Some martingale arguments

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Harmonic measure in one dimension

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Since $u(x) = x$ is discrete harmonic,

$$\begin{aligned} u(x) &= \frac{u(x-1) + u(x+1)}{2} \\ &= \frac{\frac{u(x-2)+u(x)}{2} + \frac{u(x)+u(x+2)}{2}}{2}. \end{aligned}$$

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Keep substituting this formula into itself. At each stage, you have a formula of the form

$$u(x) = \sum_{a \leq y \leq b} P_{x,y,n} \cdot u(y),$$

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Since $P_{x,y,n}$ measures the probability that SRW goes from x to y in n steps,

$$\sum_{a \leq y \leq x} P_{x,y,n} = 1, \qquad n = 0, 1, 2, 3, \ldots.$$

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tend to

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Remembering that $u(x) = x$, we obtain $h_1(x) = \mathbb{P}(S_\tau = a) = \frac{b-x}{b-a}$ and $h_2(x) = \mathbb{P}(S_\tau = b) = \frac{x-a}{b-a}$.

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Remark. This was to be expected since $h_2(x)$ should be harmonic in x and satisfy $h_2(a) = 0$ and $h_2(b) = 1$.

Occupation time in one dimension

To figure out the average amount of time it takes SRW started from x to hit either a or b, you would instead use the function $u(x) = x^2$, which is **not** harmonic:

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Question. What happens when you substitute

$$u(x) = \frac{u(x-1) + u(x+1)}{2} - 1$$

into itself many times and take $n \to \infty$?

Answer. You get

$$u(x) = \mathbb{P}(S_\tau = a)u(a) + \mathbb{P}(S_\tau = b)u(b) - \mathbb{E}\tau.$$

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If $x = 0$, this simplifies to $\mathbb{E}\tau = ab$.

Occupation time in one dimension

More generally, if $\Delta^{\mathrm{disc}} u$ was not constant, the limit would be

$$u(x) = \mathbb{P}(S_\tau = a)u(a) + \mathbb{P}(S_\tau = b)u(b) - \sum_{a<y<b} G(x,y)\Delta^{\mathrm{disc}} u(y).$$

Remark. Passing to the limit as $n \to \infty$ can be justified using the fact that

$$\mathbb{P}(\tau > n) \lesssim e^{-\beta_0 n}$$

for some $\beta_0 > 0$.

(We have to be a bit careful since Δ^{disc} can change sign and we don't want to be canceling infinities.)

Poisson-Green's formula

The continuous version of the above equation is called the Poisson-Green formula.

Suppose $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain, $u \in C^2(\Omega)$ with $\Delta u = g$ in Ω and $u = f$ on $\partial\Omega$.

$$u(x) = \int_{\partial\Omega} f(\zeta) d\omega_x(\zeta) - \frac{1}{2}\int_{\partial\Omega} g(y) G_{\text{prob}}(x,y)|dy|^n.$$

It gives the unique solution to

$$\begin{cases} \Delta u = g, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases}$$

A mathematician might want to work with the function

$$u(x,n) = u_n(x) = x^2 - n$$

since

$$u_{n-1}(x) = \frac{u_n(x-1) + u_n(x+1)}{2}.$$

Being a martingale is equivalent to

$$-\partial_n^{\mathrm{disc}} u(x,n) := u(x,n-1) - u(x,n) = \Delta_x^{\mathrm{disc}} u(x,n).$$

Optional stopping time theorem

If $u(x,n)$ solves $\partial_n^{\text{disc}} + \Delta_x^{\text{disc}} = 0$ and $|u(x,n)| \lesssim e^{\beta n}$ with $\beta < \beta_0$, we can pass to the limit to obtain

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In continuous time, $B_t^2 - t$ is a martingale since

$$u(x,t) = x^2 - t \qquad \text{satisfies} \qquad \partial_t u(x,t) = \frac{1}{2}\Delta u(x,t).$$

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This martingale can be used to compute the average amount of time BM started at x spends in an interval $[a,b]$. One obtains the same answer as in the discrete case.

Continuous Martingales

By the same reasoning,

$$B_t^3 - 3tB_t,$$

$$B_t^4 - 6tB_t^2 + 3t^2, \ldots$$

are also martingales.

These martingales can be used to compute $\mathbb{E}\tau^n$ for $n \geq 1$. For example, if you run BM started at 0 until you hit $\partial[-a, a]$,

$$\mathbb{E}^0[\tau^2] = (1/3)\,\mathbb{E}^0[-B_\tau^4 + 6\tau B_\tau^2] = (5/3)a^4.$$

Exponential Transform

Run BM started at 0 until you hit $\partial(-\infty, a]$. Since

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or $\mathbb{E}^0 \left[\exp(-\lambda\tau) \right] = \exp(-a\sqrt{2\lambda})$. Inverting the Laplace transform gives

$$\mathbb{P}^0(\tau \in ds) = (2\pi s^3)^{1/2} a e^{-a^2/2s}.$$

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(You can compute this directly or apply Green's formula with $u(z) = |z|^2$.)

Thank you for your attention!