Chapter 3

Liouville's theorem

In this chapter, we prove Liouville's theorem which says that a conformal metric of curvature -1 on a simply-connected domain Ω arises as the pullback of the hyperbolic metric by some locally univalent holomorphic map $F : \Omega \to \mathbb{D}$. We then generalize Liouville's theorem to conformal metrics of curvature -1 with isolated singularities by carefully studying the monodromy of F around each singularity.

In Liouville's theorem, F is determined uniquely up to post-composition with an element of $\operatorname{Aut}\mathbb{D}$. We first study a related object called the Schwarzian derivative which uniquely determines a holomorphic function up to $\operatorname{Aut}\hat{\mathbb{C}}$. The discrepancy between the two groups $\operatorname{Aut}\mathbb{D}$ and $\operatorname{Aut}\hat{\mathbb{C}}$ plays an important role in this chapter.

3.1 Schwarzian derivatives of functions

It is well known that a Möbius transformation $m \in \operatorname{Aut}\hat{\mathbb{C}}$ is determined by the images of three points

$$p_1 \to q_1, \quad p_2 \to q_2, \quad p_3 \to q_3.$$

One can also uniquely prescribe a Möbius transformation by specifying its 2-*jet* at a point $p \in \hat{\mathbb{C}}$, that is, the values $m(p), m'(p), m''(p)$. The values $m(p), m'(p), m''(p)$ may be prescribed arbitrarily with the provisio that $m'(p) \neq 0$ to ensure that m is locally injective. For example, the unique Möbius transformation whose power series expansion at the origin begins with

$$m = a_0 + a_1 z + a_2 z^2 + \dots$$

is

$$m(z) = \frac{(a_1^2 - a_2 a_0)z + a_0 a_1}{-a_2 z + a_1}.$$

As usual, if p or $m(p)$ is infinity, one would have to compute the derivatives in terms of the coordinates at infinity.

Suppose $\Omega \subset \mathbb{C}$ be a domain and $F : \Omega \to \hat{\mathbb{C}}$ be a locally univalent meromorphic function. Given a point $p \in \Omega$, there exists a Möbius transformation $m_p(z)$ which osculates F to order 2 at p. Loosely speaking, the Schwarzian derivative S_F measures how $m_p(z)$ varies with p. If $m_p(z)$ is the identity mapping, then $S_F(p)$ is just the third derivative $F'''(p)$. In the general case, one has the expression

$$S_F = \left(\frac{F''}{F'}\right)' - \frac{1}{2}\left(\frac{F''}{F'}\right)^2. \tag{3.1}$$

Lemma 3.1. *The Schwarzian derivative $S_F = 0$ if and only if F is a Möbius transformation.*

Proof. To see that the Schwarzian derivative annihilates Möbius transformations, note that $S_{az+b} = 0$ and $S_F = S_{1/F}$.

Conversely, suppose that $S_F = 0$. Setting $y = F''/F'$ gives $y' = y^2/2$. Integration shows that either $y(z) = 0$ or $y(z) = \frac{2}{z-c}$ for some $c \in \mathbb{C}$. In the first case, $F(z) = az+b$ is linear, while in the second case, F is a non-linear Möbius transformation. □

One of the most important properties of the Schwarzian derivative is the *cocycle condition*

$$S_{F \circ G}(z) = S_F(G(z))G'(z)^2 + S_G(z).$$

If we take $F \in \operatorname{Aut}\hat{\mathbb{C}}$, we see that post-composition with Möbius transformations does not change the Schwarzian derivative. If we instead take $G \in \operatorname{Aut}\hat{\mathbb{C}}$, then we see the Schwarzian derivative transforms like a quadratic differential.

Lemma 3.2. *If $\Omega \subset \mathbb{C}$ is simply-connected and $g : \Omega \to \mathbb{C}$ is a holomorphic function, then $S_F = g$ has a locally meromorphic solution, which is unique up to post-composition with a Möbius transformation in* $\operatorname{Aut}\hat{\mathbb{C}}$.

In other words, to recover F uniquely from its Schwarzian, it is enough to prescribe a Möbius transformation $m \in \operatorname{Aut}\hat{\mathbb{C}}$ that matches the 2-jet of F at a given point. In other words, one can prescribe $F(z_0), F'(z_0), F''(z_0)$ with $F'(z_0) \neq 0$.

Proof. A computation shows that $w = (F')^{1/2}$ satisfies the Ricatti equation

$$w'' + \frac{1}{2} \cdot gw = 0. \tag{3.2}$$

Near any point $z_0 \in \Omega$, the (3.2) has two linearly independent holomorphic solutions. There are two ways to see this: if the initial function in Picard iteration is holomorphic, then all functions in the process of Picard iteration will also be holomorphic, and the solution to the ODE will be a uniform limit of holomorphic functions. Alternatively, one can show that the ODE has two formal power series solutions $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and then verify that both formal power series have a positive radius of convergence by estimating the rate of growth of the coefficients.

Since the domain Ω is simply-connected, analytic continuation shows us that (3.2) possesses two linearly-independent global holomorphic solutions v_1, v_2. Consider the Wronskian $W = v_1'v_2 - v_1v_2'$. Since $W' = w_1''w_2 - w_1w_2'' = 0$, W is constant. As v_1, v_2 are linearly independent, $W \neq 0$.

We now check that $F = v_1/v_2$ solves $S_F = g$: since $F''/F' = -2v_2'/v_2$,

$$S_F \;=\; -2 \cdot \frac{v_2''v_2 - (v_2')^2}{v_2^2} - 2 \cdot \frac{(v_2')^2}{v_2^2} \;=\; -2 \cdot \frac{v_2''}{v_2} \;=\; g.$$

We have proved that the equation $S_F = g$ has at least one solution. Conversely, suppose that F, G are two functions with the same Schwarzian derivative g. Since g is holomorphic, F, G are locally univalent, so $G \circ F^{-1}$ is defined locally. Comparing

$$S_{G\circ F^{-1}} = S_G(F^{-1}(z))(F^{-1})'(z)^2 + S_F,$$
$$S_{F\circ F^{-1}} = S_F(F^{-1}(z))(F^{-1})'(z)^2 + S_F,$$

one sees that $G \circ F^{-1}$ is a Möbius transformation as desired. □

Remark. If F is not locally univalent, then S_F has a pole of order 2. For instance, if $F(z) = z^\alpha H(z)$ where $H(z)$ is a holomorphic function with $H(0) \neq 0$, then

$$N_F(z) \;:=\; \frac{F''(z)}{F'(z)} \;=\; \frac{\alpha(\alpha-1)z^{\alpha-2}H(z) + \dots}{\alpha z^{\alpha-1}H(z) + \dots} \;=\; \frac{\alpha - 1}{z} + \dots$$

and

$$S_F(z) = \frac{1-\alpha}{z^2} - \frac{(\alpha-1)^2}{2z^2} + \ldots = \frac{1-\alpha^2}{2} \cdot \frac{1}{z^2} + \ldots$$

3.2 Schwarzians of conformal metrics

Recall from Lemma 2.4 that solutions of the Gauss curvature equation $\Delta u = e^{2u}$ are smooth. The *Schwarzian derivative of the solution* u is defined as

$$S_u = 2\left[\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial z}\right)^2\right].$$

Using the factorization $\Delta = 4\partial\overline{\partial}$, it is easy to check that $\overline{\partial} S_u = 0$. In other words, S_u is holomorphic. Furthermore, if $u_F = \log \frac{2|F'|}{1-|F|^2}$, then the two definitions $S_{u_F} = S_F$ agree.

Lemma 3.3. *Suppose u, v are two solutions of the GCE*

$$\Delta u = e^{2u}, \qquad \text{on } \Omega,$$

which have the same Schwarzian derivative $S_u = S_v$ on Ω. If $u(z_0) = v(z_0)$ and $\frac{\partial u}{\partial z}(z_0) = \frac{\partial v}{\partial z}(z_0)$ agree at a single point $z_0 \in \Omega$ then $u = v$.

The following elegant proof comes from the survey "Conformal metrics" by D. Kraus and O. Roth [14, Lemma 5.7].

Proof. The first step is to show that u and v have the same Taylor expansion at z_0, i.e. for any $j, k \geq 0$,

$$\partial^j \overline{\partial}^k u(z_0) = \partial^j \overline{\partial}^k v(z_0). \tag{3.3}$$

By assumption, we know (3.3) for $k = 0$ and $j = 0, 1$. The equality of the Schwarzians $S_u = S_v$,

$$\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial z}\right)^2 = \frac{\partial^2 v}{\partial z^2} - \left(\frac{\partial v}{\partial z}\right)^2,$$

expresses higher ∂_z derivatives of u in terms of lower ∂_z derivatives, which gives (3.3) for $k = 0$ and all $j \geq 0$ by induction. Since u and v are real-valued, we automatically have (3.3) for $j = 0$ and all $k \geq 0$. Finally, the Gauss curvature equation $\Delta u = 4\partial\overline{\partial} u$ shows (3.3) for all other values of j, k.

The functions $v_1(z) = e^{-u(z)}$ and $v_2(z) = e^{-v(z)}$ solve

$$v_{zz} + \frac{S_u(z)}{2} v = 0.$$

Consider the analytic Wronskian

$$W(z) = \frac{\partial v_1}{\partial z}(z) v_2(z) - v_1(z) \frac{\partial v_2}{\partial z}(z).$$

Since $S_u = S_v$,

$$\frac{\partial W}{\partial z} = 0,$$

i.e. W is antiholomorphic in Ω. By the first part of the proof, $\overline{\partial}^k W(z_0) = 0$ for all $k \geq 0$. Hence, $W \equiv 0$. Hence,

$$\partial_z \left(\frac{v_1(z)}{v_2(z)} \right) = \frac{W(z)}{v_2(z)^2} = 0.$$

Since v_1/v_2 is a real-valued function, v_2 is a constant multiple of v_1. As v_1 and v_2 agree at z_0, $v_1 = v_2$. In other words, $u = v$ as desired. $\square$

Using the above lemma, we can prove Liouville's theorem for regular solutions of the GCE.

Proof of Liouville's theorem for regular solutions. Suppose u is a regular solution of the GCE in Ω. The set of F's such that $S_F = S_u$ are parametrized by $\mathrm{Aut}(\hat{\mathbb{C}})$. We need to find an $F : \Omega \to \mathbb{D}$ which corresponds to u, i.e.

$$u = u_F = \log \frac{2|F'|}{1 - |F|^2}.$$

Given a point $z_0 \in \Omega$, we claim that there exists a *unique* holomorphic function F defined in a small ball $B(z_0, r)$ centered at z_0 with $F(z_0) = 0$ and $F'(z_0) > 0$ for which $u_F = u$ in $B(z_0, r)$.

If $u = \log \frac{2|F'|}{1-|F|^2}$ in $B(z_0, r)$ with $F(z_0) = 0$ and $F'(z_0) > 0$, then

$$F(z_0) = 0, \qquad F'(z_0) = \frac{1}{2} e^{u(z_0)}, \qquad F''(z_0) = e^{-u(z_0)} \cdot \frac{\partial u}{\partial z}(z_0). \tag{3.4}$$

If F exists, then it is uniquely determined by its Schwarzian derivative and its 2-jet of F at z_0.

By Lemma 3.2, there exists a unique locally-univalent holomorphic function F with $S_F = S_u$ whose 2-jet at z_0 is given by (3.4). We restrict our attention to a small ball $B(z_0, r)$ where $|F(z)| < 1$ and $F'(z) \neq 0$. By Lemma 3.3, $u = \log \frac{2|F'|}{1-|F|^2}$. This proves existence.

Let z_0 be a fixed point in Ω, and let F be a holomorphic function defined in a small ball $B(z_0, r) \subset \Omega$ such that $S_F = S_u$. Given any path γ from z_0 to z_1, we can develop F along this path (by the method of analytic continuation) to produce a holomorphic function F_γ defined in a neighbourhood of z_1 with $S_F = S_u$. Since developing along homotopic paths leads to the same result, we can patch the elements to a global function F on Ω. Here, we crucially use the fact that Ω is simply-connected. $\square$

3.3 Dealing with singularities

Let $\Omega \subset \mathbb{C}$ be a domain and C be a discrete set. If λ is a conformal metric on $\Omega \setminus C$ of curvature -1, the Liouville map F could be a multi-valued function: when we walk around one the singularities $c_i \in C$, F could change by an element of $\operatorname{Aut} \mathbb{D}$. In this section, we show that that $u = \log \lambda$ extends to a solution of $\Delta u = e^{2\mu} + \mu$ on Ω for some measure $\mu = 2\pi \sum \alpha_i \delta_{c_i}$ with $\alpha_i \geq -1$ and the monodromy of the Liouville map F around a singularity $c_i \in C$ is trivial if and only if α_i is a non-negative integer. In particular, F is a single-valued function if and only if all the α_i are non-negative integers.

Since the monodromy around an isolated singularity is a local issue, it suffices to consider the case when λ has just one isolated singularity at $0 \in \Omega$. Write $\Omega^* = \Omega \setminus \{0\}$. If we walk around 0, F may change by a Möbius transformation $\phi \in \operatorname{Aut}(\mathbb{D})$, i.e. $F \to \phi \circ F$. An element of $\operatorname{Aut}(\mathbb{D})$ could be either hyperbolic, elliptic or parabolic. We consider these cases separately.

Hyperbolic monodromy

We show that this possibility does not occur, i.e. that ϕ cannot be a hyperbolic Möbius transformation. For this purpose, we make $F : \Omega^* \to \mathbb{D}$ single-valued by lifting it to the universal cover $\mathbb{H} \to \Omega^*$. We denote the lift by $\hat{F} : \mathbb{H} \to \mathbb{D}$. Going around a small loop $\partial B(0, \varepsilon)$ around 0 in Ω^* amounts to walking from ζ_0 to ζ_1 in the universal cover $\mathbb{H}$ which lie above $\varepsilon \in \partial B(0, \varepsilon)$. By the Schwarz lemma,

$$d_{\mathbb{H}}(\zeta_0, \zeta_1) \;\ge\; d_{\mathbb{D}}\big(\hat{F}(\zeta_0), \hat{F}(\zeta_1)\big) \;=\; d_{\mathbb{D}}\big(\hat{F}(\zeta_0), (\phi \circ \hat{F})(\zeta_0)\big). \tag{3.5}$$

Since the hyperbolic length of the loop $\partial B(0, \varepsilon)$ tends to 0 as $\varepsilon \to 0$, the left hand side of (3.5) can be made arbitrarily small. However the right hand side of (3.5) is bounded below since the hyperbolic translation length of a hyperbolic Möbius transformation is positive, i.e. $\inf_{z \in \mathbb{H}} d_{\mathbb{H}}(z, \phi(z)) > c > 0$. This gives the desired contradiction.

Elliptic monodromy

Suppose that ϕ is an elliptic transformation which rotates around the neutral point $z_0 \in \mathbb{D}$. If $\psi(z) = \frac{z - z_0}{1 - \overline{z_0} z}$ is a Möbius transformation that sends $z_0 \to 0$, then $\psi \circ F : \Omega^* \to \mathbb{D}$ is a multi-valued holomorphic function whose monodromy transformation is a rotation about 0. In other words, if we go once around the origin, $\psi \circ F$ is multiplied by a unimodular constant:

$$\psi \circ F \to e^{2\pi i \alpha}\, \psi \circ F, \quad 0 \le \alpha < 1.$$

Since the quotient $(\psi \circ F)(z)/z^{\alpha - 1}$ is bounded and single-valued near the origin, $(\psi \circ F)(z) = z^{\alpha - 1} H(z)$ where H is a holomorphic function with $H(0) = 0$. If H has a zero of order k at the origin, then $\psi \circ F = z^{\alpha + k - 1} + \dots$. Set $n = \alpha + k - 1$. From this expansion, it follows that

$$\Delta u_F = \Delta u_{\psi \circ F} = \Delta \log \frac{2|(\psi \circ F)'(z)|}{1 - |\psi \circ F(z)|^2}$$

has a singularity $(n - 1) \cdot 2\pi \delta_0$ at the origin, while

$$S_F = \frac{1 - n^2}{2} \cdot \frac{1}{z^2} + \mathcal{O}(1/z).$$

Parabolic mondromy

Suppose that F has parabolic monodromy, i.e. $\phi \in \operatorname{Aut}\mathbb{D}$ has a parabolic fixed point on the unit circle. Let $\psi : \mathbb{D} \to \mathbb{H}$ be a Möbius transformation which takes this parabolic fixed point to infinity. The monodromy of $\psi \circ F$ is a translation $z \to z + c$ with $c \in \mathbb{R}$. We scale ψ to make $c = 2\pi$. Let $\hat{F}$ be the composition

$$\Omega^* \xrightarrow{\ F\ } \mathbb{D} \xrightarrow{\ \psi\ } \mathbb{H} \xrightarrow{\ \exp(i\tau)\ } \mathbb{D}^*.$$

Thanks to the exponential, $\hat{F}$ single-valued and takes the origin to itself (the singularity is removable because the image is bounded). Inspection shows that $\gamma = \hat{F}'(0) \neq 0$. We use this information is work out the singularity of $\Delta \log \lambda$ at the origin, and the behaviour of the Schwarzian S_F near the origin.

Since $\psi : (\mathbb{D}, \rho_{\mathbb{D}}) \to (\mathbb{H}, \rho_{\mathbb{H}})$ and $e^{i\tau} : (\mathbb{H}, \rho_{\mathbb{H}}) \to (\mathbb{D}^*, \rho_{\mathbb{D}^*})$ are local isometries,

$$\frac{2|F'(z)|}{1 - |F(z)|^2} = \frac{|\hat{F}'(z)|}{|\hat{F}(z)| \log |1/\hat{F}(z)|},$$

which tells us that $u_F - u_{\mathbb{D}^*} = O(1)$ near the origin, from which we see that $\Delta u_F(\{0\}) = \Delta u_{\mathbb{D}}(\{0\}) = -2\pi$. Here, we have used that if $v \in L^1_{\mathrm{loc}}$ such that Δv is a locally finite measure then

$$\Delta v(\{a\}) = \lim_{r \to 0} \fint_{\partial B(a,r)} v(z)|dz|.$$

In particular, if v is bounded then Δv cannot charge points. (In this case, Δv is unable to charge sets of logarithmic capacity 0.)

As $S_{\exp(i\tau)} = 1/2$,

$$S_{\hat{F}} = \frac{1}{2} \cdot (\psi \circ F)'(z)^2 + S_F. \tag{3.6}$$

Under $\psi \circ F$, the circle $\partial B(0, \varepsilon)$ is approximately mapped to the horizontal line $\{\operatorname{Im} z = \log \frac{1}{\gamma\varepsilon})\}$, which is then roughly mapped to $\partial B(0, \gamma\varepsilon)$ by $e^{i\tau}$. This means that $(\psi \circ F)' \sim 1/z$ near the origin. Putting this into (3.6), we obtain:

$$S_F = -\frac{1}{2z^2} + \dots$$

In other words, S_F which has a pole of order 2 with $a_{-2} = 1/2$.