Harmonic measure and Green's functions

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Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain. If $f \in L^\infty(\partial\Omega)$, we call

$$u(x) := \mathbb{E}^x f(B_\tau).$$

the generalized solution of the Dirichlet problem.

If $A \subset \partial\Omega$, define the harmonic measure of A as viewed from x

$$\omega_x(A) := \mathbb{E}^x \chi_A(B_\tau) = \mathbb{P}^x(B_\tau \in A).$$

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Thus, $\exists!$ probability measure ω_x s.t. $u(x) = \int_{\partial\Omega} f(\zeta) d\omega_x$.

Harmonic measures on the disk

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More generally, the harmonic measure at

$$\omega_x = P_x(e^{i\theta})\frac{d\theta}{2\pi}.$$

Harmonic measure of an annulus

Let $\Omega = A(0, r, R) = \{x \in \mathbb{R}^d : r < |x| < R\}$.

Denote the boundary components by S_r and S_R.

An easy computation shows that $\log|x|$ is harmonic in dimension 2 while $\frac{1}{|x|^{d-2}}$ is harmonic for $d \geq 3$.

The harmonic measure of S_r as viewed from a point $x \in A(0, r, R)$ is

$$\frac{\log R - \log|x|}{\log R - \log r} \quad \text{or} \quad \frac{\frac{1}{R^{d-2}} - \frac{1}{|x|^{d-2}}}{\frac{1}{R^{d-2}} - \frac{1}{r^{d-2}}}.$$

Theorem. Suppose the BM particle is at x. Almost surely, BM visits $B(0, r)$, no matter how small r is.

Proof. Eventually, BM will leave the ball $B(0, R)$.

The probability that BM visits $B(x, r)$ before that is

$$\frac{\log R - \log |x|}{\log R - \log r}.$$

Taking $R \to \infty$, this quantity tends to 1.

Transience of BM in dimension $d \geq 3$

Theorem. Given any $r > 0$, almost surely, $B_t \notin B(0, r)$ for all t sufficiently large.

Observation. Suppose the BM is located at a point $x \notin B(x, r)$.

The probability that BM escapes $B(x, R)$ without ever hitting $B(0, r)$ is

$$1 - \frac{\frac{1}{R^{d-2}} - \frac{1}{r^{d-2}}}{\frac{1}{R^{d-2}} - \frac{1}{r^{d-2}}}.$$

Taking $R \to \infty$, we see that this quantity is bounded below by a definite constant independent of R.

Transience of BM in dimension $d \geq 3$

To turn this observation into a proof, define the event A_n as:

`After the first time that BM escapes` $B_{n^{10}}$`, it hits` $S_{(n+1)^{10}}$ `before hitting` S_n `if ever.`

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Lemma of many miracles: only finitely many miracles can occur.

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$$u(x) = \int_{\partial\mathbb{D}} u(e^{i\theta}) d\omega_{\mathbb{D},x}, \qquad x \in \mathbb{D}.$$

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However, we know that

$$u(x) = \int_{\partial\mathbb{D}} u(e^{i\theta}) d\omega_{A_\varepsilon,x} + \int_{|z|=\varepsilon} u(e^{i\theta}) d\omega_{A_\varepsilon,x}.$$

for all $x \in A_\varepsilon = A(0, \varepsilon, 1)$.

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for all $x \in A_\varepsilon = A(0, \varepsilon, 1)$. The second term tends to 0 as $\varepsilon \to 0$. The proof also shows that compact polar sets are removable.

More on harmonic measure

Let u be a harmonic function. I will now give another proof of the formula

$$u(x_0) = \int_{\partial\Omega} u(\zeta) d\omega_{x_0}.$$

For simplicity, we work in the discrete setting: let $\overline{\Omega}$ be a discrete domain in $\mathbb{Z}^2$, Ω denote the interior vertices and $\partial\Omega$ denote the boundary vertices.

Begin with the martingale property of SRW:

$$u(x_0) = \frac{1}{4} \sum_{x_1 \sim x_0} u(x_1).$$

If x_1 is a boundary vertex, don't touch the term $u(x_1)$.

If x_1 is an interior vertex, replace

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After repeating this procedure n times, we get an equation of the form

$$u(x_0) = \sum_{x\in\overline{\Omega}} u(x) P_{\overline{\Omega}}(x_0, x, n).$$

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The weight $0 \leq P_{\overline{\Omega}}(x_0, x, n) \leq 1$ measures the probability of going from x_0 to x in n steps. Taking $n \to \infty$ gives us the desired formula.

Green's function

Define the discrete Green's function $G(x) = G_\Omega(x_0, x)$, $x_0, x \in \Omega$ as the occupation density of SRW. More precisely, simulate SRW $x_0, x_1, x_2, \ldots$ until it hits the boundary and set

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The Green's function is uniquely determined by the following properties:

- $G(x)$ is zero on $\partial\Omega$.
- $G(x)$ is harmonic in x except at x_0.
- $\Delta^{\text{disc}} G(x) := \left\{\frac{1}{4}\sum_{x_1 \sim x_0} G(x_1)\right\} - G(x_0) = -1.$

Discrete Green's formula

For a function $u : \overline{\Omega} \to \mathbb{R}$ (not necessarily harmonic), defines its discrete Laplacian as

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Taking $n \to \infty$, we obtain

$$u(x_0) = \int_{\partial\Omega} u(\zeta) d\omega_{x_0}(\zeta) - \sum_{y \in \Omega} \Delta^{\text{disc}}(y_0) G(x_0, y).$$

Thank you for your attention!