Potential theory

Do as many questions as you can, partial answers welcome!

1. Show that $\Delta \log^+ |z|$ is Lebesgue measure on the unit circle.

2. Suppose $f_n \to f$ in $W^{1,2}(\Omega)$. Show that after passing to a subsequence, the f_n converge pointwise to f outside a polar set (i.e. outside a set of $W^{1,2}$-capacity 0).

3. Suppose $v(z)$ is a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Show that for a point $a \in \Omega$,
$$\Delta v(\{a\}) = \lim_{r\to 0} \frac{2\pi}{\log r} \fint_{\partial B(a,r)} v(z)|dz|.$$

4. Suppose F is a finite Blaschke product with $F(0) = 0$.

   (a) Use Brownian motion to show that the Lebesgue measure on the unit circle is invariant under F.

   (b) Let $G(z) = \log \frac{1}{|z|}$ be the Green's function of the unit disk with a pole at the origin. Use Brownian motion to show that
$$\sum_{F(w)=z} G(w) = G(z), \qquad z \neq 0,$$
   where the sum on the left is counted with multiplicity.

5. Show that the universal covering map $\mathbb{D} \to \mathbb{D} \setminus E$ is an inner function if and only if the set E is polar.

   Note. A holomorphic function F on the unit is called *inner* if for almost every $\theta \in [0, 2\pi)$, the radial limit
$$\lim_{z\to e^{i\theta}} F(z)$$
   exists and has absolute value 1.

6. Suppose u is a subharmonic function defined on a domain $\Omega \subset \mathbb{C}$. Show that
$$\underline{\Delta} u(z) = \liminf_{r\to 0} \frac{4}{r^2}\left(\fint_{|z|=r} u(z)|dz| - u(0)\right) \geq e^{2u(z)}, \qquad z \in \Omega,$$
   is satisfied if and only if $\Delta u \geq e^{2u}$ in the sense of distributions.

7. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n$. Call a finite measure $\mu \geq 0$ *good* if the equation

$$\Delta u = u^2 - \mu$$

has at least one solution in $B(0,1)$.

According to the method of sub- and super- solutions, there is at least one solution between any subsolution u_* and supersolution $u^* \geq u_*$.

(a) Show that if μ is good, then any measure $0 < \nu \leq \mu$ is good.

(b) Show that the sum of two good measures is good.

(c) Show that $\mu = \delta_0$ is good in dimensions $n = 2, 3$. In fact, show that any measure is good in dimensions 2 and 3.

(d) Show that $\mu = \delta_0$ is bad when $n \geq 4$.

Hint. Let $\varphi \in C_c^\infty(\mathbb{D})$ be a test function such that $\varphi(0) \neq 0$. Define $\varphi_n(x) := \varphi(nx)$. By the definition of a weak solution,

$$\int_{B(0,1)} u \Delta \varphi_n = \int_{B(0,1)} u^2 \varphi_n + \varphi(0).$$

Taking $n \to \infty$ to derive a contradiction. Note that by the definition of a weak solution u^2 is locally integrable.

(e) Show that if μ is good, then there exists a solution u_μ with zero boundary values, in the sense that the measures $u(r\zeta)dS_\zeta \to 0$ converge in the weak-$*$ topology.

8. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n$. In this problem, we study the PDE

$$\begin{cases} \Delta u = u^2, & \text{in } B(0,1), \\ u = \mu, & \text{on } \partial B(0,1), \end{cases} \tag{1}$$

where $\mu \geq 0$ is a finite measure on $\partial B(0,1)$.

For u to be qualify as a solution of (1), we require that

(i) the equation $\Delta u = u^2$ holds weakly in the sense of distributions,

(ii) the measures $u(r\zeta)dS_\zeta \to \mu$ converge in the weak-$*$ topology.

We say that $\mu \geq 0$ is a *good* measure if the PDE (1) admits at least one solution. We write P_μ for the Poisson extension of μ to the disk. In other words, P_μ is a harmonic function whose boundary measure is μ.

(a) Show that the PDE (1) admits at most one solution. If it exists, we will denote it by u_μ.

(b) Show that if $\mu_1 \geq \mu_2$ are good measures, then $u_{\mu_1} \geq u_{\mu_2}$.

(c) Check that for any positive measure $\mu \geq 0$,

(i) 0 is a subsolution and

(ii) P_μ is a supersolution of $\Delta u = u^2$.

(d) Given a positive measure $\mu \geq 0$, let u be the pointwise-maximal solution that lies below P_μ. Show that $u = u_\nu$ for some $0 \leq \nu \leq \mu$, and $\mu = \nu$ if and only if μ is good.

(e) Show that if μ is good, then any measure $0 < \nu \leq \mu$ is also good.

(f) Show that the sum of two good measures is good.