Chapter 4

Vásquez theorem

In Chapter 2, we studied the Gauss curvature equation $\Delta u = e^{2u} + \mu$ with $\mu \geq 0$. In this chapter, we consider what happens when μ is negative. We have already encountered the metrics

$$u_\alpha = \log (z^\alpha)^* \lambda_{\mathbb{D}} = \log \frac{2\alpha}{|z|^{1-\alpha}(1-|z|^{2\alpha})}, \qquad 0 < \alpha < 1,$$

and

$$u_0 = \log \lambda_{\mathbb{D}\setminus\{0\}} = \log \frac{1}{|z| \log |1/z|},$$

which solve GCE_μ with $\mu = -2\pi(1-\alpha)\delta_0$. When $\alpha = 1$, the angle at the origin is 2π. As α decreases, the cone angle at the origin shrinks. Tending $\alpha \to 0$, the u_α converge to u_0, in which case, the cone angle at the origin is zero. Can one decrease the delta mass at the origin any further? A little thought shows that one cannot: if there were a solution of GCE_μ with $\mu = -\alpha\delta_0$ with $\alpha > 2\pi$, then by the comparison principle, the maximal solution of GCE_μ would be larger than u_0 on $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, however, the hyperbolic metric of the punctured disk is the largest conformal metric on $\mathbb{D}^*$ with curvature -1.

Alternatively, one can argue as follows: if there were a solution u with $\mu(\{0\}) < -2\pi$, then u would have to grow very quickly near the origin. This would prevent the function e^{2u} from being in L^1_{loc}, which is one of the requirements for u to be considered a solution.

As a second example, take the unit disk minus a discrete set of points C. Let $\lambda_{\mathbb{D}\setminus C}$ be the hyperbolic metric on $\mathbb{D} \setminus C$ and $u_{\mathbb{D}\setminus C} = \log \lambda_{\mathbb{D}\setminus C}$. As we have seen in Chapter 3 on Liouville's theorem, $u_{\mathbb{D}\setminus C}$ satisfies

$$\Delta u = e^{2u} - 2\pi \sum_{c \in C} \delta_c.$$

In this chapter, we prove the following remarkable theorem due to Vásquez which says that GCE_μ has a solution if and only if all the measure of any point is at least -2π.

Theorem 4.1. *Let $\mu \leq 0$ be a finite measure on the unit disk. The Gauss curvature equation*

$$\begin{cases} \Delta u = e^{2u} + \mu, & \text{in } \mathbb{D}, \\ u = 0, & \text{on } \partial\mathbb{D}, \end{cases} \tag{4.1}$$

admits a solution if and only if $\mu(\{z\}) \geq -2\pi$ for all $z \in \mathbb{D}$.

The existence part of the theorem can be proved using the Schauder fixed point theorem. The argument proceeds in two steps. We first assume that $\mu(\{z\}) \geq -2\pi\alpha$ for all $z \in \mathbb{D}$ and obtain the general case with a limiting argument.

4.1 Brezis-Merle inequality

The existence of solutions of GCE_μ when $\mu(\{z\}) > -2\pi$ for every $z \in \Omega$ relies on the Brezis-Merle inequality:

Lemma 4.2 (Brezis-Merle inequality)**.** *Suppose $\mu \leq 0$ is a measure on the complex plane whose total mass $|\mu(\mathbb{C})| = \alpha < 2\pi$. If*

$$G_\mu^{\mathbb{C}}(z) = \frac{1}{2\pi} \int \log \frac{1}{|\zeta - z|} d\mu(\zeta),$$

then

$$\|e^{2G_\mu^{\mathbb{C}}}\|_{L^1(\mathbb{D})} \leq C,$$

where the constant $C > 0$ depends only on α.

Proof. By Jensen's inequality,

$$\begin{aligned} 2G_\mu^{\mathbb{C}}(z) &= \int_{\mathbb{C}} \frac{\alpha}{\pi} \log \frac{1}{|\zeta - z|} \frac{d\mu(\zeta)}{\alpha}, \\ &= \int_{\mathbb{C}} \log \frac{1}{|\zeta - z|^{\frac{\alpha}{\pi}}} \frac{d\mu(\zeta)}{\alpha}, \\ &\leq \log \int_{\mathbb{C}} \frac{1}{|\zeta - z|^{\frac{\alpha}{\pi}}} \frac{d\mu(\zeta)}{\alpha}. \end{aligned}$$

In other words,

$$e^{2G_\mu^{\mathbb{C}}(z)} \leq \int_{\mathbb{C}} \frac{1}{|\zeta - z|^{\frac{\alpha}{\pi}}} \frac{d\mu(\zeta)}{\alpha}$$

By Fubini's theorem,

$$\int_{\mathbb{D}} e^{2G_\mu(z)} |dz|^2 \leq \int_{\mathbb{C}} \left(\int_{\mathbb{D}} \frac{|dz|^2}{|\zeta - z|^{\frac{\alpha}{\pi}}} \right) \frac{d\mu(\zeta)}{\alpha} < \infty,$$

since $\alpha < 2\pi$. □

Corollary 4.3. *Suppose $\mu \leq 0$ is a finite measure on the disk such that $\mu(\{z\}) > -\alpha > -2\pi$ for every $z \in \mathbb{D}$. Then, $e^{2G_\mu^{\mathbb{C}}} \in L^1(\mathbb{D})$.*

Proof. Pick an $\alpha' \in (\alpha, 2\pi)$ and cover $\overline{\mathbb{D}}$ by finitely many balls $\{B(z_i, r_i)\}_{i=1}^n$ with

$$|\mu(B(z_i, 2r_i))| < \alpha' < 2\pi.$$

Set $\mu_i = \mu \cdot \chi_{B(z_i, 2r_i)}$. Clearly,

$$2G_\mu^{\mathbb{C}} \leq 2G_{\mu_i}^{\mathbb{C}} + C_i, \qquad z \in B(z_i, r_i).$$

where $C_i = \frac{|\mu(\mathbb{D})|}{\pi} \log^+ \frac{1}{r_i}$. Since the number of balls is finite, so is $C = \max C_i$. By the Brezis-Merle inequality,

$$\int_{\mathbb{D}} e^{2G_\mu^{\mathbb{C}}} \leq e^C \sum_{i=1}^n \int_{\mathbb{D}} e^{2G_{\mu_i}^{\mathbb{C}}} < \infty.$$

The proof is complete. □

4.2 Method of subsolutions and supersolutions

Theorem 4.4. *Let u_* be a subsolution of* GCE_μ *and u^* be a supersolution. We now show there exists a solution between $u_* \leq u \leq u^*$.*

Proof. By working in a disk $\mathbb{D}_r$ with $r < 1$ and taking $r \to 1$, we can assume that $u_*, u^* \in L^1(\partial\mathbb{D})$. Pick $u_* \leq h \leq u^*$. Consider the closed convex set

$$\mathscr{K} = \Big\{ v \in L^1(\mathbb{D}, |dz|^2),\ u_* \leq v \leq u^* \Big\} \subset L^1(\mathbb{D}, |dz|^2).$$

For $v \in \mathscr{K}$, let u be the solution of the linear Dirichlet problem $\Delta u = e^{2v} + \mu$ in $\mathbb{D}$ with $u = h$ on $\partial\mathbb{D}$ and

$$Tv(z) = \begin{cases} u_*(z), & u < u_*, \\ u(z), & u_* \leq u \leq u^*, \\ u^*(z), & u > u_*. \end{cases}$$

The definition of Tv is cooked up so that T maps $\mathscr{K}$ into itself.

Step 1. We first show that if $Tv = v$ is a fixed point, then v solves GCE_μ. Consider $g = (u - u^*)_+$. By Kato's inequality,

$$\Delta g \geq \chi_{\{u > u^*\}}(e^{2v} - e^{2u^*}).$$

From the definition of truncation, if $u > u^*$, then $v = u^*$. Since g is a non-negative subharmonic function which has zero boundary data, it is identically 0. Then, $u \leq u^*$. A similar argument shows that $u \geq u^*$.

Step 2. We now check that the operator T is continuous on $\mathscr{K}$, that is if $v_n \to v$ in $L^1(\mathbb{D}, |dz|^2)$, then $Tv_n \to Tv$. We have the estimate:

$$\|Tv_n - Tv\|_{L^1} \leq \|u_n - u\|_{L^1} \leq \|e^{2v_n} - e^{2v}\|_{L^1}.$$

Pass to a subsequence such that $v_n \to v$ converges pointwise a.e. Along this subsequence, $e^{2v_n} - e^{2v} \to 0$ and is dominated by $2e^{2u^*}$. Continuity of T now follows from the dominated convergence theorem.

Step 3. It remains to show that T is compact. Suppose $v_1, v_2, \ldots$ is a sequence in $\mathscr{K}$ which is bounded in L^1. Since

$$\|u_n\|_{W_0^{1,1}} \leq C\|e^{2v_n} + \mu\|_M \leq C\|e^{2u_n^*} + \mu\|_M \leq C'$$

and $W_0^{1,1}$ sits compactly inside L^1, we can choose a subsequence of v_n such that the correpsonding sequence of solutions u_n converges in L^1. However, then the sequence of truncations Tv_n also converge in L^1. The proof is complete. □

We can now show:

Proof of Theorem 4.1, when μ has no point masses of size -2π. It is enough to provide a subsolution and supersolution of GCE_μ with zero boundary values. For a subsolution, let u_* be the solution of the Gauss curvature equation $\Delta u = e^{2\mu}$ with zero boundary values. For a supersolution, take $u^* = G_\mu$. □

4.3 Limiting argument

Lemma 4.5. *(Contraction estimate) Let $\mu \leq 0$ be a negative measure on the unit disk and u be a solution of the Gauss curvature equation $\Delta u = e^{2u} + \mu$ with zero boundary values. Then,*

$$\|e^{2u}\|_{L^1(\mathbb{D})} \leq \|\mu\|_{M(\mathbb{D})} + C.$$

Proof. Choose a sequence of measures $\mu_n \in C_c^\infty(\mathbb{D})$ such that $\mu_n \to \mu$ weakly and $\mu_n(\Omega) \to \mu(\Omega)$. Let $f_n \geq 0$ be any sequence of smooth functions that tend to e^{2u} in $L^1(\mathbb{D})$ and $u_n \in C^\infty(\mathbb{D})$ be the solution of the linear Dirichlet problem $\Delta u_n = f_n + \mu_n$ with zero boundary values. We know that for any test function $\phi \in C_c^\infty(\mathbb{D})$,

$$\int_{\mathbb{D}} f_n \phi |dz|^2 - \int_{\mathbb{D}} u_n \Delta\phi |dz|^2 = -\int_{\mathbb{D}} \phi \, d\mu_n. \tag{4.2}$$

Let $S^\varepsilon(t)$ be a smooth increasing function that is 0 for $t < -\varepsilon$ and 1 for $t > \varepsilon$. Taking $\phi(z) = S^\varepsilon(u_n(z))$ and integrating by parts, we get

$$\int_{\mathbb{D}} f_n \phi |dz|^2 + \int_{\mathbb{D}} |\nabla u_n|^2 S'_\varepsilon(u_n(z)) |dz|^2 \leq \|\mu_n\|_{M(\mathbb{D})}. \tag{4.3}$$

Since the second term in the above equation is positive,

$$\|f_n \cdot \chi_{u>0}\|_{L^1(\mathbb{D})} \le \|\mu_n\|_{M(\mathbb{D})}. \tag{4.4}$$

Fatou's lemma shows that

$$\|e^{2u} \cdot \chi_{u>0}\|_{L^1(\mathbb{D})} \le \|\mu\|_{M(\mathbb{D})}. \tag{4.5}$$

Since $\|e^{2u} \cdot \chi_{u<0}\|_{L^1(\mathbb{D})} \le \pi$, the proof is complete. □

Proof of Theorem 4.1, when μ has point masses of size -2π. For $n \ge 1$, let $\mu_n = (1 - 1/n)\mu$ and u_n be the solution of $\Delta u = e^{2u} + \mu_n$ with zero boundary data. By the comparison principle, the u_n's form an increasing family of solutions. Let u be their pointwise limit. The contraction estimate

$$\|e^{2u_n}\|_{L^1(\mathbb{D})} \le \|\mu_n\|_{M(\mathbb{D})} + C$$

tells us that

$$\|\Delta u_n\|_{M(\mathbb{D})} \;=\; \|e^{2u_n} + \mu_n\|_{M(\mathbb{D})} \;\le\; 2\|\mu_n\|_{M(\mathbb{D})} + C.$$

Since $\|\Delta u_n\|_{M(\Omega)}$ are uniformly bounded, the $u_n = -\frac{1}{2\pi}\int_{\mathbb{D}} G(z,\zeta)\Delta u_n(\zeta)$ are bounded in L^1:

$$\sup_n \|u_n\|_{L^1(\Omega)} < \infty.$$

As the u_n form increasing sequence, $u \in L^1$ and $u_n \to u$ in L^1. The contraction estimate shows that $e^{2u} \in L^1$ and $e^{2u_n} \to e^{2u}$ in L^1. By the dominated convergence theorem (applied to the definition of a weak solution), u solves $\Delta u = e^{2u} + \mu$ as desired. □