Potentials and Energy

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Coloumb's Law

Consider two charges: q at $x \in \mathbb{R}^3$ and Q at $y \in \mathbb{R}^3$. According to Coloumb's law, the force

$$\overrightarrow{F_{x,y}} = \frac{qQ}{|y-x|^2} \cdot \widehat{y-x}.$$

The electric field of a charge x is

$$F(y) = \frac{q}{|y-x|^2} \cdot \widehat{y-x}.$$

The electric field of a configuration of N charges is

$$\sum_{i=1}^{N} \frac{q_i}{|y-x_i|^2} \cdot \widehat{y-x_i}.$$

Coloumb's Law

The work one needs to do to move a charge from y_1 to y_2 along a path $\gamma \in \mathbb{R}^n$ is

$$W = -\int_\gamma F \cdot ds$$

The potential energy $U(y)$ is defined as the amount of work one needs to do in order to bring a unit charge to y from infinity.

Since F is a conservative vector field, $U(y)$ does not depend on the choice of path γ:

$$U(y) = \sum_{i=1}^{N} q_i \cdot \frac{1}{|y - x_i|}$$

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Proof. Suppose that near $y \in \mathbb{R}^n$, the level $\{U = c\}$ looks like an $(n-1)$–dimensional manifold. Indeed, in a tangent direction $\xi \in T_x U$,

$$F \cdot \xi = D_\xi U = 0.$$

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2. To bring charge q_2 to x_2, requires the work
$$\frac{q_1 q_2}{|x_2 - x_1|}.$$
3. To bring charge q_3 to x_3, requires the work
$$\frac{q_1 q_3}{|x_3 - x_1|} + \frac{q_2 q_3}{|x_3 - x_2|}.$$

Energy

Thus, the total work required is

$$E = \sum_{i \neq j} \frac{q_i q_j}{|x_i - x_j|}.$$

The continuous analogues of the potential and energy are:

$$U_\mu = \int_{\mathbb{R}^3} \frac{1}{|x-y|} d\mu.$$

$$E_\mu = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|} d\mu d\mu.$$

Equilibrium Measures

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Equilibrium Measures

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A measure that minimizes the energy is called an equilibrium measure.

Lemma. Energy is lower semicontinuous: if $\mu_n \to \mu$ converge in weak-$*$, then $\liminf_{n\to\infty} E_{\mu_n} \geq E_\mu$.

Corollary. If E is non-polar, it admits at least one equilibrium measure.

(We will later show that the equilibrium measure is unique.)

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An optimal configuration of points is called a Fekete set; however, it may not be unique.

Some remarks

Remark 1. As $n \to \infty$,

$$\mu_n = \frac{1}{n} \sum_{x=1}^{n} \delta_{x_i}$$

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Remark 2. Since any Fekete set lies on the outer boundary of E, the same must be true for the equilibrium measure.

Remark 3. After reaching the equilibrium state, charge does not move in a conductor: $U_\mu(y) = E_\mu$ n.e.

More generally, one can study

$$U_{\mu,\alpha} = \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} d\mu.$$

$$E_{\mu,\alpha} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} d\mu d\mu.$$

When $\alpha = 0$, one instead uses

$$U_{\mu,0} = \int_{\mathbb{R}^d} \log \frac{1}{|x-y|} d\mu.$$

$$E_{\mu,0} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \log \frac{1}{|x-y|} d\mu d\mu.$$

Hausdorff measure and Hausdoff dimension

Suppose $E \subset \mathbb{R}^d$ is a compact set. Its α-dimensional Hausdorff content is given by

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Given $E \subset \mathbb{R}^d$, the Hausdorff dimension of E is the **critical** α between $H^\alpha(E) = \infty$ and $H^\alpha(E) = 0$.

Frostman's theorem

Theorem. Suppose $E \subset \mathbb{R}^d$ is a compact set. Then, $H^\alpha(E) > 0$ iff $\exists \mu \in \mathcal{M}_{\geq 0}(E)$, s.t. $\mu(B(x, r)) \lesssim r^\alpha$.

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Proof. ($\Leftarrow$) If μ exists, then $H^\alpha(E) \gtrsim \mu(E) > 0$.

($\Rightarrow$) Suppose $H^\alpha(E) > 0$. We can assume that $E \subset [0,1]^n$.

For each $m = 1, 2, 3, \ldots,$ we will construct a measure μ_m. The weak limit of these measures will be the desired measure μ.

Frostman's theorem

To define μ_m, we look at all dyadic cubes Q of size 2^{-m}.

- If $Q \cap E \neq \emptyset$, we define μ_m to be a constant multiple of Lebesgue measure so that $\mu(Q) = \ell(Q)^\alpha$.
- If $Q \cap E = \emptyset$, we define $\mu_m(Q) = 0$.

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We now look at dyadic cubes 2^{m-1}. If $\mu_m(Q) \leq \ell(Q)^\alpha$.

- If $\mu_m(Q) \leq \ell(Q)^\alpha$, don't touch $\mu_m|_Q$.
- If $\mu_m(Q) > \ell(Q)^\alpha$, redefine $\mu_m|_Q$ to be the constant multiple of Lebesgue measure so that $\mu_m(Q) = \ell(Q)^\alpha$.

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Continue to redefine μ_m by scanning the scales $2^{m-2}, 2^{m-3}, \dots, 2^0$. At each step of the process, the mass of μ_m can only decrease.

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Since

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Since

$$\|\mu_m\|_{\mathcal{M}} = \sum_{Q:\,\text{maximal}} \ell(Q)^\alpha \gtrsim H^\alpha(E),$$

$\mu \neq 0$. Since μ satisfies $\mu(Q) \leq \ell(Q)^\alpha$ on dyadic cubes, it satisfies this bound on all cubes.

Question. Suppose $\mu \in \mathcal{M}_{\geq 0}(E)$ with $\mu(B(x,r)) \lesssim r^{\alpha}$. Is it true that $E_\alpha[\mu] < \infty$?

Sets admitting measures of finite energy

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Long answer. For any $0 < \beta < \alpha$,

$$\int_E \frac{1}{|x-y|^\beta} d\mu(y) = \beta \int_0^\infty \frac{1}{r^{\beta+1}} \cdot \mu(B(x,r))dr \lesssim \|\mu\|_{\mathcal{M}},$$

so that $E_\beta[\mu] < \infty$.

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Long answer. For $M > 0$, let

$$A = \left\{ x \in E : \int_E \frac{1}{|x-y|^\alpha} d\mu(y) \leq M \right\}$$

and $\nu = \mu|_A$. Then, $\nu\big(B(x,r)\big) \lesssim r^\alpha$.

A set E of Hausdorff dimension $\geq \alpha$ may not admit a truly α-dimensional measure satisfying

$$\mu(B(x,r)) \lesssim r^{\alpha} \qquad \text{or} \qquad E_{\alpha}[\mu] < \infty,$$

but it does admit a measure satisfying these conditions for any $0 < \beta < \alpha$.

Frostman's theorem: summary

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but it does admit a measure satisfying these conditions for any $0 < \beta < \alpha$.

Conversely, if E admits a measure satisfying either of these properties, then $\operatorname{H.dim} E \geq \alpha$.

Capacity via Energy

For a compact set E, its α-capacity is defined as

$$\operatorname{cap}_\alpha E := \inf_{\mu \in \mathcal{P}(E)} \frac{1}{E_\alpha[\mu]},$$

If E does not support a measure with finite α-energy, then its α-capacity is 0.

Warning. There are a several definitions of capacity which are not exactly the same, but they do agree which sets have capacity 0.

Application: projections

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Theorem. Let $E \subset \mathbb{R}^n$ be a compact set and $0 < \alpha < m$. For any measure $\mu \in \mathcal{M}_{\geq 0}(E)$,

$$\int_{\mathbb{G}(m,n)} E_\alpha \big[(\pi_V)_* \mu \big] \, dV \lesssim E_\alpha[\mu],$$

where the implicit constant can depend on m, n, α.

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Corollary. In particular, a.e. projection has finite α-energy, so has $\mathrm{H.\,dim} \geq \alpha$.

Proof.

$$\int_{\mathbb{G}} E_\alpha\left[(\pi_V)_*\mu\right] dV = \int_{\mathbb{G}}\int_E\int_E \frac{1}{|P_V x - P_V y|^\alpha} d\mu(x) d\mu(y) dV,$$

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Proof.

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Exercise. For any two points $x, y \in \mathbb{R}^n$,

$$\int_{\mathbb{G}(m,n)} \frac{1}{|P_V(x-y)|^\alpha} dV \lesssim \frac{1}{|x-y|^\alpha}.$$

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Hence,

$$\text{LHS} \lesssim \int_E \int_E \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y) = E_\alpha[\mu].$$

Theorem. Run B_t in $\mathbb{R}^d$, $d \geq 2$. Almost surely,

$$\mathrm{H.dim}\, B([0,\infty)) = \mathrm{H.dim}\, B([0,1]) = 2.$$

Proof. $(\leq)$ The upper bound follows from that fact that Brownian motion is in $C^{1/2+\varepsilon}$ for any ε.

$(\geq)$ For the lower bound, let μ be the occupation measure on $B([0,1])$: that is $\mu(A) = \Big|\{t : B_t \in A\}\Big|$.

We want to show that a.s. for $0 < \alpha < 2$

$$E_\alpha[\mu] = \int_0^1 \int_0^1 \frac{1}{|B_s - B_t|^\alpha} dsdt < \infty.$$

Application: Brownian Motion

By the quadratic scaling of Brownian motion,

$$\mathbb{E}\big[E_\alpha[\mu]\big] = \int_0^1 \int_0^1 \mathbb{E}\left[\frac{1}{|B_s - B_t|^\alpha}\right] dsdt$$
$$= \int_0^1 \int_0^1 \mathbb{E}\left[\frac{1}{|B_1|^\alpha}\right] \cdot \frac{1}{|s-t|^{\alpha/2}} dsdt.$$

Since

$$\mathbb{E}\left[\frac{1}{|B_1|^\alpha}\right] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |x|^{-\alpha} e^{-x^2/2} < \infty,$$

$\mathbb{E}\big[E_\alpha[\mu]\big]$ is finite.

To a measure $\mu \in \mathcal{M}(S^1)$, we can associate the Fourier series

$$\mu \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

where

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L^2 theory. If $f \in L^2(S^1)$, the Fourier series converges in L^2 and

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_n|^2.$$

Motivation. If $f \in L^2$ and $f' \in L^2$ then

$$f' \sim i \sum_{n=-\infty}^{\infty} n a_n e^{in\theta},$$

In this case, $n^2 \sum |a_n|^2 < \infty$.

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Definition. For $s > 0$, the Sobolev space $H^s = W^{s,2}$ consists of all functions whose Fourier coefficients satisfy $\sum n^{2s} |a_n|^2 < \infty$.

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Warning. For $s < 0$, H^s can contain distributions that are not functions.

Convolution

Given $f, g \in L^1(S^1)$ with Fourier series

$$f \sim \sum a_n e^{in\theta}, \qquad g \sim \sum b_n e^{in\theta},$$

the convolution $f \star g$ is defined as

$$(f \star g)(x) := \frac{1}{2\pi} \int_{S^1} f(y) g(x-y) dy.$$

By Young's inequality, $f \star g \in L^1$ and

$$f \star g \sim \sum a_n b_n e^{in\theta}.$$

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The Fourier transform is essentially an isometry:

$$\|f\|_{L^2}^2 = \frac{1}{2\pi} \|\hat{f}\|_{L^2}^2.$$

The energy has a very natural interpretation in terms of Sobolev spaces:

$$
\begin{aligned}
E_{\mu,\alpha} &= \int_{\mathbb{R}^d} U_{\mu,\alpha}(z)\,d\mu(z) \\
&= c\int_{\mathbb{R}^d} \widehat{U_{\mu,\alpha}}(\xi)\widehat{d\mu}(\xi)|d\xi|^2 \\
&= c\int_{\mathbb{R}^d} |\hat{\mu}|^2|\xi|^{\alpha-d}|d\xi|^2,
\end{aligned}
$$

i.e. $E_\alpha[\mu] < \infty \iff \mu \in H^{\frac{\alpha-d}{2}}$.

Capacity via Sobolev spaces ($\alpha = d - 2$)

For a function $\phi \in H^1$, its Dirichlet energy is given by

$$\mathcal{D}(\phi) = \|\nabla \phi\|_{L^2}^2 = \int_{\mathbb{C}} |\nabla \phi|^2 \, |dx|^n.$$

For a compact set E, its H^1-capacity is defined as

$$\operatorname{cap}_{H^1} E := \inf_{\phi \in \mathcal{F}} \mathcal{D}(\phi),$$

where the infimum is taken (over the H^1-closure of)

$$\phi \in C_c^\infty, \qquad \phi \geq 0, \qquad \phi \geq 1 \text{ on } E.$$

Zero capacity

Lemma. Suppose $\text{cap}_{H^1} E = 0$. Then, it does not carry a non-trivial measure $\mu \in \mathcal{M}_{\geq 0}(E) \cap H^{-1}(\mathbb{R}^d)$.

By assumption, there exists a sequence $\phi_n \to 0$ of positive functions for which
$$\int_{\mathbb{C}} \phi_n d\mu \geq 1.$$
However, by the duality between H^1 and H^{-1},
$$\int_{\mathbb{C}} \phi_n d\mu = \langle \phi_n, \mu \rangle \lesssim \|\phi_n\|_{H^1} \cdot \|\mu\|_{H^{-1}} \to 0.$$

Positive capacity

Lemma. Suppose $\text{cap}_{H^1} E > 0$. Then, there exists a non-trivial measure $\mu \in \mathcal{M}_{\geq 0}(E) \cap H^{-1}(\mathbb{R}^d)$.

Proof idea. The extremal $\phi \in H^1$ is

1. Harmonic on $\mathbb{R}^d \setminus E$,
2. Superharmonic on all on $\mathbb{R}^d$, and
3. Tends to 0 at infinity.

Then, $\mu = -\Delta u \in \mathcal{M}_{\geq 0}(E) \cap H^{-1}(\mathbb{R}^d)$ as desired.

In fact, $\phi = \text{const} \cdot U_\mu$.

Thank you for your attention!