Subharmonic functions

Oleg Ivrii

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Background on distributions

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We say that $\varphi_n \to \varphi$ in C_c^∞ if

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A distribution T is a continuous linear functional on $C_c^\infty(\Omega)$, that is an assignment

$$T : C_c^\infty(\Omega) \to \mathbb{R}$$

which respects linearity and convergence.

Examples of distributions

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3. More specifically, any $f \in L^1_{\text{loc}}$ defines a distribution by
$$T_f[\varphi] := \int_\Omega f\varphi|dz|^n.$$
4. There are distributions that are not measures, e.g.
$$T[\varphi] = -\varphi'(p).$$

Differentiating distributions

Motivation. If $\varphi, \psi \in C_c^\infty(\Omega)$, then

$$\int_\Omega \psi'(x)\varphi(x)\,|dz|^n = -\int_\Omega \varphi'(x)\psi(x)\,|dz|^n.$$

One defines the derivative of a distribution T as

$$T'[\varphi] := -T[\varphi'].$$

Example. In $\mathbb{R}$, the distributional derivative of $|x|$ is δ_0.

Laplacians of distributions

Motivation. If $\varphi, \psi \in C_c^\infty(\Omega)$, then

$$\int_\Omega \Delta\psi(x)\varphi(x)\,|dz|^n = \int_\Omega \Delta\varphi(x)\psi(x)\,|dz|^n.$$

It follows that the distributional Laplacian of T as

$$\Delta T[\varphi] = T[\Delta\varphi].$$

Example. In $\mathbb{C}$, the distributional Laplacian of $\log|z|$ is $2\pi\delta_0$.

Positive distributions

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$$T[\varphi] \geq 0, \qquad \text{whenever } \varphi \geq 0.$$

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- Therefore, a positive distribution extends to a bounded linear functional on $C_c(\Omega)$.
- It follows that $T[\varphi] = \int_\Omega \varphi\mu$ for some positive measure μ with $\|\mu\|_{\mathcal{M}} = T[1]$.

Weyl's lemma

Lemma. A function $u : \Omega \to \mathbb{R}$ is harmonic if and only if $u \in L^1_{\text{loc}}$ and

$$\int_\Omega u \Delta \phi = 0$$

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Proof. If u was C^2, we could integrate by parts to see that

$$\int_\Omega \Delta u \cdot \phi = 0, \qquad \text{for all } \phi \in C_c^\infty(\Omega),$$

which would imply that $\Delta u = 0$.

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PDE $\Rightarrow$ Geometry. If $\Delta u = 0$ in the sense of distributions,

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As each u_ε satisfies MVP on balls, so does u.

What is a subharmonic function?

Let $\Omega \subset \mathbb{R}^n$. A function $u : \Omega \to \mathbb{R}$ is subharmonic if it upper semicontinuous and satisfies the sub mean-value property (sMVP)

$$u(x) \le \fint_{\partial B(x,r)} u(y) dS(y),$$

provided $\overline{B(x,r)} \subset \Omega$.

Alternatively, a function is subharmonic if it is in $L^1(\Omega)$ and $\Delta u \ge 0$ in the sense of distributions.

Upper semi-continuity

A function $u : \Omega \to [-\infty, \infty)$ is upper semi-continuous if it can only jump up:

$$\limsup f(y) \leq f(x), \qquad y_n \to x.$$

Remark. Upper semicontinuous functions are allowed to take the value $-\infty$.

Lemma. A function is upper semicontinuous if and only if it can be written as a **decreasing** limit of continuous functions.

The second definition means that

$$\int_{\mathbb{C}} u\Delta\varphi \geq 0, \qquad \varphi \in C_c^\infty(\Omega), \quad \varphi \geq 0.$$

As Δu is a positive distribution, it is actually a locally finite measure.

Warning. The distributional definition defines u up to a set of measure 0.

Remark. The equivalence of the two definitions means that u can be uniquely redefined on a set of measure 0 to make it upper semicontinuous.

Lebesgue set

The Lebesgue set $\mathcal{L}_f$ of $f \in L^1_{\mathrm{loc}}(\Omega)$ is the set of points z for which

$$\exists v : \qquad \lim_{z\to r} \fint_{B(z,r)} |f(z) - v|\,|dz|^2 \to 0.$$

In this case, v is called the Lebesgue value of u.

According to Lebesgue's differentiation theorem, a.e. $x \in \Omega$ is a Lebesgue point.

In the precise representative of $f \in L^1_{\mathrm{loc}}(\Omega)$, we ask that if z is a Lebesgue point, then $f(z) = v$.

Motivation from complex analysis

Suppose $f(z)$ is a holomorphic function on Ω and $B(z,r)$ is a disk compactly contained in Ω.

- If $f(z)$ is zero-free, then $\log|f(z)|$ is harmonic.
- If f has zeros at $\{a_k\} \subset B(z,r)$, Jensen's formula says

$$\frac{1}{2\pi}\int_{\partial B(z,r)} \log|f(w)|d\theta = \log|f(z)| + \sum_k \log\frac{r}{|a_k|}.$$

To prove this, assume that $B(z,r) = \mathbb{D}$ and factor

$$f(z) = g(z)\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}.$$

Motivation from complex analysis

In terms of distributions,

$$\left(\Delta \log |f(z)|\right)\Big|_{B(z,r)} = 2\pi \sum \delta_{a_k}.$$

To see this, factor,

$$f(z) = g(z) \prod_{k=1}^{n} (z - a_k)$$

so that

$$\log |f(z)| = \sum_{k=1}^{n} \log |z - a_k| + \left\{\text{a function in } C^2\right\}.$$

Potentials

Clearly linear combinations with **positive** coefficients of subharmonic functions are subharmonic:

$$f_1, f_2 \in \mathrm{sh}(\Omega), \quad \lambda_1, \lambda_2 \geq 0, \quad \Longrightarrow \quad \lambda_1 f_1 + \lambda_2 f_2 \in \mathrm{sh}(\Omega).$$

For the same reason, if μ is a finite positive measure on $\mathbb{C}$, then

$$p(z) = \int_{\mathbb{C}} \log|\zeta - z| d\mu$$

is subharmonic.

Riesz Representation Theorem

If $h(z)$ is harmonic, then

$$u(z) = h(z) + \int_{\mathbb{C}} \log|\zeta - z| d\mu$$

is a subharmonic function. The converse is also true:

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Riesz representation theorem. Locally, any subharmonic function $u(z)$ can be uniquely decomposed into a harmonic function and a potential.

Here, $\mu = \Delta u$ (distributional), i.e.

$$\int_\Omega u\Delta\varphi = \int_\Omega \varphi d\mu, \qquad \varphi \in C_c^\infty(\Omega).$$

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Basic properties of subharmonic functions

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2. Subharmonic functions lie below harmonic functions: Suppose h is harmonic and u is subharmonic on Ω. If $u \le h$ on $\partial\Omega$ in $\limsup$-sense, then $u \le h$ in Ω.

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4. Being subharmonic is a **local** property: the local s-MVP implies (1) and (3).

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4. Being subharmonic is a **local** property: the local s-MVP implies (1) and (3).
5. The set where $\{u(z) = -\infty\}$ has Lebesgue measure 0. In fact, $\mathcal{L}_u = \{u(z) \ne -\infty\}$.

Taking maxima. The advantage of working with subharmonic functions is that they are more flexible than harmonic functions:

$$u, v \in \mathrm{sh}(\Omega) \implies \max(u, v) \in \mathrm{sh}(\Omega).$$

Disk modification. Suppose $u \in C(\Omega) \cap \mathrm{sh}(\Omega)$. Given a ball B compactly contained in Ω,

$$M_B u = \begin{cases} u, & \text{in } \Omega \setminus B, \\ P[u|_{\partial B}], & \text{in } B. \end{cases} \quad \in \mathrm{sh}(\Omega).$$

Lemma. Suppose $u_n \to u$ is a **decreasing** sequence of subharmonic functions. The limit function is subharmonic.

Decreasing sequences, convex functions

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Example. As the function $g(x) = e^x$ is convex,

$$|f(z)|^p = \exp\big(p \log |f(z)|\big)$$

is subharmonic for any $0 < p < \infty$.

Increasing averages

Theorem. Suppose $u \in \mathrm{sh}(B(0,R))$. Define

$$M_r = \max_{|z|=r} u(z),$$
$$C_r = ⨍_{|z|=r} u(z)|dz|,$$
$$B_r = ⨍_{|z|<r} u(z)|dz|^2,$$

are increasing in r.

As $r \to 0$, each quantity $M_r \to u(0)$, $C_r \to u(0)$ and $B_r \to u(0)$.

Proof. The convergence of the three quantities

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- To see that C_r is increasing, apply the maximum modulus principle to the radially-invariant subharmonic function

$$v(r) = \fint_{|z|=r} u(z)|dz|.$$

- Finally, B_r is increasing since C_r is increasing.

Equivalence of definitions

Our current goal is to show the Geometric and PDE definitions of subharmonic functions are equivalent.

This is clear for C^2 functions in light of Green's formula.

PDE $\Rightarrow$ Geometry. Suppose u is subharmonic in the sense of distributions.

Then, $u_\varepsilon := u * \eta_\varepsilon$ are C^2-sh functions that tend to u in L^1_{loc}.

Since each u_ε satisfies sMVP on balls, so does u.

Equivalence of definitions

Geometry $\Rightarrow$ PDE. Conversely, suppose that u satisfies the sMVP. Then, $u_\varepsilon := u * \eta_\varepsilon$ also satisfy the sMVP. Hence,

$$\Delta(u * \eta_\varepsilon) \geq 0.$$

Since

$$\int_\Omega \Delta(u * \eta_\varepsilon) \cdot \varphi \to \int_\Omega \Delta u \cdot \varphi,$$

$\Delta u \geq 0$.

As $u * \eta_\varepsilon$ are decreasing as $\varepsilon \to 0$, u is upper semicontinuous.

Increasing sequences of subharmonic functions

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Counterexample. Consider $u_n(z) = (1/n)\log|z|$. As $n \to \infty$,

$$u_n(z) \to u(z) = \begin{cases} 0, & z \ne 0 \\ -\infty, & z = 0. \end{cases}$$

Upper semicontinous regularization

Given a function $u : \Omega \to [-\infty, \infty)$, its upper semicontinous regularization is defined as

$$u^*(z) := \limsup_{y \to x} u(y).$$

It is easily checked that $u^*(z)$ is the least USC function which is $\geq u(z)$.

In the last slide, the limit function u was not USC, but its USC-regularization (=precise repesentative) u^* differed from u on a set of measure 0.

Thank you for your attention!