Orbit counting, Pesin theory and suspension flows

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Abstract

We present an elementary framework for counting pre-images in conformal dynamical systems using Pesin theory and suspension flows. Besides being elementary, our approach only requires minimal regularity assumptions on the dynamical system. We give two applications:

(1) We show that the Orbit Counting Theorem holds up to a Cesàro average for any non-uniformly hyperbolic rational map, while the full Orbit Counting Theorem holds for all but a short list of exceptional maps. Additionally, we employ a rigidity result of A. Eremenko and S. van Strien to count pre-images when one takes into account the argument of the derivative.

(2) We show that the Orbit Counting Theorem holds for any infinite-to-one Adler map acting on the unit circle with finite Lyapunov exponent.

1 Introduction

Loosely speaking, a dynamical system $F : X \to X$ is a map which you can iterate. To state the Orbit Counting Theorem, we make several basic assumptions.

(OC1) The space $X \subset \mathbb{R}^d$ carries a probability measure m such that m a.e. F is differentiable and the derivative $DF \in \mathbb{R}_+ \cdot \mathrm{SO}(d)$. In other words $F : X \to X$ is a conformal dynamical system.

(OC2) The measure m is a conformal measure of dimension $\alpha > 0$. This means that

$$m(F(E)) = \int_E |F'(x)|^\alpha dm,$$

for any set $E \subset X$ on which F is injective. Moreover, the sets on which F is injective cover X up to measure zero, so that $|F'(x)|^\alpha$ qualifies as the measure-theoretic Jacobian of m.

(OC3) The space X carries an F-invariant ergodic probability measure $d\mu = \gamma\, dm$ which is absolutely continuous with respect to m.

(OC4) Finally, we assume that $\log|F'(x)| \in L^1(X, \mu)$ and the Lyapunov exponent

$$\chi(\mu) = \int_X \log|F'(x)| d\mu > 0.$$

At its simplest, Orbit Counting is concerned with counting pre-images of a point $x \in X$. Consider the counting function

$$n(x,T) = \#\big\{(n \ge 0,\, y \in X) \,:\, F^{\circ n}(y) = x,\, \log|(F^{\circ n})'(y)| < T\big\}.$$

We say that the *Orbit Counting Theorem* holds if for μ a.e. $x \in \partial\mathbb{D}$,

$$n(x,T) \sim e^{\alpha T} \cdot \frac{\gamma(x)}{\alpha \int_X \log|F'(x)| d\mu}, \qquad \text{as } T \to \infty.$$

We say that the *Orbit Counting Theorem holds up to a Cesàro average* if for μ a.e. $x \in \partial\mathbb{D}$,

$$\frac{1}{T}\int_0^T \frac{n(x,S)}{e^{\alpha S}} dS \to \frac{\gamma(x)}{\alpha \int_X \log|F'(x)| d\mu}, \qquad \text{as } T \to \infty.$$

In addition to the four basic assumptions above which are needed to state the Orbit Counting Theorem, we also make three additional assumptions:

(OC5) Pesin theory holds for the dynamical system $F : X \to X$ with respect to the measure μ. (The assumption will be explained in Section 4).

(OC6) The Radon-Nikodym derivative $\gamma > c > 0$ is bounded below by a positive constant μ a.e. (In particular, the measures μ and m are equivalent.)

(OC7) The restricted sum

$$\sum_{F(y)=x} |F'(y)|^{-\alpha} - \max_{F(y)=x} |F'(y)|^{-\alpha} > c' > 0 \tag{1.1}$$

is bounded below by a positive constant μ a.e. (The assumption is trivially satisfied if $|F'|$ is bounded above and a.e. point $x \in X$ has at least two pre-images.)

Theorem 1.1. *The Orbit Counting Theorem holds up to a Cesáro average under the assumptions* (OC1)–(OC7) *above.*

One ingredient in the proof of the above theorem is the ergodicity of a certain suspension flow $g_t : \widehat{X}_\rho \to \widehat{X}_\rho$ (defined in Section 2.2).

Theorem 1.2. *Suppose a dynamical system $F : X \to X$ satisfies the assumptions* (OC1)–(OC7) *above. If the suspension flow $g_t : \widehat{X}_\rho \to \widehat{X}_\rho$ is mixing, then the Orbit Counting Theorem holds.*

According to Theorem 6.1 below, the suspension flow is mixing if and only if there exists a measurable function $w : X \to \partial\mathbb{D}$ and a constant $a \in \mathbb{R} \setminus \{0\}$ such that

$$w(F(x)) = e^{ia \log |F'(x)|} w(x), \qquad \mu \text{ a.e. } x \in X. \tag{1.2}$$

Furthermore, by Lemma 6.2, any measurable solution w of (1.2) is automatically continuous on the Pesin set $X_{\text{lin}} \subset X$. The Pesin set will be defined in Section 6.1, but for now, we note that it is a relatively open subset of X with $\mu(X_{\text{lin}}) = 1$, which could potentially be all of X.

For most dynamical systems, the functional equation (1.2) does not have any non-trivial solutions (i.e. with $a \neq 0$) as it implies the alignment of inverse branches (this condition will be explained in Section 6.2). However, ruling out the existence of non-trivial eigenfunctions for particular dynamical systems is tricky and can only be done in a case-by-case basis. In this paper, we focus on rational functions acting on the Riemann sphere and Adler maps acting on the unit circle.

Remark. The assumptions (OC6) and (OC7) are used to control the number of "bad" repeated pre-images of $x \in X$. Without these assumptions, the arguments in this paper show the lower bounds in Theorems 1.1 and 1.2.

1.1 Application to rational functions

Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$ with Julia set $\mathcal{J} = \mathcal{J}(F)$. While the above list of assumptions looks long, only assumptions (OC1)–(OC4) need to be checked since the remaining assumptions are true for all rational maps or follow from these: see Section 10.

By the work of Rivera-Letelier and Przytycki [PR07], the above assumptions are satisfied for Topological Collet-Eckmann (TCE) rational maps, with $\alpha = \mathrm{H.\,dim}\, \mathcal{J}(F)$. For an even more general class of rational maps satisfying the above hypotheses, we refer the reader to the work of Graczyk and Smirnov, see [GS09, Theorem 4].

Theorem 1.3. *The Orbit Counting Theorem holds for any rational map satisfying the assumptions* (OC1)–(OC7) *above with a small number exceptions. The exceptional rational maps are the ones conjugate to:*

- $z^{\pm d}$ *for some* $d \geq 2$*,*
- $\pm T_d$ *where* T_d *is the Chebyshev polynomial of degree* $d \geq 2$*,*
- *Lattès maps.*

Motivated by the work of H. Oh and D. Winter [OW17], we examine the following counting function: for a point $x \in \mathcal{J}(F)$ and an arc $I \subset \partial\mathbb{D}$, we define $n(x, T, I)$ as

$$\#\left\{(n \geq 0,\, y \in \mathcal{J}(F)) : F^{\circ n}(y) = x,\, \log |(F^{\circ n})'(y)| < T,\, \frac{(F^{\circ n})'(y)}{|(F^{\circ n})'(y)|} \in I\right\}.$$

We say that the *Orbit Counting Theorem with rotation* holds if for any arc $I \subset \partial\mathbb{D}$,

$$n(x, T, I) \sim \frac{|I|}{2\pi} \cdot e^{\alpha T} \cdot \frac{\gamma(x)}{\alpha \int_{\mathcal{J}(F)} \log |F'(x)| d\mu}, \qquad \text{as } T \to \infty.$$

Theorem 1.4. *The Orbit Counting Theorem with rotation holds for any rational map satisfying the conditions* (OC1)–(OC7)*, albeit with a larger list of exceptions: this time, we also need to include rational maps whose Julia sets are contained in a circle or a line.*

The proof is only slightly more difficult than that of Theorem 1.3. The extra ingredient is a rigidity result of A. Eremenko and S. van Strien [EvS11]. The proofs of Theorems 1.3 and 1.4 will be presented in Sections 12 and 13 respectively.

1.2 Applications to one-dimensional dynamics

A self-map F of the unit circle is an *Adler map* if it is C^2 outside a closed subset Σ of Lebesgue measure zero (called the *singular set*) and satisfies the following conditions:

$$\sup_{x\in\partial\mathbb{D}\setminus\Sigma} |F'(x)| > 1, \qquad \limsup_{x\to\Sigma} |F'(x)| = \infty, \tag{1.3}$$

$$M := \sup_{x\in\partial\mathbb{D}\setminus\Sigma} \frac{|F''(x)|}{|F'(x)|^2} < \infty. \tag{1.4}$$

In Section 14, we show the following theorem:

Theorem 1.5. *Let F be an Adler map acting on the unit circle. If*

$$\int_{\partial\mathbb{D}} \log|F'|dm < \infty, \tag{1.5}$$

then the Orbit Counting Theorem holds up to a Cesáro average. If F is infinite-to-one, then the full Orbit Counting Theorem holds.

An alternative approach to orbit counting via thermodynamic formalism and Tauberian theory was pioneered by S. P. Lalley [Lal89] and further developed in [PoU17, IU23]. While this approach has its own advantages, it requires a slightly stronger integrability assumption than necessary:

$$\int_0^1 (\log|F'|)^{1+\varepsilon} dm < \infty, \qquad \text{for some } \varepsilon > 0.$$

By contrast, the technique in this paper gives the optimal result.

1.3 On periodic orbits and critical points

Lemma 1.6. *Suppose μ is an ergodic probability measure on X.*

(i) *If the Lyapunov exponent of μ is positive, then μ does not charge the union of the non-repelling periodic orbits.*

(ii) *If $\log|F'(x)| \in L^1(X, \mu)$, then μ does not charge $\{x \in X : F'(x) = 0\}$.*

Proof. (i) For $q \geq 1$, let NR_q denote the set of periodic points of period q whose multiplier has absolute value at most 1. If $\mu(\mathrm{NR}_q) > 0$ for some $q \geq 1$, then by the ergodic theorem, there would exist a point $x_0 \in \mathrm{NR}_q$ such that

$$\frac{1}{q} \log|(F^{\circ q})'(x_0)| = \int_X \log|(F^{\circ n})'(x)| d\mu(x).$$

This is a contradiction since the left hand side is ≤ 0 while the right hand side is > 0 by the assumption. As a result, μ does not charge any of the sets $\mu(\mathrm{NR}_q)$.

(ii) is immediate from the assumption. □

1.4 Notes and references

The general strategy was first utilized in the work [IU24] in the context of inner functions with derivative in the Nevanlinna class. The case of inner functions is in some ways easier, but in some ways harder than what is presented here: One thing that makes inner functions simpler is that they are expanding on the unit circle. One thing that makes inner functions more complicated is that in general, the geodesic flow on a Riemann surface lamination associated to an inner function cannot be represented as a suspension flow over the solenoid. Thus, while the present work bears some similarities to [IU24], the two works are somewhat orthogonal with different sets of challenges.

The work of Oh and Winter [OW17] discusses the Orbit Counting Theorem with rotation for hyperbolic rational maps. Although they work with a rather special class of well-behaved rational maps, they provide an $O(e^{(\alpha-\varepsilon)T})$ error estimate by employing intricate machinery due to Dolgopyat [Dol98]. This approach relies on the fact that the dynamical zeta function admits a meromorphic extension to a slightly larger half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha - \varepsilon\}$. Somewhat remarkably, Z. Li and J. Rivera-Letelier

[LR25] extended the Oh-Winter result to TCE rational maps, which could be less well-behaved. We note that for general conformal dynamical systems, the meromorphic extension requires some extra regularity hypotheses, e.g. for Adler maps, one would ask that $\int_{\partial\mathbb{D}} |F'|^p dm < \infty$ for some $p > 0$. It is possible that the Orbit Counting Theorem does not hold with the above error term for a general Adler map satisfying (1.5).

In an important work, M. Lyubich and Y. Minsky [LM97] associated natural affine and hyperbolic laminations, $\mathcal{A}_F$ and $\mathcal{H}_F/\hat{F}$, to a rational map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$. In [Glu10a, Glu10b], A. Glutsyuk studied the density and ergodic properties of horospheres in $\mathcal{H}_F/\hat{F}$ for convex co-compact rational maps (a rational map is *convex co-compact* if it is critically non-recurrent and has no parabolic points). An important tool in these works is a rigidity result from [KL05, Theorem 3.54] which says that $\mathcal{A}_F$ is Euclidean if and only if F is one of the exceptional rational maps from Theorem 1.3. Glutsyuk's work closely aligns with the approach in this paper, although it is expressed in different terminology. In a classical argument due to E. Hopf, one shows the mixing of the geodesic flow on the unit tangent bundle of a compact hyperbolic surface by using the ergodicity of the horocyclic flow. Although horospheres appear implicitly in Section 6, we do not use the horospherical flow – instead, we rely on an argument of M. Babillot [Bab02] to give a direct proof of the mixing of the geodesic or suspension flow.

The arguments in this paper show that under the assumptions of Theorem 1.2 on the conformal dynamical system (F, X, μ), for μ a.e. $x \in X$, as $T \to \infty$, the $n(x,T)$ repeated pre-images of x with $\log |(F^{\circ n})'(y)| < T$ equidistribute with respect to the measure μ. It is possible that with a little bit more work, one can conclude equidistribution under the less restrictive hypotheses of Theorem 1.1. One can compare this with a celebrated result of Lyubich [Lyu83] which says that for any rational map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$, for all but two points $x \in \widehat{\mathbb{C}}$, as $n \to \infty$, the d^n pre-images of x of order n equidistribute with respect to the measure of maximal entropy of F. It is worth mentioning that the case of polynomials is substantially simpler and has been worked out by H. Brolin [Bro65] almost twenty years earlier.

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Part I

2 Preliminaries

In this section, we define the fundamental notions that will be used throughout this paper: natural extensions, suspension flows and almost-invariant functions. We then discuss the consequences of ergodicity and mixing of the suspension flow.

2.1 Natural extensions

Let $F : X \to X$ be a dynamical system and μ be an invariant measure supported on X. The *natural extension* of (X, F, μ) is a triple $(\widehat{X}, \widehat{F}, \widehat{\mu})$, where:

1. The space $\widehat{X}$ consists of all backward orbits

$$\cdots \to x_{-2} \to_F x_{-1} \to_F x_0$$

under the dynamics of F.

2. The map $\widehat{F} : \widehat{X} \to \widehat{X}$ applies F to each coordinate:

$$(x_{-n}) \to (F(x_{-n})).$$

3. The measure $\widehat{\mu}$ is the unique $\widehat{F}$-invariant measure which projects to μ on each coordinate.

Remark. We will sometimes index backwards orbits $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ by the integers, rather than by the natural numbers. The difference is not essential since for $n > 0$, $x_n = F^{\circ n}(x_0)$ is uniquely determined by x_0.

Given a set $A \subset X$, we write $\widehat{A} \subset \widehat{X}$ for the set of inverse orbits $\mathbf{x} = (x_{-n})$ with $x_0 \in A$. From the definition of the natural extension measure, it is clear that $\widehat{\mu}(\widehat{A}) = \mu(A)$.

It is well known that if μ is ergodic with respect to $F : X \to X$, then the natural extension measure $\widehat{\mu}$ is ergodic with respect to $\widehat{F} : \widehat{X} \to \widehat{X}$.

2.2 The suspension flow

We now define the suspension flow $g_s : \widehat{X}_\rho \to \widehat{X}_\rho$ with respect to the roof function $\rho(\mathbf{x}) = \log |F'(x_0)|$.

1. The suspension space $\widehat{X}_\rho$ is formed by taking the quotient of $\widehat{X} \times \mathbb{R}_+$ with respect to the equivalence relation $(\mathbf{x}, t) \sim \widehat{F}(\mathbf{x}, t) = (\widehat{F}(\mathbf{x}), |F'(x_0)|\, t)$.
2. The product measure $\widehat{\mu} \times (dt/t)$ on $\widehat{X} \times \mathbb{R}_+$ descends to an invariant measure $\widehat{\mu}_\rho$ on the suspension space.
3. The suspension or geodesic flow $g_s : \widehat{X}_\rho \to \widehat{X}_\rho$ is given by $(\mathbf{x}, t) \to (\widehat{F}(\mathbf{x}), e^s \cdot t)$.

It is not difficult to see that if $\widehat{\mu}$ is ergodic for $\widehat{F} : \widehat{X} \to \widehat{X}$, then $\widehat{\mu}_\rho$ is ergodic with respect to the suspension flow $g_s : \widehat{X}_\rho \to \widehat{X}_\rho$.

We write $F(x, t) = (F(x), |F'(x)|t)$ for the action on $X \times \mathbb{R}_+$ induced by F, and $\widehat{F}(\mathbf{x}, t) = (\widehat{F}(\mathbf{x}), |F'(x_0)|t)$ for the action on $\widehat{X} \times \mathbb{R}_+$ induced by $\widehat{F}$. For a point $(\mathbf{x}, r) \in \widehat{X} \times \mathbb{R}_+$, it is convenient to write

$$r_n = |(F^{\circ n})'(x_0)|r, \qquad n \in \mathbb{Z},$$

so that $(\mathbf{x}, \mathbf{r}) = (x_n, r_n)_{n=-\infty}^{\infty}$ is a bi-infinite orbit under $\widehat{F} : \widehat{X} \times \mathbb{R}_+ \to \widehat{X} \times \mathbb{R}_+$.

Lemma 2.1. *The total mass of the measure $\widehat{\mu}_\rho$ on $\widehat{X}_\rho$ is*

$$\int_X \log |F'(x)| d\mu(x).$$

Proof. To prove the lemma, we describe a fundamental domain E for the action of $\widehat{F}$ on $\widehat{X} \times \mathbb{R}_+$ and compute its area. Here, by a fundamental domain, we mean a set

$E \subset \widehat{X} \times \mathbb{R}_+$ such that almost any orbit in $\widehat{X} \times \mathbb{R}_+$ is equivalent to a unique point in E.

Let $A \subset X \times \mathbb{R}_+$ be the set of points (x,t) in $X \times (0,1]$ whose image under $\widehat{F}$ is contained in $X \times (1,\infty)$ and $B \subset X \times \mathbb{R}_+$ be the set of points (x,t) in $X \times (1,\infty)$ whose image is contained in $X \times (0,1]$. In other words, A is the set of points which leave $X \times (0,1]$ while B is the set of points which enter $X \times (0,1]$.

The saturated sets $\widehat{A}, \widehat{B} \subset \widehat{X} \times \mathbb{R}_+$ have areas

$$(\widehat{\mu} \times (dt/t))(\widehat{A}) = \int_X \log^+ |F'(x)| d\mu(x),$$
$$(\widehat{\mu} \times (dt/t))(\widehat{B}) = \int_X \log^- |F'(x)| d\mu(x).$$

By the ergodic theorem, for almost every bi-infinite orbit $(\mathbf{x}, \mathbf{t}) = (x_n, t_n)_{n \in \mathbb{Z}}$ in $X \times \mathbb{R}_+$, we have $t_{-n} \to 0$ and $t_n \to \infty$. As a result, under forward iteration by $\widehat{F}$, almost every orbit passes through A, while almost every point in B lands in A in a finite number of steps. We can therefore decompose

$$B = \bigsqcup_{i=1}^{\infty} B_i \sqcup N,$$

where $B_i \subset B$ is the subset of points whose minimal iterate that lies in A is $\widehat{F}^{\circ i}$ and N has measure zero. From the above remarks, it follows that

$$E = \widehat{A} \setminus \bigsqcup_{i=1}^{\infty} \widehat{F}^{\circ i}(\widehat{B}_i)$$

is a fundamental domain for the action of $\widehat{F}$ on $\widehat{X} \times \mathbb{R}_+$ with

$$\begin{aligned}(\widehat{\mu} \times (dt/t))(E) &= (\widehat{\mu} \times (dt/t))(\widehat{A}) - (\widehat{\mu} \times (dt/t))(\widehat{B}) \\ &= \int_X \log |F'(x)| d\mu(x).\end{aligned}$$

The proof is complete. $\square$

2.3 Almost-invariant functions

Following [McM08, IU24], we say that a function $h(x,t) : X \times \mathbb{R}_+ \to \mathbb{R}$ is *weakly almost-invariant* if for $\widehat{\mu} \times (dt/t)$ a.e. $(\mathbf{x}, t) \in \widehat{X} \times \mathbb{R}_+$, the limit

$$\widehat{h}(\mathbf{x}, t) := \lim_{n\to\infty} h\big(x_{-n}, |(F^{\circ n})'(x_{-n})|^{-1}\, t\big) \tag{2.1}$$

exists. The limit $\widehat{h}(\mathbf{x}, t)$ can be viewed as a function defined almost everywhere on the suspension space $\widehat{X}_\rho$. We refer to the function $\widehat{h}(\mathbf{x}, t)$ as the *natural extension* of $h(x, t)$.

Theorem 2.2. *Suppose $h : X \times \mathbb{R}_+ \to \mathbb{R}$ is a bounded weakly almost-invariant function. Then for μ almost every $x \in X$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T h(x, e^{-t})dt = \frac{1}{\int_X \log|F'(x)|d\mu} \int_{\widehat{X}_\rho} \widehat{h}(\mathbf{x}, t) d\widehat{\mu}_\rho.$$

Proof. By the ergodic theorem applied to the flow $g_{-t} : \widehat{X}_\rho \to \widehat{X}_\rho$, for $\widehat{\mu} \times (dt/t)$ a.e. $(\mathbf{x}, r) \in \widehat{X}_\rho$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \widehat{h}(\mathbf{x}, re^{-t})dt = \frac{1}{\int_X \log|F'(x)|d\mu} \int_{\widehat{X}_\rho} \widehat{h}(\mathbf{x}, t) d\widehat{\mu}_\rho.$$

As the set of pairs $(\mathbf{x}, r) \in \widehat{X}_\rho$ for which the above equation holds is invariant under the suspension flow, we might as well assume that $r = 1$. To complete the proof of the theorem, we need to show that for $\widehat{\mu}$ a.e. inverse orbit $\mathbf{x} = (x_{-n})_{n=0}^{\infty} \in \widehat{X}$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T h(x_0, e^{-t})dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \widehat{h}(\mathbf{x}, e^{-t})dt. \tag{2.2}$$

To that end, we check that

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |h(x_0, e^{-t}) - \widehat{h}(\mathbf{x}, e^{-t})|dt = 0. \tag{2.3}$$

Given an $\varepsilon > 0$ and a $\rho > 0$, we define $A(\varepsilon, \rho) \subset \widehat{X}_\rho$ as the set of pairs $(\mathbf{x}, r) \in \widehat{X}_\rho = (\widehat{X} \times \mathbb{R}_+)/\sim$ for which

$$|h(x_n, r_n) - \widehat{h}(\mathbf{x}, r)| < \varepsilon, \qquad \text{for all } n \in \mathbb{Z} \text{ with } r_n \le \rho.$$

In other words, we look at the points of $\widehat{X}_\rho$, for which the value of $\widehat{h}$ is determined within ε by the part of the orbit that lies in $X \times (0, \rho]$.

By the definition of a weakly almost invariant function, for any fixed $\varepsilon > 0$,

$$\widehat{\mu}_\rho(A(\varepsilon, \rho)^c) \to 0, \qquad \text{as } \rho \to 0.$$

We may therefore choose $\rho = \rho(\varepsilon)$ so that $\widehat{\mu}_\rho(A(\varepsilon, \rho)^c) < \varepsilon$.

By the ergodic theorem, a generic backward trajectory $\{(\mathbf{x}, re^{-t}) : t > 0\}$ spends little time in $A(\varepsilon, \rho)^c$, i.e.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_{A(\varepsilon,\rho)^c}(\mathbf{x}, re^{-t})\, dt < \frac{\varepsilon}{\int_X \log |F'(x)| d\mu},$$

and as before, we may assume that $r = 1$. Consequently,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |h(x_0, e^{-t}) - \widehat{h}(\mathbf{x}, e^{-t})| dt \lesssim \varepsilon + \frac{2\varepsilon \|h\|_\infty}{\int_X \log |F'(x)| d\mu},$$

which can be made arbitrarily small by requesting that $\varepsilon > 0$ is small. This concludes the proof of (2.3) and hence of (2.2). □

Theorem 2.3. *Suppose $h : X \times \mathbb{R}_+ \to \mathbb{R}$ is a bounded weakly almost-invariant function. If the suspension flow g_t on $\widehat{X}_\rho$ is mixing, then for any window size $\delta > 0$,*

$$\lim_{T\to\infty} \frac{1}{\delta} \int_T^{T+\delta} \int_X h(x, e^{-t})\, d\mu\, dt = \frac{1}{\int_X \log |F'(x)| d\mu} \int_{\widehat{X}_\rho} \widehat{h}(\mathbf{x}, t) d\widehat{\mu}_\rho \tag{2.4}$$

and

$$\lim_{T\to\infty} \frac{1}{\delta} \int_T^{T+\delta} \int_X h(x, e^{-t})\, dm\, dt = \frac{1}{\int_X \log |F'(x)| d\mu} \int_{\widehat{X}_\rho} \widehat{h}(\mathbf{x}, t) d\widehat{\mu}_\rho. \tag{2.5}$$

Proof. Consider the set $X_\delta = X \times [e^{-\delta}, 1]$. As usual, we write $\widehat{X}_\delta$ for the collection of inverse orbits $(\mathbf{x}, t) \in \widehat{X} \times \mathbb{R}_+$ with $(x_0, t) \in X_\delta$. Pushing forward the measure $\widehat{\mu} \times (dt/t)|_{\widehat{X}_\delta}$ of mass δ under the natural projection map $\widehat{X} \times \mathbb{R}_+ \to \widehat{X}_\rho$, we get a measure $M(\mathbf{x}, t)\widehat{\mu}_\rho$ on $\widehat{X}_\rho$, where the integer-valued density function $M(\mathbf{x}, t)$ records the number of backward orbits in each equivalence class contained in $\widehat{X}_\delta$. By the mixing of the suspension flow $g_t : \widehat{X}_\rho \to \widehat{X}_\rho$, as $T \to \infty$,

$$\left\langle \widehat{h}, g_{-T}[M(\mathbf{x}, t)] \right\rangle \to \frac{\delta}{\int_X \log |F'(x)| d\mu} \int_{\widehat{X}_\rho} \widehat{h}(\mathbf{x}, t)\, d\widehat{\mu}_\rho$$

or

$$\frac{1}{\delta}\int_{\widehat{X}}\int_{T}^{T+\delta}\widehat{h}(\mathbf{x},e^{-t})\,d\widehat{\mu}\,dt \to \frac{1}{\int_X \log|F'(x)|d\mu}\int_{\widehat{X}_\rho}\widehat{h}(\mathbf{x},t)\,d\widehat{\mu}_\rho.$$

Using the sets $A(\varepsilon,\rho)\subset\widehat{X}_\rho$ as in the previous proof, one can show that

$$\lim_{T\to\infty}\frac{1}{\delta}\left|\int_{\widehat{X}}\int_{T}^{T+\delta}\widehat{h}(\mathbf{x},e^{-t})\,d\widehat{\mu}\,dt - \int_{\widehat{X}}\int_{T}^{T+\delta}h(x_0,e^{-t})\,d\widehat{\mu}\,dt\right| = 0,$$

from which (2.4) follows. The argument for (2.5) is similar, except that we use the pushforward of the measure $\gamma(x_0)^{-1}\cdot\widehat{\mu}\times(dt/t)|_{\widehat{X}_\delta}$, which also has mass δ. □

We define the *hyperbolic metric* on $X\times\mathbb{R}_+$ by

$$d_{X\times\mathbb{R}_+}\big((x,t_1),(y,t_2)\big) := d_{\mathbb{H}}(it_1, |x-y|+it_2),$$

where $d_{\mathbb{H}}$ denotes the hyperbolic distance in the upper half-plane $\mathbb{H}$. In particular,

$$d_{X\times\mathbb{R}_+}\big((x,t_1),(x,t_2)\big) = \log|t_2/t_1|.$$

Under a mild continuity hypothesis on h, averaging over windows becomes unnecessary:

Corollary 2.4. *Suppose $h: X\times\mathbb{R}_+\to\mathbb{R}$ is a bounded weakly almost-invariant function which is uniformly continuous in the hyperbolic metric of $X\times\mathbb{R}_+$ in the t variable. If the suspension flow g_t on $\widehat{X}_\rho$ is mixing, then*

$$\lim_{T\to\infty}\int_X h(x,e^{-T})d\mu = \frac{1}{\int_X\log|F'(x)|d\mu}\int_{\widehat{X}_\rho}\widehat{h}(\mathbf{x},t)d\widehat{\mu}_\rho.$$

and

$$\lim_{T\to\infty}\int_X h(x,e^{-T})dm = \frac{1}{\int_X\log|F'(x)|d\mu}\int_{\widehat{X}_\rho}\widehat{h}(\mathbf{x},t)d\widehat{\mu}_\rho.$$

3 Linear distortion

For simplicity of exposition, we work with holomorphic mappings in the complex plane, but the arguments are applicable in $\mathbb{R}^d$ for any $d\geq 1$.

Let $\Omega \subset \mathbb{C}$ be a domain in the plane and $F : \Omega \to \mathbb{C}$ be a (possibly discontinuous) map. In this section, we describe one possible way of quantifying how far away F is from a linear map in a ball $B(x, r) \subset \Omega$. We equip the punctured plane $\mathbb{C} \setminus \{0\}$ with the metric $|dz|/|z|$.

If F is holomorphic on $B(x, r)$ and F' does not vanish on $B(x, r)$, then we define the *linear distortion* of F near x at scale r as

$$\delta_F(x, r) := \min\Big(1/10,\ \mathrm{diam}_{\mathbb{C}\setminus\{0\}}\big\{F'(z) : z \in B(x, r)\big\}\Big).$$

Otherwise, we simply set $\delta_F(x, r) := 1/10$. For $0 < \varepsilon < 1/10$, we say that F is ε-linear on $B(x, r)$ if $\delta_F(x, r) < \varepsilon$. The constant $1/10$ has been chosen so that if F is ε-linear on $B(x, r)$ then F is injective on $B(x, r)$. The following lemma summarizes the basic properties of linear distortion:

Lemma 3.1. *Linear distortion satisfies the following properties:*

(i) $\delta_F(x, r) = 0$ *if and only if* F *is linear on* $B(x, r)$*.*

(ii) *If* $B(y, s) \subset B(x, r)$ *then* $\delta_F(y, s) \le \delta_F(x, r)$*.*

(iii) *If* F *is conformal at* x*, then* $\delta_F(x, r) \to 0$ *as* $r \to 0$*.*

(iv) *The image of a ball* $B(x, r)$ *is close to a ball centered at* $F(x)$*:*

$$B\big(F(x), e^{-\varepsilon}|F'(x)|r\big) \ \subset\ F(B(x, r)) \ \subset\ B\big(F(x), e^{\varepsilon}|F'(x)|r\big).$$

Lemma 3.2. *Suppose* $F_1, F_2, \ldots, F_n$ *is a sequence of maps. For a point* $x = x_0 \in \mathbb{C}$ *and real number* $r = r_0 > 0$*, we have*

$$\delta_{F_n \circ F_{n-1} \circ \cdots \circ F_1}(x, r) \le \sum_{j=1}^{n} \delta_{F_j}(x_j, 2r_j), \tag{3.1}$$

where $x_{j+1} = F_{j+1}(x_j)$ *and* $r_{j+1} = |F'_{j+1}(x_j)|r_j$ *for* $j = 0, 1, \ldots, n-1$*.*

The motivation behind the definition of r_j is that if F were a linear map at each step, then F would map $B(x_j, r_j)$ to $B(x_{j+1}, r_{j+1})$.

Lemma 3.3. *For any complex numbers* $z_j, w_j \in \mathbb{C} \setminus \{0\}$*,* $j = 1, 2, \ldots, n$*, we have*

$$d_{\mathbb{C}\setminus\{0\}}(z_1 z_2 \cdots z_n, w_1 w_2 \cdots w_n) \ \le\ \sum_{j=1}^{n} d_{\mathbb{C}\setminus\{0\}}(z_j, w_j). \tag{3.2}$$

Proof. It suffices to consider the case when $n = 2$, as the general case follows from induction. First note that $d_{\mathbb{C}\setminus\{0\}}$ is homogenous in the sense that if $a \in \mathbb{C} \setminus \{0\}$, then

$$d_{\mathbb{C}\setminus\{0\}}(az, aw) = d_{\mathbb{C}\setminus\{0\}}(z, w).$$

Using the above fact together with the triangle inequality, we get

$$\begin{aligned} d_{\mathbb{C}\setminus\{0\}}(z_1z_2, w_1w_2) &\le d_{\mathbb{C}\setminus\{0\}}(z_1z_2, w_1z_2) + d_{\mathbb{C}\setminus\{0\}}(w_1z_2, w_1w_2) \\ &= d_{\mathbb{C}\setminus\{0\}}(z_1, w_1) + d_{\mathbb{C}\setminus\{0\}}(z_2, w_2), \end{aligned}$$

as desired. □

Proof of Lemma 3.2. If the right hand side of (3.1) exceeds 1/10, the inequality is trivial. Otherwise, by Lemma 3.1(iv), under iteration, the image of the ball $B(x, r)$ is contained inside the balls $B(x_j, 2r_j)$, $j = 1, 2, \dots, n$. From here, (3.1) follows from Lemma 3.3. □

4 Background on Pesin theory

In this section, we give a gentle introduction to Pesin theory for conformal dynamical systems. As a byproduct of our discussion, we will give a new (and perhaps simpler) proof of a result by Przytycki and Urbanski [PrU10, Theorem 10.2.3] which says that if F is a rational map and μ is an invariant measure supported on $\mathcal{J}(F)$ with positive Lyapunov exponent, then $(F, \mathcal{J}(F), \mu)$ is non-uniformly hyperbolic.

4.1 Hyberbolic dynamical systems

Suppose $X \subset \mathbb{C}$ is a closed set in the complex plane. In a *hyperbolic dynamical system* $F : X \to X$, the limit set X is self-similar with bounded distortion. This means that a little piece of X looks like a slightly distorted copy of a piece of X of definite size. More precisely, for each $x \in X$ and $0 < r < \operatorname{diam} X$, there is an iterate $F^{\circ n}$ which maps $B(x, r)$ injectively onto a set whose diameter is comparable to the diameter of X such that

$$1/C \le \frac{|(F^{\circ n})'(x_1)|}{|(F^{\circ n})'(x_2)|} \le C, \qquad x_1, x_2 \in B(x, r).$$

The characteristic property of a hyperbolic dynamical system is that for any inverse orbit $\mathbf{x} = (x_{-n})$, the dynamics is *univalent asymptotically linear* (UAL):

1. There exists an $r(\mathbf{x}) > 0$ so that that $F^{\circ n}$ has an inverse branch defined on $B(x_0, r)$ which takes x_0 to x_{-n}.

2. The diameters of $F^{-n}(B(x_0, r))$ shrink to 0.

3. The sequence of normalized maps

$$F_{\mathbf{x},-n}(z) \;=\; L_{\mathbf{x},-n} \circ F^{-n} \;=\; \frac{F^{-n}(z) - x_{-n}}{|(F^{-n})'(x_0)|}$$

   converges uniformly on compact subsets of $B(x_0, r)$ to some function $F_{\mathbf{x}}^{-\infty}$. Here, the sequence of affine maps $L_{\mathbf{x},-n}(z) = A_{\mathbf{x},-n}z + B_{\mathbf{x},-n}$ has been chosen so that

$$F_{\mathbf{x},-n}(x_0) = 0 \qquad \text{and} \qquad (F_{\mathbf{x},-n})'(x_0) = 1.$$

As $n \to \infty$, the sequence of entire functions $\{F^{\circ n} \circ L_{\mathbf{x},-n}^{-1}\}$ converges uniformly on compact subsets of the complex plane. The limit of these maps $F_{\mathbf{x}}^{\infty}$ is the inverse of $F_{\mathbf{x}}^{-\infty}$ on $B(x_0, r)$.

4.2 Non-uniform hyperbolicity

In this paper, we work with a less stringent class of dynamical systems, where the dynamics is *non-uniformly hyperbolic*. Below, we assume that the dynamical system $F : X \to X$ satisfies the following basic assumptions:

P1. F possesses an ergodic probability measure μ.

P2. F is differentiable μ almost everywhere.

P3. The Lyapunov exponent

$$0 < \int_X \log |F'(x)| d\mu < \infty.$$

We say that that a dynamical system $F : X \to X$ is *non-uniformly hyperbolic* if backward iteration is asymptotically linear for $\widehat{\mu}$ almost every inverse orbit $\mathbf{x} \in \widehat{X}$.

Theorem 4.1. *Suppose that (F, X, μ) is a dynamical system which satisfies the Assumptions P1–P3 above. If*

$$\Delta = \int_X \int_0^1 \delta_F(x, r) \, \frac{d\mu \, dr}{r} < \infty,$$

then (F, X, μ) is non-uniformly hyperbolic.

The above condition says that if we want to show that Pesin theory holds for a particular dynamical system, then its enough to check that the total amount of linear distortion present at scales less than 1 is finite.

4.3 Cumulative distortion

Suppose $\mathbf{x} = (x_{-n})_{n=0}^{\infty}$ is a backward orbit. We define the *cumulative linear distortion* of F along $\mathbf{x}$ at scale $r > 0$ as

$$\widehat{\delta}_F(\mathbf{x}, r) = \sum_{n=1}^{\infty} \delta_F(x_{-n}, 2r_{-n}), \qquad r_{-n} = |F^{\circ n}(x_{-n})|^{-1} r.$$

Lemma 4.2. *Let $0 < \varepsilon < 1/10$. If $\widehat{\delta}_F(\mathbf{x}, r) < \varepsilon$, then for any $n \geq 0$, F^{-n} admits an ε-linear inverse branch on $B(x_0, r)$ which takes x_0 to x_{-n}.*

Corollary 4.3. *If $\widehat{\delta}_F(\mathbf{x}, r) < \infty$, then for any $\varepsilon > 0$ and $n \geq n_0(\varepsilon)$ sufficiently large, F admits ε-linear inverse branches on $B(x_{-n}, r_{-n})$ along $\mathbf{x}$.*

Proof of Theorem 4.1. By the ergodic theorem and the positivity of the Lyapunov exponent, for $\widehat{\mu}$ almost every inverse orbit $\mathbf{x} \in \widehat{X}$,

$$(F^{-n})'(x_0) \to 0,$$

so that $\widehat{\mu} \times (dr/r)$ almost every backward orbit of the map

$$F : (x, r) \to (F(x), |F'(x)| r)$$

eventually lies inside $X \times (0, 1]$. In other words, $X \times (0, 1]$ is a backwards absorbing set. Since

$$\widehat{\delta}_1(\mathbf{x}, \mathbf{r}) = \sum_{n=-\infty}^{\infty} \chi_{\{r_n < 1\}} \cdot \delta(x_n, r_n)$$

measures the cumulative distortion along the part of a bi-infinite orbit $(\mathbf{x}, \mathbf{r}) = (x_n, r_n)_{n=-\infty}^{\infty} \in \widehat{X}_\rho$ contained in $X \times (0, 1]$,

$$\Delta = \int_X \int_0^1 \delta_F(x, r) \, \frac{d\mu \, dr}{r} = \int_{\widehat{X}_\rho} \widehat{\delta}_1(\mathbf{x}, \mathbf{r}) d\widehat{\mu}_\rho.$$

Consequently, if $\Delta < \infty$, then for a.e. $(\mathbf{x}, \mathbf{r}) \in \widehat{X}_\rho$, the cumulative linear distortion over the part of $(\mathbf{x}, \mathbf{r})$ contained in $X \times (0, 1]$ is finite. The theorem now follows from Corollary 4.3. $\square$

4.4 Example: Gauss map

We first consider a one-dimensional example. By identifying the endpoints, we can think of the interval $[0, 1]$ as a circle. We write $\{x\} = x - \lfloor x \rfloor$ for the fractional part of x. The Gauss map $F : (0, 1] \to (0, 1]$ is given by $x \to \{1/x\}$. The Lebesgue measure $m = dx$ is a conformal measure of dimension 1, while

$$\mu = \frac{1}{\log 2} \cdot \frac{dx}{1 + x}$$

is an ergodic absolutely continuous probability measure. In this example, the Radon-Nikodym derivative $\gamma = d\mu/dm$ is bounded above and below.

We may express $(0, 1)$ as a union of countably many intervals $I_n = [1/(n+1), 1/n)$, with $n = 1, 2, \ldots$, each of which get mapped bijectively onto $[0, 1)$. It is not difficult to see that

$$\delta_F(x, r) \lesssim \begin{cases} 1, & r \geq |I_n|, \\ r/|I_n|, & r \leq |I_n|. \end{cases}$$

As the length $|I_n|$ of I_n is comparable to $1/n^2$, we have $\Delta \asymp |I_n| \log \frac{1}{|I_n|} < \infty$, and so the Gauss map is non-uniformly hyperbolic by Theorem 4.1.

4.5 Example: Rational maps

Let $F : \mathbb{C} \to \mathbb{C}$ be a rational map and μ be an invariant measure supported on the Julia set of F whose Lyapunov exponent is finite and positive:

$$0 < \int_{\mathcal{J}(F)} \log |F'(x)| d\mu < \infty.$$

For a point $x \in \mathcal{J}(F)$, let $\operatorname{Inj}(x)$ denote the radius of the largest ball centered at x on which F is injective. (If x is a critical point of F, we set $\operatorname{Inj}(x)$ to 0.) By Koebe's distortion theorem,

$$\delta_F(x,r) \lesssim \begin{cases} 1, & r \geq \operatorname{Inj}(x), \\ r/\operatorname{Inj}(x), & r \leq \operatorname{Inj}(x), \end{cases}$$

which leads to the estimate

$$\Delta \lesssim \int_{\mathcal{J}(F)} \log^+ \frac{1}{\operatorname{Inj} x}\, d\mu(x). \tag{4.1}$$

Let $d(x, \mathrm{CV})$ be the Euclidean distance from x to the set of the critical values of F. With help of Koebe's distortion theorem, it is not difficult to see that

$$\frac{d(F(x), \mathrm{CV})}{|F'(x)|} \lesssim \operatorname{Inj} x.$$

Consequently, the finiteness of the integral

$$\int_{\mathcal{J}(F)} \log^+ \frac{1}{d(x, \mathrm{CV})}\, d\mu(x) \tag{4.2}$$

guarantees that $\Delta < \infty$. As rational maps have finitely many critical values, Δ is finite.

The above conditions are also applicable to the dynamics of entire functions, provided one replaces critical values with singular values. Note however that an entire function can have infinitely many singular values, Δ need not be finite. For applications, see [Jov24].

5 A measurable atlas

By a *measurable atlas*, we mean a collection of measurable sets that cover a measure space up to a set of measure zero. Below, we describe convenient collections of UAL charts for $(\widehat{X}, \widehat{\mu})$ and $(\widehat{X}_\rho, \widehat{\mu}_\rho)$, which we will use throughout this paper.

Suppose $x \in X$ does not belong to a grand orbit of a critical point or a periodic point such that $|(F^{\circ n})'(x)| \to \infty$ as $n \to \infty$. (By Lemma 1.6, we are excluding a set of points of measure zero.) We write

$$\bigcirc = \bigcirc_x = B(x,\eta) \cap X \subset X$$

and

$$\square = \square_{x,t} = (B(x,\eta) \cap X) \times (e^{-\eta}t, t) \subset X \times \mathbb{R}_+.$$

Due to the above restrictions on the point $x \in X$, when the *scale* $0 < \eta < \eta_0(x)$ is sufficiently small,

$$F^{\circ n}(\square) \cap \square = \emptyset, \qquad \text{for any } n \geq 0, \tag{5.1}$$

and the sets $\{F^{-n}(\square) : n \geq 0\}$ are disjoint.

Let y be a repeated pre-image of x, i.e. a point in X such that $F^{\circ n}(y) = x$ for some $n \geq 0$. We write $\bigcirc_y$ for the connected component of $F^{-n}(\bigcirc)$ which contains y and $\square_y$ for the connected component of $f^{-n}(\square)$ which contains $(y, |(F^{\circ n})'(y)|^{-1})$. We say that y is "ε-linear at scale η" or"good" if for any $0 \leq k \leq n$, the branch of F^{-k} which takes $x \to F^{\circ(n-k)}y$ defines an ε-linear homeomorphism from $\bigcirc_x$ to $\bigcirc_{F^{\circ(n-k)}y}$. We denote the union of the boxes $\square_y$ associated to good repeated pre-images of x by $\widetilde{\square}_{\varepsilon,\eta} \subset X \times \mathbb{R}$.

Similarly, we say that a backward orbit $\mathbf{x}$ with $x_0 = x$ is "ε-linear at scale η" or "good" if for any $n \geq 0$, F^{-n} defines an ε-linear homeomorphism from $\bigcirc_{x_0}$ to $\bigcirc_{x_{-n}}$. We define $\bigcirc_{\mathbf{x}} \subset \widehat{X}$ as the set of inverse orbits $\mathbf{z}$ with $z_{-n} \in \bigcirc_{x_{-n}}$ and $\square_{\mathbf{x}} \subset \widehat{X} \times \mathbb{R}_+$ as the set of inverse orbits $(\mathbf{z}, \mathbf{t})$ with $(z_{-n}, t_{-n}) \in \square_{x_{-n}}$. Finally, we define $\widehat{\bigcirc}_{\varepsilon,\eta} \subset \widehat{\bigcirc}$ and $\widehat{\square}_{\varepsilon,\eta} \subset \widehat{\square}$ as unions of $\bigcirc_{\mathbf{x}}$ and $\square_{\mathbf{x}}$ over all good inverse orbits that start in $\bigcirc_x$. In view of (5.1), the natural projection from $\widehat{X} \times \mathbb{R}_+$ to $\widehat{X}_\rho$ is injective on $\widehat{\square}_{\varepsilon,\eta}$, so that $\widehat{\square}_{\varepsilon,\eta}$ is naturally a subset of $\widehat{X}_\rho$.

To summarize, for any fixed $\varepsilon > 0$, the sets $\{\widehat{\bigcirc}_{x_j,\eta_j}\}$ cover $\widehat{X}$ up to $\widehat{\mu}$ measure zero, while the sets $\{\widehat{\square}_{x_j,t_j,\eta_j}\}$ satisfying (5.1) cover $\widehat{X}_\rho$ up to $\widehat{\mu}_\rho$ measure zero.

5.1 Abundance of good pre-images

We now show that when the scale $\eta > 0$ is small, most pre-images are good:

Lemma 5.1. *Suppose $0 < \varepsilon < 1/10$ is fixed. For μ a.e. $x \in X$, we have*

$$\lim_{\eta \to 0} \frac{\widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta})}{\widehat{\mu}(\widehat{\bigcirc})} = 1,$$

where $\bigcirc = B(x,\eta) \cap X$ and $\widehat{\bigcirc}_{\varepsilon,\eta} \subset \widehat{X}$ is the union of all ε-linear inverse branches defined on $\bigcirc$.

Proof. For $r > 0$, we define $\mathcal{G}_r \subset \widehat{X}$ as the set of inverse orbits $\mathbf{y} = (y_{-n})_{n=0}^{\infty} \in \widehat{X}$ for which inverse iteration is ε-linear on $B(y_0, r)$. It is clear from the definitions that the sets $\mathcal{G}_r$ are increasing as $r \to 0^+$. By non-uniform hyperbolicity, the union $\bigcup_{r>0} \mathcal{G}_r$ has full $\widehat{\mu}$ measure.

Restricting $\widehat{\mu}$ to $\mathcal{G}_r$ and projecting onto the 0-th coordinate, we obtain the measures $\nu_r = (\pi_0)_* \widehat{\mu}|_{\mathcal{G}_r}$, $r > 0$, on X. Evidently, as $r \to 0$, the measures ν_r increase to $(\pi_0)_*\widehat{\mu} = \mu$. In other words, for μ a.e. $x \in X$, the Radon-Nikodym derivatives $(d\nu_r/d\mu)(x)$ increase to 1.

By Lebesgue's differentiation theorem, for μ a.e. $x \in X$ and $\delta > 0$, there exists an $r = r(x,\delta) > 0$ so that

$$\lim_{\eta \to 0} \frac{\widehat{\mu}(\mathcal{G}_r \cap \widehat{\bigcirc})}{\mu(\bigcirc)} = \lim_{\eta \to 0} \frac{\nu_r(\bigcirc)}{\mu(\bigcirc)} = \frac{d\nu_r}{d\mu}(x) > 1 - \delta. \tag{5.2}$$

From the definitions, it is clear that if $0 < \eta < r/2$ and $\mathbf{y}$ is an inverse orbit with $y_0 \in \bigcirc$ such that inverse iteration under F is ε-linear on $B(y_0, r)$ along $\mathbf{y}$, then $\mathbf{y} \in \widehat{\bigcirc}_{\varepsilon,\eta}$. Consequently, for any $0 < \eta < r/2$ sufficiently small, we have

$$\frac{\widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta})}{\mu(\bigcirc)} > 1 - \delta.$$

The lemma follows since $\delta > 0$ was arbitrary. $\square$

Remark. We say that $x \in X$ is a *ε-Pesin point* if

$$\lim_{\eta \to 0} \frac{\widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta})}{\widehat{\mu}(\widehat{\bigcirc})} = 1$$

and a *strong Pesin point* if the above limit is 1 for any $\varepsilon > 0$. The above lemma shows that μ a.e. $x \in X$ is a strong Pesin point.

5.2 Uniformly continuous functions

In Section 2.3, we defined a hyperbolic metric on $X \times \mathbb{R}_+$. We now define a leafwise hyperbolic metric on $\widehat{X} \times \mathbb{R}_+$. Namely, suppose that $\mathbf{z}(\cdot)$ is a UAL inverse branch on U, $x_0, y_0 \in U$ and $\mathbf{x} = \mathbf{z}(x_0), \mathbf{x} = \mathbf{z}(y_0)$. We set

$$\begin{aligned} d_{\widehat{X}\times\mathbb{R}_+}\big((\mathbf{x},\mathbf{s}),\,(\mathbf{y},\mathbf{t})\big) &:= \lim_{n\to\infty} d_{X\times\mathbb{R}_+}\big((x_{-n}, s_{-n}),\,(y_{-n}, t_{-n})\big) \\ &= \lim_{n\to\infty} d_{\mathbb{H}}(is_{-n}, |x_{-n} - y_{-n}| + it_{-n}). \end{aligned}$$

Lemma 5.2. *Consider the set of functions $h : X \times \mathbb{R}_+ \to \mathbb{R}$ which are uniformly continuous in the hyperbolic metric of $X \times \mathbb{R}_+$. Their natural extensions $\widehat{h}$ to $\widehat{X}_\rho$ are dense in $L^2(\widehat{X}_\rho, \widehat{\mu}_\rho)$.*

From the definition above, it is clear that the natural extensions $\widehat{h}$ are uniformly continuous functions on the leaves of $\widehat{X}_\rho$.

Proof. Since $\widehat{X}_\rho$ is covered by countably many product charts $\widehat{\square}_{\varepsilon,\eta}$ up to a set of measure zero, it suffices to approximate any function $h \in L^2(\widehat{X}_\rho, \widehat{\mu})$ which is supported on finitely many such product charts, and thus any function $h \in L^2(\widehat{X}_\rho, \widehat{\mu})$ supported on a single product chart $\widehat{\square}_{\varepsilon,\eta}$.

For a point $(w, r) \in X \times \mathbb{R}_+$, we write

$$T(w,r) \;=\; \big\{(\mathbf{w},\mathbf{r}) \,:\, w_0 = w,\, r_0 = r\big\} \;\in\; \widehat{X} \times \mathbb{R}$$

for the "fiber" of inverse orbits under $\widehat{F}$ which start at (w, r). Naturally, $T_{\varepsilon,\eta}(w, r)$ denotes the intersection of $T(w, r)$ and $\widehat{\square}_{\varepsilon,\eta}$.

Since $\widehat{\square}_{\varepsilon,\eta}$ is a product over $\square$, the measure $\widehat{\mu}|_{\widehat{\square}_{\varepsilon,\eta}}$ disintegrates into conditional measures $\nu_{w,r}$ on the fibers $T_{\varepsilon,\eta}(w, r)$, indexed by $(w, r) \in \square$. This means that

$$\widehat{\mu}|_{\widehat{\square}_{\varepsilon,\eta}}(E) = \int_\square \nu_{w,r}(E \cap T(w,r))\, \frac{d\mu\, dt}{t},$$

for any measurable set $E \subset \widehat{\square}_{\varepsilon,\eta}$. More generally, for a good repeated pre-image y of x, we write $\widehat{\square}_{y,\varepsilon,\eta} \subset \widehat{\square}_{\varepsilon,\eta}$ for the subset of inverse orbits that pass through $\square_y$. By the same reasoning, the measure $\widehat{\mu}|_{\widehat{\square}_{y,\varepsilon,\eta}}$ disintegrates into the conditional measures

$\nu_{w,r}$ indexed by $(w,r) \in \square_y$ on the fibers $T_{\varepsilon,\eta}(w,r)$, which consist of inverse orbits in $\widehat{\square}_{\varepsilon,\eta}$ that pass through (w,r).

We define a sequence of weakly almost-invariant functions $h_n : X \times \mathbb{R}_+ \to \mathbb{R}$, supported on $\widetilde{\square}_{\varepsilon,\eta}$. We first define h_n on the boxes $\square_y$, where y ranges over the repeated pre-images of x of order n, by averaging h over good inverse branches:

$$h_n(w,r) = \frac{1}{\nu_{w,r}(T_{\varepsilon,\eta}(w,r))} \int_{T_{\varepsilon,\eta}(\xi,r)} h(\mathbf{w},r) d\nu_{w,r}, \qquad (w,r) \in \square_y.$$

We then extend h_n by backwards invariance to the pre-images $\square_z$ of $\square_y$ which belong to $\widetilde{\square}_{\varepsilon,\eta}$. By the L^2 martingale convergence theorem, the natural extensions $\widehat{h}_n(z,r)$ convege to $\widehat{h}$.

To get an approximation by uniformly continuous functions, we first use Lusin's theorem to tweak h_n on the boxes $\square_y$ before extending h_n to the boxes $\square_z$. $\square$

6 When is the suspension flow mixing?

In this section, we show the following theorem:

Theorem 6.1. *Let (X, F, μ) be a non-uniformly hyperbolic dynamical system, where μ is an ergodic probability measure on X with a positive Lyapunov exponent. The suspension flow $g_t : \widehat{X}_\rho \to \widehat{X}_\rho$ is mixing if and only if there exists a measurable function $w : X \to \partial\mathbb{D}$ and an $a \in \mathbb{R} \setminus \{0\}$ such that*

$$w(F(x)) = e^{ia \log |F'(x)|} w(x), \qquad \mu \text{ a.e. } x \in X. \tag{6.1}$$

6.1 Basic observations

Let U be a connected open set which intersects X. We define $\widehat{U}_{\mathrm{UAL}} \subset \widehat{U}$ as the union of the inverse branches on which backward iteration is univalent and asymptotically linear. Similarly, for $0 < \varepsilon < 1/10$, we define $\widehat{U}_{\varepsilon\text{-linear}} \subset \widehat{U}_{\mathrm{UAL}}$ as the union of the inverse branches on which backward iteration is ε-linear. We define the *Pesin set* $X_{\mathrm{lin}} \subset X$ as the union of relatively open subsets $X \cap U \subset X$ for which $\widehat{\mu}(\widehat{U}_{\mathrm{UAL}}) > 0$. Alternatively, the Pesin set could have been defined using ε-linearity. From Section 5, we know that these cover X up to a set of measure 0, i.e. $\mu(X_{\mathrm{lin}}) = \mu(X) = 1$.

The following lemma is essentially due to A. Zdunik [Zdu90, Lemma 2], see also [FU00, Lemma 3]:

Lemma 6.2. *After redefining on a set of μ measure zero, any solution w of the functional equation (6.1) is continuous on the Pesin set.*

Sketch of proof. Let U be a connected open set such that $\widehat{\mu}(\widehat{U}_{\mathrm{UAL}}) > 0$. Pick an arbitrary point $x \in X \cap U$. For a UAL inverse branch $\mathbf{z} = (z_{-n}(\cdot))_{n=0}^{\infty}$ defined on U, form the rescaling limit $\Phi_{\mathbf{x}}$, normalized along the inverse orbit $\mathbf{x} = \mathbf{z}(x)$. By the functional equation, for any $y \in X \cap U$, $\mathbf{y} = \mathbf{z}(y)$ and $n \geq 0$, we have

$$\arg \frac{w(y)}{w(x)} = \arg \frac{w(y_{-n})}{w(x_{-n})} + a \cdot \log \frac{|(F^{\circ n})'(y_{-n})|}{|(F^{\circ n})'(x_{-n})|}.$$

An argument involving Lusin's theorem (see the references) shows that there is a subsequence of integers $n_j \to \infty$ such that $\arg \frac{w(y_{-n_j})}{w(x_{-n_j})} \to 0$. Taking the limit along this subsequence, we see that

$$\arg \frac{w(y)}{w(x)} \;=\; a \cdot \log \frac{|\Phi'_{\mathbf{x}}(y)|}{|\Phi'_{\mathbf{x}}(x)|} \;=\; a \cdot \log |\Phi'_{\mathbf{x}}(y)|.$$

From the above equation, it is clear that $w(y)$ is continuous on $X \cap U$. $\square$

Lemma 6.3. *If the functional equation (6.1) has a solution, then the geodesic flow $g_t : \widehat{X}_\rho \to \widehat{X}_\rho$ is not mixing.*

Proof. Since the value of $\arg w(x)$ is defined modulo 2π, the value of $(1/a) \arg w(x)$ is defined modulo $2\pi/a$. By the functional equation for w, the set

$$A = \big\{(x,t) \in X \times \mathbb{R}_+ : (1/a) \arg w(x) - \log t \in [0, \pi/a] \mod 2\pi/a\big\}$$

is invariant under $\widehat{F} : X \times \mathbb{R}_+ \to X \times \mathbb{R}_+$. An examination of the proof of Theorem 2.3 shows that if $g_t : \widehat{X}_\rho \to \widehat{X}_\rho$ is mixing, then for any connected open set $U \subset \mathbb{C}$ and $\delta > 0$, the limit

$$\lim_{T \to \infty} \frac{1}{\delta} \int_T^{T+\delta} \int_{X \cap U} \chi_A(x, e^{-T}) d\mu \tag{6.2}$$

exists. It is clear from the definition of the set A that the function

$$T \to \int_{X \cap U} \chi_A(x, e^{-T}) d\mu$$

is periodic with period $2\pi/a$.

By choosing U appropriately, we can ensure that the limit in (6.2) does not exist. For this purpose, let V be a connected set such that $\widehat{\mu}(\widehat{V}_{\mathrm{UAL}}) > 0$. Since w is continuous on $X \cap V$ by Lemma 6.2, we can take U to be an open subset of V such that $\mu(\widehat{U}_{\mathrm{UAL}}) > 0$ and $|w(x_1) - w(x_2)| < \pi/(5a)$ for $x_1, x_2 \in X \cap U$. If $\delta = \pi/(5a)$, then for some values of T,

$$\frac{1}{\delta} \int_T^{T+\delta} \int_{X \cap U} \chi_A(x, e^{-T}) d\mu$$

will be zero, while for other values of T, it will be positive (and strictly bounded below by periodicity). □

6.2 Non-alignment implies mixing

Suppose $\mathbf{z}$ and $\mathbf{z}'$ are two univalent asympotically linear (UAL) inverse branches defined on a connected open set U which intersects X. We say that $\mathbf{z}$ and $\mathbf{z}'$ are *aligned* on X if the rescaling maps $F_{\mathbf{x}}^{-\infty}(z) = F_{\mathbf{x}'}^{-\infty}(z)$, $z \in X \cap U$, where $\mathbf{x} = \mathbf{z}(x_0)$ and $\mathbf{x}' = \mathbf{z}'(x_0)$ are inverse orbits that start at $x_0 \in X \cap U$. We say that $\mathbf{z}$ and $\mathbf{z}'$ are *absolute-value aligned* if the weaker condition $|F_{\mathbf{x}}^{-\infty}(z)| = |F_{\mathbf{x}'}^{-\infty}(z)|$ holds for $z \in X \cap U$. In this section, we show:

Theorem 6.4. *If a positive measure of UAL inverse branches are not absolute-value aligned, then the suspension flow on $\widehat{X}_\rho$ is mixing.*

To prove Theorem 6.4, we use an argument due to M. Babillot based on [Bab02, Lemma 1]. In our setting, Babillot's lemma can be stated as follows: let $h \in L^2(\widehat{X}_\rho, \widehat{\mu}_\rho)$. If $h \circ g_t$ does not converge weakly to a constant function (as t tends to $\pm\infty$), then there is a sequence of times $t_n \to \infty$ and a non-constant function $\psi \in L^2(\widehat{X}_\rho)$ so that

$$h \circ g_{t_n} \to \psi \qquad \text{and} \qquad h \circ g_{-t_n} \to \psi,$$

weakly in $L^2(\widehat{X}_\rho, \widehat{\mu}_\rho)$ as $n \to \infty$.

In light of the aforementioned density result, Theorem 6.4 reduces to showing the following statement:

Lemma 6.5. *In the setting of Theorem 6.4, suppose that $\widehat{h} : \widehat{X}_\rho \to \mathbb{R}$ is the natural extension of a function $h : X \times \mathbb{R}_+ \to \mathbb{R}$ that is uniformly continuous in the hyperbolic metric of $\widehat{X} \times \mathbb{R}_+$. If for some sequence of real numbers $t_n \to \infty$ and $\psi \in L^2(\widehat{X}_\rho, \widehat{\mu}_\rho)$,*

$$\widehat{h} \circ g_{t_n} \to \psi \qquad \textit{and} \qquad \widehat{h} \circ g_{-t_n} \to \psi,$$

then ψ is constant.

Let U be a connected open set which intersects X with $\widehat{\mu}(\widehat{U}_{\mathrm{UAL}}) > 0$. Below, x_0, y_0 will denote two points in $X \cap U$. We will generally fix x_0 and let y_0 vary. For a UAL inverse branch $\mathbf{z}(\cdot) = (z_{-n}(\cdot))_{n=0}^{\infty}$ defined on U, we write $\mathbf{x} = \mathbf{z}(x_0)$, $\mathbf{y} = \mathbf{z}(y_0)$ and $\Delta_{\mathbf{x}}(y_0) = |(F_{\mathbf{x}}^{-\infty})'(y_0)|$. By construction, the function $y_0 \to \Delta_{\mathbf{x}}(y_0)$ is continuous on $X \cap U$ and is normalized so that $\Delta_{\mathbf{x}}(x_0) = 1$.

Lemma 6.6. *Let $\{t_n\}$ be the sequence of real numbers and ψ be the limit function from Lemma 6.5. After redefining ψ on a set of $\widehat{\mu}$ measure zero if necessary, the following holds: for any univalent inverse branches $\mathbf{z}, \mathbf{z}' \subset \widehat{U}_{\mathrm{UAL}}$, we have:*

$$\psi\left(\mathbf{x}, \frac{\Delta_{\mathbf{x}'}(y_0)}{\Delta_{\mathbf{x}}(y_0)}\, t\right) = \psi(\mathbf{x}, t), \qquad t > 0, \qquad x_0, y_0 \in X \cap U. \tag{6.3}$$

Proof. Step 1. Using the forward geodesic flow. If $\widehat{h}$ is leafwise uniformly continuous in the hyperbolic metric on $\widehat{X}_\rho$, then so are the functions $\widehat{h} \circ g_{\pm t_n}$. After redefining on a set of measure zero, the limit ψ is also leafwise uniformly continuous with the same modulus of continuity. As the points $(\mathbf{x}, t)$ and $(\mathbf{y}, \Delta_{\mathbf{x}}(y_0)t)$ are forward-asymptotic under the geodesic flow:

$$\lim_{s \to \infty} d_{\mathrm{hyp}}\big((\mathbf{x}, e^s t),\, (\mathbf{y}, \Delta_{\mathbf{x}}(y_0) e^s t)\big) = 0,$$

we have

$$\psi(\mathbf{x}, t) = \psi(\mathbf{y}, \Delta_{\mathbf{x}}(y_0)t), \qquad t > 0. \tag{6.4}$$

A similar argument shows that

$$\psi(\mathbf{x}', t) = \psi(\mathbf{y}', \Delta_{\mathbf{x}'}(y_0)t), \qquad t > 0. \tag{6.5}$$

Step 2. Using the backward geodesic flow. With help of the sets $A(\varepsilon,\rho) \subset \widehat{X}_\rho$ from Section 2.2, it is not difficult to show that for $\widehat{\mu} \times (dt/t)$ a.e. $(\mathbf{x}, r) \in \widehat{X} \times \mathbb{R}_+$,

$$\lim_{n\to\infty} g_{-t_n}\widehat{h}(\mathbf{x}, r) = \lim_{n\to\infty} \widehat{h}(\mathbf{x}, re^{-t_n}) = \lim_{n\to\infty} h(x, re^{-t_n}).$$

In other words, except on a set of $\widehat{\mu} \times (dt/t)$ measure zero,

$$\psi(\mathbf{x}, r) = \psi(x_0, r) \tag{6.6}$$

only depends on the 0-th coordinate of $\mathbf{x}$.

Step 3. Conclusion. Putting (6.4),(6.5) and (6.6) together, we obtain

$$\psi\left(\mathbf{x}, \frac{\Delta_{\mathbf{x}'}(y_0)}{\Delta_{\mathbf{x}}(y_0)}\, t\right) \;=\; \psi\big(\mathbf{y}, \Delta_{\mathbf{x}'}(y_0)\, t\big) \;=\; \psi\big(\mathbf{y}', \Delta_{\mathbf{x}'}(y_0)\, t\big) \;=\; \psi(\mathbf{x}', t) \;=\; \psi(\mathbf{x}, t),$$

which is what we wanted to show. $\square$

We are now ready to show Theorem 6.4:

Proof of Theorem 6.4. Let $x_0 \in X$ and suppose that the inverse branch $\mathbf{z}'$ is not absolute-valued aligned with $\mathbf{z}$ on any neighbourhood U of x_0. As the function $y_0 \to \frac{\Delta_{\mathbf{x}'}(y_0)}{\Delta_{\mathbf{x}}(y_0)}$ attains values arbitrarily close but not equal to 1, the periods

$$\frac{\Delta_{\mathbf{x}'}(y_0)}{\Delta_{\mathbf{x}}(y_0)}, \qquad y_0 \in U,$$

generate a dense subgroup group $(\mathbb{R}_+, \cdot)$. Lemma 6.6 tells us that ψ is constant on the fiber

$$(\widehat{U}_{\mathrm{UAL}} \times \mathbb{R}_+) \cap \big\{(\mathbf{z}, t) \in \widehat{X}_\rho : z_0 = x_0,\, t \in \mathbb{R}_+\big\}.$$

By (6.4), ψ is constant on $\widehat{U}_{\mathrm{UAL}} \times \mathbb{R}_+$, while by (6.6), ψ is constant on $\widehat{U} \times \mathbb{R}_+$. Since $\widehat{\mu}_\rho$ a.e. point in $\widehat{X}_\rho$ is equivalent to a point in $\widehat{U} \times \mathbb{R}_+$, ψ is constant on $\widehat{X}_\rho$ as desired. $\square$

6.3 Mixing and the functional equation

Proof of Theorem 6.1. Lemma 6.3 says that if the functional equation (6.1) has a solution, then the flow $g_t : \widehat{X}_\rho \to \widehat{X}_\rho$ is not mixing. It remains to show the converse: namely, if the flow is not mixing, then the functional equation (6.1) has a solution $w : X \to \partial\mathbb{D}$.

Let ψ be the limit function from Lemma 6.5 as before. We lift $\psi : \widehat{X}_\rho \to \mathbb{R}$ to a function defined on $\widehat{X} \times \mathbb{R}_+$. We say that $\lambda > 0$ is a *period* of ψ on the fiber $\{\mathbf{x}\} \times \mathbb{R}_+ \subset \widehat{X} \times \mathbb{R}_+$ if

$$\psi(\mathbf{x}, \lambda t) = \psi(\mathbf{x}, t), \qquad \text{for all } t > 0.$$

We write $\mathrm{Per}(\mathbf{x}) \subset \mathbb{R}_+$ for the set of periods of ψ on $\{\mathbf{x}\} \times \mathbb{R}_+$. Since ψ is continuous, $\mathrm{Per}(\mathbf{x})$ forms a closed multiplicative subgroup of $\mathbb{R}_+$. We say that $\lambda > 0$ is an *essential period* of $\psi : \widehat{X} \times \mathbb{R}_+ \to \mathbb{R}$ if the above equation holds for $\widehat{\mu}$ a.e. inverse orbit $\mathbf{x} \in \widehat{X}$. We denote the essential periods of ψ by $\mathrm{Per}(\psi)$.

Step 1. Suppose U is a connected open set such that $\widehat{\mu}(\widehat{U}_{\varepsilon\text{-linear}}) > 0$. We use the following notation: $\mathbf{z}, \mathbf{z}'$ are ε-linear inverse branches defined on U, x_0, y_0 are two points in $X \cap U$ and $\mathbf{x} = \mathbf{z}(x_0), \mathbf{y} = \mathbf{z}(y_0), \mathbf{x}' = \mathbf{z}'(x_0), \mathbf{y}' = \mathbf{z}'(y_0)$. We make the following simple observations:

1. Since $\psi : \widehat{X} \times \mathbb{R}_+ \to \mathbb{R}$ is $\widehat{F}$-invariant, $\mathrm{Per}(\mathbf{x}) = \mathrm{Per}(\widehat{F}(\mathbf{x}))$.
2. With help of the forward geodesic flow (6.4), it readily follows that $\mathrm{Per}(\mathbf{x}) = \mathrm{Per}(\mathbf{y})$.
3. Meanwhile, the backward geodesic flow (6.6) tells us that for $\widehat{\mu}$ a.e. ε-linear inverse branches $\mathbf{z}, \mathbf{z}'$ defined on U, $\mathrm{Per}(\mathbf{x}) = \mathrm{Per}(\mathbf{x}')$.

Observations 2 and 3 say that $\mathrm{Per}(\mathbf{x})$ is the same for $\widehat{\mu}$ a.e. $\mathbf{x} \in \widehat{\mu}(\widehat{U}_{\varepsilon\text{-linear}})$. By Observation 1 and ergodicity, $\mathrm{Per}(\mathbf{x})$ is the same for $\widehat{\mu}$ a.e. $\mathbf{x} \in \widehat{X}$.

Step 2. Let $\mathbf{z}$ be an ε-linear inverse branch defined on U. By the $\widehat{F}$-invariance of $\psi : \widehat{X} \times \mathbb{R}_+ \to \mathbb{R}$ and (6.4), it follows that if $x_0, x_{-n} = z_{-n}(x_0) \in X \cap U$ then

$$\Delta_{\mathbf{x}}(x_{-n}) |(F^{\circ n})'(x_{-n})| \in \mathrm{Per}(\mathbf{x}). \tag{6.7}$$

In this step, we show that ψ has a non-trivial essential period. By the Poincaré recurrence theorem, for $\widehat{\mu}$ a.e. $\mathbf{x} \in \widehat{U}_{\varepsilon\text{-linear}}$, there is an increasing sequence of integers $n_j \to \infty$ such that $\widehat{F}^{-n_j}(\mathbf{x})$ lands in $\widehat{U}_{\varepsilon\text{-linear}}$. As one of the requirements in the definition of an ε-linear inverse branch, the derivative $|(F^{\circ n})'(x_{-n})| \to \infty$ as $n \to \infty$. Since $e^{1-\varepsilon} < \Delta_{\mathbf{x}}(x_{-n}) < e^{1+\varepsilon}$, the period constructed in (6.7) is non-trivial when n_j is large.

Step 3. If the set of essential periods of ψ is all of $\mathbb{R}_+$, then ψ is constant, being a uniformly continuous function. Otherwise, the set of essential periods is $\{\lambda^n : n \in \mathbb{Z}\}$ for some $\lambda > 1$. We choose $a > 0$ so that $\lambda = \exp(2\pi/a)$. To construct a solution of the functional equation (6.1), we fix a connected open set U such that $\widehat{\mu}(\widehat{U}_{\varepsilon\text{-linear}}) > 0$ and a point $x_0 \in X \cap U$. We first define w on $X \cap U$ by

$$w(y_0) = e^{ia\Delta_{\mathbf{x}}(y_0)}$$

and then use the functional equation to extend w to $\bigcup_{n\geq 0} F^{-n}(X \cap U) \subset X$. By ergodicity, the inverse images cover a set of full μ measure. Equation (6.7) guarantees that w is well-defined. □

Part II

7 A rough estimate

In this section, we give a rough upper bound for the number of repeated pre-images of a point $x \in X$:

Lemma 7.1. *Under the assumptions* (OC6) *and* (OC7)*, there exists a constant $C > 0$ so that*

$$n(x, T) \leq C e^{\alpha T} \gamma(x),$$

for any point $x \in X$ with $\gamma(x) < \infty$ and $T > 0$.

Proof. For the convenience of exposition, we assume that x does not belong to a grand orbit of a periodic point, so that the repeated pre-images of x form a tree, which we call $\mathscr{T}(x)$.

For a vertex $y \in \mathscr{T}(x)$, we write N_y for the iterate of F which takes y to x, so that $F^{\circ N_y}(y) = x$. Fix a $T > 0$. We say that a vertex y is *heavy* if $|(F^{\circ N_y})'(y)| < e^T$ and *light* otherwise. Our goal is to show the number of heavy vertices is bounded by $Ce^{\alpha T}\gamma(x)$.

Let $\mathscr{T}(x,T) \subset \mathscr{T}(x)$ be the minimal subtree which contains the root x and the heavy vertices. We classify vertices $y \in \mathscr{T}(x,T)$ as leaves, ordinary vertices and joints:

- y is a *leaf* if it has no heavy descendants.
- y is an *ordinary* vertex if it has only one child with heavy descendants.
- y is a *joint* if it has at least two children which have heavy descendants.

It is well known (and not difficult to see) that the Radon-Nikodym derivative γ of invariant and conformal measures satisfies the transfer identity

$$\gamma(y) = \sum_{F(z)=y} |F'(z)|^{-\alpha}\gamma(z). \tag{7.1}$$

In terms of the function $\Gamma(y) := |(F^{\circ N_y})'(y)|^{-\alpha}\gamma(y)$, the above identity takes the form

$$\Gamma(y) = \sum_{F(z)=y} \Gamma(z). \tag{7.2}$$

Repeatedly expanding (7.2) shows

$$\gamma(x) \;=\; \Gamma(x) \;\le \sum_{y\in\partial\mathscr{T}(x,T)} \Gamma(y),$$

so that the number of heavy leaves is at most $e^{\alpha T}\gamma(x)/c$, where c is the constant from (OC6). Inspection shows that the number of joints is bounded above by the number of heavy leaves minus 1, and so is also $\le e^{\alpha T}\gamma(x)/c$.

It remains to estimate the number of ordinary heavy vertices. Let y be an ordinary heavy vertex and z be the (unique) child of y which has a heavy descendent. The equation

$$\Gamma(y) = \Gamma(z) + |(F^{\circ N_y})'(y)|^{-\alpha} \sum_{\substack{F(w)=y \\ w\neq z}} |F'(w)|^{-\alpha}\gamma(w)$$

implies that $\Gamma(y) \geq \Gamma(z) + Ke^{-\alpha T}$, where $K = cc' > 0$ is the product of the constants from the assumptions (OC6) and (OC7). An induction argument shows that $\gamma(x) = \Gamma(x)$ is at least $Ke^{-\alpha T}$ times the number of heavy ordinary vertices. Consequently, the number of heavy ordinary vertices is bounded above by $e^{\alpha T}\gamma(x)/K$. Putting the above estimates together proves the lemma. □

8 Orbit Counting up to a Cesàro average

In this section, we show Theorem 1.1, which says that if a dynamical system $F : X \to X$ satisfies the hypotheses (OC1)–(OC7) from the introduction, then the Orbit Counting Theorem holds up to a Cesàro average. The idea is to count good and bad pre-images separately. Given $0 < \varepsilon < 1/10$, $\eta > 0$ and $x \in X$,

$$n_{\varepsilon,\eta}(x,T) = \#\big\{(n \geq 0,\, y \in X) \,:\, F^{\circ n}(y) = x,\, \log|(F^{\circ n})'(y)| < T,\, (\varepsilon,\eta)\text{-good}\big\},$$

counts repeated pre-images of x that are ε-linear at scale η. Naturally, we set $n_{(\varepsilon,\eta)\text{-bad}}(x,T) := n(x,T) - n_{\varepsilon,\eta}(x,T)$. The proof of Theorem 1.1 splits into two lemmas:

Lemma 8.1. *Fix the threshold of distortion $0 < \varepsilon < 1/10$. For μ a.e. $x \in X$, when the scale $0 < \eta < \eta_0(x,\varepsilon)$ is sufficiently small,*

$$n_{\varepsilon,\eta}(x,T) \sim_\varepsilon \frac{\gamma(x)}{\int_X \log|F'(x)|d\mu} \cdot e^{\alpha T},$$

for any $T > 0$ sufficiently large.

The notation "$A \sim_\varepsilon B$" indicates that there exists a constant $C > 0$, independent of $\varepsilon > 0$, so that $(1 - C\varepsilon)A \,\leq\, B \,\leq\, (1 + C\varepsilon)A$.

Lemma 8.2. *Fix the threshold of distortion $0 < \varepsilon < 1/10$. For μ a.e. $x \in X$ and $\theta > 0$, when the scale $0 < \eta < \eta_0(x,\varepsilon,\theta)$ is sufficiently small,*

$$n_{(\varepsilon,\eta)\text{-bad}}(x,T) \leq \theta \cdot \gamma(x)e^{\alpha T}, \qquad T > 0.$$

8.1 Some assumptions on $x \in X$

Since we only want to prove the Orbit Counting Theorem for μ a.e. $x \in X$, we may make the following assumptions on x:

A. By Lemma 1.6, x does not belong to the grand orbit of a critical or periodic point.

B. By (OC4) and the ergodic theorem, the derivative

$$(F^{\circ n})'(x) \to \infty, \qquad \text{as } n \to \infty. \tag{8.1}$$

C. By Lebesgue's differentiation theorem,

$$\lim_{\eta \to 0} \frac{\mu(\bigcirc)}{m(\bigcirc)} = \gamma(x). \tag{8.2}$$

D. By Lemma 5.1, x is an ε-Pesin point:

$$\lim_{\eta \to 0} \frac{\widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta})}{\mu(\bigcirc)} = 1.$$

8.2 Asymptotic counting of good pre-images

We define $h : X \times \mathbb{R}_+ \to \mathbb{R}$ to be 1 on the ε-linear repeated pre-images of $\square$ under the dynamics of $F : X \times \mathbb{R}_+ \to X \times \mathbb{R}_+$ and 0 otherwise. From the definition, it is clear that $h(x,t)$ is a bounded weakly almost-invariant function, supported on the set $\widetilde{\square}_{\varepsilon,\eta}$. We denote its natural extension to $\widehat{X}_\rho$, described in (2.1), by $\widehat{h}(\mathbf{x},t)$. Since we have chosen $\eta > 0$ sufficiently small so that (5.1) holds, we have

$$\int_{\widehat{X}_\rho} \widehat{h}(\mathbf{x},t) d\widehat{\mu}_\rho = \widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta}) \cdot \eta.$$

Lemma 8.3. *The modified counting function*

$$\overline{n}_{\varepsilon,\eta}(x,T) := \sum_{\substack{n \geq 0,\, F^{\circ n}(y) = x,\, (\varepsilon,\eta)\text{-good} \\ \log|(F^{\circ n})'(y)| < T}} |(F^{\circ n})'(y)|^{-\alpha} \sim_\varepsilon \frac{\gamma(x)}{\alpha \int_X \log|F'(x)| d\mu}, \tag{8.3}$$

for any $T > 0$ sufficiently large.

Proof. Step 1. Using ergodicity. By Theorem 2.2, for μ a.e. $x \in X$, we have

$$\frac{1}{T}\int_0^T h(x, e^{-t})\, dt \to \frac{\widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta}) \cdot \eta}{\int_X \log|F'(x)| d\mu}, \qquad \text{as } T \to \infty.$$

As the measures μ and m are equivalent, the above identity also holds for m a.e. $x \in X$. Integrating with respect to m (and using the bounded convergence theorem), we get

$$\frac{1}{T}\int_X \int_0^T h(x, e^{-t})\, dt\, dm \to \frac{\widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta}) \cdot \eta}{\int_X \log|F'(x)| d\mu}, \qquad \text{as } T \to \infty. \tag{8.4}$$

Step 2. Left hand side of (8.4). From the definition of ε-linearity and the description of the function h, it is clear that the left hand side of (8.4) is bounded below by

$$\frac{\eta}{T} \sum_{\substack{n\geq 0,\, F^{\circ n}(y)=x,\, (\varepsilon,\eta)\text{-good} \\ \log|(F^{\circ n})'(y)|<T-\varepsilon-\eta}} m(\bigcirc_y)$$

and bounded above by

$$\frac{\eta}{T} \sum_{\substack{n\geq 0,\, F^{\circ n}(y)=x,\, (\varepsilon,\eta)\text{-good} \\ \log|(F^{\circ n})'(y)|<T+\varepsilon+\eta}} m(\bigcirc_y).$$

Since m is a conformal measure of dimension α, we have

$$m(\bigcirc_y) = |(F^{\circ n})'(y)|^{-\alpha} \cdot m(\bigcirc_x).$$

As a result,

$$\frac{\overline{n}_{(\varepsilon,\eta)\text{-good}}(x, T-\varepsilon-\eta)}{T} \leq \frac{\text{LHS}}{\eta \cdot m(\bigcirc_x)} \leq \frac{\overline{n}_{(\varepsilon,\eta)\text{-good}}(x, T+\varepsilon+\eta)}{T},$$

for any $T > T_0(x, \varepsilon, \eta)$ sufficiently large.

Step 3. Right hand side of (8.4). By assumptions C and D on the point $x \in X$, by asking $\eta > 0$ to be sufficiently small,

$$\frac{\text{RHS}}{\eta \cdot m(\bigcirc_x)} = \frac{\mu(\bigcirc_x)/m(\bigcirc_x)}{\int_X \log|F'(x)| d\mu} \cdot \frac{\widehat{\mu}(\widehat{\bigcirc}_{x,\epsilon,\eta})}{\mu(\bigcirc_x)}$$

can be made as close as we want to

$$\frac{\gamma(x)}{\int_X \log|F'(x)|d\mu}.$$

The proof is complete. $\square$

Proof of Lemma 8.1. By the lemma above,

$$\begin{aligned}
\frac{1}{T}\int_0^T \frac{n_{\varepsilon,\eta}(x,S)}{e^{\alpha S}}dS &= \frac{1}{T}\sum_{\substack{n\geq 0,\, F^{\circ n}(y)=x,\, (\varepsilon,\eta)\text{-good} \\ \log|(F^{\circ n})'(y)|<T}} \int_{\log|(F^{\circ N_y})'(y)|}^T e^{-\alpha S}dS \\
&= \frac{1}{\alpha T}\sum_{\substack{n\geq 0,\, F^{\circ n}(y)=x,\, (\varepsilon,\eta)\text{-good} \\ \log|(F^{\circ n})'(y)|<T}} (e^{-\alpha\log|(F^{\circ N_y})'(y)|} - e^{-\alpha T}) \\
&= \frac{1}{\alpha T}\sum_{\substack{n\geq 0,\, F^{\circ n}(y)=x,\, (\varepsilon,\eta)\text{-good} \\ \log|(F^{\circ n})'(y)|<T}} e^{-\alpha\log|(F^{\circ N_y})'(y)|} + o(1),
\end{aligned}$$

where in the last step we used the a priori estimate (Lemma 7.1) to estimate the number of terms. $\square$

8.3 Estimating the number of bad pre-images

As in Section 7, we denote the set of repeated pre-images y of x by $\mathscr{T}(x)$. Naturally, we denote the subset of repeated pre-images of x which are ε-linear at scale η by $\mathscr{T}_{\varepsilon,\eta}(x)$. The set $\mathscr{T}_{\varepsilon,\eta}(x)$ may be alternatively described as

$$\mathscr{T}_{\varepsilon,\eta}(x) = \mathscr{T}(x) \setminus \bigsqcup_{y\in\text{max-bad}(x)} \mathscr{T}(y),$$

where the sum is over the maximal bad repeated pre-images (we say that y is a *maximal bad repeated pre-image* of x if y is a bad pre-image but $F(y)$ is a good pre-image).

Lemma 8.4. *Fix the threshold of distortion $0 < \varepsilon < 1/10$. For μ a.e. $x \in X$ and $\delta > 0$, we can choose the scale $\eta = \eta(x,\delta) > 0$ sufficiently small, so that the ratio*

$$\frac{\sum_{y\in\text{max-bad}(x)}\Gamma(y)}{\Gamma(x)} = \frac{1}{\gamma(x)}\cdot\sum_{y\in\text{max-bad}(x)}\gamma(y)|F'(y)|^{-\alpha} < \delta. \qquad (8.5)$$

Proof. For $x \in X$, let $F_x \subset \widehat{X}$ be the fiber (or transversal) of inverse orbits $\mathbf{x} = (x_{-n})_{n=0}^{\infty}$ that start at $x_0 = x$. If y is a repeated pre-image of x, we write $F_{x,y} \subset F_x$ for the set of inverse orbits that start at x and pass through y. For m a.e. $x \in X$, we define a measure γ_x on $F_x \subset \widehat{X}$ of mass $\gamma(x) = \Gamma(x)$ such that $\gamma_x(F_{x,y}) = \Gamma(y)$.

For a point $x \in X$ and $\varepsilon, \eta > 0$, we have

$$\begin{aligned}\widehat{\mu}(\widehat{\bigcirc}_{x,\varepsilon,\eta}) &= \lim_{n\to\infty} \sum_{F^{\circ n}(y)=x,\,(\varepsilon,\eta)\text{-good}} \mu(\bigcirc_y) \\ &= \int_{\bigcirc_x} \gamma_{x'}(F_{x'} \cap \widehat{\bigcirc}_{x,\varepsilon,\eta})dm(x').\end{aligned}$$

Meanwhile,

$$\mu(\bigcirc_x) \;=\; \widehat{\mu}(\widehat{\bigcirc}_x) \;=\; \int_{\bigcirc_x} \gamma_{x'}(F_{x'})dm(x').$$

Together with Condition D, this implies that for most (with respect to m) $x' \in \bigcirc_x$, we have

$$\frac{\gamma_{x'}(F_{x'} \cap \widehat{\bigcirc}_{x,\varepsilon,\eta})}{\gamma_{x'}(F_{x'})} \approx 1,$$

i.e.

$$\frac{\sum_{y'\in\text{max-bad}(x')} \Gamma(y')}{\Gamma(x')} \;=\; \frac{\gamma_{x'}(F_{x'} \setminus \widehat{\bigcirc}_{x,\varepsilon,\eta})}{\gamma_{x'}(F_{x'})} \;\approx\; 0,$$

as desired. ∎

Proof of Lemma 8.2. By Lemmas 7.1 and 8.4, we have

$$\begin{aligned}n_{(\varepsilon,\eta)\text{-bad}}(x,T) &= \sum_{y\in\text{max-bad}(x)} n\big(y, T - \log|(F^{\circ N_y})'(y)|\big) \\ &\le C \sum_{y\in\text{max-bad}(x)} \gamma(y)e^{\alpha(T-\log|(F^{\circ N_y})'(y)|)} \\ &= Ce^{\alpha T} \sum_{y\in\text{max-bad}(x)} \gamma(y)|(F^{\circ N_y})'(y)|^{-\alpha} \\ &= Ce^{\alpha T} \sum_{y\in\text{max-bad}(x)} \Gamma(y) \\ &= C\delta\cdot\gamma(x)e^{\alpha T},\end{aligned}$$

as desired. ∎

9 Mixing implies Orbit Counting

In this section, we show:

Theorem 9.1. *Let $F : X \to X$ be a dynamical system that satisfies the hypotheses* (OC1)–(OC7) *from the introduction. If the suspension flow g_t on $\widehat{X}_\rho$ is mixing with respect to the product measure $\widehat{\mu} \times (dt/t)$, then the Orbit Counting Theorem holds.*

For a point $x \in X$ and real numbers $0 < T_1 < T_2$, we write

$$n(x, T_1, T_2) = \#\big\{(n \geq 0,\, y \in X) \,:\, F^{\circ n}(y) = x,\, T_1 < \log|(f^{\circ n})'(y)| < T_2\big\}.$$

To prove the Orbit Counting Theorem, we show for a fixed window size $\delta > 0$ and μ a.e. $x \in X$,

$$n(x, T, T+\delta) \sim \left\{\int_T^{T+\delta} e^{\alpha t} dt\right\} \cdot \frac{\gamma(x)}{\int_X \log|F'(x)| d\mu}, \qquad \text{as } T \to \infty.$$

In fact, by cutting up the interval of length δ into smaller pieces, it is enough to prove the seemingly weaker statement

$$n(x, T, T+\delta) \sim_\delta \frac{\gamma(x)}{\int_X \log|F'(x)| d\mu} \cdot \delta \cdot e^{\alpha T}, \qquad \text{as } T \to \infty, \tag{9.1}$$

where the notation $\sim_\delta$ means that the ratio between the two quantities tends to 1 as $\delta \to 0^+$.

As before, we assume that the point x satisfies the Conditions A, B, C and D from Section 8.1. When applying the mixing of the suspension flow, we fix two constants $0 < \varepsilon, \eta \ll \delta$. Eventually, we will take $\varepsilon, \eta \to 0$.

Proof of Theorem 9.1. Applying Theorem 2.3 to the function $h(x,t)$ from Section 8.2, we get

$$(m \times dt/t)\big((X \times [e^{-(T+\delta)}, e^{-T}]) \cap \widetilde{\square}_{(\varepsilon,\eta)\text{-good}}\big) \to \frac{\delta \cdot \eta}{\int_X \log|F'(x)| d\mu} \cdot \widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta}), \tag{9.2}$$

as $T \to \infty$. In view of ε-linearity, the left hand side of (9.2) is bounded below by

$$\sum_{\substack{n \geq 0,\, F^{\circ n}(y) = x,\, (\varepsilon,\eta)\text{-good} \\ T+\varepsilon+\eta < \log|(F^{\circ n})'(y)| < (T+\delta)-\varepsilon-\eta}} m(\bigcirc_y)$$

and bounded above by

$$\sum_{\substack{n\geq 0,\, F^{\circ n}(y)=x,\, (\varepsilon,\eta)\text{-good} \\ T-\varepsilon-\eta<\log|(F^{\circ n})'(y)|<(T+\delta)+\varepsilon+\eta}} m(\bigcirc_y).$$

Since m is a conformal measure of dimension α,

$$m(\bigcirc_y) \sim_\delta e^{-\alpha T} m(\bigcirc_x),$$

for all repeated pre-images y involved in these expressions. We will be slightly imprecise and say that the left hand side of (9.2) is roughly

$$n_{\varepsilon,\eta}(x, T, T+\delta) \cdot \eta \cdot e^{-\alpha T} \cdot m(\bigcirc_x).$$

Since the point $x \in X$ satisfies Conditions C and D, when $\eta > 0$ is small,

$$\widehat{\mu}(\widehat{\bigcirc}_{\varepsilon,\eta}) \;\approx\; \mu(\bigcirc_x) \;\approx\; \gamma(x) m(\bigcirc_x).$$

Putting the above equations together, we obtain

$$n_{\varepsilon,\eta}(x, T, T+\delta) \sim_\delta \frac{\gamma(x)}{\int_X \log|F'(x)|d\mu} \cdot \delta \cdot e^{\alpha T}.$$

As in the case of the Orbit Counting up to a Cesàro average, we may use the a priori estimate (Lemma 7.1) to show that the contribution of the bad pre-images is negligible, which proves (9.1). □

Part III

10 On the Radon-Nikodym derivative

In this section, we show that if F is a rational function which satisfies (OC1)–(OC4), then it automatically satisfies the (OC6). In Section 4.5, we saw that any rational function F which has an invariant probability measure μ with a positive Lyapunov exponent satisfies (OC5), while (OC7) holds for any rational function as

$|F'|$ is bounded above. Consequently, if a rational function satisfies (OC1)–(OC4), then it also satisfies (OC5)– (OC7).

The property (OC6) was originally proved by N. Dobbs [Dob12, Proposition 34]. For the convenience of the reader, we provide a slightly different argument, which is perhaps more elementary.

Lemma 10.1. *Suppose m is a conformal measure and $\mu = \gamma\, dm$ is an ergodic absolutely continuous invariant measure. For any measurable set $E \subset X$, we have*

$$\mu(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} m(F^{-j}(E)).$$

Proof. Applying the ergodic theorem to the map $\widehat{F}^{-1} : \widehat{X} \to \widehat{X}$ and the function $\mathbf{x} = (x_n)_{n=-\infty}^{0} \to \gamma^{-1}(x_0)$ shows that for $\widehat{\mu}$ a.e. $\mathbf{x} \in \widehat{X}$,

$$\frac{1}{n} \sum_{j=0}^{n-1} \gamma^{-1}(x_{-j}) \;\to\; \int_{\widehat{X}} \gamma^{-1}(x_0) d\widehat{\mu}(\mathbf{x}) \;=\; 1.$$

Integrating over $\widehat{E} = \left\{\mathbf{x} = (x_{-n})_{n=0}^{\infty} \in \widehat{X} : x_0 \in E\right\}$, we get

$$\frac{1}{n} \sum_{j=0}^{n-1} \int_{\widehat{E}} \gamma^{-1}(x_{-j}) d\widehat{\mu} \;\to\; \widehat{\mu}(\widehat{E}) \;=\; \mu(E).$$

By the $\widehat{F}$-invariance of $\widehat{\mu}$, for any non-negative integer $j \geq 0$, we have

$$\int_{\widehat{E}} \gamma^{-1}(x_{-j}) d\widehat{\mu} \;=\; \int_{\widehat{F^{-j}(E)}} \gamma^{-1}(x_0) d\widehat{\mu} \;=\; \int_{F^{-j}(E)} \gamma^{-1}(x_0) d\mu \;=\; m(F^{-j}(E)).$$

Substituting the above expressions for the integrals in the equation above proves the lemma. □

Lemma 10.2. *Suppose $F : X \to X$ is* eventually onto *in the sense that for any relatively open set $U \subset X$, there is an $n \geq 0$ so that $F^{\circ n}(U) = X$. Then, any conformal probability measure m gives positive mass to every open set.*

Proof. The lemma follows from the inequality $m(X) \leq \int_U |(F^{\circ n})'(x)|^{\alpha} dm$. □

The following lemma is taken from [Dob12, Lemma 29]:

Lemma 10.3. *Suppose m is a conformal measure and $\mu = \gamma\, dm$ is an absolutely continuous invariant measure. Assume that F is eventually onto and (X, F, μ) is non-uniformly hyperbolic. If m gives positive mass to open sets, then the measures μ and m are equivalent.*

Proof. Assume for the sake of contradiction that there exists a measurable set B such that $m(B) > 0$ and $\mu(B) = 0$. Then,

$$X' = X \setminus \bigcup_{j \geq 0} F^{-j}(B)$$

is a forward-invariant set such that $\mu(X') = 1$ but $m(X') < 1$.

Fix an $0 < \varepsilon < 1/10$ and pick an open ball U such that $\widehat{\mu}(\widehat{U}_{\varepsilon\text{-linear}}) > 0$. By ergodicity, for $\widehat{\mu}$ a.e. $\mathbf{x} \in \widehat{X}$, there is an increasing sequence of integers $n_j \to \infty$ such that $\widehat{F}^{\circ n_j}(\mathbf{x})$ lands in the positive measure set $\widehat{U}_{\varepsilon\text{-linear}}$. Consequently, for μ a.e. $x \in X'$, there exists sequences of open sets U_j containing x and positive integers $n_j \to \infty$ such that $F^{-n_j} : U \to U_j$ is ε-linear. By Lebesgue's density point theorem, e.g. see [Mat95, Corollary 2.14], for m a.e. $x \in X'$,

$$\lim_{r \to 0^+} \frac{m(X' \cap B(x, r))}{m(B(x, r))} = 1,$$

and consequently, for μ a.e. $x \in X'$,

$$\lim_{j \to \infty} \frac{m(X' \cap U_j)}{m(U_j)} = 1,$$

since the sets U_j are approximately round balls. In view of the conformality of the measure m and the uniform bound on the distortion, this implies that

$$m(U) = m(X' \cap U) \qquad \implies \qquad m(U \setminus X') = 0.$$

By the eventually onto property, $m(X \setminus X') = 0$, which contradicts that $m(B) > 0$. $\square$

Lemma 10.4. *Fix an $0 < \varepsilon < 1/10$. If m gives positive mass to open sets and (X, F, μ) is non-uniformly hyperbolic, then the Radon-Nikodym derivative $\gamma > c(U) > 0$ is bounded below on any open set $U \subset X$ such that $\widehat{\mu}(\widehat{U}_{\varepsilon\text{-linear}}) > 0$.*

Proof. Since $\mu(U) = \widehat{\mu}(\widehat{U}) \geq \widehat{\mu}(\widehat{U}_{\varepsilon\text{-linear}})$, the open set U has positive μ measure. Consequently, by Lemma 10.3, U also has positive m measure. For a set measurable set $B \subset U$, let

$$\mu_{\text{good}}(B) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \sum_{\substack{B_j \subset F^{-j}(B) \\ \varepsilon\text{-linear}}} m(B_j),$$

where the inner sum is over the sets of the form $B_j = F^{-j}(B) \cap U_j$, where U_j is an ε-linear repeated pre-image of U of order j. By ε-linearity and the fact that m is conformal measure of dimension α, we have

$$\frac{\mu_{\text{good}}(B)}{\mu_{\text{good}}(U)} \geq c \cdot \frac{m(B)}{m(U)}, \qquad c = e^{-2\alpha\varepsilon}.$$

Thus,

$$\mu(B) \;\geq\; \mu_{\text{good}}(B) \;\geq\; c \cdot \frac{\mu_{\text{good}}(U)}{m(U)} \cdot m(B).$$

Since $B \subset U$ was an arbitrary measurable set, the Radon-Nikodym derivative

$$\gamma(x) > c \cdot \frac{\mu_{\text{good}}(U)}{m(U)}$$

is bounded below on U. $\square$

Lemma 10.5. *In the setting of the above lemma, suppose that we also know that the derivative $|F'|$ is bounded above m a.e. on X and F is eventually onto. Then, $\gamma > c > 0$ on all of X.*

Proof. Since m is a conformal measure and μ is an invariant measure, for a point $x \in X$, we have

$$\gamma(x) = \sum_{F^{\circ n}(y)=x} \gamma(y)\, |(F^{\circ n})'(y)|^{-\alpha}.$$

Consequently, if γ is bounded below on a relatively open subset $U \subset X$, then it is also bounded below on X. $\square$

11 On alignment of inverse branches

The following lemma says that in the case of rational functions, if backward iteration along two inverse branches is not aligned (the definition was given in Section 6.2), then misalignment occurs on a positive measure set of inverse branches:

Lemma 11.1. *Let F be a rational function acting on its Julia set. Suppose that some two UAL inverse branches $\mathbf{z}$ and $\mathbf{z}'$ are not aligned. If the diameters of $z_{-n}(U)$ and $z'_{-n}(U)$ shrink to 0, then there exist positive $\widehat{\mu}$ measure sets of inverse branches $\mathcal{Z}, \mathcal{Z}'$ defined on U, so that any inverse branch in $\mathcal{Z}$ is not aligned with any inverse branch in $\mathcal{Z}'$. The same statement also holds for absolute-value alignment in place of alignment.*

Proof. By non-uniform hyperbolicity, for any $\varepsilon > 0$, there is a ball $B(x, \rho)$ centered at a point $x \in X$ so that the union of the ε-linear inverse branches defined on $B(x, \rho)$ has positive $\widehat{\mu}$ measure.

For each point $y \in X$, there is an $r_y > 0$ and an inverse iterate defined on $B(y, r_y)$ with $F^{-N_y}(B(y, r_y)) \subset B(x, \rho)$. By compactness, one can cover $X = \mathcal{J}(F)$ by finitely many such balls $B(y, r_y)$. We refer to these inverse iterates as "transitions" to $B(x, \rho)$.

Now, when n is large, $z_{-n}(U)$ belongs to one of these balls $B(y, r_y)$. Given an inverse branch $\mathbf{z}$, we produce a positive $\widehat{\mu}$ measure $\mathcal{Z}_{n,\varepsilon}$ of inverse branches which have rescaling limits close to that of $\mathbf{z}$ as follows: we first follow $\mathbf{z}$ for n steps, then we transition to $B(x, \rho)$ and finally, we follow a Pesin branch from there. Similarly, we can produce a family $\mathcal{Z}'_{n,\varepsilon}$ of inverse branches which have approximately the same rescaling limit as $\mathbf{z}'$. When $n > 0$ is large and $\varepsilon > 0$ is small, the families $\mathcal{Z}_{n,\varepsilon}$ and $\mathcal{Z}'_{n,\varepsilon}$ will have disjoint sets of rescaling limits. $\square$

12 Alignment implies rigidity

To prove the Orbit Counting Theorem for rational maps, we show the following theorem:

Theorem 12.1. *Let F be a rational function satisfying the conditions* (OC1)–(OC7). *If all univalent inverse branches are absolute-value aligned, then F is one of the exceptional rational functions listed in the introduction.*

We follow an argument of V. Mayer [May02], almost verbatim. We recall the main techniques. A holomorphic function $\Psi : \mathbb{C} \to \widehat{\mathbb{C}}$ is *automorphic* with respect to a discrete group $\Gamma \subset \mathrm{Isom}^+ \mathbb{C}$ if:

1. $\Psi \circ \gamma = \Psi$ for every $\gamma \in \Gamma$,
2. Γ acts transitively on the fibers, i.e. if $\Psi(z_1) = \Psi(z_2)$, then there is an element $\gamma \in \Gamma$ which takes z_1 to z_2.

As explained in [May02, Lemma 2.5], in order to check that the transitivity of Γ on the fibers, it is enough to check that Γ acts transitively on some regular fiber, i.e. there exists a point $w \in \Psi(\mathbb{C})$ which is not a critical value of Ψ, such that for any two pre-images $z_1, z_2 \in \Psi^{-1}(w)$, there is an element of Γ which takes z_1 to z_2.

The following lemma due to Ritt is at the core of Mayer's argument:

Lemma 12.2. *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. Suppose there exists an entire function Ψ such that*

$$F(\Psi(z)) = \Psi(\lambda z), \qquad z \in \mathbb{C},$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. If Ψ is automorphic with respect to a co-compact group $\Gamma \subset \mathrm{Isom}^+ \mathbb{C}$, then F is a Lattès map, while if Ψ is automorphic with respect to $\mathbb{Z}$, then F is either a power map of a Chebyshev polynomial.

The following two elementary lemmas are implicit in Mayer's paper. We state them explicitly for convenience:

Lemma 12.3. *Suppose $U_0 \subset \mathbb{C}$ is a domain in the plane which contains the origin and $g : U_0 \to \mathbb{C}$ is an analytic function which is not identically zero. If the analytic set $Z = \{\mathrm{Re}\, g = 0\}$ is invariant under $z \to \lambda z$ with $|\lambda| > 1$, then $\lambda \in \mathbb{R}$ and Z is contained in a union of finitely many lines through the origin.*

Lemma 12.4. *Let $S \subset \mathbb{C}$ be a discrete set in the plane which contains at least 2 points. If $\gamma \in \mathrm{Aut}\, \mathbb{C}$ maps S to S, then $\gamma \in \mathrm{Isom}^+ \mathbb{C}$.*

With the above preparations, we are now ready to prove Theorem 12.1:

Proof of Theorem 12.1. For simplicity, we assume that there is a repelling fixed point $r \in X$, which is not contained in the grand orbit of any critical point of F. (In the general case involving a repelling periodic orbit, an additional argument is required. We refer to Mayer's paper for details.) We denote the multiplier of the repelling fixed point by $\lambda = F'(r)$. Let $\mathbf{r}$ be the constant inverse orbit associated to r, i.e. $r_{-n} = r$ for all $n \geq 0$, and $\Psi(\zeta) = F_{\mathbf{r}}^{-\infty}(\zeta)$ be the rescaling limit at $\mathbf{r}$, defined in Section 4.1. The map Ψ is just the inverse of the linearizing coordinate at r.

Being the Julia set of a rational function, $\mathcal{J} = \mathcal{J}(F)$ is a perfect set. Consequently, so is $\mathcal{K} = \Psi^{-1}(\mathcal{J})$. In view of the normalization of Ψ, the point $0 \in \Psi^{-1}(r) \subset \mathcal{K}$. The relation

$$F(\Psi(w)) = \Psi(\lambda w), \qquad w \in \mathbb{C},$$

tells us that the set $\mathcal{K}$ is invariant under multiplication by λ. By assumption on the point r, the fiber $\Psi^{-1}(r)$ is discrete and Ψ is a local homeomorphism from a neighbourhood U_ξ of any $\xi \in \Psi^{-1}(r)$ to a neighbourhood V of $r \in X$. (In fact, a point in $\Psi^{-1}(r)$ corresponds to a "homoclinic" orbit $\mathbf{x} = \widehat{\mathcal{J}}$ with $x_0 = r$ and $x_{-n} \to r$ as $n \to \infty$.)

Consequently, for any $\xi \in \Psi^{-1}(r)$, the composition $\gamma = \gamma_\xi = \Phi^{-1} \circ \Phi$ is a conformal map from U_0 to U_ξ. Absolute-value alignment implies

$$\left|\frac{\Psi'(\zeta)}{\Psi'(0)}\right| = \left|\frac{(\Psi \circ \gamma)'(\zeta)}{(\Psi \circ \gamma)'(0)}\right|, \qquad \zeta \in \mathcal{K} \cap U_0.$$

By the chain rule, there exists a real constant $C \in \mathbb{R}$ so that

$$\log|\gamma'(\zeta)| = C, \qquad \zeta \in \mathcal{K} \cap U_0.$$

As $\log|\gamma'(\zeta)|$ is a harmonic function on U_0, the set where it is equal to C is an analytic set. Consequently, $\log|\gamma'(\zeta)| = C$ holds on the minimal analytic set containing $\mathcal{K} \cap U_0$. There are two cases:

1. The set $\mathcal{K} \cap U_0$ is not contained in a proper analytic set near the origin. In this case, $\log|\gamma'(\zeta)| = C$ on all of U_0 and $\gamma(\zeta)$ is an affine map.

2. Suppose that $\mathcal{K}$ is contained in a proper analytic set near the origin. As $\mathcal{K}$ is invariant under multiplication by λ, by Lemma 12.3, the multiplier λ is real and $\mathcal{K}$ is contained in a union of finitely many lines passing through the origin. Actually, as γ is invertible near the origin, $\mathcal{K} \subset \{\log|\gamma'(\xi)| = C\}$ must be contained in a single line near the origin. As $\mathcal{K}$ is a perfect set, $\log|\gamma'(\xi)| = C$ on this line, and so $\gamma(\zeta)$ is affine on U_0 in this case as well.

To summarize, we have proved that every "local symmetry" γ of $\mathcal{K}$ which maps a neighbourhood of $0 \in \mathcal{K}$ to a neighbourhood of $\xi \in \Psi^{-1}(r)$ extends to a "global symmetry" defined on all of $\mathbb{C}$. In other words, Ψ is automorphic with respect to the group of automorphisms

$$\Gamma = \{\gamma \text{ affine with } \Psi \circ \gamma = \Psi\}.$$

By Lemma 12.4, Γ must actually be a group of isometries. In view of Lemma 12.2, F must be one of the exceptional rational maps listed in the introduction. □

13 Orbit counting with rotation

In this section, we prove Theorem 1.4. As the strategy is similar to that of Theorem 1.3, we only sketch the differences. We equip the space

$$\widehat{\mathcal{J}} \times \mathbb{R}_+ \times \partial\mathbb{D},$$

with the measure $\widehat{\mu} \times (dt/t) \times (d\theta/2\pi)$ and consider the quotient $\widehat{\mathcal{J}}_{\rho,\theta}$ with respect to the equivalence relation

$$(\mathbf{x}, t, e^{i\theta}) \to \left(\widehat{F}(\mathbf{x}),\, |F'(x_0)|t,\, \frac{F'(x_0)}{|F'(x_0)|} \cdot e^{i\theta}\right).$$

Theorem 13.1. *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map which satisfies the hypotheses in the introduction. The suspension flow $g_s : (\mathbf{x}, t, e^{i\theta}) \to (\mathbf{x}, e^s t, e^{i\theta})$ is mixing on $\widehat{\mathcal{J}}_{\rho,\theta}$ unless F is in the extended list of exceptional list of rational maps. In this case, the Orbit Counting Theorem with Rotation holds.*

For brevity, we write

$$\alpha_{\mathbf{x}}(y_0) = \log |(F_{\mathbf{x}}^{-\infty})'(y_0)|, \qquad \beta_{\mathbf{x}}(y_0) = \arg(F_{\mathbf{x}}^{-\infty})'(y_0).$$

We say that two UAL inverse branches $\mathbf{z}(\cdot)$ and $\mathbf{z}'(\cdot)$ are *weakly aligned* on U if there are constants $C_1, C_2 \in \mathbb{R}$ (not both zero) such that

$$C_1\alpha_{\mathbf{x}}(y_0) + C_2\beta_{\mathbf{x}}(y_0) = C_1\alpha_{\mathbf{x}'}(y_0) + C_2\beta_{\mathbf{x}'}(y_0), \qquad y_0 \in U,$$

i.e. if the linear span of the vectors

$$\big(\alpha_{\mathbf{x}}(y_0) - \alpha_{\mathbf{x}'}(y_0),\, \beta_{\mathbf{x}}(y_0) - \beta_{\mathbf{x}'}(y_0)\big), \tag{13.1}$$

as y_0 varies over U is all of $\mathbb{R}^2$.

Following the discussion in Section 12, we see that if all UAL inverse branches are weakly aligned, then the local symmetries $\gamma : U_0 \to U_\xi$ of the set $\mathcal{K}$ satisfy

$$C_1 \log |\gamma'(\zeta)| + C_2 \arg \gamma'(\zeta) = C, \qquad z \in \mathcal{K} \cap U_0. \tag{13.2}$$

As $C_1 \log |\gamma'(\zeta)| + C_2 \arg \gamma'(\zeta)$ is a harmonic function on U_0, (13.2) should continue to hold on the minimal analytic set containing $\mathcal{K} \cap U_0$. As before, we consider two cases:

1. If $\mathcal{K}$ is not contained in a proper analytic set near 0, then

   $$C_1 \log |\gamma'(\zeta)| + C_2 \arg \gamma'(\zeta) = C, \qquad \zeta \in U_0,$$

   which forces $\gamma'(\zeta)$ to be constant on U_0 by Lemma 13.2 below. In other words, γ is an affine mapping. (The proof now continues as in Section 12).

2. Suppose that $\mathcal{K}$ is contained in a proper analytic set near the origin. Since $\mathcal{K}$ is invariant under multiplication by λ, the multiplier λ is real and $\mathcal{K}$ is contained in a union of finitely many lines passing through the origin. As Ψ is a local homeomorphism between a neighbourhood of $0 \in \mathcal{K}$ and a neighbourhood of $r \in \mathcal{J}$, we see that $\mathcal{J}$ contains a real analytic curve as a relatively open subset. By the paragraph below the statement of [EvS11, Corollary 1.1], the Julia set $\mathcal{J}$ is contained in a circle or a line.

We now show the elementary lemma used in the above proof:

Lemma 13.2. *Let $u : \Omega \to \mathbb{R}$ be a harmonic function and v be its harmonic conjugate. Suppose that*

$$C_1 u(z) + C_2 v(z) = K, \tag{13.3}$$

for some real constants $C_1, C_2, K \in \mathbb{R}$. If C_1, C_2 are not both zero, then u and v are constant functions.

Proof. If C_2 is not zero, then $i(C_1/C_2)u + iv$ and $u + iv$ are analytic functions, which implies that $(iC_1/C_2 - 1)u$ is analytic. Consequently, u itself is analytic. Since u is real-valued, it is constant. As the harmonic conjugate of a constant function is constant, v is also constant. The case that $C_1 \neq 0$ is similar. $\square$

We now comment on why weak non-alignment implies mixing. This time, h is a uniformly continuous function on $\widehat{\mathcal{J}}_{\rho,\theta}$. As in Section 6.2, we assume that $t_n \to \infty$ is a sequence of increasing real numbers such that $g_{\pm t_n} h \to \psi$ weakly. We want to show that ψ is a constant function.

Using the backward geodesic flow as before shows that the limiting function ψ depends only on the zero-th coordinate of $\mathbf{x} \in \widehat{\mathcal{J}}$. The argument with the forward geodesic flow shows that if the vectors in (13.1) span the plane, as y_0 varies over U, then $g_t : \widehat{\mathcal{J}}_{\rho,\theta} \to \widehat{\mathcal{J}}_{\rho,\theta}$ is mixing.

Part IV

14 Orbit counting for Adler maps

We now turn to one-dimensional dynamics. In this section, we show Theorem 1.5 on orbit counting for Adler maps acting on the unit circle. (Adler maps were defined in Section 1.2.)

14.1 Anatomy on an Adler map

Adler maps resemble one component inner functions, see [IU23, Section 7], but are more general since there are no "holomorphy" requirements. Nevertheless, Adler maps possess many of the same properties that one component inner functions do, with identical proofs. For the convenience of the reader, we repeat some of the arguments to give a self-contained exposition. Below, we write $I(x, r)$ for the arc of the unit circle centered at x of length $2r$.

Lemma 14.1. *Let F be an Adler map. For any $\varepsilon > 0$, there exists a constant $c = c(M, \varepsilon) > 0$ such that if $x \in \partial\mathbb{D} \setminus \Sigma$ is a point on the unit circle, then F is injective on $I(x, c/|F'(x)|)$ and*

$$e^{-\varepsilon}|F'(x)| \le |F'(y)| \le e^{\varepsilon}|F'(x)|, \qquad y \in I\left(x, \frac{c}{|F'(x)|}\right).$$

In particular, $F\big(I(x, c/|F'(x)|)\big)$ is an arc on the unit circle of length $\asymp 1$.

Proof. The lemma follows from Grönwall's inequality applied to F'. $\square$

The above lemma implies that there exists a constant $r_0 > 0$ so that if $F(y) = x$, $y \in \partial\mathbb{D} \setminus \Sigma$, then F^{-1} admits a continuous inverse branch on $I(x, r_0)$ which takes $x \to y$. Consequently, F is a covering map from $\partial\mathbb{D} \setminus \Sigma \to \partial\mathbb{D}$. Furthermore, if the singular set Σ is empty, then F is a finite covering map, while if Σ is not empty, then for any $p \in \partial\mathbb{D}$, the set $E_p = F^{-1}(p)$ divides each connected component $J \subset \partial\mathbb{D} \setminus \Sigma$ into countably many subarcs.

Since F is expanding on the unit circle, $F^{-1}(I(x, r_0)) \subset I(y, r_0)$, allowing us to take an arbitrary number of inverse iterates:

Corollary 14.2. *For any $\varepsilon > 0$, there exists an $r_0 > 0$ so that inverse iteration is ε-linear on $I(x, r_0)$ along any inverse orbit $\mathbf{x} = (x_{-n})_{n=0}^{\infty}$ with $x_0 = x$.*

As the unit circle can be covered by finitely many arcs of the form $I(x, r_0)$, the above lemma has the following consequence:

Corollary 14.3. *For any $p \in \partial\mathbb{D}$, the set $E_p = F^{-1}(p)$ partitions $\partial\mathbb{D} \setminus \Sigma$ into countably many arcs I_k, each of which is mapped homeomorphically onto $\partial\mathbb{D} \setminus \{p\}$ and*

$$|F'(x)| \asymp 1/|I_k|, \qquad x \in I_k,$$

where the implicit constant depends only on F.

In particular, Corollary 14.3 implies that

$$\int_{\partial\mathbb{D}} \log|F'|dm < \infty \qquad \Longleftrightarrow \qquad \sum |I_j| \log \frac{1}{|I_j|} < \infty, \tag{14.1}$$

where the sum ranges over the complementary arcs in $\partial\mathbb{D} \setminus E_p$.

14.2 Conformal and invariant measures

Clearly, the normalized Lebesgue measure m on the unit circle is a 1-dimensional conformal measure for $F : \partial\mathbb{D} \to \partial\mathbb{D}$. Using thermodynamic formalism, one can construct an ergodic absolutely continuous invariant probability measure $\mu = \gamma\, dm$, e.g. see [Rych83] or [URM22, Theorem 14.3.1]. As explained in these references, the Radon-Nikodym derivative γ is a non-negative Hölder continuous function on the unit circle. It is not difficult to see that γ does not vanish: the transfer relation

$$\gamma(x) = \sum_{F(y)=x} |F'(y)|^{-1}\gamma(y)$$

implies that the set $Z = \{x \in \partial\mathbb{D} : \gamma(x) = 0\}$ is backward-invariant. If Z were not empty, ergodicity would imply that Z is dense in the unit circle, and since γ is continuous, γ would be identically zero, which contradicts the fact that μ is a probability measure.

14.3 Adler maps satisfy (OC1)–(OC7)

Let F be an Adler map satisfying (1.5). We take a moment to verify that F satisfies the conditions (OC1)–(OC7):

- (OC1), (OC2) hold trivially since m is a 1-dimensional conformal measure.

- For (OC3), (OC6), we have constructed an absolutely continuous invariant measure μ using thermodynamic formalism. We saw that the density $\gamma \in d\mu/dm$ was continuous on X and bounded away from zero.
- Since the density γ is bounded above, we have $\log|F'(x)| \in L^1(X,\mu)$. As $|F'(x)| > 1$ for Lebesgue a.e. $x \in \partial\mathbb{D}$, the Lyapunov exponent is positive, so (OC4) holds.
- (OC5) holds in view of the uniform hyperbolicity of F (Corollary 14.2).
- Since Adler maps are expanding, they cannot be homeomorphisms of the unit circle. Consequently, there exist at least two intervals I_1, I_2 which get mapped to the unit circle under F. By Corollary 14.3, $|F'|$ is bounded above on $I_1 \cup I_2$. Consequently,

$$\sum_{F(y)=x} |F'(y)|^{-\alpha} - \max_{F(y)=x} |F'(y)|^{-\alpha} > \min_{\substack{F(y)=x \\ y\in I_1\cup I_2}} |F'(y)|^{-\alpha} > 0.$$

Having checked the above properties, we see that the Orbit Counting Theorem holds up to a Cesàro average for any Adler map satisfying (1.5).

14.4 Mixing of the suspension flow

To complete the proof of Theorem 1.5, it remains to show the Orbit Counting Theorem holds for any infinite-to-one Adler map. By Theorem 1.2, one needs to check that the suspension flow $\widehat{X}_\rho$ is mixing. Since the Pesin set $X_{\text{lin}} = \partial\mathbb{D}$, this amounts to showing the following statement:

Lemma 14.4. *Let F be an infinite-degree Adler map. For any $a \in \mathbb{R} \setminus \{0\}$,*

$$w(F(x)) = e^{ia\log|F'(x)|} \cdot w(x). \tag{14.2}$$

has no continuous solutions $w : \partial\mathbb{D} \to \partial\mathbb{D}$.

Proof. Let d be the topological degree of w. Since F is an infinite Adler map, the singular set $\Sigma \neq \emptyset$. Pick an arbitrary complementary arc $J = \bigcup_{k\in\mathbb{Z}} I_k$ in $\partial\mathbb{D} \setminus \Sigma$.

Since J is simply-connected, the arguments of $w(x)$ and $w(F(x))$ admit continuous branches on J. We consider two cases:

Case I. $d = 0$. Let $I_k = [z_k, z_{k+1}]$. As z moves from z_k to z_{k+1}, the argument of w does not change. Consequently, $\arg w(F(z))$ is bounded on J. As $\arg w(x)$ is bounded on J, equation (14.2) implies that $a \log |F'(z)|$ must also be bounded on J. This contradicts the definition of an Adler map, which asks that $|F'(z)| \to \infty$ as $z \to \Sigma$.

Case II. $d \neq 0$. Below, we assume that $d > 0$, as the case $d < 0$ is similar. As z moves from z_k to z_{k+1}, the argument of $w(F(z))$ increases by d, so that

$$\arg w(F(z_k)) - \arg w(F(z_0)) = kd + O(1).$$

In particular, this implies that $\log |F'(z_k)| \to -\infty$ as $k \to -\infty$. This once again contradicts the definition of an Adler map. $\square$

References

[Bab02] M. Babillot, *On the mixing property for hyperbolic systems*, Israel J. Math 129 (2002), 61–76.

[Bro65] H. Brolin, *Invariant sets under iteration of rational functions* Ark. Mat. 6 (1965), no. 2, 103–144.

[Dob12] N. Dobbs, *Measures with positive Lyapunov exponent and conformal measures in rational dynamics*, Trans. Amer. Math. Soc. 364 (2012), no. 6, 2803–2824.

[Dol98] D. Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. 147 (1998), 357–390.

[EvS11] A. Eremenko, S. van Strien, *Rational maps with real multipliers*, Trans. Amer. Math. Soc. 363 (2011), no. 12, 6453–63.

[FU00] A. M. Fisher, M. Urbański, *On Invariant Line Fields*, Bull. London. Math. Soc. 32 (2000), no. 5, 555–570.

[Glu10a] A. Glutsyuk, *On density of horospheres in dynamical laminations*, Moscow Math. J. 10 (2010), no. 3, 547–591.

[Glu10b] A. Glutsyuk, *Unique ergodicity of horospheric foliations revisited*, J. Fixed Point Theory Appl. 8 (2010), no. 1, 113–139.

[GS09] J. Graczyk, S. Smirnov, *Non-uniform hyperbolicity in complex dynamics*, Invent. Math. 175 (2009), no. 2, 335-415.

[IU23] O. Ivrii, M. Urbański, *Inner functions and thermodynamic formalism*, preprint, 2024. arXiv:2308.16063

[IU24] O. Ivrii, M. Urbański, *Inner functions and laminations*, preprint, 2024. arXiv:2405.02878

[Jov24] A. Jové, *Pesin theory for transcendental maps and applications*, preprint, 2024. arXiv:2410.19703

[KL05] V. A. Kaimanovich, M. Lyubich, *Conformal and Harmonic Measures on Laminations Associated with Rational Maps*, Mem. Amer. Math. Soc. 173 (2005), no. 820.

[Lal89] S. P. Lalley, *Renewal theorems in symbolic dynamics, with applications to geodesic flows, noneuclidean tessellations and their fractal limits*, Acta Math. 163 (1989), 1–55.

[LR25] Z. Li, J. Rivera-Letelier, *Prime orbit theorems for Collet–Eckmann maps*, in preparation, 2025.

[Lyu83] M. Ju. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergod. Th. & Dyn. Sys. 3 (1983), no. 3, 351–385.

[LM97] M. Lyubich, Y. Minsky, *Laminations in holomorphic dynamics*, J. Differential Geom. 47 (1997), 17–94.

[Mat95] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, 1995.

[May02] V. Mayer, *Comparing measures and invariant line fields*, Ergodic Theory Dynam. Systems 22 (2002) 555–570.

[McM08] C. T. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric*, Invent. Math. 173 (2008), 365–425.

[OW17] H. Oh, D. Winter, *Prime number theorems and holonomies for hyperbolic rational maps*, Invent. Math. 208 (2017), 401–440.

[PoU17] M. Pollicott and M. Urbański, *Asymptotic counting in conformal dynamical systems*, Memoirs A.M.S., to appear. arXiv:1704.06896.

[PR07] F. Przytycki, J. Rivera-Letelier, *Statistical properties of topological Collet–Eckmann maps*, Ann. Sci. Éc. Norm. Supér. 40 (2007), no. 1, 135–178.

[PrU10] F. Przytycki, M. Urbański, *Conformal Fractals: Ergodic Theory Methods*, Cambridge University Press, Cambridge, 2010.

[Rych83] M. Rychlik, *Bounded variation and invariant measures*, Studia Math. 76 (1983), 69–80.

[URM22] M. Urbański, M. Roy, S. Munday, *Non-Invertible Dynamical Systems: Finer Thermodynamic Formalism – Distance Expanding Maps and Countable State Subshifts of Finite Type, Conformal GDMSs, Lasota-Yorke Maps and Fractal Geometry*, de Gruyter Expositions in Mathematics 69, Vol. 2, De Gruyter, Berlin, 2022.

[Zdu90] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Math. 99 (1990), 627–649.