Frostman's theorem and applications

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Potentials and energy

In these slides, we will be following the book of Thomas Ransford, who uses the notation:

$$p_\mu(z) = \int_{\mathbb{C}} \log|z - w| d\mu(w).$$

$$I_\mu(z) = \int_{\mathbb{C}} \int_{\mathbb{C}} \log|z - w| d\mu(z) d\mu(w).$$

- Potentials are subharmonic functions.
- A set is polar if it only supports measures with energy $-\infty$.
- Equilibrium measure maximizes the energy.

Continuity Principle

Theorem. Let K be a compact set and $\mu \in \mathcal{M}_{\geq 0}(K)$.

1. For $\zeta_0 \in K$,

$$\liminf_{z \to \zeta_0} p_\mu(z) = \liminf_{z \to \zeta_0,\, z \in K} p_\mu(z).$$

2. In particular, if

$$\liminf_{z \to \zeta_0,\, z \in K} p_\mu(z) = p_\mu(z_0),$$

then p_μ is continuous at ζ_0.

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Case I. If $p_\mu(\zeta_0) = -\infty$, then

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For $z \in \mathbb{C}$,

$$p_\mu(z) = p_\mu(\zeta) - \int_K \log\left|\frac{\zeta - w}{z - w}\right| d\mu(w).$$

Pick $\zeta \in K$ *to minimize* $\operatorname{dist}(z, K)$ *!*

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Since

$$|\zeta - w| \;\leq\; |\zeta - z| + |z - w| \;\leq\; 2|z - w|,$$

$$\int_\Delta \log\left|\frac{\zeta - w}{z - w}\right| d\mu(w) \leq \varepsilon \log 2.$$

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Theorem. Let K be a compact set and $\mu \in \mathcal{M}_{\geq 0}(K)$. If $p_\mu \geq M$ on K, then $p_\mu \geq M$ on all of $\mathbb{C}$.

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Proof. Since p_μ is harmonic on $\mathbb{C} \setminus K$, the minimum value is achieved on $\partial K \cup \{\infty\}$ in $\liminf$-sense.

However, for any $\zeta \in \partial K$,

$$\liminf_{z \to \zeta_0} p_\mu(\zeta) \geq M,$$

while

$$\lim_{z \to \infty} p_\mu(\zeta) = \infty.$$

Frostman's Theorem

Theorem. Let $K \subset \mathbb{C}$ be a compact set and ν be an equilibrium measure for K. Then:

- $p_\nu \geq I(\nu)$ on $\mathbb{C}$.
- $p_\nu = I(\nu)$ n.e. on K.

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To prove Frostman's theorem, it suffices to show:

$$K_n = \{z \in K : p_\nu(z) \geq I(\nu) + 1/n\}. \qquad \textit{polar?}$$

$$L_n = \{z \in \operatorname{supp} \nu : p_\nu(z) < I(\nu) - 1/n\}. \qquad \textit{empty?}$$

Claim. For any $n = 1, 2, \ldots,$ the set

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$\exists z_0 \in \operatorname{supp} \nu$ s.t. $p_\nu(z_0) \leq I(\nu)$.

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1. USC $\implies$ $p_\nu < I(\nu) + \frac{1}{2n}$ on some ball $\overline{\Delta} = \overline{B(z_0, r)}$.
2. $a = \nu(\overline{\Delta}) > 0$.

Frostman's Theorem

Define a path of probability measures $[0, \varepsilon) \to \mathcal{P}(K)$:

$$\nu_t = \nu + t \begin{cases} \mu, & \text{on } K_n, \\ -\nu/a, & \text{on } \overline{\Delta}, \\ 0, & \text{otherwise.} \end{cases}$$

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$$\begin{aligned} \frac{I(\nu_t) - I(\nu)}{t} &= 2\left[\int_{K_n} p_\nu(z) d\mu(z) - \int_{\overline{\Delta}} p_\nu(z) \frac{d\nu(z)}{a}\right] \\ &\geq 2\left[\left(I(\nu) + \frac{1}{n}\right) - \left(I(\nu) + \frac{1}{2n}\right)\right] \\ &> 0. \end{aligned}$$

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Claim. For any $n = 1, 2, \ldots,$ the set

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is empty.

Proof. If $\exists z_0 \in L_n$, then by USC

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By the first part of the proof,

$$I(\nu) = \int_{K \cap \overline{\Delta}} p_\nu d\nu + \int_{K \setminus \overline{\Delta}} p_\nu d\nu < I(\nu).$$

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Then, $u^* \in \mathrm{sh}(\Omega)$ and $\{z \in \Omega : u(z) < u^*(z)\}$ is polar.

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Remark 1. Only the last assertion is difficult.

Remark 2. One can also take lim sup instead of sup:

$$\limsup u_n = \inf_n \Big(\sup_{m>n} u_m \Big).$$

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To show that the set $\{z : u(z) < u^*(z)\}$ is polar, it suffices to prove:

Claim. For any disk $\overline{\Delta} \subset \Omega$ and $\beta \in \mathbb{Q}$, the set

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Proof (by contradiction). If E is not polar, it contains a compact subset $K \subset \Delta$ with a non-trivial equilibrium measure ν.

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This contradicts Frostman's theorem "$p_\nu = I(\nu)$ n.e. on K."

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E is G_δ.

Let

$$v(z) = \limsup_{n\to\infty} \frac{u(z)}{n} = \begin{cases} 0, & z \in \Omega \setminus E, \\ -\infty, & z \in E. \end{cases}$$

Since $v^*(z) = 0$, E is polar.

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Set

$$u(z) = \sum 2^{-n} p_{\nu_n}(z).$$

Theorem. Let $u \in \mathrm{sh}(\Omega \setminus E)$ where $E \subset \Omega$ be a closed polar set. If u is locally bounded above, then u extends to $\mathrm{sh}(\Omega)$.

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Proof. We extend u to E by upper semi-continuity:

$$u(w) = \limsup_{z \to w} u(z).$$

Let $\overline{\Delta} = \overline{B(z_0, r)} \subset \Omega$ be a ball and h is a harmonic function on $\overline{\Delta}$ such that $u \leq h$ on $\partial\Delta$ in limsup-sense. To show that $u \leq h$ in Δ, note that

$$u(z) - \varepsilon v_E(z) \leq h(z), \qquad \varepsilon > 0.$$

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Recall that $A(r) = \frac{1}{2\pi} \int_{|z|=r} f(re^{i\theta}) d\theta$ is an increasing function of r. It is actually logarithmically convex, i.e. a convex function of $\log r$. Hint: Use the decomposition $u = p_\mu + h$.

Extended Liovuille's theorem

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Clearly, E_ε is a closed polar set.

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$$v = \begin{cases} \max(u, \varepsilon), & \text{on } \Omega, \\ \varepsilon, & \text{on } (\mathbb{C} \setminus \Omega) \cup (\partial\Omega \setminus E_\varepsilon). \end{cases}$$

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Thank you for your attention!