbolic metric

$$\lambda_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}.$$

Then $\mathrm{Aut}(\mathbb{D})$ *acts isometrically on* $(\mathbb{D}, \lambda_{\mathbb{D}})$*, while holomorphic mappings* $f : \mathbb{D} \to \mathbb{D}$ *are contractions.*

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The polynomial version of this theorem is rather silly. To construct a polynomial $p(z)$ with critical set C, first construct a polynomial $q(z)$ with zero set C and integrate: $p(z) = \int_0^z q(w)dw$.

Inner functions

An inner function is a holomorphic self-map of $\mathbb{D}$ such that for almost every $\theta \in [0, 2\pi)$, the radial limit

$$\lim_{r \to 1} F(re^{i\theta})$$

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Different inner functions can have the same critical set. For example, $F_1(z) = z$ and $F_2(z) = \exp(\frac{z+1}{z-1})$ have no critical points.

BS decomposition

An inner function can be represented as a (possibly infinite) Blaschke product $\times$ singular inner function:

$$B(z) = e^{i\alpha} \prod_i -\frac{\overline{a_i}}{|a_i|} \cdot \frac{z - a_i}{1 - \overline{a_i} z}, \quad a_i \in \mathbb{D}, \quad \sum_i (1 - |a_i|) < \infty.$$

$$S(z) = \exp\left(- \int_{\mathbb{S}^1} \frac{\zeta + z}{\zeta - z} \, d\mu_\zeta \right), \quad \mu \perp m, \quad \mu \geq 0.$$

Here, B records the zero set, while S records the boundary zero structure.

Inner functions of finite entropy

We will also be concerned with the subclass $\mathscr{J}$ of inner functions whose derivative lies in the Nevanlinna class:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|F'(e^{i\theta})|d\theta < \infty.$$

In 1974, P. Ahern and D. Clark showed that F' admits a BSO decomposition, allowing us to define $\operatorname{Inn} F' := BS$, where B records the critical set of F and S records the boundary critical structure.

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Theorem. (I, 2017) An inner function in $\mathscr{J}$ is uniquely determined by its critical structure up to post-composition with a Möbius transformation.

An inner function BS_μ can be represented as $\operatorname{Inn} F'$ for some $F \in \mathscr{J}$ if and only if μ is sufficiently concentrated (lives on a countable union of Beurling-Carleson sets).

Beurling-Carleson sets

Definition. A Beurling-Carleson set E is a closed subset of the unit circle which has measure 0 such that

$$\sum |I_j| \cdot \log \frac{1}{|I_j|} < \infty,$$

where $\{I_j\}$ are the complementary intervals.

[Measures which do not charge Beurling-Carleson sets also occur in the description of cyclic functions in Bergman spaces given indepedently by Korenblum (1977) and Roberts (1979).]

Limits of critical structures

Suppose that F_n is a finite Blaschke product of degree $n+1$ which has a critical point at $1-1/n$ of multiplicity n, and is normalized so that $F_n(0)=0$, $F_n'(0)>0$.

The F_n converge to the unique inner function F_{δ_1} with critical structure $S_{\delta_1} = \exp\left(\frac{z+1}{z-1}\right)$.

In the above example, the critical structure **survives** in the limit, that is, $\operatorname{Inn} F' = \lim_{n\to\infty}(\operatorname{Inn} F_n')$. In general, part of the critical structure may be **lost**.

Limits of critical structures

More generally, suppose that F_n has n critical points at

$$c_k = (1 - 1/n)e^{ik\theta_n}, \qquad k = 1, 2, \dots, n.$$

- If $n\theta_n \log \frac{1}{\theta_n} \to 0$, then the F_n converge to F_{δ_1}.
- If $n\theta_n \log \frac{1}{\theta_n} \to \infty$, then the F_n converge to the identity.
- For any $0 < c < 1$, one can choose θ_n appropriately so that the F_n converge to $F_{c\delta_1}$.

Stable topology on inner functions

Endow $\mathscr{J} \,/\, \mathrm{Aut}\,\mathbb{D}$ with the stable topology where $F_n \to F$ if

- The convergence is uniform on compact subsets of the disk,
- The Nevanlinna splitting is stable in the limit:

$$\mathrm{Inn}\, F_n' \to \mathrm{Inn}\, F', \qquad \mathrm{Out}\, F_n' \to \mathrm{Out}\, F'.$$

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Theorem. (I, 2018) This happens if and only if the "critical structures" of the F_n are uniformly concentrated on Korenblum stars.

Background on conformal metrics

The curvature of a conformal metric $\lambda(z)|dz|$ is given by

$$k_\lambda = -\frac{\Delta \log \lambda}{\lambda^2}.$$

Examples. The hyperbolic metric

$$\lambda_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}$$

has curvature $\equiv -1$,

while the Euclidean metric $|dz|$ has curvature $\equiv 0$.

Liouvillean correspondence

Since curvature is a conformal invariant, if $F : \mathbb{D} \to \mathbb{D}$ is a holomorphic map then

$$\lambda_F = F^* \lambda_{\mathbb{D}} = \frac{2|F'|}{1 - |F|^2}$$

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Liouville observed that there is a natural bijection between $\mathrm{Hol}(\mathbb{D}, \mathbb{D}) / \mathrm{Aut}\, \mathbb{D}$ and pseudometrics of constant curvature -1 with integral singularities.

Nearly-maximal solutions

Consider the Gauss curvature equation

$$\Delta u = e^{2u}, \qquad u : \mathbb{D} \to \mathbb{R}.$$

It has a unique maximal solution $u_{\max} = \log \lambda_{\mathbb{D}}$ which tends to infinity as $|z| \to 1$.

We are interested in solutions close to maximal in the sense that

$$\limsup_{r \to 1} \int_{|z|=r} (u_{\max} - u) d\theta < \infty.$$

[Equivalently, we want $u_{\max} - u$ to have a harmonic majorant.]

Embedding into the space of measures

For each $0 < r < 1$, we may view

$$(u_{\max} - u)d\theta$$

as a positive measure on the circle of radius r.

Subharmonicity $\implies \exists$ weak limit as $r \to 1$, which we denote $\mu[u]$.

It turns out that the measure μ uniquely determines the solution u. Thus, the question becomes: which measures occur?

Constructible measures

Theorem. (I, 2017) Any measure μ on the unit circle can be uniquely decomposed into a constructible part and an invisible part:

$$\mu = \mu_{\text{con}} + \mu_{\text{inv}}.$$

In fact, $u_{\mu_{\text{con}}}$ is the **minimal solution** which exceeds the subsolution $u_{\max} - P_\mu$ (Poisson extension).

Remark. The above theorem holds for other PDEs of the form $\Delta u = g(u)$, any smooth bounded domain, and is valid in higher dimensions.

Cullen's Theorem

Theorem. (M. Cullen, 1971) If a measure μ is supported on a Beurling-Carleson set, then $S'_\mu \in \mathcal{N}$.

In particular,

$$u = \log \frac{2|S'_\mu|}{1 - |S_\mu|^2} \quad \text{is} \quad \textbf{nearly-maximal},$$

i.e. μ is constructible.

From my work, it follows that Cullen's theorem is essentially sharp: if $S'_\mu \in \mathcal{N}$, then μ lives on a countable union of Beurling-Carleson sets. Artur Nicolau gave an elementary proof of this fact.

Roberts' decompositions

Claim. If $\omega_\mu(t) \leq c \cdot t \log(1/t)$, then μ is invisible.

[The modulus of continuity $\omega_\mu(t) = \sup_{I \subset \mathbb{S}^1} \mu(I)$, with the supremum taken over all intervals of length t.]

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Theorem. (J. Roberts, 1979) Suppose μ does not charge Beurling-Carleson sets. Given a real number $c > 0$ and integer $j_0 \ge 1$, μ can be expressed as a countable sum

$$\mu = \sum_{j=1}^{\infty} \mu_j,$$

where

$$\omega_{\mu_j}(1/n_j) \le \frac{c}{n_j} \cdot \log n_j, \qquad n_j := 2^{2^{j+j_0}}.$$

On L^1 bounded solutions

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^N$. Consider the differential equation

$$\Delta u = u^q, \qquad u : \mathbb{B} \to [0, \infty), \quad q > 1.$$

We say that u is an L^1 bounded solution if

$$\limsup_{r \to 1} \int_{\partial \mathbb{B}} u(r\xi) dS < \infty.$$

Taking the weak limit of $u(r\xi)\, dS$ as $r \to 1$, one obtains an embedding of L^1 bounded solutions into $\mathcal{M}(\partial \mathbb{B})$.

Question. Which measures occur (are constructible)?

On L^1 bounded solutions

Theorem. (A. Gmira & L. Véron, 1991) In the **subcritical case**, $q < q_c = \frac{N+1}{N-1}$, all measures are constructible.

Theorem. In the **supercritical case**, $q \geq q_c$, a measure is constructible iff it is diffuse with respect to $\mathrm{cap}_{W^{2/q,q'}}$.

This was proved by:

- J. F. Le Gall, $q = 2$ (1993),
- E. B. Dynkin & S. E. Kuznestov, $q_c \leq q \leq 2$ (1996),
- M. Marcus & L. Véron, $q > 2$ (1998).

Entropy of universal covering maps

Let $P \subset \mathbb{D} \setminus \{0\}$ be a finite set.

Consider the universal covering map $\mathcal{U}_P : \mathbb{D} \to \mathbb{D} \setminus P$, normalized so that $\mathcal{U}_P(0) = 0$ and $\mathcal{U}_P'(0) > 0$.

With this normalization, $m = d\theta/2\pi$ is invariant under $\mathcal{U}_P$.

Theorem. (Pommerenke, 1976 & I, 2018)

$$\int_{|z|=1} \log |\mathcal{U}_P'(z)| dm = \sum_{\text{punctures}} \log \frac{1}{|p_i|} - \sum_{\text{zeros}} \log \frac{1}{|z_i|},$$

where we don't sum over the trivial zero at the origin.

Critical structures of inner functions

Consider the weighted Bergman space $A^2_1(\mathbb{D})$ which consists of all holomorphic functions on the unit disk for which

$$\|f\|_{A^2_1} = \left(\int_{\mathbb{D}} |f(z)|^2 \cdot (1 - |z|)|dz|^2 \right)^{1/2} < \infty.$$

Theorem. (D. Kraus, 2007) Critical sets of inner functions = Zero sets of the weighted Bergman space A^2_1.

Conjecture. $\mathrm{Inn} \,/\, \mathrm{Aut}\, \mathbb{D} \cong \overline{\{\text{zero-based subspaces}\}}$.

Thank you for your attention!