Perron's Method

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Dirichlet's problem. Suppose Ω is a bounded domain. Given a continuous function $f \in C(\partial\Omega)$, solve

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Perron's method. Take

$$u(z) := H[f](z) = \sup_{v \in \Phi} v(z)$$

where Φ is the collection of all subharmonic functions on Ω that are $\leq f$ on $\partial\Omega$ in $\limsup$-sense.

Theorem. The Perron solution $H[f]$ is a harmonic function, which matches f nearly everywhere (except on a polar set).

The key is that Φ is a Perron family, i.e. a **non-empty** collection of subharmonic functions which satisfies

1. If $u, v \in \mathcal{F}$, then $\max(u, v) \in \Phi$.
2. If $u \in \Phi$ and $B \subset\subset \Omega$, then $M_B u \in \Phi$.

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Proof. To show that a function is harmonic, we need to show that it is harmonic on a neighbourhood $B = B(z_0, r) \subset \Omega$.

1. Pick a maximizing sequence $u_n(z_0) \to u(z_0)$.
2. Replace u_n with $M_B u_n$.
3. Replace u_n with $\max(u_1, u_2, \ldots, u_n)$.

The function $\tilde{u} = \lim_{n\to\infty} u_n$ has the following properties:

$$\tilde{u} \le u_\Phi \text{ on } \Omega, \qquad \tilde{u}(z_0) = u_\Phi(z_0), \qquad \tilde{u} \text{ is harmonic on } B.$$

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Claim. $\tilde{u}(z_1) = \tilde{u}_\Phi(z_1)$, where $z_1 \in B$.

1. Pick a maximizing sequence $v_n(z_1) \to u_\Phi(z_1)$.
2. Replace v_n with $M_B v_n$.
3. Replace v_n with $\max(u_1, u_2, \ldots, u_n; v_1, v_2, \ldots v_n)$.

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Since $\tilde{v} - \tilde{u}$ is harmonic on B, $\tilde{u} = \tilde{v}$ on B. In particular, $\tilde{u}(z_1) = u_\Phi(z_1)$ as desired.

A weak local barrier at $\zeta \in \partial\Omega$ is a subharmonic function defined on $\Omega \cap N$ s.t.

$$b < 0 \quad \text{and} \quad \lim_{z\to\zeta} b(z) = 0.$$

Theorem. If Ω possesses a barrier at $\zeta \in \partial\Omega$, then $H[f]$ is continuous at ζ.

The proof relies on two lemmas:

Lemma 1. $H[f] + H[-f] \leq 0$ on Ω.

Bouligand's Lemma

Lemma 2. Suppose Ω has a weak local barrier $\zeta \in \partial\Omega$. Then ζ admits a strong global barrier $b_\varepsilon \in \mathsf{sh}(\Omega)$: $\forall N'$,

$$b_\varepsilon \le 0, \qquad b_\varepsilon \le -1 \text{ on } \Omega \setminus N', \qquad \liminf_{z\to\zeta} b_\varepsilon(z) \ge -\varepsilon.$$

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Let K be a compact set in $\Omega \cap \partial\Delta$ s.t. $\bigl|(\Omega \cap \partial\Delta) \setminus K\bigr| < \varepsilon$.

Consider

$$b^*_\varepsilon(z) = \frac{b(z)}{m} - P_\Delta[\chi_L](z), \qquad \text{on } \Omega \cap \Delta,$$

where $m > 0$ is chosen so that $b^*_\varepsilon \le -1$ on $\Omega \cap \partial\Delta$.

We have thus constructed a **negative** subharmonic function

$$b^*_\varepsilon(z) = \frac{b(z)}{m} - P_\Delta[\chi_L](z), \qquad \text{on } \Omega \cap \Delta$$

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$$b_\varepsilon = \max(b_\varepsilon^*, -1), \qquad \text{on } \Omega \cap \Delta.$$

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$$b_\varepsilon = \max(b^*_\varepsilon, -1), \qquad \text{on } \Omega \cap \Delta.$$

This glues to -1 on $\Omega \setminus \Delta$ to form a subharmonic function on Ω.

Since $f : \partial\Omega \to \mathbb{R}$ is continuous at ζ_0, there exists $\Delta = B(\zeta_0, r)$ so that $|f(\zeta) - f(\zeta_0)| < \varepsilon$ for all $\zeta \in \partial\Omega \cap \Delta$.

Continuity at a regular boundary point

Since $f : \partial\Omega \to \mathbb{R}$ is continuous at ζ_0, there exists $\Delta = B(\zeta_0, r)$ so that $|f(\zeta) - f(\zeta_0)| < \varepsilon$ for all $\zeta \in \partial\Omega \cap \Delta$.

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Hence,

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Hence,

$$\liminf_{z\to\zeta_0} H[f](\zeta_0) \geq f(\zeta_0).$$

Similarly,

$$-\limsup_{z\to\zeta_0} H[-f](\zeta_0) \ \geq\ \liminf_{z\to\zeta_0} H[-f](\zeta_0) \ \geq\ -f(\zeta_0).$$

Barriers for Jordan domains

Theorem. If Ω is a simply-connected domain bounded by a Jordan curve, then every $\zeta \in \partial\Omega$ admits a barrier.

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Corollary. Dirichlet's problem can always be solved in a Jordan domain.

Proof. As $\log(z - \zeta)$ maps $B(\zeta, 1)$ to the left half-plane, the function

$$b(z) = \operatorname{Re} \frac{1}{\log(z - \zeta)}$$

defines a barrier at ζ, with $N = B(\zeta, 1)$.

Thank you for your attention!