# Brownian motion

Oleg Ivrii

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- The construction can be used to show that one can solve the continuous Dirichlet problem by discrete approximation.

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- Brownian motion can be constructed as the scaling limit of simple random walk on $\delta\mathbb{Z}^n \subset \mathbb{R}^n$.
- The construction can be used to show that one can solve the continuous Dirichlet problem by discrete approximation.
- Last class, we did not use this construction of Brownian motion in the proofs. Instead, we relied on the fact that Brownian motion is conformally invariant.

# Conformal invariance

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- The internal clock of the image of Brownian motion is given by
$$\int_0^t |f'(B_s)|^2 ds.$$
- For some applications like to Liouville's theorem, this whole time change business is important. For many other things, it is not: we only care about the destination and not the journey.

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- In dimension 2, one argues as follows: suppose you are located on $S_n = \{y : |y - x| = 2^n\}$. Brownian motion cannot stay in $\{y : 2^{n-1} < |y - x| < 2^{n+1}\}$ forever, so it must eventually hit either $S_{n-1}$ or $S_{n+1}$.

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- The annulus $\{y:2^{n-1}<|y-x|<2^{n+1}\}$ has a conformal involution which changes the two boundary components and fixes $S_n$.

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- This reduces BM in dimension 2 to simple random walk on $\mathbb{Z}$.
- Since simple random walk on the integer line hits arbitrarily large negative numbers infinitely often, BM in dimension 2 eventually hits $B(x, r)$ no matter how small $r$ is.

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- Since probability is a measure, BM misses countable sets.
- A set is called polar if BM misses it almost surely. Countable sets are polar.
- Suppose $f$ is a holomorphic function. Since BM almost surely misses the critical points of $f$, BM is actually invariant under all holomorphic maps.

# Dirichlet's problem

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- Let $x \in \Omega$. To define $u(x)$, we run BM started at $x$ until we hit $\partial\Omega$. We record the value of $f(B_\tau)$ at the point where we hit $\partial\Omega$.
- We run this experiment 1000000 times and take the average of the values we have written down. Taking 1000000 to infinity, we get:

$$u(x) := \mathbb{E}^x f(B_\tau).$$

  The subscript $x$ refers to the fact that BM is started at $x$.

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- The definition
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always lead to a harmonic function since it is easy to test the mean-value property.
- The issue is that $u(x)$ may not have the right boundary values. I gave the example of $\Omega = \mathbb{D} \setminus \{0\}$, $f(0) = 1$ and $f(z) = 1$ on the unit circle. In fact, the Dirichlet's problem does not have a solution in this case.

# Dirichlet's problem

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- Suppose $\text{dist}(x, \partial\Omega) = r$ where $r = \varepsilon/2^n$. We want to show that if $n$ is large, then the probability that BM escapes $B(x, \varepsilon)$ before hitting $\partial\Omega$ is small.

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- In order for BM to escape $B(x, \varepsilon)$ and not hit $\partial\Omega$ it must accomplish many miracles: it must pass through $n$ dyadic annuli without hitting the fence $\partial\Omega$.
- But there is always a definite chance that BM does hit $\partial\Omega$ when crossing each of the dyadic annuli. This makes the probability of escape small.

Thank you for your attention!