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The escaping set of $z^2 + 1/4$ is quasiconvex

Thesis submitted in partial fulfillment of the requirements for the M.Sc. degree in the School of Mathematical Sciences, Tel-Aviv University

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Abstract

Consider the sequence of complex numbers z_n defined by the rule

$$z_{n+1} = z_n^2 + 1/4.$$

For which initial values $z_0 \in \mathbb{C}$ do we have $z_n \to \infty$?

This work proves that the set of such values of z_0 is *quasiconvex*. This means that every two points in it can be connected by a curve which is comparable in length to a straight line segment.

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1 Introduction

Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial. Its *filled Julia set* consists of points in the complex plane with bounded orbit under iteration by f_c:

$$\mathcal{K}_c = \big\{ z \in \mathbb{C} : \sup_{n \geq 0} f_c^{\circ n}(z) < \infty \big\}.$$

Its boundary $\mathcal{J}_c = \partial \mathcal{K}(f_c)$ is known as the *Julia set*, and its complement Exterior$(\mathcal{J}_c) = \mathbb{C} \setminus \mathcal{K}(f_c)$ forms the *attracting basin of infinity*, also called the *escaping set*. The sets $\mathcal{J}_c$ and $\mathcal{K}_c$ are compact and are both forward and backward invariant under the dynamics of f_c.

The *main cardioid*

$$\heartsuit = \big\{ c \in \mathbb{C} : c = \lambda/2 - \lambda^2/4,\ \lambda \in \mathbb{D} \big\}$$

is the set of parameters $c \in \mathbb{C}$ for which f_c has an attracting fixed point. When $c \in \heartsuit$, the filled Julia set $\mathcal{K}_c$ is a *quasidisk*, the image of a round disk under a quasiconformal mapping of the plane, e.g. see [Gam03, Theorem VI.2.1]. This intuitively means that $\mathcal{K}_c$ has no "cusps".

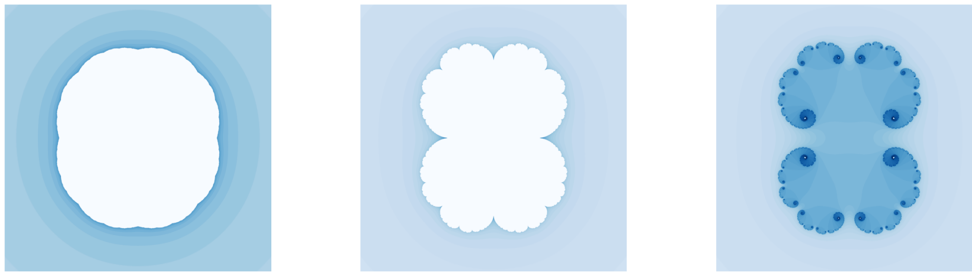
In this work we take $c = 1/4$, which lies on the boundary of $\heartsuit$. The filled Julia set $\mathcal{K}_{1/4}$, also called the *Cauliflower*, is a Jordan domain with an inward-pointing cusp at the point $p = 1/2$. See Figure 1.

According to a theorem of Carleson, Jones and Yoccoz [CJY94, Theorem 6.1], the Cauliflower is a *John domain*, a condition which rules out "outward-pointing cusps". Formally, a domain Ω is John if there exists a "center" $z_0 \in \Omega$ that can be connected to any other point $z_1 \in \Omega$ by a curve γ which does not get too close to the boundary:

$$\mathrm{dist}(z, \partial\Omega) \gtrsim |z_1 - z|, \tag{1.1}$$

for all $z \in \gamma$.

A set $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \overline{\Omega}$ are connected by a rectifiable path η_{z_1,z_2} :



(a) $c = 0.1$. (b) $c = 1/4$.
The Cauliflower (c) $c = 0.3$.

Figure 1: Julia sets $\mathcal{J}_c$ for different values of c. When $c > 1/4$, the Julia set is no longer connected.

$[0, 1] \to \overline{\Omega}$ which satisfies

$$\text{Length}(\eta_{z_1,z_2}) \leq A \cdot |z_1 - z_2| \tag{1.2}$$

and $\eta_{z_1,z_2}(w) \in \Omega$ for all $w \in (0, 1)$. We refer to a family of paths η_{z_1,z_2} satisfying Equation (1.2) with a uniform constant A as *quasiconvexity certificates*.

If Ω is a quasiconvex Jordan domain, then its complement has a John interior; see [HH08, Corollary 3.4] for a proof. In this work, we strengthen the result of [CJY94, Theorem 6.1] by showing:

Theorem 1.1. *The exterior of the Cauliflower is quasiconvex.*

Our result also has a function-theoretic interpretation. For a planar domain $\Omega \subset \mathbb{R}^2$, the *Sobolev space* $W^{1,1}(\Omega)$ is the set of functions $u \in L^1(\Omega)$ for which the distributional derivatives $\partial_1 u, \partial_2 u$ exist and are in $L^1(\Omega)$. We call Ω a *$W^{1,1}$ extension domain* if every $u \in W^{1,1}(\Omega)$ extends to a function in $W^{1,1}(\mathbb{C})$. In [GBR22, Equation (1.1)], it is proved that a bounded Jordan domain is a $W^{1,1}$ extension domain if and only if its complement is quasiconvex. Thus our result can be rephrased as follows:

Theorem 1.2. *The Cauliflower is a $W^{1,1}$ extension domain.*

1.1 Sketch of the argument

By [HH08, Corollary F], to show that a Jordan domain Ω is quasiconvex, it is enough to find certificates for points z_1, z_2 that lie on the boundary curve $\partial\Omega$.

We show quasiconvexity by explicitly constructing the certificates that connect pairs of points on the Julia set. We first build the certificates in the exterior unit disk $\mathbb{D}^*$ in an $f_0 : z \mapsto z^2$ invariant manner, and then transport them to the exterior of the Cauliflower by the Riemann map $\psi : \mathbb{D}^* \to \text{Exterior}(\mathcal{J}_{1/4})$, which conjugates the dynamics of f_0 and $f_{1/4}$. As the certificates in the exterior of the Cauliflower possess an invariance property under $f_{1/4}$, they may be analyzed using a parabolic variant of the principle of the conformal elevator.

In the hyperbolic setting ($c \in \heartsuit$), the *principle of the conformal elevator* says that small balls centered at points of $\mathcal{J}_c$ can be blown up to a definite size by repeatedly applying f_c, while roughly retaining their shape. Put more colloquially, Julia sets of hyperbolic polynomials are self-similar with bounded distortion. We use this self-similarity to analyze certificates that connect nearby points by using certificates that connect their iterated images located a definite distance apart.

In the parabolic setting ($c = 1/4$), we can only blow up balls to definite size as long as they stay away from the parabolic point. Nevertheless, we are able to control the certificates by exploiting the geometry of the cusp.

To facilitate the reading, we first demonstrate the proof in the hyperbolic case of maps $f_c(z) = z^2 + c$ with $c \in \heartsuit$, in which the usual conformal elevator applies, and subsequently treat the parabolic case with $c = 1/4$.

2 Preliminaries

In this section, we gather a number of tools from complex analysis and dynamics that will be used throughout this work.

2.1 The hyperbolic metric

The *hyperbolic metric* on the unit disk $\mathbb{D}$ is given by

$$\rho = \frac{2|dz|}{1-|z|^2}. \tag{2.1}$$

It is the unique Riemannian metric on the unit disk, up to multiplication by a positive constant, which is invariant under conformal automorphisms in $\operatorname{Aut}\mathbb{D}$. The factor 2 makes the curvature -1 instead of -4. One may transfer the hyperbolic metric to any simply connected domain in the plane using the Riemann map.

Definition 2.1. A domain $U \subset \widehat{\mathbb{C}}$ is *hyperbolic* if its universal covering space $\widetilde{U}$ is biholomorphic to $\mathbb{D}$.

It is well known that if $U \subset \widehat{\mathbb{C}}$ is a domain which omits at least three points, then it is hyperbolic. One may define the *hyperbolic metric* on any hyperbolic domain U by asking that the projection $\mathbb{D} \cong \widetilde{U} \to U$ is a local isometry.

Theorem 2.2 (The Schwarz-Pick theorem)**.** *Let $f : U_1 \to U_2$ be a holomorphic map between two hyperbolic domains $U_1, U_2 \subset \widehat{\mathbb{C}}$. Then f is a hyperbolic contraction, meaning that*

$$\operatorname{dist}_{U_2}(f(z), f(w)) \leq \operatorname{dist}_{U_1}(z, w), \tag{2.2}$$

for all $z, w \in U_1$. If f is not a covering map, then the inequality is strict for all $z \neq w$.

Proof. ([Mil06], Theorem 2.11). The classical Schwarz-Pick lemma is the case when $f : \mathbb{D} \to \mathbb{D}$. The general case follows after lifting f to a map between the universal covering spaces of U_1 and U_2. □

Definition 2.3. Let $f : U_1 \to U_2$ be holomorphic map between hyperbolic domains. The *hyperbolic derivative* of f at a point $z \in U_1$ is given by

$$\|f'(z)\|_{\text{hyp}} = \frac{\|Df(z)(v)\|_{\text{hyp}(U_2)}}{\|v\|_{\text{hyp}(U_1)}} = \frac{\rho_{U_2}(f(z))}{\rho_{U_1}(z)} \cdot |f'(z)|, \tag{2.3}$$

where v is a nonzero tangent vector at the point z.

2.2 Koebe's distortion theorem

One important tool in complex dynamics is Koebe's distortion theorem, which says that on a compact set, a conformal map resembles a linear map:

Theorem 2.4 (Koebe distortion theorem)**.** *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a univalent map. There exist $C_1, C_2 > 0$ so that*

$$C_1 \, |\varphi'(0)| \;\le\; |\varphi'(z)| \;\le\; C_2 \, |\varphi'(0)|, \qquad \text{for any } z \in B(0, 1/2).$$

In particular,

$$\frac{|\varphi(y) - \varphi(z)|}{|y - z|} \asymp |\varphi'(x)| \tag{2.4}$$

for all $x, y, z \in B(0, 1/2)$.

2.3 Relative distance

The following notion will be used for showing the existence of Koebe space, i.e. room to apply Koebe's distortion theorem:

Definition 2.5. The *relative distance* between two sets $E, F \subset \mathbb{C}$ is

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min(\operatorname{diam}(E), \operatorname{diam}(F))}, \tag{2.5}$$

and $\Delta(E, F) = \infty$ if the denominator vanishes. If $\Delta(E, F) \ge \eta$, we say that E and F are *η-relatively separated.*

Lemma 2.6. *Suppose γ is a rectifiable curve which efficiently connects two points $w_1, w_2 \in \mathbb{C}$, i.e. satisfies*

$$\operatorname{Length}(\gamma) \le C|w_1 - w_2|.$$

If γ is contained in a simply connected domain U with

$$\operatorname{dist}(\gamma, \partial U) \ge \eta |w_1 - w_2|$$

and $\varphi : U \to \mathbb{C}$ is a univalent map, then its image $\varphi\circ\gamma$ connects the points $z_1 = \varphi(w_1)$ and $z_2 = \varphi(w_2)$ efficiently:

$$\mathrm{Length}(\varphi \circ \gamma) \le A|z_1 - z_2|,$$

for some constant $A > 0$ which depends only on $C, \eta > 0$.

Proof. We can cover γ by at most $M = \lceil 2C/\eta \rceil$ balls of radius $(\eta/2)|w_1 - w_2|$, centered at points of γ. Applying Koebe's distortion theorem on each ball shows that $|\varphi'(p)| \asymp |\varphi'(q)|$ for any two points $p, q \in \gamma$. In particular,

$$\mathrm{Length}(\varphi \circ \gamma) \;\asymp\; |\varphi'(w_1)| \cdot \mathrm{Length}(\gamma) \;\asymp\; |\varphi'(w_1)| \cdot |w_1 - w_2|. \tag{2.6}$$

Without loss of generality, we may assume that $\eta < 1$. Since the ball

$$B\big(w_1, (\eta/2)|w_1 - w_2|\big)$$

does not contain w_2, Koebe's distortion theorem implies that

$$|z_1 - z_2| \gtrsim |\varphi'(w_1)| \cdot |w_1 - w_2|. \tag{2.7}$$

Putting the statements (2.6) and (2.7) together completes the proof. $\square$

The following corollary is essentially a variation of Lemma 2.6:

Corollary 2.7. *Let K be a compact subset of a domain $U \subset \mathbb{C}$, P be a compact set disjoint from U, and g be a function univalent in U.*

(i) *The sets $g(K)$ and $g(P)$ are η-relatively separated, for some $\eta = \eta(K)$ independent of g.*

(ii) *For rectifiable curves $\gamma_1, \gamma_2 \subset K$,*

$$\frac{\mathrm{Length}(g(\gamma_1))}{\mathrm{Length}(g(\gamma_2))} \asymp \frac{\mathrm{Length}(\gamma_1)}{\mathrm{Length}(\gamma_2)},$$

where the implicit constant depends on K but not on g.

2.4 The post-critical set

Let $f : z \mapsto z^2 + c$ be a quadratical polynomial. Since our argument will involve applying Koebe's distortion theorem to the iterates of f^{-1}, we will need to exclude points around which some iterate $f^{\circ n}$ does not have a holomorphic inverse:

Definition 2.8. The *post-critical set* of f is the closure of the forward orbits of the critical points,

$$\mathcal{P} = \overline{\{f^{\circ n}(0) : n \geq 1\} \cup \{\infty\}}.$$

If $c \neq 0$, then the post-critical set $\mathcal{P}$ contains at least 3 points, e.g. 0, c and ∞, and consequently its complement $\widehat{\mathbb{C}} \setminus \mathcal{P}$ admits a hyperbolic metric.

Theorem 2.9. *For $c \neq 0$, the map $f : \widehat{\mathbb{C}} \setminus \mathcal{P} \to \widehat{\mathbb{C}} \setminus \mathcal{P}$ is strictly expanding with respect to the hyperbolic metric of $\widehat{\mathbb{C}} \setminus \mathcal{P}$, in the sense that the hyperbolic derivative satisfies*

$$\|f'(z)\|_{\text{hyp}} > 1, \qquad z \in \widehat{\mathbb{C}} \setminus \mathcal{P}\,. \tag{2.8}$$

Proof. ([McM94], Theorem 3.5). To see the theorem, notice that

$$f : \widehat{\mathbb{C}} \setminus f^{-1}(\mathcal{P}) \to \widehat{\mathbb{C}} \setminus \mathcal{P},$$

is a local isometry, while the inclusion $\widehat{\mathbb{C}} \setminus f^{-1}(\mathcal{P}) \hookrightarrow \widehat{\mathbb{C}} \setminus \mathcal{P}$ is a strict contraction by Theorem 2.2. □

A map f is *hyperbolic* if its post-critical set $\mathcal{P}$ is disjoint from its Julia set $\mathcal{J}$. By the above theorem, this is equivalent to f being expanding on $\mathcal{J}$:

$$\|f'(z)\|_{\text{hyp}} \geq \kappa, \qquad z \in \mathcal{J}, \tag{2.9}$$

for some constant $\kappa > 1$.

Corollary 2.10. *Let f be a hyperbolic quadratic map. There exists an $\epsilon > 0$ such that every pair of points $z, w \in \mathcal{J}$ has a forward iterate $f^{\circ n}$ for which*

$$|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon.$$

Proof. By (2.9), for every iterate $f^{\circ j}$, we have

$$\left\|(f^{\circ j})'(z)\right\|_{\mathrm{hyp}} \geq \kappa^j, \qquad z \in \mathcal{J}. \tag{2.10}$$

As the hyperbolic metric and the Euclidean metric are equivalent on $\mathcal{J}$, we may take j large enough so that for $g = f^{\circ j}$, the Euclidean derivative

$$|g'(z)| > \mu, \qquad z \in \mathcal{J},$$

for some constant $\mu > 1$. By compactness, there exists an $\epsilon > 0$ such that for $z, w \in \mathcal{J}$ with $|z - w| < \epsilon$ on $\mathcal{J}$, we have $|g(z) - g(w)| \geq \mu|z - w|$. The claim follows with this value of ϵ by iterating g. □

3 The exterior disk

We write $\mathbb{D}^* = \mathbb{C} \setminus \mathbb{D}$ for the exterior of the unit disk. We connect any two boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ by a path in $\mathbb{D}^*$ in a manner that respects the map $f_0 : \zeta \mapsto \zeta^2$. We describe these paths using the metaphor of a passenger who travels by train:

Definition 3.1. *Stations* are the points in $\mathbb{D}^*$ of the form

$$s_{n,k} = 2^{1/2^n} \exp\left(2\pi i \cdot \frac{k}{2^n}\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

See Figure 2. These are the iterated preimages of the *central station* $s_{0,0} = 2$ under the map f_0. We refer to n as the *generation* of the station $s_{n,k}$. The 2^n stations of generation n are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks", which we use to travel between stations.

Definition 3.2. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are the two stations $s_{n,(k\pm1)(\mathrm{mod}\ 2^n)}$ adjacent to $s_{n,k}$ on C_n.

2. The *peripheral track* $\gamma_{s,s'}$ from s to a peripheral neighbor s' is the shorter arc of the circle C_n connecting s to s'.

3. The *radial successor* of s is $\mathrm{RadialSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.

4. The *express track* $\gamma_{s,s'}$ from s to its radial successor s' is the radial segment $[s, s']$.

Notice that the tracks respect the dynamics: applying f_0 to a track gives a track of the previous generation.

When a passenger travels between two stations s_1 and s_2, he or she must follow a particular itinerary from s_1 to s_2. If s_1 is the central station, then this itinerary is determined by the rule that the passenger stays as close as possible to its destination s_2 in the angular distance. This also determines how to travel from the central station to a boundary point $\zeta \in \partial\mathbb{D}^*$, by continuity. See Figure 2 and the next definition.

Definition 3.3. Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *central itinerary* of ζ is a path $\eta_\zeta = \gamma_{\sigma_0,\sigma_1} + \gamma_{\sigma_1,\sigma_2} + \dots$ from the central station to ζ, made of tracks between the stations $\sigma_0, \sigma_1, \dots$. It is defined inductively as follows:

Start at the central station $\sigma_0 = s_{0,0}$. Suppose that we already chose $\sigma_0, \dots, \sigma_k$. If there is a peripheral neighbor σ of σ_k that is closer peripherally to ζ, meaning that

$$|\operatorname{Arg}(\zeta) - \operatorname{Arg}(\sigma)| < |\operatorname{Arg}(\zeta) - \operatorname{Arg}(\sigma_k)|,$$

then take $\sigma_{k+1} = \sigma$. Otherwise, take $\sigma_{k+1} = \mathrm{RadialSuccessor}(\sigma)$.

We identify the central itinerary η_ζ with its sequence of stations $(\sigma_0, \dots)$. We record two properties of central itineraries:

- No central itinerary η_ζ uses two consecutive peripheral tracks. In particular,

$$\mathrm{Generation}(\sigma_k) \geq \frac{k}{2}; \tag{3.1}$$

- Central itineraries are essentially equivariant under f_0, in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

  for every $\zeta \in \partial\mathbb{D}^*$.

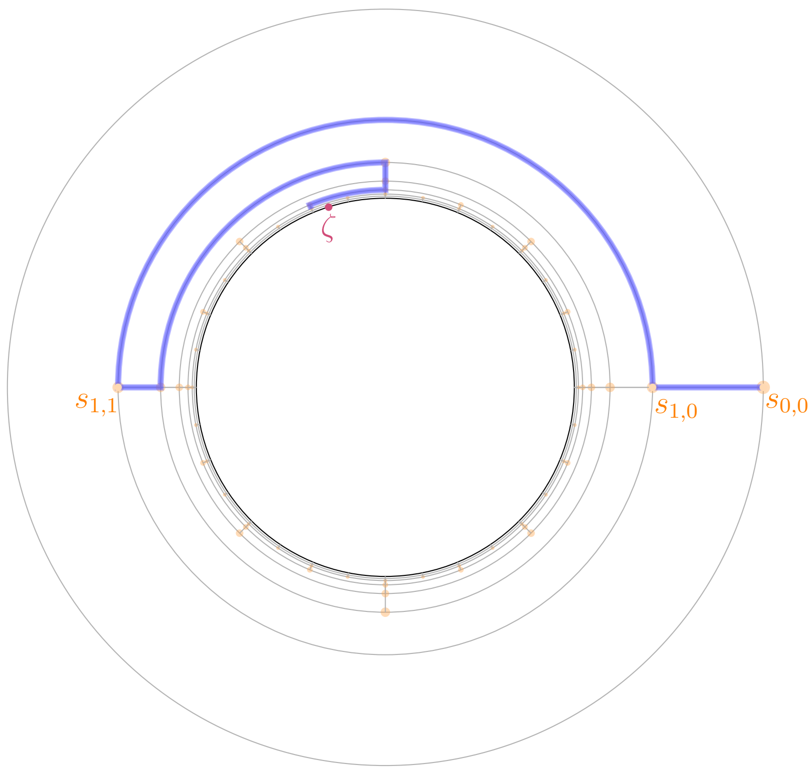


Figure 2: The central itinerary to a point ζ.

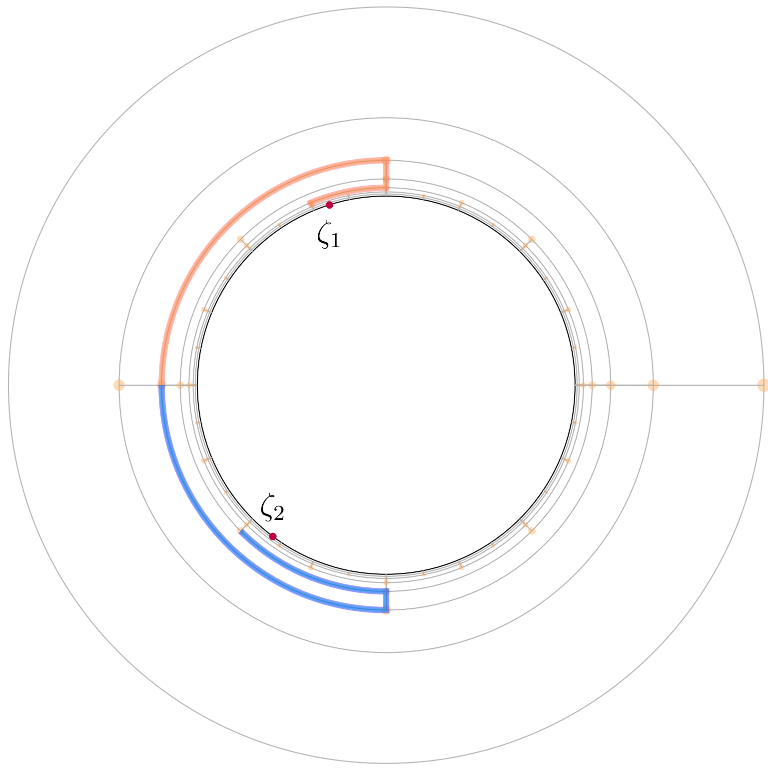


Figure 3: A quasiconvexity certificate between two points ζ_1, ζ_2 in $\mathbb{D}^*$. Only the first two steps are shown.

Definition 3.4. Given two distinct boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$, form the central itineraries $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^{\infty}$ and $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^{\infty}$ and let $\sigma = \sigma_i^1 = \sigma_j^2$ be the last station that is in both η_{ζ_1} and η_{ζ_2}. We define the *itinerary* between ζ_1 and ζ_2 to be the path

$$\eta_{\zeta_1,\zeta_2} = (\ldots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \ldots).$$

This is a simple bi-infinite path connecting ζ_1 and ζ_2, see Figure 3. Note that itineraries are equivariant under the dynamics: we have

$$f(\eta_{\zeta_1,\zeta_2}) = \eta_{f(\zeta_1),f(\zeta_2)} \tag{3.2}$$

for every pair of boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ with $|\zeta_1 - \zeta_2| < \sqrt{2}$.

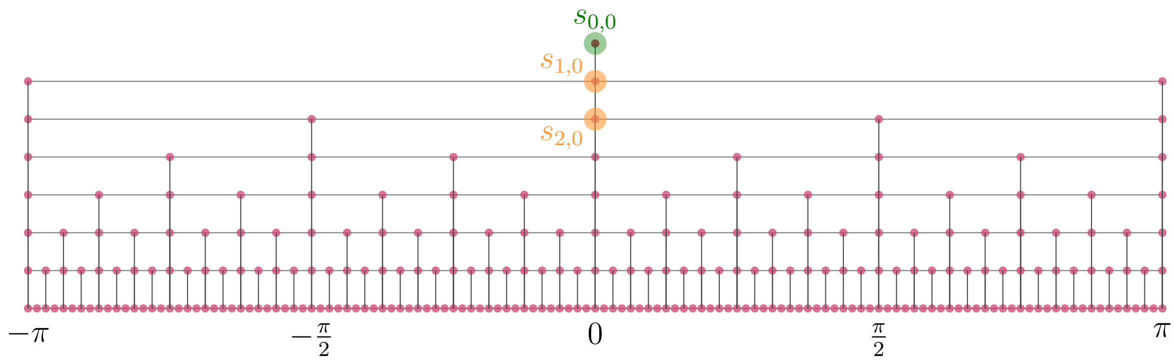


Figure 4: A convenient representation of the dyadic grid in the Böttcher coordinates. The horizontal axis is the external angle $\mathrm{Arg}(\psi^{-1}(z))$, and the vertical axis is the equipotential $|\psi^{-1}(z)|$, plotted on a log scale. The rightmost edge is glued to the leftmost edge. Stations are marked in red, and the segments connecting adjacent stations are tracks. An express track is a vertical segment, while a peripheral track is a horizontal segment. The central station is highlighted in green. The bottom row represents the Julia set $\mathcal{J}$.

4 Transporting the tracks

For $c \in \heartsuit$, the Julia set of $f_c : z \mapsto z^2 + c$ is a Jordan curve. The *Böttcher coordinate* $\psi = \psi_c$ is the unique conformal map $\mathbb{D}^* \to \mathrm{Exterior}(\mathcal{J}_c)$ which fixes ∞ and satisfies the conjugacy relation

$$f_c \circ \psi = \psi \circ f_0.$$

The Böttcher coordinate ψ extends to a homeomorphism between the unit circle $\partial\mathbb{D}$ and $\mathcal{J}_c$ by Carathéodory's theorem. See [Mil06, Theorem 9.5] for a proof of existence, relying on the explicit construction

$$\psi(z) \;=\; \lim_{n\to\infty} (f_0)^{\circ(-n)} \circ f_c^{\circ n} \;=\; \lim_{n\to\infty} (f_c^{\circ n})^{1/2^n}. \tag{4.1}$$

We apply ψ to the tracks that we constructed in $\mathbb{D}^*$ to obtain the corresponding tracks in $\mathrm{Exterior}(\mathcal{J}_c)$:

Definition 4.1.

1. The *stations* of f_c are the points $\psi(s_{n,k})$.

2. The *tracks* of f_c are the curves of the form $\psi(\gamma_{s,s'})$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$. Express tracks lie on the external rays of the filled Julia set $\mathcal{K}_c$, while peripheral tracks lie on the equipotentials of $\mathcal{K}_c$.

3. The *itinerary* between a pair of points (z_1, z_2) on $\mathcal{J}_c$ is $\eta_{z_1,z_2} = \psi(\eta_{\zeta_1,\zeta_2})$, where $\zeta_i = \psi^{-1}(z_i)$ are the corresponding points on $\partial\mathbb{D}^*$.

Notice that by symmetry $\psi((1,\infty)) \subseteq \mathbb{R}$. In particular, the central station $\psi(s_{0,0})$ lies on the real axis.

We henceforth omit c and ψ from the notation for ease of reading. It will be clear from the context whether we work in $\mathbb{D}^*$ or in Exterior$(\mathcal{J})$.

5 Hyperbolic maps

In this section we prove the quasiconvexity of Exterior$(\mathcal{J})$ for parameters c in the main cardioid $\heartsuit$. For $c \in \heartsuit$, the dynamics is hyperbolic since the critical point at 0 converges to an attracting fixed point under iteration.

Definition 5.1. A point $z \in \mathcal{J}$ is *rectifiably accessible* from Exterior$(\mathcal{J})$ if there is a rectifiable curve $\gamma : [0,1) \to$ Exterior$(\mathcal{J})$ such that $\gamma(t) \to z$ as $t \to 1$.

We are now ready to show quasiconvexity in the hyperbolic case:

Theorem 5.2. *Let $f : z \mapsto z^2 + c$ be a quadratic map with $c \in \heartsuit$.*

(i) *Given $z \in \mathcal{J}$ decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots.$$

We have the estimate

$$\mathrm{Length}(\gamma_k) \lesssim \theta^k,$$

uniformly in z, for some constant $\theta = \theta(c) < 1$. In particular, any point on $\mathcal{J}$ is rectifiably accessible.

(ii) *The domain* Exterior$(\mathcal{J})$ *is quasiconvex with the itineraries* η_{z_1,z_2} *as certificates.*

Proof. (i) For $c = 0$, this is a direct computation. Suppose $c \neq 0$, and let $\mathcal{P}$ be the post-critical set of f. Recall that $f : \widehat{\mathbb{C}} \setminus \mathcal{P} \to \widehat{\mathbb{C}} \setminus \mathcal{P}$ is expanding in the hyperbolic metric by Theorem 2.9.

Let $B(0, R) \subset \mathbb{C}$ be a ball large enough that it contains every central itinerary. By hyperbolicity, Exterior$(\mathcal{J}) \cap B(0, R)$ is compactly contained in $\widehat{\mathbb{C}} \setminus f(\mathcal{P})$, and there is a constant $\theta < 1$ such that $\|(f^{-1})'\|_{\text{hyp}} < \theta$ on Exterior$(\mathcal{J}) \cap B(0, R)$. Therefore,

$$\begin{aligned}\text{HypLength}(\gamma_k) &\leq \theta \cdot \text{HypLength}(f(\gamma_k))\\ &\leq \dots\\ &\leq \theta^k \cdot \text{HypLength}(f^{\circ k}(\gamma_k)),\\ &\lesssim \theta^k,\end{aligned}$$

where the last inequality holds since $f^{\circ k}(\gamma_k)$ lies on the real axis in case γ_k is an express track, or on the equipotential $\psi\big(\{|z| = \sqrt{2}\}\big)$ otherwise.

As the Euclidean metric is bounded above by a constant multiple of the hyperbolic metric, we conclude that Length$(\gamma_k) \lesssim \theta^k$ as well. Thus any point on $\mathcal{J}$ can be reached from the central station $s_{0,0}$ by a curve of bounded length.

(ii) By (2.10), there exists an $\epsilon > 0$ such that any two points are ϵ-apart under some iterate of f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$, we are done since the length of η_{z_1,z_2} is uniformly bounded above by part (i). On the other hand, if $|z_1 - z_2| < \epsilon$, then there is an iterate $f^{\circ n}$ for which

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon, \tag{5.1}$$

so that η_{w_1,w_2} is a quasiconvexity certificate. As η_{w_1,w_2} is contained in Exterior$(\mathcal{J}) \cap B(0, R)$, it is relatively separated from the post-critical set. By Lemma 2.6, η_{z_1,z_2} is also a quasiconvexity certificate. $\square$

6 The Cauliflower

In this section, $c = \frac{1}{4}$ and $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$. Our goal is to prove the quasiconvexity of Exterior($\mathcal{J}$). This is more complicated than the hyperbolic case, because the post-critical set $\mathcal{P} \subset [0, 1/2] \cup \{\infty\}$ of f accumulates at the parabolic fixed point $p = \frac{1}{2}$. One no longer has a uniform bound on the distortion of inverse iterates, and we cannot immediately deduce the quasiconvexity of the itinerary η_{z_1,z_2} from the quasiconvexity of η_{w_1,w_2} using Koebe's distortion theorem. As a substitute, we present an analogue of the principle of the conformal elevator in this parabolic setting.

6.1 Itineraries have finite length

We first show that each itinerary η_{z_1,z_2} has a finite length. We will in fact show an exponential decay of the lengths of the constituent tracks. For this to hold it is necessary to glue together consecutive express tracks: for example, the tracks in the central itinerary that lies on the real axis, $\eta_{1/2}$, have only a quadratic rate of length decay. To fix this, we introduce:

Definition 6.1. The *reduced decomposition* of an itinerary η is the unique decomposition $\eta = \gamma_1 + \delta_1 + \dots$ where each γ_i is a concatenation of express tracks and δ_i consists of a single peripheral track.

Proposition 6.2. *Let $z \in \mathcal{J}$, and let $\eta_z = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of its central itinerary. Then* $\text{Length}(\gamma_k) \lesssim \theta^k$ *and* $\text{Length}(\delta_k) \lesssim \theta^k$ *for some $\theta < 1$. Thus* $\text{Length}(\eta_z)$ *is finite and uniformly bounded over $z \in \mathcal{J}$, and all points $z \in \mathcal{J}$ are rectifiably accessible.*

For the proof, let $\mathcal{U}_{-1}$ be the Jordan domain enclosed by the Julia set $\mathcal{J}$, the rightmost itinerary starting from the pre-central station and the leftmost one. See Figure 5. This domain is constructed so that it contains all itineraries that start at the station $s_{1,1} = \psi(-\sqrt{2})$, the preimage of the central station under f. Notice that $\mathcal{U}_{-1}$ is compactly contained in $\widehat{\mathbb{C}} \setminus \mathcal{P}$, because the post-critical set $\mathcal{P} \subset [0, 1/2] \cup \{\infty\}$ touches the Julia set $\mathcal{J}$ only at $p = 1/2$, which is not contained in $\overline{\mathcal{U}_{-1}}$.

The crucial property of the domain $\mathcal{U}_{-1}$ is:

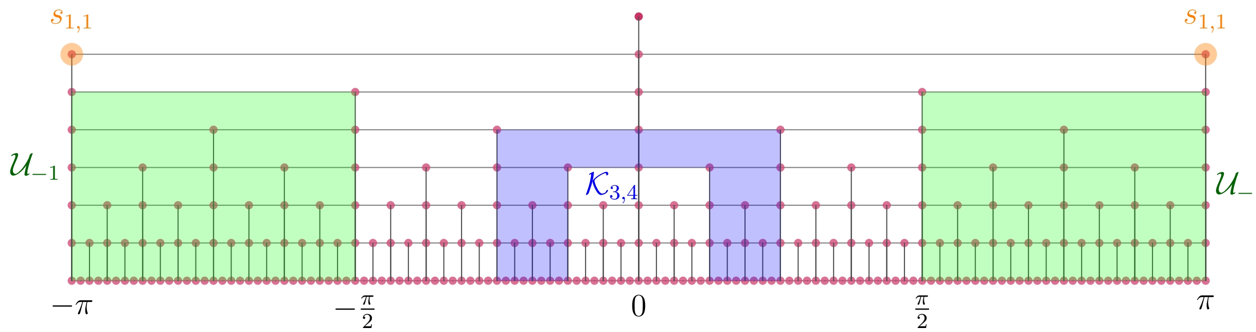


Figure 5: Using the dyadic representation from Figure 4, the domain $\mathcal{U}_{-1}$ is highlighted in green, and the domain $\mathcal{K}_{3,4}$ is highlighted in purple. The station $s_{1,1}$ is highlighted in orange; recall that points with external angle $\pm\pi$ appear twice in this representation.

Lemma 6.3. *Let $\gamma = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of an itinerary γ. Then for every $k > 1$, there exist $k-1$ iterates $n_1 < \dots < n_{k-1}$ such that $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$.*

Proof. Every station $s \notin (0, \infty)$ has a first iterate $f^{\circ n_s}(s)$ lying on the negative real axis $(-\infty, 0)$. For any $i \in \{2, \dots, k-1\}$, let s_i be the first station of γ_i and take $n_i := n_{s_i}$. By the definition of $\mathcal{U}_{-1}$, the itinerary $f^{\circ n_i}(\gamma)$ is contained in $\mathcal{U}_{-1}$ from the station $f^{\circ n_i}(s_i)$ onwards, and in particular $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$. $\square$

Proof (Proposition 6.2). By Theorem 2.9, there is a uniform bound $\|(f^{-1})'\|_{\text{hyp}} < \theta < 1$ on $\mathcal{U}_{-1}$ with respect to the hyperbolic metric of the domain $\widehat{\mathbb{C}} \setminus \mathcal{P}$, for both branches of f^{-1} defined in $\mathcal{U}_{-1}$.

In the notation of Lemma 6.3, we then have

$$\begin{aligned}
\text{HypLength}(\gamma_k) &\leq \text{HypLength}(f^{\circ(n_1-1)}(\gamma_k)) \\
&\leq \theta \cdot \text{HypLength}(f^{\circ n_1}(\gamma_k)) \\
&\leq \dots \\
&\leq \theta^k \cdot \text{HypLength}(f^{\circ n_k}(\gamma_k)) \\
&\lesssim \theta^k.
\end{aligned}$$

As the Euclidean metric is bounded above by a constant multiple of the hyperbolic metric, we infer that $\mathrm{Length}(\gamma_k) \lesssim \theta^k$. $\square$

6.2 Some estimates and notation

To estimate the length of express tracks, we introduce the notations $s_n := s_{n,0}$ and

$$\ell_n := \mathrm{Length}([s_n, s_{n+1}]) = s_n - s_{n+1}. \tag{6.1}$$

From the definition of the stations s_n, we have $f(s_{n+1}) = s_n$ for all $n \geq 0$. Since the Cauliflower is symmetric with respect to the real line, the s_n are positive real numbers converging to the cusp at $p = \frac{1}{2}$.

Lemma 6.4. *The lengths ℓ_n satisfy:*

(i)
$$\frac{|p - s_n|}{\ell_n} \to \infty, \tag{6.2}$$

(ii)
$$\frac{\ell_n}{\ell_{n+1}} \to 1. \tag{6.3}$$

In particular, for any $C > 0$, there is a sufficiently large integer d such that

$$\ell_m + \ldots + \ell_n \geq C(\ell_m + \ell_n)$$

whenever $|m - n| \geq d$.

Sketch of proof. Recall that s_n satisfy the recurrence

$$s_{n+1}^2 + 1/4 = f(s_{n+1}) = s_n. \tag{6.4}$$

Equivalently, the sequence of positive numbers

$$a_n = s_n - \frac{1}{2}$$

satisfies the recurrence

$$a_n = a_{n+1}^2 + a_{n+1}. \tag{6.5}$$

An induction argument shows that $a_n \asymp 1/n$, and consequently, $\ell_n = a_n - a_{n+1} = a_{n+1}^2 \asymp 1/n^2$. □

Lemma 6.5. *There exists a constant $k > 0$ such that for any pair of points $z_1, z_2 \in \mathcal{J}$, we have $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$.*

Proof. We have

$$|f(z_1) - f(z_2)| = \left| \int_{z_1}^{z_2} f'(z) dz \right| \leq k|z_1 - z_2| \tag{6.6}$$

for $k = \max_{z \in B} |f'(z)|$, where B is any ball containing $\mathcal{J}$. □

Lemma 6.6. *There exists an $\epsilon > 0$ such that every pair of points $z, w \in \mathcal{J}$ has a forward iterate $f^{\circ n}$ for which $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$.*

The proof is similar to that of Corollary 2.10. We leave the details to the reader.

6.3 Dynamics near the parabolic fixed point

The purpose of the following definition is to organize points on the Julia set $\mathcal{J}$ according to their distance from the main cusp $p = 1/2$ in an f-invariant way. We decompose the points of $\mathcal{J}$ according to the first *departure* : the first station at which the central itinerary makes a turn.

Definition 6.7. Let $n \in \mathbb{N}$. We define the n-th *departure set* $I_{n,\mathbb{D}} \subset \partial\mathbb{D}^*$ to be the set of points $\zeta \in \partial\mathbb{D}^*$ whose central itinerary η_ζ starts with n express tracks, followed by a peripheral track. See Figure 7.

This decomposition is invariant under f_0 in the sense that $f_0(I_{n+1,\mathbb{D}}) = I_{n,\mathbb{D}}$, because of the invariance of η_ζ. Applying the Böttcher map ψ, we obtain a corresponding departure decomposition $I_n = \psi(I_{n,\mathbb{D}})$ of $\mathcal{J}$ that is invariant under f.

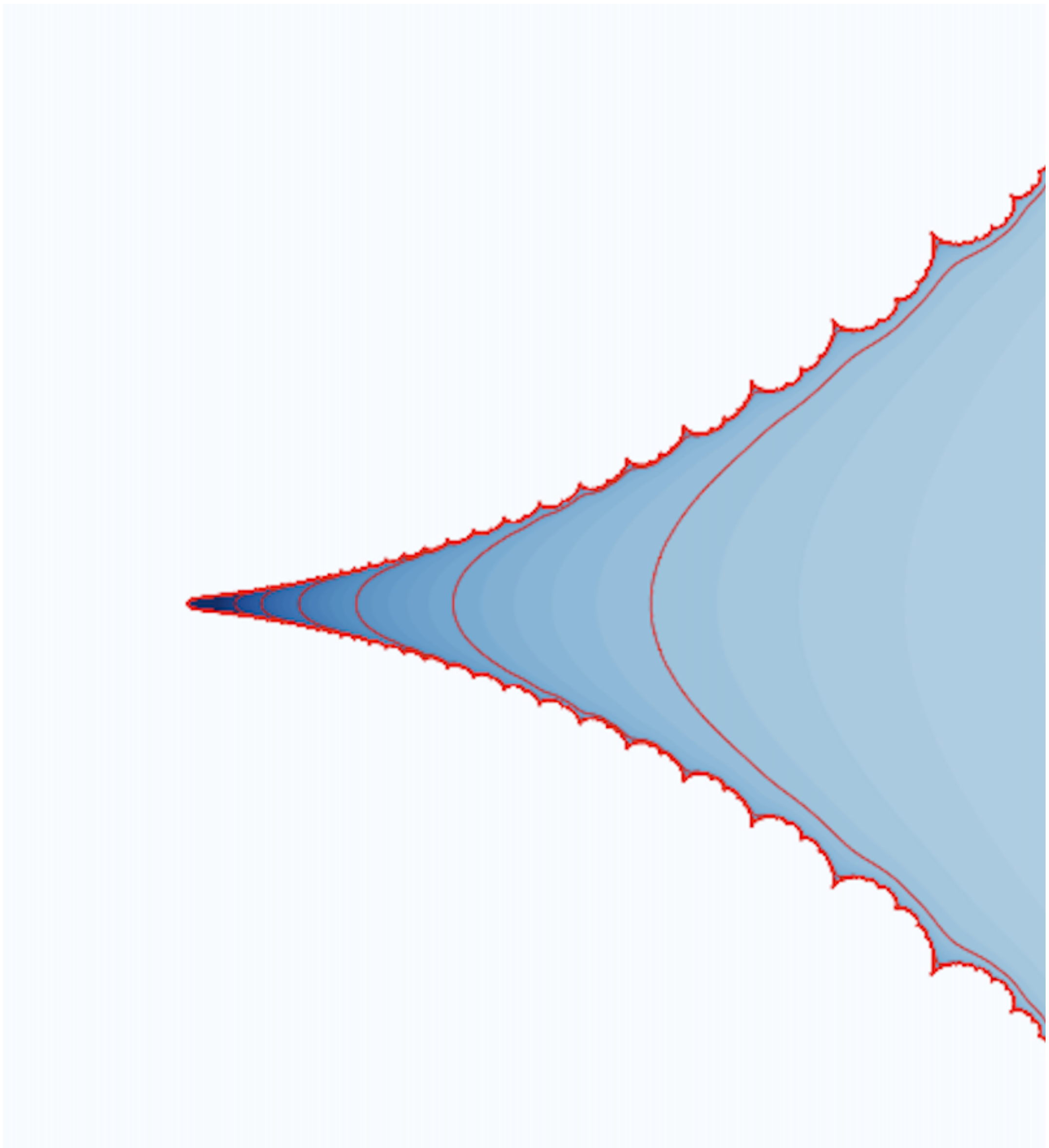


Figure 6: The Cauliflower near the parabolic point $p = 1/2$.

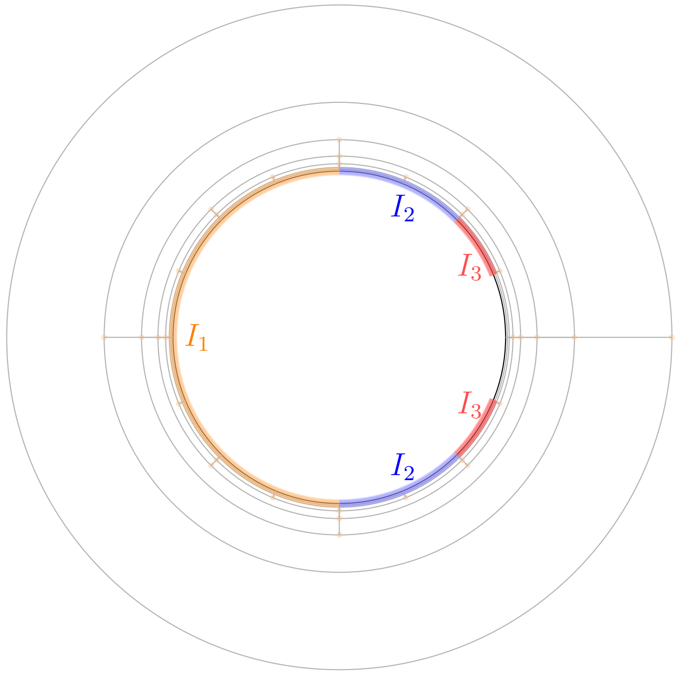


Figure 7: First few parts of the departure decomposition I_m of the circle.

We now use this decomposition to analyze the case where the points w_1, w_2 lie in "well-separated cusps". Namely, suppose that

$$w_1 \in I_n, \quad w_2 \in I_m, \quad m - n > d, \tag{6.7}$$

where d is a sufficiently large integer, to be chosen later. This will give some control from below on $|w_1 - w_2|$. We represent the itinerary $\eta = \eta_{w_1,w_2}$ as a concatenation of three paths: the radial segment $\gamma_{m,n} = [s_{m,0}, s_{n,0}]$ and the two other components, γ_m and γ_n. See Figure 8 for the picture in the exterior unit disk. Thus we have

$$\begin{aligned} \text{Length}(\eta) &= \text{Length}(\gamma_m) + \text{Length}(\gamma_{m,n}) + \text{Length}(\gamma_n) \\ &= \text{Length}(\gamma_m) + \ell_m + \cdots + \ell_{n-1} + \text{Length}(\gamma_n). \end{aligned} \tag{6.8}$$

The condition $m - n \geq d$ prevents the line segment $\gamma_{m,n}$ from being small in comparison to γ_m and γ_n:

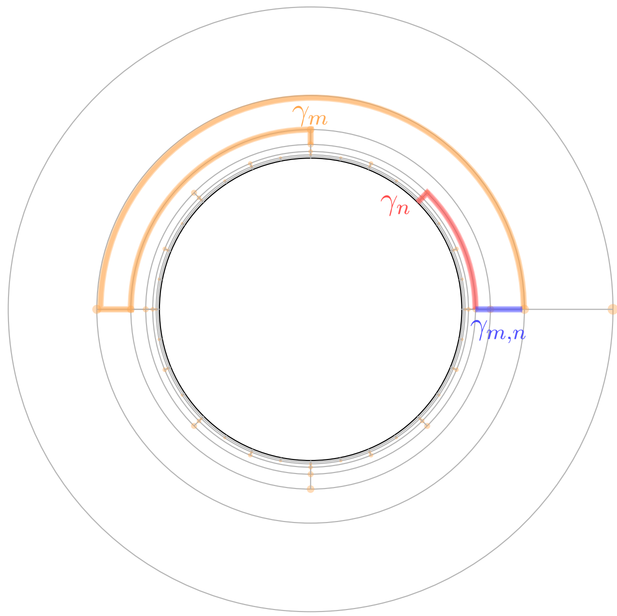


Figure 8: The three parts of an itinerary η.

Proposition 6.8. *There exists a sufficiently large integer d so that for every pair of points $w_1, w_2 \in \mathcal{J}$ satisfying $w_1 \in I_m$ and $w_2 \in I_n$ with $m - n \geq d$, we have*

$$\text{Length}(\gamma_{m,n}) \asymp |w_1 - w_2|. \tag{6.9}$$

Moreover, the itinerary η_{w_1,w_2} is a quasiconvexity certificate.

We henceforth fix a value of d as in the proposition.

Proof. We first make two elementary observations. Koebe's distortion theorem applied to the iterates of f^{-1} shows that $\text{Length}(\gamma_m) \asymp \ell_m$. In particular,

$$\text{Length}(\gamma_m) \leq C\ell_m, \tag{6.10}$$

for some constant $C \geq 0$. Notice that (6.10) holds for $m = 1$ by Proposition 6.2, which gives a uniform bound on the length of an itinerary.

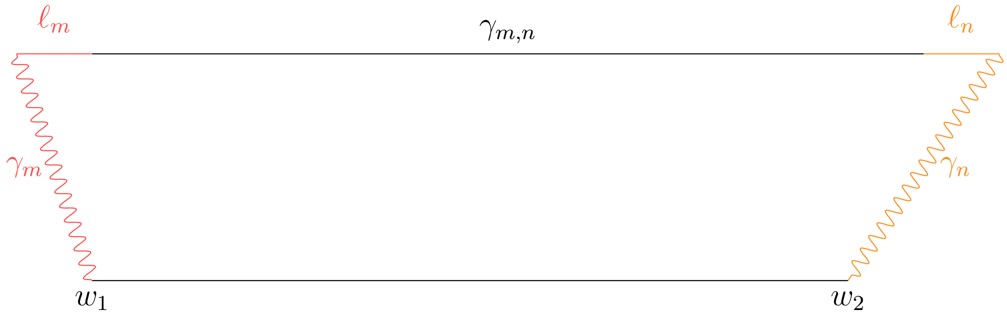


Figure 9: Schematic illustration of the proof of Proposition 6.8.

Meanwhile, by Lemma 6.4, there exists an integer d such that

$$C(\ell_m + \ell_n) \leq \frac{\text{Length}(\gamma_{m,n})}{2} \tag{6.11}$$

whenever $m - n \geq d$.

By the triangle inequality, we have

$$\begin{aligned} \big|\text{Length}(\gamma_{m,n}) - |w_1 - w_2|\big| &\leq \text{Length}(\gamma_m) + \text{Length}(\gamma_n) \\ &\leq \frac{\text{Length}(\gamma_{m,n})}{2}, \end{aligned}$$

which clearly implies (6.9) and thereby that η_{w_1,w_2} is a certificate. $\square$

6.4 Three special cases

We now show that the itineraries η_{w_1,w_2} are certificates in three special cases. To state them, we introduce some notation.

6.4.1 Notation

For each n, we denote by α_n the union of the two outermost tracks emanating from the station $s_{n,0}$. Notice that the curves α_n are pairwise disjoint since this is true for their pullbacks to the exterior unit disk.

We define the constants C_1, C_2, ϵ as follows. We first choose $C_1 \geq 2$, then we let $C_2 = C_1 + d + 2$ and choose $\epsilon > 0$ small enough so that we have

$$\mathrm{dist}(\alpha_{C_2}, \alpha_{C_1}) \geq k \cdot \epsilon, \tag{6.12}$$

where the constant k was defined in Lemma 6.5. The constant C_2 was chosen so that for any pair (m, n) of integers, we have at least one of the following three cases: either m, n are both greater than C_1, or both are smaller than C_2, or $|m - n| > d$.

6.4.2 The special cases

In this section we treat the following special cases:

1. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad 2 \leq m, n < C_2;$
2. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n > C_1;$
3. $|w_1 - w_2| \leq k\epsilon, \quad |m - n| \geq d.$

Notice that Case 2 overlaps with Case 1. We denote the domain enclosed by α_m, α_n and $\mathcal{J}$ by $\mathcal{K}_{m,n}$, and the domain enclosed by $\mathcal{J}$ and α_n by $\mathcal{K}_{n,\infty}$. See Figure 5.

Lemma 6.9. *Let $w_1 \in I_m$ and $w_2 \in I_n$, for $n \geq m \geq 2$. Then the itinerary η_{w_1,w_2} is contained in the domain $\mathcal{K}_{m,n+1}$.*

Proof. This follows from the combinatorics of the construction in the exterior unit disk. □

Lemma 6.10. *Let $n \geq m$ be integers satisfying $n - m \leq d$. The domain $\mathcal{K}_{m,n}$ is η-relatively separated from $\mathcal{P}$, for some $\eta = \eta(d) > 0$ independent of m, n.*

Proof. For convenience of exposition, we assume that $n = m + 1$. The general case follows from the special case.

Let $U \subset \mathbb{C} \setminus \mathcal{P}$ be any simply-connected domain which contains $\overline{K_{1,2}}$. Since U is disjoint from the post-critical set, for any $m \geq 1$, there exists a branch g of the inverse of $f^{\circ(m-1)}$ which takes s_1 to s_m. Since the sets $K_{m,m+1}$ have been defined in an f-equivariant way, g takes $K_{1,2}$ to $K_{m,m+1}$. The lemma follows after applying Koebe's distortion theorem to g. □

Lemma 6.11. *Let $w_1, w_2 \in \mathcal{J}$. In Cases 1 and 2, η_{w_1,w_2} is relatively separated from the post-critical set $\mathcal{P}$. In each of the three special cases, the itinerary η_{w_1,w_2} is a quasiconvexity certificate.*

Proof. In Cases 1 and 2, the Euclidean distance $|w_1 - w_2| \geq \varepsilon$ is bounded from below, so that η_{w_1,w_2} is a quasiconvexity certificate by Proposition 6.2.

Case 1. In this case, the itinerary is contained in the domain $\mathcal{K}_{2,C_2+1}$. Since $\mathrm{dist}(\mathcal{K}_{2,C_2+1}, \mathcal{P}) > 0$, η_{w_1,w_2} is relatively separated from $\mathcal{P}$.

Case 2. Assuming without loss of generality that $n \geq m$, the itinerary is contained in $\mathcal{K}_{m,n+1}$. By Lemma 6.10, η_{w_1,w_2} is relatively separated from the post-critical set.

Case 3 is the content of Proposition 6.8. □

6.5 The general case

We apply a stopping time argument to promote the quasiconvexity of η_{w_1,w_2} to the quasiconvexity of η_{z_1,z_2}:

Theorem 6.12. *The domain* Exterior$(\mathcal{J})$ *is quasiconvex, with the itineraries η_{z_1,z_2} as certificates.*

Proof. (Parabolic Conformal Elevator on $\mathcal{J}$). Let (z_1, z_2) be a pair of points on $\mathcal{J}$. If $|z_1 - z_2| \geq \epsilon$, we are done since the length of η_{z_1,z_2} is uniformly bounded above by Proposition 6.2.

It remains to treat the case when $|z_1 - z_2| \leq \epsilon$. By Lemmas 6.5 and 6.6, we may repeatedly apply f to (z_1, z_2) until either of the three special cases occurs. Denote by $w_i = f^{\circ N}(z_i)$ the resulting points. We have already proved that the itinerary η_{w_1,w_2} satisfies

$$\mathrm{Length}(\eta_{w_1,w_2}) \leq A|w_1 - w_2|,$$

for some $A > 0$. We now show that the original pair of points (z_1, z_2) enjoys a similar estimate,

$$\mathrm{Length}(\eta_{z_1,z_2}) \leq C|z_1 - z_2|,$$

where C depends only on A.

In Cases 1 and 2, we are done by Lemma 6.11. In Case 3, the itinerary η_{w_1,w_2} is contained in $\mathcal{K}_{2,\infty}$. Let $\mathcal{K}_{-2,\infty}$ be the preimage of $\mathcal{K}_{2,\infty}$ under f that contains the negative preimage $f^{-1}(p) = -\frac{1}{2}$ of the cusp p. As the domain $\mathcal{K}_{-2,\infty}$ is relatively separated from $\mathcal{P}$ and contains the curve $f^{\circ(N-1)}(\eta_{z_1,z_2}) = \eta_{f^{-1}(w_1),f^{-1}(w_2)}$, we may use Koebe's distortion theorem and Lemma 2.6 to conclude that

$$\frac{\text{Length}(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{f^{-1}(w_1),f^{-1}(w_2)})}{|f^{-1}(w_1) - f^{-1}(w_2)|} \asymp \frac{\text{Length}(\eta_{w_1,w_2})}{|w_1 - w_2|} \tag{6.13}$$

as desired. □

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Nomenclature

α_n The union of the two outermost tracks emanating from the station $s_{n,0}$.

$\Delta(\gamma, \mathcal{P})$ The relative distance to the post-critical set.

ℓ_n Length($[s_n, s_{n+1}]$) $= s_{n,0} - s_{n+1,0}$.

η_{z_1,z_2} The itinerary connecting two points. When z_1 and z_2 are stations, η_{z_1,z_2} coincides with γ_{z_1,z_2}.

γ_{s_1,s_2} The track connecting stations s_1 and s_2. It can be either angular ("peripheral") or radial ("express").

$\mathbb{D}^*$ The exterior of the unit disk.

$\mathcal{J}$ The Julia set of f.

$\mathcal{K}$ The filled Julia set of f.

$\mathcal{K}_{m,\infty}$ The domain enclosed by α_m and $\mathcal{J}$.

$\mathcal{K}_{m,n}$ The domain enclosed by α_m, α_n and $\mathcal{J}$.

Exterior($\mathcal{J}$), $A_\infty(f)$ The exterior of the Julia set of f.

$\mathcal{P}$ The post-critical set, the closure of the forward critical orbits. See Definition 2.8.

ψ The Böttcher coordinate $\mathbb{D}^* \to$ Exterior($\mathcal{J}$) conjugating f_0 and f.

C_1, C_2, ϵ Constants defined in Section 6.4.1.

d An integer for which itineraries connecting $w_1 \in I_m$ with $w_2 \in I_n$ for $m-n \geq d$ are certificates. Defined in Proposition 6.8.

f, f_c The map $z \mapsto z^2 + c$.

I_n The n-th departure set, see Definition 6.7.

k The constant guaranteed by Lemma 6.5.

$s_{n,k}$ A station in $\mathbb{D}^*$ or its image under ψ.