Harmonic functions and Brownian Motion

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What is potential theory?

Potential theory lies in the intersection of

- complex analysis,
- probability,
- partial differential equations.

The main objects we will study are

- harmonic and subharmonic functions,
- Brownian motion,
- Sobolev spaces.

Harmonic functions

Let $\Omega \subset \mathbb{R}^n$. A function $u : \Omega \to \mathbb{R}$ is harmonic if it continuous and satisfies the mean-value property (MVP)

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y),$$

provided $\overline{B(x,r)} \subset \Omega$.

Alternatively, a function is harmonic if it is C^2 and satisfies

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Remark. There are 6 different versions of the MVP. One can require the MVP on

1. Spheres,
2. Small spheres,
3. A sequence of arbitrarily small spheres,
4. Balls,
5. Small balls,
6. A sequence of arbitrarily small balls.

In fact, all 6 versions of the MVP are equivalent.

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Proof. Given $\varepsilon > 0$, construct a bump function

$$\phi \in C^\infty, \qquad \phi(x) \geq 0, \qquad \phi(x) \text{ only depends on } |x|,$$

$$\operatorname{supp}\phi \subset B(0,\varepsilon), \qquad \int_{\mathbb{R}^n} \phi(x)\,|dx|^n = 1.$$

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If $\operatorname{dist}(x, \partial\Omega) > \varepsilon$,

$$u(x) \;=\; \int_{\mathbb{R}^n} u(y)\phi(x-y)\,|dy|^n \;\in\; C^\infty.$$

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Harmonic functions

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Green's formula. If $u \in C^2\big(\overline{B(x,r)}\big)$, then

$$\fint_{\partial B(x,r)} u(y)dS(y) - u(x) = \frac{1}{2\pi}\int_{B(x,r)} \Delta u(y)\log\frac{r}{|y-x|}|dy|^2$$

(A similar formula holds in higher dimensions.)

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Exercise. $\Delta u(y) = \lim_{r\to 0}(4/r^2)\cdot\Big\{\fint_{\partial B(x,r)} u(y)dS(y) - u(x)\Big\}$.

Subharmonic functions

The same argument shows that for C^2 functions, the sub-mean value property

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Warning! Typically when discussing subharmonic functions, one asks that they are merely upper semi-continuous. Thus, the equation $\Delta u \geq 0$ needs to be understood weakly in the sense of distributions.

Maximum modulus principle. Suppose $u : \Omega \to \mathbb{R}$ is a harmonic function. If it achieves its maximum value in the interior, then u is constant.

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The maximum-modulus principle shows that if the solution exists, then it is unique.

Poisson Integral Formula. If $u : B(0,1) \to \mathbb{R}$ is harmonic and continuous up to the boundary, then

$$u(x) = \fint_{\partial B(0,1)} u(\zeta) \frac{1 - |x|^2}{|x - \zeta|^n} dS(\zeta).$$

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Dirichlet's problem on the ball. Since the Poisson kernel is harmonic, for any cts. function $f : \partial B(0,1) \to \mathbb{R}$,

$$u(x) := P[f](x) = \fint_{\partial B(0,1)} f(\zeta)\, \frac{1 - |x|^2}{|x - \zeta|^n} dS(\zeta)$$

defines a harmonic function.

Properties of the Poisson kernel

To see that $P[f]$ extends f, notice that the Poisson kernel

$$P_x(\zeta) = \frac{1 - |x|^2}{|x - \zeta|^n}$$

satisfies the following properties:

- $P_x(\zeta) > 0$,
- $⨍_{\partial B(0,1)} P_x(r\zeta) dS(\zeta) = 1$ for $0 < r < 1$,
- $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$P_x(\zeta) < \varepsilon,$$

  provided $1 - |x| < \delta$ and $|x - \zeta| > \delta$.

Harnack's inequality

Suppose u is a positive harmonic function defined on a neighbourhood of $\overline{B(0,1)}$. Since

$$C_1 < \frac{1-|x|^2}{|x-\zeta|^n} < C_2, \qquad x \in B(0,1), \quad \zeta \in \partial B(0,1),$$

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there exists universal constants $c, C > 0$ such that

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If u is harmonic on a domain Ω and K is a compact subset, then

$$c(K,\Omega) < \frac{u(x_1)}{u(x_2)} < C(K,\Omega), \qquad x_1, x_2 \in K.$$

Theorem. Suppose

$$u_1 \leq u_2 \leq \cdots \leq u_n \leq \ldots$$

is an increasing sequence of positive harmonic functions defined on a domain Ω. Either they converge uniformly on compact subsets of Ω or they converge to $+\infty$.

The proof is left as an exercise.

Green's identities

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary. Let η be the outward pointing normal. If $u \in C^1(\overline{\Omega})$, then

$$\int_\Omega u_{x_i}\, dV = \int_{\partial\Omega} u\eta^i\, dS, \qquad i = 1, 2, \ldots n.$$

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To see this, just apply Stokes theorem

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

to the differential form

$$\omega = u(x)\, dx_1\, dx_2 \dots \widehat{dx_i} \dots dx_n.$$

Substituting uv instead of u into

$$\int_{\Omega} u_{x_i}\,dV = \int_{\partial\Omega} u\eta^i\,dS,$$

we get the integration-by-parts formula:

$$\int_{\Omega} (u_{x_i}v + v_{x_i}u)\,dV = \int_{\partial\Omega} uv\eta^i\,dS.$$

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If $u \in C^2(\overline{\Omega})$, we can plug in u_{x_i} instead of u:

$$\int_{\Omega} (u_{x_i x_i} v + v_{x_i} u_{x_i})\, dV = \int_{\partial\Omega} u_{x_i} v\eta^i\, dS.$$

Summing over $i = 1, 2, \dots, n$, we get

$$\int_{\Omega} (\Delta u \cdot v + \nabla u \cdot \nabla v) dV = \int_{\partial\Omega} D_{\eta} u \cdot v \, dS.$$

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Subtracting, we arrive at:

$$\int_\Omega (\Delta u \cdot v - \Delta v \cdot u) dV = \int_{\partial\Omega} (D_\eta u \cdot v - D_\eta v \cdot u) \, dS.$$

Green's formula

We now apply Green's identity in $n = 2$, $\Omega = \mathbb{D} \setminus B(0, \varepsilon)$, $v(z) = \log \frac{1}{|z|}$ and take $\varepsilon \to 0^+$:

$$\int_{\varepsilon<|z|<1} (\Delta u \cdot v - \Delta v \cdot u)|dz|^2 = \int_{\varepsilon<|z|<1} \Delta u \cdot \log \frac{1}{|z|} |dz|^2. \quad (1)$$

As $\varepsilon \to 0^+$, this tends to

$$\int_{\mathbb{D}} \Delta u \cdot \log \frac{1}{|z|} |dz|^2.$$

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$$\int_{\partial\mathbb{D}} (D_\eta u \cdot v - D_\eta v \cdot u)\,|dz| = \int_{\partial\mathbb{D}} u(z)\,|dz|. \quad (2)$$

Green's formula

Finally,

$$\int_{\partial B(0,\varepsilon)} (D_\eta u \cdot v - D_\eta v \cdot u)\,|dz| = o(1) - \frac{1}{\varepsilon}\int_{\partial B(0,\varepsilon)} u(z)\,|dz| \quad (3)$$

tends to $-2\pi\, u(0)$ as $\varepsilon \to 0^+$.

Finally,

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Putting everything together, we arrive at:

$$\int_{\mathbb{D}} \Delta u \cdot \log\frac{1}{|z|} |dz|^2 = \int_{\partial\mathbb{D}} u(z)\,|dz| - 2\pi\, u(0).$$

$\Delta \log |z| = 2\pi\delta_0$

If Ω is an arbitrary domain in the plane with C^1 boundary, the above reasoning shows:

$$\int_\Omega \Delta u \log \frac{1}{|z|} dV = \int_{\partial\Omega} \left\{ D_\eta u \cdot \log \frac{1}{|z|} - D_\eta \log \frac{1}{|z|} \cdot u \right\} dS - 2\pi\, u(0).$$

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If $u=\phi\in C_c^\infty(\mathbb{C})$, then

$$\int_{\mathbb{C}} \Delta u \log\frac{1}{|z|}\,dV = -2\pi\,u(0).$$

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One summarizes this as $\Delta \log |z| = 2\pi\delta_0$ in the sense of distributions.

Green's formula via Brownian Motion

Run Brownian motion starting from the origin, stopped at time τ when it hits the unit circle.

The Green's function has the probabilistic interpretation as the occupation density of Brownian motion.

That is, for any measurable set $E \subset \mathbb{D}$,

$$\mathbb{E}^0(\text{time BM spends in } E) = \int_E \frac{1}{\pi} \log \frac{1}{|z|} |dz|^2.$$

Remark. $\int_{\mathbb{D}} \frac{1}{\pi} \log \frac{1}{|z|} |dz|^2 = \frac{1}{2}$.

Green's formula via Brownian Motion

If f were harmonic then $f(B_t)$ would be a martingale with respect to the Brownian filtration, so that

$$\frac{1}{2\pi}\int f(\zeta)|d\zeta| - f(0) = \mathbb{E}f(B_\tau) - f(0) = 0.$$

Of course, if f is not harmonic, then $f(B_t)$ isn't martingale. The correct formula is:

$$\frac{1}{2\pi}\int f(\zeta)|d\zeta| - f(0) = \mathbb{E}f(B_\tau) - f(0) = \mathbb{E}\int_0^t \frac{1}{2}\Delta f(B_t)dt$$

$$= \frac{1}{2}\int_{\mathbb{D}} \Delta f \cdot \frac{1}{\pi}\log\frac{1}{|z|}\,|dz|^2 = \frac{1}{2\pi}\int_{\mathbb{D}} \Delta f \cdot \log\frac{1}{|z|}\,|dz|^2.$$

Greens functions for C^1 domains

In the **analytic** proof of Green's formula, we used that the Green's function with singularity at z_0 satisfies the following properties:

1. $G(z)$ is zero on $\partial\Omega$.
2. $G(z)$ on $\Omega \setminus \{z_0\}$ is harmonic.
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By the maximum-modulus principle, if such a function exists, then it is unique.

Conversely, let $u(z)$ be the solution of Dirichlet's problem with boundary values $\log|z - z_0|$. Then, $G(z) = u(z) + \log \frac{1}{|z-z_0|}$ satisfies the above properties.

Conformal invariance of Green's function

Lemma. If $f : \Omega \to \Omega'$ is holomorphic and $u : \Omega' \to \mathbb{R}$ is harmonic, then $u \circ f : \Omega \to \mathbb{R}$ is also harmonic.

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Examples.

$$G_{\mathbb{D}}(z,w)=\log\left|\frac{1-z\overline{w}}{z-w}\right|,\qquad G_{\mathbb{H}}(z,w)=\log\left|\frac{w-\overline{z}}{w-z}\right|=\log\left|\frac{z-\overline{w}}{z-w}\right|.$$

Subordination principle

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Proof. For a fixed $z \in \Omega$, the difference

$$u(w) = G_\Omega(z, w) - G_{\Omega'}\big(f(z), f(w)\big).$$

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Corollary. If $\Omega \subset \Omega'$, then $G_\Omega(z, w) \leq G_{\Omega'}(z, w)$.

Lemma. For any domain $\Omega \subset \mathbb{C}$ with C^1 boundary,

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Hence, $G_\Omega(z, w) \leq G_\Omega(w, z)$.

For a function u on the disk, write $u_r(\zeta) := u(r\zeta)$.

For $f(\zeta) \in C(\partial\mathbb{D})$, form the Poisson extension

$$u(x) = P[f](x) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1 - |x|^2}{|x - \zeta|^2} f(\zeta)|d\zeta|.$$

Define $P_r[f] := u_r$.

Lemma. For $f \in C(\partial\mathbb{D})$,

$$P_r[f] \to f, \quad \text{as } r \to 1^-, \quad \text{uniformly on } \partial\mathbb{D}.$$

Positive Harmonic Functions

Theorem. There is a bijection between positive harmonic functions and measures on the unit circle given by integration against the Poisson kernel:

$$u(x) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1 - |x|^2}{|x - \zeta|^2} \, d\mu(\zeta)$$

Examples. Lebesgue measure corresponds to the function 1, while the Poisson kernel $P_\zeta(x)$ corresponds to $2\pi\,\delta_\zeta$.

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1. For $0 < r < 1$, consider the probability measure $\mu_r := u_r(\theta)(d\theta/2\pi)$.
2. Since the space of probability measures is weak-$*$ compact, there exist $r_k \to 1^-$ such that $\mu_r \to \mu$.

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2. Since the space of probability measures is weak-$*$ compact, there exist $r_k \to 1^-$ such that $\mu_r \to \mu$.
3. Weak-$*$ convergence implies that for any $x \in \mathbb{D}$,

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4. As $P[\mu_r] = u(rx)$, we see that $P[\mu](x) = u(x)$ as desired.

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$$\int f \cdot P_r[\mu] \cdot \frac{d\theta}{2\pi} \to \int f \, d\mu, \qquad f \in C(\partial\mathbb{D}).$$

By Fubini's theorem,

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Remark. We have seen any positive harmonic function u can be represented as $P[\mu]$ for some measure μ. The above convergence result shows that the measure μ is unique.

Weyl's lemma

Lemma. A function $u : \Omega \to \mathbb{R}$ is harmonic if and only if $u \in L^1_{\text{loc}}$ and

$$\int_\Omega u\Delta\phi = 0$$

for all $\phi \in C_c^\infty(\Omega)$.

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Proof. If u was C^2, we could integrate by parts to see that

$$\int_\Omega \Delta u \cdot \phi = 0, \qquad \text{for all } \phi \in C_c^\infty(\Omega),$$

which would imply that $\Delta u = 0$.

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Since

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$$= \int_\Omega u \cdot \Delta(\varphi * \eta^\varepsilon) = 0,$$

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each u_ε satisfies the MVP on balls.

Therefore, u itself satisfies the MVP on balls.

Thank you for your attention!