Dimensions of *sparse* quasicircles

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Dimensions of Quasicircles

Find $D(k)$, the maximal dimension of a k-quasicircle, the image of $\mathbb{S}^1$ under a k-quasiconformal mapping of the plane,

$$\text{homeomorphism}, \quad \overline{\partial} w^{\mu}(z) = \mu(z) \cdot \partial w^{\mu}(z), \quad \|\mu\|_{\infty} \leq k.$$

Theorem: (Becker-Pommerenke, 1987)

$$D(k) \leq 1 + 36\, k^2 + \mathcal{O}(k^3).$$

Astala's conjecture: (proved by Smirnov)

$$D(k) \leq 1 + k^2, \qquad \text{for } 0 < k < 1.$$

Bloch functions

Let b be a Bloch function on $\mathbb{D}$, i.e. a holomorphic function satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|b'(z)| < \infty.$$

Examples:

$$\log f', \qquad f:\mathbb{D}\to\mathbb{C} \text{ conformal},$$

$$P\mu = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\mu(w)}{(1-z\overline{w})^2}|dw|^2, \quad \mu \in L^\infty(\mathbb{D}).$$

Lacunary series:

$$z + z^2 + z^4 + z^8 + \dots$$

Asymptotic variance

For a Bloch function, define its asymptotic variance by

$$\sigma^2(b) = \limsup_{r\to 1} \frac{1}{2\pi d_{\mathbb{D}}(0,r)} \int_{|z|=r} |b(z)|^2 \, |dz|.$$

Set

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(AIPP) $\quad 0.879 \le \Sigma^2 \le 1,\qquad$ (Hedenmalm) $\quad \Sigma^2 < 1,$

$$D(k) = 1 + k^2\Sigma^2 + \mathcal{O}(k^{8/3-\varepsilon}),$$

(Prause-Smirnov) $\quad D(k) < 1 + k^2 \quad$ for all $0 < k < 1$.

McMullen's identity

Suppose μ is a **dynamical** Beltrami coefficient on the disk, either

- invariant under a co-compact Fuchsian group Γ,
- or eventually invariant under a Blaschke product $f(z)$.

Then,

$$\begin{aligned} 2\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{M.dim}\, w^{t\mu}(\mathbb{S}^1) &= \sigma^2\left(\frac{d}{dt}\bigg|_{t=0} \log f'\right), \\ &= \sigma^2(P\mu), \\ &= \|\cdot\|^2_{\mathrm{WP}}, \end{aligned}$$

where $\|\cdot\|^2_{\mathrm{WP}}$ is the Weil-Petersson metric.

Integral means spectra

For a conformal map $f : \mathbb{D} \to \Omega$, the integral means spectrum is given by

$$\beta_f(p) := \limsup_{r\to 1^-} \frac{\log \int_{|z|=r} |f'(z)^p|\,|dz|}{d_{\mathbb{D}}(0,r)}, \qquad p \in \mathbb{C}.$$

Fact. For $f \in \mathbf{S}_k$, we have $\beta(p) = p - 1 \iff p = \mathrm{M.}\dim f(\mathbb{S}^1)$.

Question: What if $\operatorname{supp}\mu$ is sparse?

Suppose $\mu \in M(\mathbb{D})$ is a Beltrami coefficient with $\|\mu\|_\infty \leq 1$ whose support is contained in a "garden" $\mathcal{G} = \bigcup_{j=1}^{\infty} B_j$.

Separation condition: hyperbolic distance $d_{\mathbb{D}}(B_i, B_j) > R$, $i \neq j$.

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Theorem A (I. 2016)

For sparse μ, we have the improved bounds

$$\beta_{w^{k\mu^+}}(p) \le Ce^{-R/2}k^2|p|^2/4, \qquad 0 < k < k_1, \quad k|p| < k_2(R),$$

$$\mathrm{M.}\dim w^{k\mu^+}(\mathbb{S}^1) \le 1 + Ce^{-R/2}k^2, \quad k < k_2(R).$$

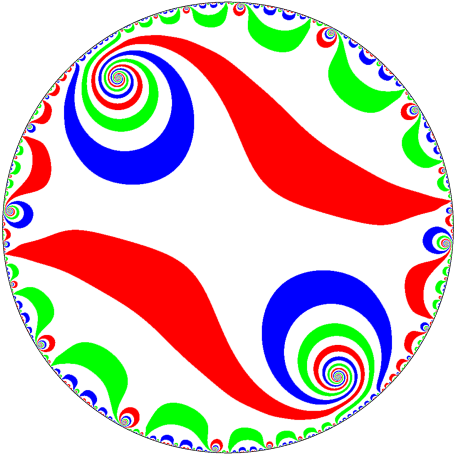
Dynamical case

Using McMullen's identity, one can show that for **dynamical** Beltrami coefficients μ, we have

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{M.dim}\, w^{t\mu}(\mathbb{S}^1) \lesssim \lim_{r \to 1^-} |\mathcal{G} \cap S_r|,$$
$$\lesssim \limsup_{r \to 1^-} \frac{1}{|\log(1-r)|} \int_0^r |\mathcal{G} \cap S_s| \frac{ds}{1-s}.$$

For a garden $\mathcal{G}$ which is the union of horoballs a hyperbolic distance R apart, this is $\leq Ce^{-R/2}$.

A set invariant under some degree 2 Blaschke product



Change of gears: Feynman-Kac formula

Consider a potential $V : \mathbb{D} \to \mathbb{R}$, which we assume to be positive, bounded and continuous.

We are interested in studying the growth of solutions of

$$\frac{\partial u}{\partial t} = \Delta_{\mathrm{hyp}} u + V(x)u(x),$$

where the initial condition $u(x,0) \geq 0$ & has compact support.

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Feynman-Kac formula

$$u(x,t) = \mathbb{E}_x \left\{ u(B_t, 0) \exp \int_0^t V(B_s) ds \right\}.$$

Growth of solutions

Let

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Theorem

The rate of growth of the solution is given by

$$\beta_V = \limsup_{t\to\infty} \frac{1}{t} \cdot \mathbb{E}_0 \left\{ \exp \int_0^t V(B_s)\, ds \right\}.$$

As a function of $p \geq 0$, $\beta_V(p) := \beta_{pV}$ *is increasing and convex.*

Sparsely supported potentials

Theorem B (I. 2016)

Suppose $V = \chi_{\mathcal{G}} = \bigcup B_j$*, where* B_j *are disjoint horoballs, satisfying the separation condition* $d_{\mathbb{D}}(B_i, B_j) > R$*. Then,*

$$\beta_V(p) := \beta_{pV} \le Ce^{-R/2}p^2,$$

for $p < p_0(R)$ *sufficiently small.*

Non-concentration estimate

Integrating the PDE, we obtain

$$\frac{d}{dt}\int_{\mathbb{D}} u_t(x)dA_{\mathrm{hyp}}(x) = \int_{\mathbb{D}} V(x)u_t(x)dA_{\mathrm{hyp}}(x).$$

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Therefore, it suffices to show the non-concentration estimate

$$\int_B u_t(x)dA_{\text{hyp}}(x) \leq Ce^{-R/2}\int_{B^*} u_t(x)dA_{\text{hyp}}(x), \quad \forall t > 0,$$

where $B_j^* = \{z \in \mathbb{D} : d_{\mathbb{D}}(z, B_j) \leq R/2\}$ are disjoint.

Brownian motion escapes from horoballs

Lemma

Consider hyperbolic Brownian motion started at a point $z_0 \in \partial B$. Let

$$\ell(B) = \int_0^\infty \chi_B(B_s) ds$$

denote the amount of time Brownian motion spends in B.

(i) $\mathbb{E}\big(\ell(B)\big) = O(1)$.

(ii) $\mathbb{P}\big(\ell(B) > t\big) < Ce^{-\gamma t}, \qquad \gamma > 0.$

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Lemma

Consider hyperbolic Brownian motion started at a point $z_0 \in \partial B^*$. *Then,* $\mathbb{E}\big(\ell(B)\big) = O(e^{-R/2})$.

Back to conformal mappings

Fix $p \in \mathbb{C} \setminus \{0\}$ and consider the functions

$$u_t(x) = |f'(x)^p| \cdot p_t(x), \qquad p_t(x) = p_t(0, x).$$

In view of the maximal modulus principle and the fact that the probability $\mathbb{P}\big(d_{\mathbb{D}}(0, B_t) > 0.99\, t\big) > c$,

$$\int_{\mathbb{D}} u_t(x) dA_{\text{hyp}}(x) \geq c \int_{|z|=r} |f'(z)^p| \, |dz|,$$

where $d_{\mathbb{D}}(0, r) = 0.99\, t$.

Growth of solutions: "Brownian spectra"

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{D}} |f'(x)^p| \cdot p_t(x) dA_{\mathrm{hyp}}(x) &= \int_{\mathbb{D}} |f'(x)^p| \cdot \Delta_{\mathrm{hyp}}[p_t(x)] dA_{\mathrm{hyp}}(x), \\ &= \int_{\mathbb{D}} \Delta_{\mathrm{hyp}} |f'(x)^p| \cdot p_t(x) dA_{\mathrm{hyp}}(x), \\ &= \int_{\mathbb{D}} V(x) |f'(x)^p| \cdot p_t(x) dA_{\mathrm{hyp}}(x).\end{aligned}$$

potential $\quad V = |p|^2 |n_f/\rho|^2$ $\qquad$ non-linearity $\quad n_f = f''/f'$

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$$= \int_{\mathbb{D}} \Delta_{\mathrm{hyp}} |f'(x)^p| \cdot p_t(x) dA_{\mathrm{hyp}}(x),$$
$$= \int_{\mathbb{D}} V(x) |f'(x)^p| \cdot p_t(x) dA_{\mathrm{hyp}}(x).$$

potential $\quad V = |p|^2 |n_f/\rho|^2 \qquad$ non-linearity $\quad n_f = f''/f'$

E. Dyn'kin's estimate:

$$V(z) \le Ck^2 |p|^2 e^{-(2-2k-\varepsilon)S}, \qquad S = d_{\mathbb{D}}(z, \mathcal{G}),$$

We are fine as long as $k < 1/2 - \varepsilon$ so that $2 - 2k - \varepsilon > 1$.

Thank you for your attention!