Sobolev spaces

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Let $\Omega \subset \mathbb{R}^n$. The Sobolev space $W^{k,p}(\Omega)$ consists of functions with k distributional derivatives in L^p:

$$\|f\|_{W^{k,p}} := \left(\|f\|_p^p + \|Df\|_p^p + \cdots + \|D^k f\|_p^p\right)^{1/p}.$$

It is easy to see that $W^{k,p}(\Omega)$ is a Banach space.

Lemma. The space $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.

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- If $p < n \le \infty$ then $u \in C^{1-n/p}(\overline{\Omega})$.

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Sobolev compactness theorem. For $1 \le q < p^*$, the space $W^{1,q}$ sits compactly in L^p, that is, any bounded sequence u_m in $W^{1,q}$ has a subsequence which converges in L^p.

Density of smooth functions

Corollary. If Ω has C^1 boundary, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$.

Warning 1. The above theorem fails for arbitrary domains Ω.

Warning 2. $C_c^\infty(\Omega)$ is usually not dense in $W^{k,p}(\Omega)$.

It really suffices to treat the case when $\Omega = \mathbb{H}^n = \{x_n > 0\}$ and then use a partition of unity argument.

Extensions

Theorem. Any function in $W^{1,p}(\mathbb{H}^n)$ extends to a function in $W^{1,p}(\mathbb{R}^n)$. The extension operator $E : W^{1,p}(\mathbb{H}^n) \to W^{1,p}(\mathbb{R}^n)$ can be chosen to be linear and bounded.

Proof. Suppose $u \in C^1(\overline{\mathbb{H}^n})$. On the lower half plane, define

$$Eu(x) = -3u(x_1, x_2, \dots, x_{n-1}) + 4u\Big(x_1, x_2, \dots, x_{n-1}, -\frac{x_n}{2}\Big).$$

Then, $Eu \in C^1(\mathbb{R}^n)$ with $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p}}$.

This **crucial** estimate allows us to extend Eu to $W^{1,p}(\mathbb{H}^n)$ by continuity. By a partition of unity argument, this construction extends to domains bounded by C^1 curves.

Traces

Theorem. Suppose Ω is a bounded domain with C^1 boundary. Any function in $W^{1,p}(\Omega)$ restricts to a function in $L^p(\partial\Omega)$:

$$\|f\|_{L^p(\partial\Omega)} \lesssim \|f\|_{W^{1,p}(\Omega)}.$$

Proof. By a partition of unity argument, we can transfer to the upper half-space.

Suppose $u \in C^1(\mathbb{R}^n)$ and $\zeta(x)$ is a bump function supported in a ball $B(w,1)$, $w \in \mathbb{R}^{n-1}$. By Stokes theorem,

$$\int_{\mathbb{R}^{n-1}} |u|^p \zeta \, dx' = \int_{B^+(w,1)} \partial_{x_n} \Bigl(|u|^p \zeta\Bigr) \, dx \lesssim \|u\|_{W^{1,p}(B^+(w,1))}.$$

Trace zero functions

Suppose Ω is a bounded domain with C^1 boundary. Let $W_0^{1,p}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{1,p}$.

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Corollary. In particular, a function in $W_0^{1,p}(\Omega)$ extends by 0 to a function in $W_0^{1,p}(\mathbb{R}^n)$.

Sobolev inequalities in dimension 1

Theorem. Suppose $1 < p < \infty$. If $f \in W^{1,p}$ then $f \in C^{1-1/p}$.

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Proof. Suppose $f \in C^\infty$. By the fundamental theorem of calculus,

$$|f(y) - f(x)| \leq \int_x^y |f'(y)| dy \leq \left(\int_x^y |f'(y)|^p \right)^{1/p} |y - x|^{1/q}.$$

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Since $1/q = 1 - 1/p$, we see that

$$\|f\|_{C^{1-1/p}} \lesssim \|f\|_{W^{1,p}}.$$

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Since $C^\infty(\mathbb{R})$ is dense in $W^{1,p}$, the inclusion map extends to $W^{1,p}$.

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$$|f(y) - f(x)| \le \int_x^y |f'(t)| dt$$

Integrating over y, we get

$$\begin{aligned} \fint_{x-h}^{x+h} |f(y) - f(x)| dy &\le \fint_{x-h}^{x+h} |f'(t)| \big(h - |x - t| \big) dt \\ &\le \frac{1}{2} \int_{x-h}^{x+h} |f'(t)| dt \\ &\le \frac{1}{2} \|f\|_{W^{1,1}(\mathbb{R})}. \end{aligned}$$

Sobolev inequalities: $1 \le p < n$

Theorem. Suppose $1 \le p < n$. For $1/p^* = 1/p - 1/n$, then

$$\|u\|_{L^{p^*}} \lesssim \|Du\|_{L^p}$$

for $u \in C_c^\infty(\mathbb{R}^n)$.

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Proof ($p = 1$). By the fundamental theorem of calculus,

$$|u(x_1, x_2)| \le \int_{\mathbb{R}} |Du(y_1, x_2)| dy_1.$$

Switching the roles of the two variables and multiplying, we get

$$|u(x)|^2 \le \left(\int_{\mathbb{R}} |Du(y_1, x_2)| dy_1\right)\left(\int_{\mathbb{R}} |Du(x_1, y_2)| dy_2\right).$$

Integrating over x_1:

$$\int_{\mathbb{R}} |u|^2 dx_1 \le \Big(\int_{\mathbb{R}} |Du| dy_1\Big) \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, y_2)| dy_2 dx_1.$$

Integrating over x_2:

$$\int_{\mathbb{R}^2} |u|^2 dx \le \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, y_2)| dy_2 dx_1 \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, y_2)| dy_2 dx_1.$$

The RHS is of course $\|Du\|^2_{L^1(\mathbb{R}^2)}$.

Sobolev inequalities: $1 \le p < n$

For $1 < p < n$, we apply Sobolev's inequality to $|u|^\gamma$ with $\gamma > 1$ to be chosen.

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} \bigl|D|u|^\gamma\bigr|\,dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}|Du|\,dx$$

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Applying Hölder's inequality, we get

$$\le \gamma\left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\,dx\right)^{\frac{1}{p}}$$

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Choose γ s.t. $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} \quad\iff\quad \gamma = \frac{p(n-1)}{n-p} > 1$.

$$\Bigl(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\Bigr)^{\frac{n-1}{n}-\frac{p-1}{p}} \lesssim \Bigl(\int_{\mathbb{R}^n} |Du|^p\,dx\Bigr)^{\frac{1}{p}}.$$

Lemma. For $u \in W^{1,p}$,

$$\fint_{B(x,r)} |u(y) - u(x)|\,dy \lesssim \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy.$$

Sobolev inequalities: $p < n < \infty$

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If $n < p < \infty$, we can apply Hölder's inequality:

$$\text{RHS} \le \left(\int_{B(x,r)} \frac{dy}{|y-x|^{(n-1)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \|Du\|_{L^p} \lesssim r^{1-\frac{n}{p}} \|Du\|_{L^p}.$$

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Proof. For $\xi \in B(x,r)$,

$$|u(\xi) - u(x)| \le \int_x^\xi |Du|,$$

where we integrate over a straight line from x to ξ.

Sobolev inequalities: $p < n < \infty$

Theorem. Any $u \in W^{1,p}$ with $n < p < \infty$ is bounded:

$$\|u\|_{L^\infty} \le C\|u\|_{W^{1,p}}.$$

Proof. By the above lemma,

$$\begin{aligned} |u(x)| &\le \fint_{B(x,1)} |u(x) - u(y)|dy + \fint_{B(x,1)} |u(y)|dy \\ &\lesssim \|Du\|_{L^p} + \|u\|_{L^p}. \end{aligned}$$

Theorem. Any $u \in W^{1,p}$ with $n < p < \infty$ is Hölder continuous:

$$\|u\|_{C^{1-\frac{n}{p}}} \le C\|u\|_{W^{1,p}}.$$

Proof. If $r = |x-y|$ and $W = B(x,r) \cap B(y,r)$,

$$\begin{aligned}|u(x)-u(y)| &\le \fint_W |u(x)-u(z)|dy + \fint_W |u(y)-u(z)|dy\\ &\le \fint_{B(x,r)} |u(x)-u(z)|dy + \fint_{B(y,r)} |u(y)-u(z)|dy\\ &\lesssim r^{1-\frac{n}{p}}\|Du\|_{L^p}.\end{aligned}$$

Theorem. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary. If $1 \le p < n$ and $q < p^*$, the inclusion

$$i : W^{1,q}(\Omega) \subset L^p(\Omega)$$

is compact.

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We may assume that each function u_m is defined on all of $\mathbb{R}^n$, $\|u_m\|_{W^{1,q}(\mathbb{R}^n)} < C$ are uniformly bounded and that u_m are supported on a fixed ball $B(0, R) \supset \Omega$.

Sobolev compactness

Set $u_m^\varepsilon = \eta_\varepsilon * u_m$. It is easy to see that for each $m = 1, 2, \ldots,$

$$u_m^\varepsilon \to u_m, \qquad \text{in } L^q(B(0,R))$$

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By interpolation, we only need to show that

$$\sup_m \|u_m^\varepsilon - u_m\|_{L^1(B(0,R))} \to 0, \qquad \text{as } \varepsilon \to 0.$$

Claim. Suppose $v(x) \in W_0^{1,p}(B(0,R))$ is a smooth function. Then,

$$\int_{B(0,R)} |v^\varepsilon - v| \le \varepsilon \int_{B(0,R)} |Dv|.$$

Claim. Suppose $v(x) \in W_0^{1,p}(B(0,R))$ is a smooth function. Then,

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Sobolev embedding: an improvement

Theorem. Suppose $u \in H^1(\mathbb{D})$. Then $h = u|_{\partial\mathbb{D}} \in H^{1/2}(\partial\mathbb{D})$. Conversely, any function in $H^{1/2}(\partial\mathbb{D})$ is the trace of a unique harmonic function $u \in H^1(\mathbb{D})$.

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Proof. Suppose $h(\theta) = \sum a_n e^{in\theta}$ with $\sum |n||a_n|^2 < \infty$. Set

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The function u is harmonic and

$$\|\nabla u\|^2_{L^2(\mathbb{D})} = \pi \sum_{n\neq 0} |n||a_n|^2.$$

Dirichlet's principle

Given $h \in C^1(\partial\mathbb{D})$, consider all $H^1(\mathbb{D}) = W^{1,2}(\mathbb{D})$ functions on the unit disk whose Sobolev trace is $u|_{\partial\mathbb{D}}$.

Problem. Minimize the energy $\mathcal{E}[u] = \int_{\mathbb{D}} |\nabla u|^2 dx$.

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Minimizers are harmonic: If $\phi \in C_c^\infty(\mathbb{D})$, then $\mathcal{E}[u + t\phi] \leq \mathcal{E}[u]$.

This shows that $\int_{\mathbb{D}} \nabla u \cdot \nabla \phi = 0$ for all ϕ, or $-\int_{\mathbb{D}} u \cdot \Delta \phi = 0$.

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Existence: If u_n is a bounded sequence in $W^{1,2}(\mathbb{D})$, there exists a subsequence which converges strongly in $L^2(\Omega)$. We will show that the energy is lower continuous:

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Suppose $\Phi \in C^\infty(\overline{\mathbb{D}}, \mathbb{R}^n)$.

$$\left| \int_{\mathbb{D}} u_n \operatorname{div} \Phi \right| = \left| \int_{\mathbb{D}} \nabla u_n \cdot \Phi \right| \leq \mathcal{E}[u_n] \cdot \|\Phi\|_{L^2(\Omega)}.$$

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Taking $n \to \infty$,

$$\left|\int_{\mathbb{D}} u \operatorname{div} \Phi\right| \le \left(\liminf_{n\to\infty} \mathcal{E}[u_n]\right) \cdot \|\Phi\|_{L^2(\mathbb{D})}.$$

Since $C^\infty(\overline{\mathbb{D}}, \mathbb{R}^n)$ is dense in $L^2(\mathbb{D}, \mathbb{R}^n)$, the linear functional

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$$\int_{\mathbb{D}} u \operatorname{div} \Phi = \int G \cdot \Phi.$$

Therefore, $u \in W_0^{1,q}$ with $\Delta u = -G$. Then,

$$E[u] = \|G\|^2_{L^2(\mathbb{D})} \leq \liminf_{n\to\infty} E[u_n].$$

An obstacle problem

Let Ω be a smoothly bounded domain. For $u \in L^1(\Omega)$, define

$$\mathcal{S}_u = \{v \in W_0^{1,2}(\Omega), v \geq u \text{ a.e. in } \Omega\}.$$

If $\mathcal{S}_u$ is not empty, then $\inf_{v \in \mathcal{S}_\mu} \mathcal{E}[v]$ is achieved by the smallest superharmonic function in $\mathcal{S}_u$.

Existence and unique proceed as before.

Why is $\tilde{u}$ superharmonic?

Euler-Lagrange equation. For any $v \in \mathcal{S}_u$, we have

$$\int_\Omega \nabla \tilde{u} \cdot \nabla (v - \tilde{u}) \geq 0.$$

To see this, note that the derivative function

$$t \to \int_\Omega |\nabla (tv + (1-t)\tilde{u})|^2, \qquad t \in [0,1].$$

is non-negative at $t = 0$.

To see that $\tilde{u}$ is subharmonic, plug in $v = \tilde{u} + \varphi$ where $\varphi \in C_c^\infty$ is a non-negative function.

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Euler-Lagrange equation. For any $v \in \mathcal{S}_u$, we have

$$\int_\Omega \nabla \tilde{u} \cdot \nabla (v - \tilde{u}) \geq 0.$$

To see this, note that the derivative function

$$t \to \int_\Omega |\nabla (tv + (1-t)\tilde{u})|^2, \qquad t \in [0,1].$$

is non-negative at $t = 0$.

To see that $\tilde{u}$ is subharmonic, plug in $v = \tilde{u} + \varphi$ where $\varphi \in C_c^\infty$ is a non-negative function.

Why is $\tilde{u}$ minimal?

Suppose $w \in \mathcal{S}_u$. We want to show that $(\tilde{u} - w)^+ = 0$.

$$\begin{aligned}\int_\Omega |\nabla(\tilde{u} - w)^+|^2 &= \int_\Omega \nabla(\tilde{u} - w)\nabla(\tilde{u} - w)^+ \\ &= \int_\Omega \nabla\tilde{u} \cdot \nabla(\tilde{u} - w)^+ - \int_\Omega \nabla w \cdot \nabla(\tilde{u} - w)^+.\end{aligned}$$

Suppose $w \in \mathcal{S}_u$. We want to show that $(\tilde{u} - w)^+ = 0$.

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$$= \int_\Omega \nabla\tilde{u} \cdot \nabla(\tilde{u} - w)^+ - \int_\Omega \nabla w \cdot \nabla(\tilde{u} - w)^+.$$

We will show that both terms in the RHS are ≤ 0:

$$\int_\Omega \nabla\tilde{u} \cdot \nabla(\tilde{u} - w)^+ \; = \; -\int_\Omega \nabla\tilde{u} \cdot \big(\min\{w, \tilde{u}\} - \tilde{u}\big) \; \leq \; 0.$$

Thank you for your attention!