Decorated rescaling limits of Blaschke products

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Abstract

We analyze rescaling limits of hyperbolic Blaschke products that have one critical cluster. We show that under rescaling, horocyclic degenerations of *hyperbolic* Blaschke products converge to *parabolic* Blaschke products indexed by the *Epstein phase* and the *configuration of critical points*. We also also study rescaling limits on the level of holomorphic motions, which forces us to pay attention to the *Lavaurs phase*. Using rescaling limits, we endow the space of decorated parabolic Blaschke products with a Weil-Petersson metric which we show to be incomplete.

1 Introduction

Let $\mathcal{B}_d$ be the space of *hyperbolic* Blaschke products of degree $d \geq 2$ which have an attracting fixed point at the origin:

$$z \to f_{\mathbf{a}}(z) = z \cdot \prod_{i=1}^{d-1} \frac{z + a_i}{1 + \overline{a_i} z}, \qquad a_1, a_2, \dots, a_{d-1} \in \mathbb{D}. \tag{1.1}$$

Let $a = f'_{\mathbf{a}}(0) = a_1 a_2 \cdots a_{d-1}$. In this work, we are interested in extracting limits of degenerating sequences $\{f_n\} \subset \mathcal{B}_d$ for which $|a| \to 1$. One way to do this would be to simply take the limit of the maps f_n themselves; however, this would be uninteresting because the limiting maps would be Möbius transformations. By "rescaling" the f_n, i.e. by conjugating f_n with elements of $\mathrm{Aut}(\mathbb{D})$, one can extract more elaborate limits which have topological degree greater than 1.

Problem 1. (Rescaling limits – convergence of mappings) Understand the possible limits of $M_n \circ f_n \circ M_n^{-1}$ or more generally of $M_n \circ f_n^{\circ q} \circ M_n^{-1}$.

Problem 2. (Decorated rescaling limits – convergence of deformations) Find all sequences $\{f_n, M_n, \mu_n\}$ with Beltrami coefficients $\mu_n \in M(\mathbb{D})^{f_n}$ so that:

(i) $\lim M_n \circ f_n^{\circ q} \circ M_n^{-1} = g$ exists,

(ii) the rescaled versions $\tilde{\mu}_n := M_n^*(\mu_n)$ converges weakly in the sense of distributions to a Beltrami coefficient $\tilde{\mu} \in M(\mathbb{D})^g$.

Brief answers. The first problem leads to the space $\mathcal{PB}_d$ of *parabolic Blaschke products of degree d* while the answer to the second problem involves the space $\mathcal{LE}_d$ of decorated parabolic Blaschke products, which we also call the *Lavaurs-Epstein space*, in honour of P. Lavaurs and A. Epstein.

Note. In Section 1.3, we will give an alternative formulation of Problem 2 in terms of the convergence of the renormalized linearizing coordinates.

Applications and motivation

For a Beltrami coefficient μ with $\|\mu\|_\infty < 1$, we use w_μ to denote the solution of the Beltrami equation $\partial w = \mu \overline{\partial} w$, which fixes the points $0, 1, \infty$. The (weak) convergence of Beltrami coefficients $\tilde{\mu}_n \to \tilde{\mu}$ implies the convergence of holomorphic motions $w_{t\tilde{\mu}_n}(z) \to w_{t\tilde{\mu}}(z)$, where $t \in \mathbb{D}$ is a complex time parameter. In particular, the Julia sets

$$w_{t\tilde{\mu}_n}(\mathbb{S}^1) = \mathcal{J}\Big(w_{t\tilde{\mu}_n} \circ \big[M_n \circ f_n^{\circ q} \circ M_n^{-1}\big] \circ (w_{t\tilde{\mu}_n})^{-1}\Big)$$

converge to $w_{t\tilde{\mu}}(\mathbb{S}^1)$ in the Hausdorff topology as $n \to \infty$.

The convergence $\tilde{\mu}_n \to \tilde{\mu}$ can also be used to show that the Weil-Petersson metric on $\mathcal{B}_d$ has asymptotic symmetries. This is reminiscent of the invariance of the classical Weil-Petersson metric on Teichmüller space with respect to the mapping class group Mod_g. However, unlike Teichmüller space, $\mathcal{B}_d$ does not possess an infinite mapping class group, so these "symmetries" are necessarily asymptotic and we think of them as *ghosts of the mapping class group*. Further, these considerations lead us to define a Weil-Petersson metric on the space of decorated rescaling limits $\mathcal{LE}_d$.

We briefly recall the definition of the Weil-Petersson metric in complex dynamics due to McMullen. As is well known, an invariant Beltrami coefficient $\mu \in M(\mathbb{D})^f$ defines a (possibly trivial) tangent vector in $T_f\mathcal{B}_d$ represented by the path

$$f_t = w^{t\mu} \circ f \circ (w^{t\mu})^{-1}, \qquad t \in (-\varepsilon, \varepsilon), \tag{1.2}$$

where $w^{t\mu}$ is the quasiconformal map that has dilatation $t\mu$ on the unit disk and is symmetric with respect to the unit circle. Following McMullen, the Weil-Petersson metric on $\mathcal{B}_d$ is defined as

$$\|\mu\|^2_{\mathrm{WP}} := \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{H.}\dim w_{t\mu}(\mathbb{S}^1). \tag{1.3}$$

Actually, this definition does not cover all tangent vectors in $T_f\mathcal{B}_d$ when f has critical relations. We refer to [McM1] for a more complete definition.

In a previous work [Ivr], the author proved that the Weil-Petersson metric on $\mathcal{B}_d$ is incomplete. We will use this to show that the Weil-Petersson metric on $\mathcal{LE}_d$ is also incomplete.

1.1 Generalities on rescaling limits

We now describe the basic ideas and definitions that will help illuminate Problem 1. In order for the rescaling limit to be interesting, the limiting map must have at least one critical point. Therefore, we must rescale f_n at a "recurrent critical cluster." For a degenerating sequence of Blaschke products, a *critical cluster* is a maximal collection of critical points

$$C(f_n) = \{c_1(f_n), c_2(f_n), \dots, c_k(f_n)\}$$

that stay within a bounded hyperbolic distance of each other. A critical cluster is *recurrent* if for some $q \geq 1$, the hyperbolic distance $d_{\mathbb{D}}(C, f_n^{\circ q}(C))$ remains bounded as $n \to \infty$.

The operation of *rescaling at a critical cluster* means that we use Möbius transformations $M_n = e^{i\theta} \cdot \frac{z - Q_n}{1 - \overline{Q_n} z}$ with $Q_n \in \mathbb{D}$ and $d_{\mathbb{D}}(Q_n, C(f_n))$ bounded. We call two rescaling limits *equivalent* if one can be obtained from the other by a Möbius conjugacy. It is easy to see that choosing a different sequence Q'_n with $d_{\mathbb{D}}(Q_n, Q'_n)$ bounded can only lead to an equivalent rescaling limit. In particular, the number of (inequivalent) rescaling limits is bounded by the number of critical clusters.

Assumption. In this paper, we only consider rescaling limits that have *one* critical cluster, i.e. rescaling limits of Blaschke products for which the critical points remain within a bounded distance of each other.

Reductions. By passing to subsequences, we may assume that the clusters $C(f_n)$ converge to a point on the unit circle and $\theta_n \to \theta_0$ in $M_n = e^{i\theta} \cdot \frac{z - Q_n}{1 - \overline{Q_n} z}$. Therefore, we may restrict our attention to straight rescalings with $\theta = 0$ and take $Q_n = c_1(f_n)$.

We will show that in the above setting, if the rescaling limit exists, then it is necessarily a parabolic Blaschke product, i.e. a Blaschke product which has a fixed point p on the unit circle with $g'(p) = 1$; in fact, $p = \lim M_n(0)$. As we shall see in Section 4, parabolic Blaschke products are *not* uniquely determined (up go conjugacy) by the locations of their critical points – one must also specify an additional real parameter $T \in \mathbb{R}$, which we call the *Epstein phase*. To see that this data may be prescribed arbitrarily, we appeal to a beautiful theorem due to Heins:

Theorem 1.1 (Heins)**.** *Given a finite set $\mathcal{C}$ of points in the disk counted with multiplicity, there exists a Blaschke product $f_{\mathcal{C}}$ of degree $d = |\mathcal{C}| + 1$ for which*

$$\mathcal{C} = \{z \in \mathbb{D} : f'(z) = 0\}.$$

Furthermore, $f_{\mathcal{C}}$ is unique up to post-composition with a Möbius transformation.

For a discussion of Heins' theorem, see [KR]. Since our problem concerns Blaschke products up to Möbius conjugacy, one has to translate Heins' result carefully. We have:

Theorem 1.2. *For any set $\mathcal{C}$ consisting of $(d-1)$ points in the disk counted with multiplicity, and a number $T \in \mathbb{R}$, there exists a unique parabolic Blaschke product with critical set $\mathcal{C}$ and Epstein phase T. Furthermore, every parabolic Blaschke product arises as a rescaling limit of* some *sequence of hyperbolic Blaschke products.*

The parabolic fixed point $p \in \mathbb{S}^1$ of a parabolic Blaschke product may be either singly-parabolic or doubly-parabolic. In terms of the expansion $f(z) = z + a_k(z-p)^k + \dots$, p is *singly-parabolic* if the first non-zero coefficient is a_2; and is *doubly-parabolic* if the first non-zero coefficient is a_3. The dichotomy between singly-parabolic and doubly-parabolic Blaschke products will play an important role in this work. In the singly-parabolic case ($\Leftrightarrow T \neq 0$), the Epstein phase carries the same information as the index of the parabolic fixed point.

1.2 Rescaling Limits in Degree 2

We now describe the answer to Problem 1 in degree 2. The answer has been first worked out by A. Epstein [E] who worked in the generality of quadratic rational maps. Since a holomorphic degree 2 self-map of the disk has a unique critical point, it is natural to rescale there. We are therefore led to consider the "critically centered version" of f_a:

$$\tilde{f}_a := m_{c\to 0} \circ f_a \circ m_{0\to c}, \tag{1.4}$$

where $m_{0\to c} = \frac{z+c}{1+\bar{c}z}$ and $m_{c\to 0} = \frac{z-c}{1-\bar{c}z}$.

Degenerating $a \to 1$ through horocycles.

Suppose $a \to 1$ radially or along another path that is eventually contained in arbitrary small horoballs which rest on $1 \in \mathbb{S}^1$. In this case, the limit of $\tilde{f}_a(z)$ exists and is equal to $g_0(z) = \frac{z^2+1/3}{1+1/3z^2}$. Conjugating $g_0(z)$ to the upper half-plane, we get the simple expression $w \to G_0(w) = w - 1/w$. During rescaling, all three fixed points of f_a collide to form a doubly-parabolic fixed point at infinity.

Degenerating $a \to 1$ along horocycles.

Consider a sequence of Blaschke products f_n with $a_n \to 1$ (asymptotically) along a horocycle, i.e.

$$a_n = \mathtt{HtoD}_1(iy_n + x_n), \qquad \text{with} \quad y_n \to y, \quad x_n \to \pm\infty, \tag{1.5}$$

where $\texttt{HtoD}_1$ is the conformal map $(\mathbb{H}, i, \infty) \to (\mathbb{D}, 0, 1)$. When we rescale by an automorphism of the disk, the attracting fixed points at 0 and ∞ continue to be mirror images. As $a_n \to 1$, they collide to form a parabolic fixed point. However, this time, the third (repelling) fixed point does not collide with the other two, and the rescaling limit is only a singly-parabolic Blaschke product. In the upper half-plane, a singly-parabolic Blaschke of degree 2 has the form

$$G_T(w) = w - 1/w + T, \qquad T \in \mathbb{R} \setminus \{0\}.$$

Using the holomorphic index theorem (Theorem 3.1), one discovers that $T^2 = 1/y$. The sign of T depends on whether $a \to 1$ clockwise or counter-clockwise, and determines whether ∞ is attracting from the "positive" direction or from the "negative" direction. While the maps G_T and G_{-T} are not conjugate as maps of the upper half-plane, there is a symmetry: $(G_{-T}, \mathbb{H})$ is conjugate to $(G_T, \overline{\mathbb{H}})$ via $w \to -w$. Summarizing, *one has a bijection between horocycles and rescaling limits*.

Degenerating $a \to e(p/q)$.

When discussing degenerations with $a \to e(p/q)$, we instead consider limits of $\tilde{f}_a^{\circ q}$. Remarkably, the $\tilde{f}_a^{\circ q}$ converge to exactly the same class of degree 2 parabolic Blaschke products. In particular, the degree drops in the limit (the degree of $\tilde{f}_a^{\circ q}$ is 2^q). For a point $\zeta \in \mathbb{S}^1$, let $\texttt{HtoD}_\zeta$ be the conformal map $(\mathbb{H}, i, \infty) \to (\mathbb{D}, 0, \zeta)$. For $y \geq 0$, let

$$\mathcal{H}_{p/q}(y) := \texttt{HtoD}_{e(p/q)}\big(\{w : \operatorname{Im} w = y \cdot q^2\}\big).$$

With this natural parametrization of horocycles, the rescaling limits along the horocycles $\mathcal{H}_{p/q}(y)$ with different p/q and same y coincide.

1.3 Generalities on decorated rescaling limits

We now give another perspective on Problem 2 in terms of the convergence of linearizing maps. Suppose $f_a \in \mathcal{B}_2$ is a Blaschke product for which the multiplier of the attracting fixed point $a \neq 0$. In this case, one can define the linearizing coordinate

$$\varphi_a(z) := \lim_{n \to \infty} a^{-n} \cdot f_a^{\circ n}(z), \qquad z \in \mathbb{D}, \tag{1.6}$$

which conjugates f_a with multiplication by a:

$$\varphi_a(f_a(z)) = a \cdot \varphi_a(z). \tag{1.7}$$

In fact, (1.7) determines φ_a uniquely with the normalization $\varphi_a'(0) = 1$. Consider the domain Ω_{f_a} which is the unit disk with the grand orbits of the attracting fixed point and critical points removed. From the existence of the linearizing coordinate, it is easy to see that the quotient $\hat{\varphi}_a : \Omega \to T_a^\times := \Omega/(f_a)$ is a torus with at most $d-1$ punctures, but there could be less in the presence of critical relations. Let $T_a \supset T_a^\times$ denote the underlying closed torus.

Problem 2b. (Convergence of renormalized linearizing maps) Find all sequences $\{f_n\} \subset \mathcal{B}_d$ such that:

(i) $\lim M_n \circ f_n^{\circ q} \circ M_n^{-1} = g$ exists,

(ii) the renormalized linearizing maps $\tilde{\psi}_n = C_n \cdot \tilde{\varphi}_n^{\frac{1}{\log a_n^q}}$ converge to a holomorphic function $\tilde{\psi}$.

Above, $\tilde{\varphi}_n = \varphi_n \circ M_n$ is the linearizing map of $\tilde{f}_n$, the exponent $\frac{1}{\log a_n^q}$ ensures that $\tilde{\psi}_n(\tilde{f}^{\circ q}(z)) = e \cdot \tilde{\psi}_n(z)$, while C_n is chosen so that $\tilde{\psi}_n(c_1) = 1$.

In case of a doubly-parabolic rescaling limit, we have:

Theorem 1.3. *Suppose f_n is a sequence of Blaschke products with a doubly-parabolic rescaling limit g_0. Then, the $\tilde{\psi}_n$ converge to a non-vanishing holomorphic function $\tilde{\psi}$ which is uniquely determined by the conditions $\tilde{\psi}(g_0(z)) = e \cdot \tilde{\psi}(z)$ and $\tilde{\psi}(c_1) = 1$.*

Here, the quotient $\Omega_{g_0}/(g_0)$ is a cylinder with punctures and $h = \log \tilde{\psi}$ is the Fatou coordinate which is uniquely determined by the functional equation $h(g_0(z)) = h(z)+1$ with the normalization $h(c_1) = 0$, e.g. see [Mil1]. Therefore, e^h is the only possible accumulation point of $\tilde{\psi}_n$. A normal families argument now gives the convergence. Note that while the functions $\tilde{\psi}_n$ are multi-valued, branching over the grand orbit of the attracting fixed point, their limit is single-valued on the entire disk. Indeed, during rescaling, the grand orbit of the attracting fixed point is exiled to the unit circle.

The case of a singly-parabolic rescaling limit is more complicated because the convergence is only subsequential. We prove the following theorem:

Theorem 1.4. *Suppose f_n is a sequence of Blaschke products with a singly-parabolic rescaling limit g_T. The following conditions are equivalent:*

(i) *The closed tori T_{f_n} converge to $X = \mathbb{C} \setminus \langle 1, \tau \rangle$ in the moduli space $\mathcal{M}_1$. The imaginary part of τ is determined by the index of the parabolic fixed point p of g_T.*

(ii) *There exist $N_n \to +\infty$ so that* (a) *$\tilde{f}_n^{\circ N_n}$ converge to a non-constant holomorphic function g_X* and (b) *the semigroups $\langle \tilde{f}_n^{\circ q} \rangle$ converge geometrically to*

$$\mathcal{G}(g_T, X) := \{g_T^{\circ k},\, k \geq 1\} \cup \{g_T^{\circ k} \circ g_X^{\circ \ell},\, k \in \mathbb{Z}, \ell \geq 1\}. \tag{1.8}$$

Note that we allow inverse iteration by g_T provided we do at least one step by g_X.

Under these assumptions, the $\tilde{\psi}_n$ converge to a non-vanishing holomorphic function $\tilde{\psi}$ which is uniquely determined by the conditions

$$\tilde{\psi}(g_T(z)) = e \cdot \tilde{\psi}(z), \qquad \tilde{\psi}(g_X(z)) = e' \cdot \tilde{\psi}(z), \qquad \tilde{\psi}(c_1) = 1.$$

Here, g_X is called the Lavaurs map, *it commutes with g_T; and extends to a holomorphic function on $\mathbb{S}^2 \setminus \mathcal{J}(g_X)$ whose Denjoy-Wolff point is also at p. The quotient of $\Omega(\mathcal{G})$, the disk with the grand orbits of the critical points removed, by the action of $\mathcal{G}$ is also a torus with punctures – after filling in the punctures, its representative in moduli space is X.*

We further note that any pair (g, X), consisting of a singly-parabolic Blaschke product g and a torus $X = \mathbb{C} \backslash \langle 1, \tau \rangle$ where τ has the correct imaginary part determined by g, can be realized as a decorated rescaling limit of some sequence of hyperbolic Blaschke products. We denote the space of such pairs by $\mathcal{LE}^1$.

The space $\mathcal{LE}^1 = \mathcal{LE}^{1,+} \cup \mathcal{LE}^{1,-}$ has two connected components associated to clockwise degenerations ($T > 0$) and counter-clockwise degenerations ($T < 0$) respectively. These two components are canonically isomorphic and are trivial fiber bundles over $\mathcal{PB}^1$ with fiber $\mathbb{S}^1$. Note that by Heins' theorem, the base $\mathcal{PB}^1$ is contractible. We may form the space $\mathcal{LE}$ by attaching the doubly-parabolic Blaschke products $\mathcal{PB}^2$ to $\mathcal{LE}^{1,+}$, with $(g_n, X) \to g_0$ if $g_n \to g_0$ and the Epstein phase $T(g_n) \to 0$. Clearly, $\mathcal{LE}$ is contractible since it retracts to $\mathcal{PB}^2$. We emphasize that the Lavaurs-Epstein spaces constructed for different $e(p/q) = \lim a_n$ are identical since the correspondence $(g_T, X) \to g_X$ does not depend on $e(p/q) \in \mathbb{S}^1$.

Convergence of Beltrami coefficients

In order for sequences of rescaled Beltrami coefficients $\tilde{\mu}_n$ to converge, they need to be "systematically defined." Suppose Λ_n are lattices in the plane converging to Λ. Furthermore, assume that $\nu_n \in M(\mathbb{C})^{\Lambda_n}$ are Beltrami coefficients, invariant under translations by elements of Λ_n. For applications to complex dynamics, L^∞ convergence is much too strong, so we only require weak convergence: $\int_{\mathbb{C}} \nu_n \phi \to \int_{\mathbb{C}} \nu \phi$ for $\phi \in C_c^\infty(\mathbb{C})$. If $\tilde{\Psi}_n : \Omega_n \to \mathbb{C}$ are holomorphic mappings that converge to $\tilde{\Psi} : \Omega \to \mathbb{C}$ uniformly on compact sets, then the Beltrami coefficients $\tilde{\mu}_n = (\tilde{\Psi}_n)^* \nu_n \in M(\mathbb{D})$ converge weakly to $\mu = \tilde{\Psi}^* \nu$. More generally, we can assume that $\tilde{\Psi}_n$ are multi-valued holomorphic mappings with periods in Λ_n.

However, this is precisely the setting of Theorem 1.4, where one needs only to consider $\Psi_n = \log \psi_n$ to create invariant Beltrami coefficients which have a rescaling limit. Similar considerations with cylinders instead of tori provide converging sequences of Beltrami coefficients in the doubly-parabolic case.

1.4 Visualizing Rescaling Limits in Degree 2

We now discuss one possible way to visualize the main theorems of this paper in degree 2. For this purpose, we draw pictures of invariant sets called *half-flower gardens*

$\mathcal{G}_{p/q}(f_n) \subset \mathbb{D}$. Since we draw these pictures for intuition, we only give a loose description and refer the reader to [Ivr, Sections 3 – 5] for details.

In essence, one takes a simple closed curve of rotation number p/q in the quotient torus, thickens it up to an annulus like in Figure 1, and lifts it to the unit disk. Note that a Blaschke product has many gardens, but when studying degenerations $a \to e(p/q)$, the garden $\mathcal{G}_{p/q}(f_n)$ is the most appropriate. The *flower* is a forward-invariant region which is the union of the q connected components which emanate from the origin; the individual components are called *petals*. The garden is composed of the flower and its inverse iterates or *pre-flowers*.

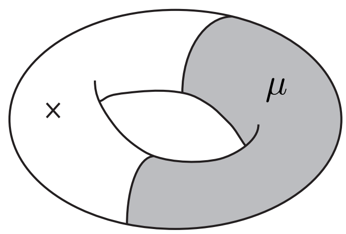

Figure 1: Half of the quotient torus away from the critical point.

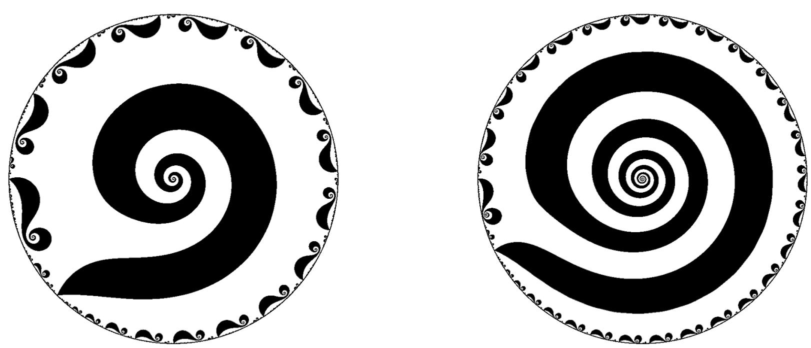

Figure 2: Gardens $\mathcal{G}_{0/1}(f_a)$ with $a = 0.5 + 0.5e^{i\theta}$ for $\theta = 2\pi/8$ and $2\pi/14$.

By an *optimal* Beltrami coefficient in $M(\mathbb{D})$, we mean a Teichmüller coefficient on the closed quotient torus, lifted to the disk. To form the *half-optimal* coefficient, we multiply the optimal coefficient by the characteristic function of the garden.

Consider the case when $a_n \to 1^{\circlearrowright}$ clockwise along a horocycle, i.e. with $a_n = \mathtt{HtoD}_1(iy + x_n)$, $x_n \to -\infty$. For each f_n, let μ_n be the half-optimal coefficient which represents the clockwise direction tangent to the horocycle in $T_{f_n}\mathcal{B}_2$. As Figures 2 and 3 show, the half-petals $\mathcal{P}(\tilde{f}_n)$ converge in the Hausdorff topology, giving a visual

Figure 3: Critically-centered versions of the gardens above.

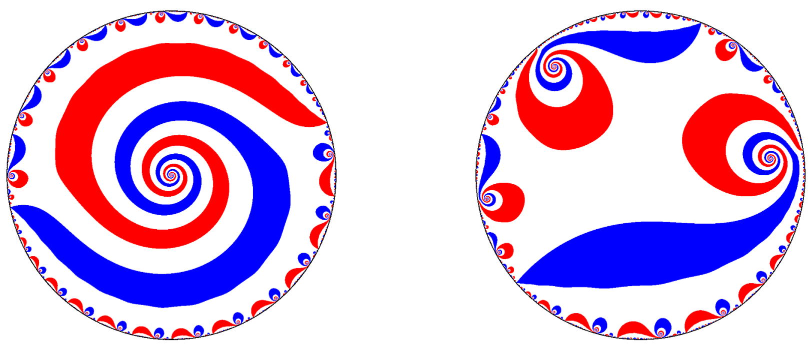

Figure 4: A garden $\mathcal{G}_{1/2}(f_a)$ with $a \approx -1$, and its critically-centered version.

confirmation of Epstein's theorem on rescaling limits. However, the limit set $\mathcal{P}(g_T)$ is disconnected (it has countably many components) since $\mathcal{P}(g_T)$ thins out near the parabolic fixed point. Therefore, one cannot use the Carathéodory convergence theorem to conclude the convergence of the rescaled half-optimal Beltrami coefficients $\tilde{\mu}_n$. In fact, the convergence is only subsequential: we need to keep track of how the high iterates $\tilde{f}_n^{\circ N_n}$ pass through the thin part of $\mathcal{P}(\tilde{f}_n)$. This is related to the phenomenon of *parabolic implosion* which has to do with the fact that the Julia set $\mathcal{J}(g_T(z))$ is a proper subset of the unit circle (it is a Cantor set).

For the convergence of Beltrami coefficients, we need the convergence of the quotient tori $T^{\times}_{a_n} \to X$ in moduli space, which occurs if $\sigma := \lim x_n \pmod{\pi^{-1}} \in \mathbb{R}/\pi^{-1}\mathbb{Z}$ exists. Therefore, the Lavaurs-Epstein space $\mathcal{LE}^1 \cong \mathbb{H}/\pi^{-1}\mathbb{Z}$. This presentation endows $\mathcal{LE}^{1,+}$ with a complex structure and a Teichmüller-Kobayashi metric. More canonically, one may describe the Lavaurs-Epstein space as the quotient of $\mathcal{T}_{1,1}$ by a Dehn twist with respect to a simple closed curve of slope 0/1. In particular, $\mathcal{LE}^{1,+}$ is conformally equivalent to a punctured disk. One can associate to the puncture the doubly-parabolic Blaschke product $g_0(z) = \frac{z^2+1/3}{1+(1/3)z^2}$. Since the Julia set $\mathcal{J}(g_0(z))$ is the entire circle, the dynamics of $g_0(z)$ cannot be enriched by Lavaurs maps. If we fill in this puncture, we obtain a disk $\mathcal{LE} := \mathcal{LE}^{1,+} \cup \{z \to g_0(z)\}$. Needless to say, the results are identical for counter-clockwise degenerations, except we work with $\mathcal{LE}^{1,-}$ instead of $\mathcal{LE}^{1,+}$.

If we instead take $a_n \to e(p/q)^{\circlearrowright}$ clockwise along a horocycle, i.e. if we consider

$$a_n = \mathtt{HtoD}_{e(p/q)}\big((iy + x_n)q^2\big), \qquad x_n \to -\infty \quad \text{and} \quad T_{a_n} \to X \in \mathcal{M}_1,$$

then the semigroups $\langle \tilde{f}_{a_n}^{\circ q} \rangle$ converges to the *same* semigroup $\mathcal{G}(g_T, X) = \mathcal{G}_{T,\sigma}$ that was described in the $a_n \to 1$ case. The Lavaurs-Epstein disk associated to horocyclic degenerations with $a \to e(p/q)$ is naturally the quotient of $\mathcal{T}_{1,1}$ by a Dehn twist by a simple closed curve of slope p/q. In particular, it has the same complex structure as the Lavaurs-Epstein disk associated to horocyclic degenerations with $a \to 1^{\circlearrowright}$.

1.5 Notes and references

In [E], Epstein studied rescaling limits of general degree 2 rational maps using normal forms. His methods are algebraic in nature. Our approach to rescaling limits leverages the hyperbolic geometry of Blaschke products, using the tools from [McM3] and [McM4]. While we only work with Blaschke products, we do allow the degree to vary.

The theory of parabolic implosion began with the work of Lavaurs. See [Dou] for a survey; another good reference is [S].

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2 Anatomy of a Blaschke Product

For a degree 2 Blaschke product $f_a \in \mathcal{B}_2$, the critical point c is the midpoint of the hyperbolic geodesic $[0, -a]$. In particular, $(1 - |c|) \sim \sqrt{2(1 - |a|)}$ as $|a| \to 1$. The following lemma says that Blaschke products for which the critical points are contained in a ball of a fixed hyperbolic radius behave similarly to degree 2 Blaschke products:

Lemma 2.1. *Suppose that $f_{\mathbf{a}} \in \mathcal{B}_d$ is a Blaschke product for which the set of critical points lies within a disk of hyperbolic radius R. Let $\delta_c := 1 - \min_{c \in C(f_{\mathbf{a}})} |c|$. There exists constants $C, K_0 \geq 1$, depending on R and d so that:*

(i) *The map $f_{\mathbf{a}}(z)$ is injective on $\mathbb{S}^2 \setminus B(\widehat{c_1}, K_0 \cdot \delta_c)$.*

(ii) *The zeros and critical points of* $f_{\mathbf{a}}$ *are contained in* $B(\widehat{c_1}, K_0 \cdot \delta_c)$ *and*

$$1 - |a_i| \asymp 1 - |a| \asymp \sqrt{\delta_c}.$$

(iii) *For* $K \geq K_0$*, we have the estimate*

$$|f_{\mathbf{a}}(z) - az| \leq (C/K)\sqrt{1 - |a|}, \qquad z \in \mathbb{D} \setminus B(0, 1 - K \cdot \delta_c).$$

(iv) *On the unit circle, the derivative satisfies:*

$$\begin{cases} |f'_{\mathbf{a}}(\zeta)| > 1 + \varepsilon, & \text{for } \zeta \in \mathbb{S}^1 \cap B(\widehat{c_1}, K_0 \cdot \delta_c), \\ |f'_{\mathbf{a}}(\zeta)| - 1 \asymp \frac{\delta_c^2}{|\zeta - \widehat{c_1}|^2}, & \text{for } \zeta \in \mathbb{S}^1 \setminus B(\widehat{c_1}, K_0 \cdot \delta_c). \end{cases}$$

The above lemma will be fundamental for proving results on rescaling limits and decorated rescaling limits.

2.1 Derivatives of Blaschke Products

Before proving Lemma 2.1, we first gather some well known facts about Blaschke products. We mostly cite McMullen's works although the lemmas are classical. We begin with a formula for the derivative of a Blaschke product on the unit circle:

Lemma 2.2 (Equation (3.1) of [McM3])**.** *Given a Blaschke product* $f_{\mathbf{a}} \in \mathcal{B}_d$ *and a point* ζ *on the unit circle,*

$$|f'_{\mathbf{a}}(\zeta)| = 1 + \sum_{i=1}^{d-1} \frac{1 - |a_i|^2}{|\zeta + a_i|^2}. \tag{2.1}$$

The proof of (2.1) follows from logarithmic differentiation. The utility of Lemma 2.2 comes from the fact that it allows one to estimate the location of a point $\zeta \in \mathbb{S}^1$ in terms of the derivative $|f'_{\mathbf{a}}(\zeta)|$. Inside the unit disk, the derivative of a Blaschke product is controlled by its behaviour on the unit circle:

Lemma 2.3 (Proposition 3.2 in [McM3])**.** *Given a Blaschke product* $f \in \mathcal{B}_d$*, for a point* $\zeta \in \mathbb{S}^1$*, we have*

$$\max_{0 \leq r \leq 1} |f'(r\zeta)| \leq 4|f'(\zeta)|. \tag{2.2}$$

Let $G(z) = \log \frac{1}{|z|}$ be the Green's function of the disk with a pole at the origin. It is uniquely characterized by three properties:

(i) $G(z)$ is harmonic on the punctured disk,
(ii) $G(z)$ tends to 0 as $|z| \to 1$,
(iii) $G(z) - \log \frac{1}{|z|}$ is harmonic near 0.

Lemma 2.4. *For a Blaschke product $f \in \mathcal{B}_d$, we have*

$$\sum_{f(w_i)=z} G(w_i) = G(z), \qquad z \in \mathbb{D}. \tag{2.3}$$

In particular, the Lebesgue measure on the unit circle is invariant under f.

To prove (2.3), one can check that $\sum_{f(w_i)=z} G(w_i)$ also satisfies the three properties above. We leave the verification to the reader. To deduce the invariance of Lebesgue measure, fix a point $x \in \mathbb{S}^1$ and consider (2.3) with $z = rx$. Dividing both sides by $(1-r)$ and taking $r \to 1$ gives $\sum_{f(y)=x} |f(y)|^{-1} = 1$ as desired.

Corollary 2.1 (Proposition 4.4 in [McM3])**.** *A Blaschke product $f \in \mathcal{B}_d$ is injective on the set*

$$\mathbb{S}^1_{\text{thin}} := \{\zeta \in \mathbb{S}^1 : |f'(\zeta)| < 2\}.$$

2.2 Approximate isometries

The next lemma says that a Blaschke product $f_{\mathbf{a}} \in \mathcal{B}_d$ is close to multiplication by a in the "critical disk" $B\big(0, 1 - K\sqrt{1-|a|}\big)$:

Lemma 2.5. *Suppose $f_{\mathbf{a}}(z) = z \prod \frac{z+a_i}{1+\overline{a_i}z} \in \mathcal{B}_d$ with $|a| = |f'_{\mathbf{a}}(0)| > 1/2$. For a point $z \in B\big(0, 1-K\sqrt{1-|a|}\big)$ with $K \geq 1$, the hyperbolic distance $d_{\mathbb{D}}(f_{\mathbf{a}}(z), az) < C/K^2$.*

Proof. The map $z \to \frac{z+a_i}{1+\overline{a_i}z}$ takes the ball $B\big(0, 1-K\sqrt{1-|a|}\big)$ inside the ball

$$B\left(a_i,\ (C_1/K) \cdot \sqrt{1-|a|} \cdot \frac{1-|a_i|}{1-|a|}\right).$$

Multiplying over $i = 1, 2, \ldots, d-1$, we see that $\prod \frac{z+a_i}{1+\overline{a_i}z} \in B\Big(a, (C_2/K)\sqrt{1-|a|}\Big)$ which shows that $|f_{\mathbf{a}}(z) - az| \leq (C/K)\sqrt{1-|a|}$ as desired. ☐

Together with the argument principle, the above lemma implies that $f_{\mathbf{a}}$ is injective in a disk $B\big(0, 1-K_0\sqrt{1-|a|}\big)$ for some constant $K_0 \geq 1$ sufficiently large.

Applying Lemma 2.5 to $m_{f_{\mathbf{a}}(z_0)\to z_0} \circ f_{\mathbf{a}}(z)$ where $m_{f_{\mathbf{a}}(z_0)\to z_0}$ is a Möbius transformation that takes $f_{\mathbf{a}}(z_0)$ to z_0, it follows that Blaschke products are close to hyperbolic isometries away from the critical points. This principle was formulated in a convenient form in [McM4]:

Lemma 2.6 (Theorem 10.11 in [McM4])**.** *There is a constant $R > 0$ such that for any proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ of degree d,*

1. *If $d_{\mathbb{D}}([a,b], C(f)) > R$, then $d_{\mathbb{D}}(f(a), f(b)) = d_{\mathbb{D}}(a,b) + O(1)$.*
2. *If $d_{\mathbb{D}}([a,b], f(C(f))) > R$, then $d_{\mathbb{D}}(f^{-1}(a), f^{-1}(b)) = d_{\mathbb{D}}(a,b) + O(1)$ where f^{-1} is any branch of the inverse chosen continuously along $[a,b]$.*

(We use the notation $[a, b]$ to denotes the hyperbolic geodesic segment joining a and b in the unit disk.)

Let us show that $\delta_c \geq \sqrt{1 - |a|}$ for any Blaschke product $f \in \mathcal{B}_d$, justifying our definition of the critical disk. Consider the geodesic segment $[0, z_i]$ joining the origin with the nearest zero of f. By Lemma 2.6, $[0, z_i]$ needs to pass by a critical point before its midpoint so that its image under f can reverse direction.

Proof of Lemma 2.1. Lemma 2.6 gives part (i), which implies the statement (ii) on the location of the zeros. Part (iv) now follows from Lemma 2.2. An examination of the proof of Lemma 2.5 shows that in the setup of Lemma 2.1, one can replace $B\big(0, 1 - K\sqrt{1 - |a|}\big)$ with $\mathbb{S}^1 \setminus B(0, 1 - K \cdot \delta_c)$, thus proving (iii). □

3 Dynamics on the circle

In this section, we gather further preliminaries dealing with dynamics of Blaschke products on the unit circle.

3.1 Simple Cycles

On the unit circle, a Blaschke product has many repelling periodic orbits or cycles. Since all Blaschke products of degree d are quasisymmetrically conjugate on $\mathbb{S}^1$, we can label the periodic orbits of $f \in \mathcal{B}_d$ by the corresponding periodic orbits of $z \to z^d$. A cycle is *simple* if it f preserves its cyclic ordering. In this case, we say that $\langle \xi_1, \xi_2, \dots, \xi_q \rangle$ has *rotation number* p/q if $f(\xi_i) = \xi_{i+p \pmod q}$.

Given a cycle $\langle \xi_1, \xi_2, \dots, \xi_q \rangle$ of period q, its *multiplier* is defined as $(f^{\circ q})'(\xi_1)$. It is easy to see that the multiplier is a positive real number (greater than 1) since Blaschke products preserve the unit circle. Given a sequence of Blaschke products $\{f_n\} \subset \mathcal{B}_d$, we say that a cycle *degenerates* if its multiplier tends to 1.

According to [McM3, Theorem 6.1], only simple cycles can degenerate; in fact, the multiplier of a non-simple cycle is bounded from below by a constant $C_d > 1$ that only depends on the degree d. Furthermore, the only way that the multiplier of a simple cycle of rotation number p/q can approach 1 is if $a \to e(p/q)$ through horocycles. We will need the following lemma which is a variant of [McM3, Corollary 3.3]:

Lemma 3.1. *Suppose $\{f_n\}$ is a degenerating sequence of Blaschke products, tending pointwise to a rotation $z \to \omega z$, $\omega \in \mathbb{S}^1$. If $\langle \xi_0, \xi_1, \xi_2, \dots, \xi_{q-1} \rangle$ is a repelling periodic orbit whose multiplier remains bounded as $n \to \infty$, then $\xi_j(f_n)/\xi_0(f_n) \to \omega^j$ as $n \to \infty$.*

Proof. It suffices to check that $\xi_{j+1} \approx \omega \cdot \xi_j$. For this purpose, consider the geodesic ray $[0, \xi_j)$. By Lemma 2.5, f is approximately rotation by ω on $[0, \xi_j(1 - K \cdot \delta_c)]$. We can control the length of $f\big([\xi_j(1-K\cdot\delta_c),\xi_j)\big)$ using Lemma 2.3. Thus, f is nearly a rotation on the complete ray $[0,\xi_j)$ as desired. □

3.2 Holomorphic Index Theorem

If $g : U \to \mathbb{C}$ is a holomorphic map, and $g(\zeta) = \zeta$ for some point $\zeta \in U$, the *fixed point index* of ζ is defined as

$$I_\zeta := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-g(z)}, \tag{3.1}$$

where γ is a small counter-clockwise loop around ζ. If the multiplier $\lambda = g'(\zeta) \neq 1$, then the above definition reduces to $I_\zeta = \frac{1}{1-\lambda}$. The Residue theorem shows:

Theorem 3.1 (Holomorphic Index Theorem)**.** *Suppose $R(z)$ is a rational function and $\{\zeta_i\}$ is the set of its fixed points. Then, $\sum I_{\zeta_i} = 1$.*

For a Blaschke product $f \in \mathcal{B}_d$, the holomorphic index theorem tells us that

$$\sum \frac{1}{\lambda-1} = \frac{1-|a|^2}{|1-a|^2}, \tag{3.2}$$

where the sum ranges over the repelling fixed points on the unit circle.

4 Parabolic Blaschke Products

In this section, we define the Epstein phase of a parabolic Blaschke product and discuss the dichotomy between singly-parabolic and doubly-parabolic Blaschke products, which will play an important role in the study of horocyclic degenerations.

4.1 Normal form

It is easier to consider parabolic Blaschke products as endomorphisms of the upper half-plane, with the parabolic fixed point normalized at infinity. We will use the convention that lower case letters like f and g refer to Blaschke products that are self-maps of the disk, while capital letters refer to self-maps of the upper half-plane.

One advantage of working in the upper half-plane is that it is easy to write down the general expression for a parabolic Blaschke product, namely

$$F(w) = w - \sum_{i=1}^{d-1} \frac{a_i}{w-b_i} + T, \tag{4.1}$$

where $\{b_i\}$ and T are real and $\{a_i\}$ are positive. To see why this is indeed the general expression, consider the (finite) pre-images $\{b_i\}$ of infinity. In order for the upper half-plane to be completely-invariant, each b_i must lie on the real axis and be a simple pole with a negative residue. Clearly, any function of the form (4.1) is an endomorphism of the upper half-plane, being the sum of endomorphisms.

Let us now show that a parabolic Blaschke product is determined by the set of its critical points up to an additive constant. Indeed, if two Blaschke products F_1, F_2 are of the form (4.1) and have the same critical points, then by Heins' theorem from the introduction,

$$F_1 = L \circ F_2, \qquad \text{where } L(w) = Sw + T, \quad S > 0, \ T \in \mathbb{R}.$$

Since $\lim_{w\to\infty} F_i'(w) = 1$, $i = 1, 2$, we see that $S = 1$. Therefore, $F_1 = F_2 + T$ as desired.

Remark. Of course, if one interested in maps up to conjugacy, then (4.1) is a normal only up to conjugation by

$$w \to Aw + B, \qquad A \geq 0, \quad B \in \mathbb{R}. \tag{4.2}$$

4.2 Epstein phase

Write $F_T(w) = w + T + c_{-1}/w + c_{-2}/w^2 + \dots$ Using an affine conjugacy (4.2), we can make $c_{-1} = -1$. In this case, we refer to T as the *Epstein phase*.

Inspection shows that a parabolic Blaschke product is singly-parabolic if $T \neq 0$ and doubly-parabolic if $T = 0$. In the singly-parabolic case, a simple computation shows that the index of the parabolic fixed point at infinity is $1 - 1/T^2 \in (1, \infty)$.

Thus, for singly-parabolic Blaschke products, the index of the parabolic fixed point carries the same information as the Epstein phase.

Conversely, given a set of $(d-1)$ points in the upper half-plane and a number $T \in \mathbb{R}$, there is a unique parabolic Blaschke product with this set of critical points and whose Epstein phase is T. This fact follows easily from the theorem of Heins.

4.3 The Dichotomy

The dichotomy between singly-parabolic and doubly-parabolic Blaschke products can be expressed in terms of many different dynamical properties, for instance:

Theorem 4.1. *For a parabolic Blaschke product F considered as a map from the upper half-plane to itself, the following are equivalent:*

- (a) *The Blaschke product F has a doubly-parabolic fixed point at infinity,*
- (b) *The Julia set $J(F) = \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$,*
- (c) *The Lebesgue measure on $\mathbb{R}$ is ergodic under F,*

(*d*) *The parabolic Blaschke product F is rigid; i.e. given any other parabolic Blaschke product G conjugate to F, the conjugacy is not absolutely continuous.*

Proof. Under the dynamics of F, points outside the real line iterate towards ∞, therefore the Julia $J(F) \subseteq \overline{\mathbb{R}}$. However, it is possible that some points on the real line lie in the basin of ∞ as well. This happens when ∞ is attracting from at least one side. From the normal form (4.1), this happens precisely when $T \neq 0$, i.e. when the parabolic fixed point is not degenerate. Thus, we have shown the equivalence $(a) \Leftrightarrow (b)$.

The equivalences $(b) \Leftrightarrow (c) \Leftrightarrow (d)$ can be found in [H]. □

5 Horocyclic approach: $a \to 1$

Suppose $\{f_n\} \subset \mathcal{B}_d$ is a degenerating sequence of Blaschke products with one critical cluster for which $\tilde{f}_n := M_n \circ f_n \circ M_n^{-1} \to g$. By Hurwitz' theorem, the limiting map $g(z)$ cannot have any fixed points inside $\mathbb{D}$ since $\tilde{f}_n(z) - z$ is zero-free on any disk $B(0, r) \subset \mathbb{D}$, $0 < r < 1$, for sufficiently large n. By the Denjoy-Wolff theorem, g must have either an attracting or a parabolic fixed point on the unit circle.

Lemma 5.1. *Suppose f_n is a degenerating sequence of Blaschke products with one critical cluster for which*

$$M_n \circ f_n \circ M_n^{-1} \to g, \qquad M_n(z) = \frac{z - c_1}{1 - \overline{c_1} z}.$$

Then, a_n is eventually contained within some horoball resting on $1 \in \mathbb{S}^1$, $g(z)$ is a parabolic Blaschke product, and its parabolic fixed point $p = \lim M_n(0)$. Furthermore, the following dichotomy holds:

(i) *If the sequence a_n is eventually contained in arbitrarily small horoballs resting on $1 \in \mathbb{S}^1$, then the rescaling limit is doubly-parabolic.*

(ii) *Otherwise a_n is eventually contained in a region between two horocycles $\mathcal{H}_1(\eta_1)$, $\mathcal{H}_1(\eta_2)$, and the rescaling limit is singly-parabolic. In this case, $a_n \to 1$ clockwise or counter-clockwise and the choice of access determines the attracting side of p on the unit circle.*

Sketch of proof. By Lemma 2.5, for $z \in B(0, 1 - K\delta_c)$, $K \geq 1$, one has $d_{\mathbb{D}}(f_n(z), a_n z) < C/K^2$. Consider the auxiliary point

$$c_1^K := (1 - K\delta_c) \cdot \widehat{c_1}, \qquad \widehat{c_1} := c_1/|c_1|. \tag{5.1}$$

The existence of the rescaling limit implies that $d_{\mathbb{D}}(c_1, f_n(c_1))$ is bounded as $n \to \infty$. For fixed $K > 1$, this is equivalent to the boundedness of

$$d_{\mathbb{D}}\big(c_1^K, f_n(c_1^K)\big) \approx d_{\mathbb{D}}(c_1^K, a_n \cdot c_1^K). \tag{5.2}$$

In turn, this is equivalent to the boundedness of

$$\sup_n \frac{\arg a_n}{\sqrt{1-|a_n|}} < \infty, \tag{5.3}$$

i.e. a_n eventually lies within *some* horoball based at 1.

Under rescaling, $m_{c_1\to 0}\big(B(0, 1-K\delta_c)\big)$ converge to a horoball $\mathcal{H}_p$ resting on $p = \lim m_{c_1\to 0}(0)$. Therefore, up to small error in the hyperbolic metric,

$$g(z) \approx m_{c_1\to 0} \circ (\cdot\, a) \circ m_{0\to c_1}(z), \qquad z \in \mathcal{H}_p. \tag{5.4}$$

The lemma now follows by examining the different possibilities for the multipliers a_n and comparing them to the different possibilities for the local behaviour of g near p. For instance, inspecting the relative locations of c_1^K and $f_n(c_1^K)$ with K large, we see that the local translation length $\liminf_{z\to p} d_{\mathbb{D}}(z, g(z)) = 0$ and therefore, p must be a parabolic fixed point. A finer examination reveals whether or not p is singly-parabolic and so forth. □

5.1 What is the parabolic index?

After rescaling, the attracting fixed points at $M_n(0)$ and at $M_n(\infty)$ continue to be mirror images and therefore are guaranteed to collide. Assuming that they are the *only* fixed points that collide, they will form a parabolic fixed point of index

$$I_p = \lim_{n\to\infty} \Big(\frac{1}{1-a_n} + \frac{1}{1-\overline{a_n}} \Big). \tag{5.5}$$

Lemma 5.2. *Suppose* $a \in \mathcal{H}_1(\eta)$. *Then* $1/(1-a) + 1/(1-\overline{a}) = 1 + \eta$.

Proof. The lemma follows from the observation that $a \to \frac{1}{1-a}$ takes the unit disk to the half-plane $\{w : \operatorname{Re} w \geq \frac{1}{2}\}$, with the point 0 mapping to 1. □

5.2 Do repelling fixed points participate?

Of course, we must decide whether repelling fixed points take part in the formation of the parabolic during the rescaling limit.

Theorem 5.1. *Suppose* $\{f_n\} \subset \mathcal{B}_d$ *is a sequence of of Blaschke products with one critical cluster and* $a_n \to 1$. *If the rescaling limit* $g = \lim \tilde{f}_n$ *exists, then either:*

(i) *A repelling fixed point collides with the two attracting fixed points if and only if its multiplier tends to 1. In this case,* $a_n \to 1$ *through horocycles, i.e. eventually* a_n *lies in arbitrarily small horoballs and the rescaling limit is doubly-parabolic.*

(ii) *Alternatively, if no repelling fixed point collides with the two attracting fixed points, then* $a_n \to 1$ *(asymptotically) along some horocycle* $\mathcal{H}_1(\eta)$ *and the index of the parabolic fixed point is* $1+\eta$*. In this case, the rescaling limit is singly-parabolic.*

Furthermore, in the first case, only a single repelling fixed point can take part in the parabolic index.

Proof. For a fixed n, the critical cluster $\{c_1, c_2, \dots, c_k\} \subset B(\widehat{c_1}, C \cdot \delta_c)$. When we rescale by $M_n = m_{c_1(f_n) \to 0}$, the attracting fixed points $M_n(0)$ and $M_n(\infty)$ will be contained in $B(-\widehat{c_1}, C \cdot \delta_c)$.

Suppose ξ is a repelling fixed point whose multiplier $m = f'(\xi)$ is bounded away from 1. By Lemma 2.2, $|\xi - \widehat{c_1}| \le C_2 \cdot \delta_c$, which implies that $M_n(\xi)$ is a definite distance away from $-\widehat{c_1}$. Therefore, ξ cannot collide with $M_n(0)$ and $M_n(\infty)$.

Suppose instead that for some repelling fixed point ξ, the multiplier $m = f'(\xi)$ tends to 1. Since $\frac{1}{1-m} \to -\infty$, the holomorphic index theorem implies that $\frac{1}{1-a} + \frac{1}{1-\overline{a}} \to \infty$. In other words, $a \to 1$ through horocycles. By reversing the reasoning, we see that the converse is true: if $a \to 1$ through horocycles, then the multiplier of some repelling fixed point must tends to 1.

To see that at most one repelling fixed point can take part in the parabolic index, note that by Lemma 2.1, f is injective on the arc

$$J = \mathbb{S}^1 \setminus B(\widehat{c_1}, C_3 \cdot \delta_c)$$

for $C_3 > 0$ sufficiently large. Topological considerations show that f has at most one fixed point on $\mathbb{S}^1 \setminus B(\widehat{c_1}, C_3 \cdot \delta_c)$. However, by Lemma 2.2, on the complementary arc, the derivative $|f'(\zeta)|$ is bounded away from 1. ☐

5.3 Realizing rescaling limits

From the above discussion, it follows that:

Theorem 5.2. *A sequence* $\tilde{f}_n$ *of hyperbolic Blaschke products of degree* d *converges uniformly on compact subsets of* $\mathbb{D}$ *to* $g \in \mathcal{PB}_d$ *if and only if*

(i) *the attracting fixed points of* $\tilde{f}$ *converge to the parabolic fixed point of* g*,*

(ii) *the critical configurations* $C(\tilde{f}_n) \to C(g)$ *converge and*

(iii) *either* $a_n \to 1$ *through horocycles, or asymptotically along a horocycle.*

Conversely, one has:

Theorem 5.3. *Every parabolic Blaschke product* g *is the limit of some sequence of Blaschke products that have attracting fixed points in the unit disk.*

Proof. As mentioned earlier, the translation length of g is 0. For $n \geq 1$, we choose a point $z_n \in \mathbb{D}$ such that $d_{\mathbb{D}}(z_n, g(z_n)) < 1/n$. Let $g_n(z) = m_{g(z_n)\to z_n} \circ g(z)$ so that z_n is a fixed point of g_n. Clearly, $g_n \to g$ as desired. □

The above proof shows that g is the rescaling limit of $f_n := m_{z_n \to 0} \circ g_n \circ m_{0\to z_n}$.

6 Horocyclic approach: $a \to e(p/q)$

We now turn our attention to studying limits of $\tilde{f}_{\mathbf{a}}^{\circ q}$ when $a \to e(p/q)$ along a horocycle. From the holomorphic index theorem (3.2) applied to $f_{\mathbf{a}}^{\circ r}$, it follows that multipliers of all cycles of period $r < q$ go to infinity. In fact, from Lemma 3.1, it follows that multipliers of all cycles of period q that do not have rotation number p/q must also go to infinity: otherwise, we would be able to extract a cycle $\langle \xi_0, \xi_1, \dots, \xi_{q-1}\rangle$ as a subsequential limit, which would be invariant under the limiting map $z \to e(p/q)z$, yet by construction, its rotation number is *not* p/q. This is clearly absurd. With this in mind, the holomorphic index theorem (3.2) applied to $f_{\mathbf{a}}^{\circ q}$ tells us that

$$\frac{1}{1-a^q} + \frac{1}{1-\overline{a}^q} + \sum_{C(p/q)} \frac{q}{1-r_i} \approx 1 \tag{6.1}$$

where we sum over the multipliers of (p/q)-cycles. We have:

Theorem 6.1. *Suppose $\{f_n\} \subset \mathcal{B}_d$ is a sequence of of Blaschke products with one critical cluster and $a_n \to e(p/q)$. If the rescaling limit $g = \lim \tilde{f}_n^{\circ q}$ exists, then either:*

(i) *A repelling periodic orbit collides with the two attracting fixed points if and only if its multiplier tends to 1. In this case, $a_n \to e(p/q)$ through horocycles, i.e. eventually a_n lies in arbitrarily small horoballs and the rescaling limit is doubly-parabolic.*

(ii) *Alternatively, if no repelling periodic orbit collides with the two attracting fixed points, then $a_n \to e(p/q)$ (asymptotically) along some horocycle $\mathcal{H}_{p/q}(\eta)$ and the index of the parabolic fixed point is $1+\eta$. In this case, the rescaling limit is singly-parabolic.*

In the first case, only a single repelling periodic orbit can take part in the parabolic index. In this case, it is necessarily a simple cycle of rotation number p/q.

Plan. The proof of Theorems 6.1 is nearly identical to the proofs of Theorems 5.1 in the last section. Therefore, we only highlight the differences: we explain why the degree drops under rescaling and how to compute the index of the parabolic fixed point.

6.1 Why does the degree drop?

While the degree of $\tilde{f}_{\mathbf{a}}^{\circ q}$ is d^q, the rescaling limit has degree only d. To see this, we will show that the rescaling limit has $(d-1)$ critical points. The critical points of $f_{\mathbf{a}}^{\circ q}$ include the critical points of $f_{\mathbf{a}}$ and their $1, 2, \ldots, (q-1)$-fold pre-images. They split into two groups: the dominant critical points and subordinate critical points. The *dominant critical points* are the *actual critical points* $c_i = c_{i,0}$ of $f_{\mathbf{a}}(z)$ and their *shadows*: since the map $f_{\mathbf{a}}$ is nearly a rotation $z \to e(p/q)z$, there exist critical points $c_{i,j}$ near $c_i \cdot e(-j \cdot p/q)$ for $j = 1, 2, \ldots, q-1$. We will refer to all remaining critical points as the *subordinate critical points*.

Clearly, the shadows of the critical points are far away from the critical cluster at which we rescale. From formula (2.3), it follows that the "heights" of the subordinate critical points are insignificant compared to the heights of the dominant critical points. This tells us that the subordinate critical points are also far away from the critical cluster. Therefore, the rescaling limit will only have $(d-1)$ critical points, and so has degree d. From this analysis, it follows that in order for the maps $\{f_n\}$ to have a rescaling limit, it is necessary for the critical clusters of $C(\tilde{f}_n)$ to converge, in which case, the critical cluster $C(g)$ is the limit of $C(\tilde{f}_n)$.

6.2 Computing the parabolic index

We now compute the index of the parabolic fixed point provided that $\frac{1}{1-a^q} + \frac{1}{1-\overline{a}^q}$ remains bounded as $n \to \infty$. In this case, the multipliers of p/q-cycles are bounded away from 1. In view of Lemma 3.1, if a sequence $\{f_n\} \subset \mathcal{B}_d$ has one critical cluster, we can order the repelling periodic orbit $\langle \xi_0, \xi_1, \xi_2, \ldots, \xi_{q-1} \rangle$ so that $\xi_j \approx \widehat{c_1} \cdot \omega^j$. It follows that ξ_0 carries most of the multiplier of the repelling periodic orbit, i.e. $|f'(\xi_0)| \approx |(f^{\circ q})'(\xi_0)|$ while $|f'(\xi_i)| \approx 1$ for $i \neq 0$.

We now apply this observation to study rescaling limits. If the multiplier $(f^{\circ q})'(\xi_0)$ is bounded away from 1, when we rescale, $\widetilde{\xi_0(f_n)}$ stays a definite distance away from $-\widehat{c_1}$ while the rest of the orbit collapses into the parabolic fixed point.

Therefore, the index of the parabolic fixed point equals to

$$I_p = \lim \left\{ \frac{1}{1-a^q} + \frac{1}{1-\overline{a}^q} + \sum_{C(p/q)} \frac{q-1}{1-r_i} \right\}. \tag{6.2}$$

Lemma 6.1. *Suppose $a \to e(p/q)$ along $\mathcal{H}_{p/q}(\eta)$. Then,*

$$\frac{1}{1-a^q} + \frac{1}{1-\overline{a}^q} + \sum_{C(p/q)} \frac{q-1}{1-r_i} \to 1 + \eta. \tag{6.3}$$

Proof. By Lemma 5.2,

$$\frac{1}{1-a^q} + \frac{1}{1-\overline{a}^q} \to 1 + q \cdot \eta.$$

Together with (6.1), this shows

$$\sum_{C(p/q)} \frac{1}{1-r_i} \to -\eta.$$

Hence,

$$\frac{1}{1-a^q} + \frac{1}{1-\overline{a}^q} + \sum_{C(p/q)} \frac{q-1}{1-r_i} \to 1+\eta$$

as desired. □

7 Lavaurs maps: $a \to 1$

We now turn to the second part of the paper, dealing with Problem 2. We explain the procedure of "renormalizing the quotient torus" and use it to study the dynamics of $f_{\mathbf{a}} \in \mathcal{B}_d$ in the critical disk $B(0, 1-K_0\delta_c)$. As before, we first examine horocyclic degenerations with $a \to 1$, and discuss the modifications that need to be made for the $a \to e(p/q)$ case later.

7.1 Renormalizing the quotient torus

Without loss of generality, let us consider a sequence $\{a_n\} \subset \mathbb{D}$ which approaches $1^{\circlearrowright}$ clockwise along a horocycle, i.e. $a_n = \mathtt{HtoD}_1(iy + x_n) \to 1$ with $x_n \to -\infty$. In this case, $\arg a$ is positive and therefore multiplication by a is a counter-clockwise rotation. (For counter-clockwise degenerations, multiplication by a is a clockwise rotation.) As discussed in the introduction, even though the quotient tori $T_{a_n} = \mathbb{C}^*/(\cdot\, a_n)$ diverge in Teichmüller space, they are recurrent in moduli space.

One fundamental domain for $(\mathbb{C}^*, \cdot\, a)$ is the annulus $\{z : |a| < |z| < 1\}$. We now describe another fundamental domain for $T_{a_n} = \mathbb{C}^*/(\cdot\, a_n)$. To construct it, pick an arbitrary point $z_0 \in \mathbb{C}$ and consider its orbit $z_0, z_1, z_2, \dots$ under multiplication by a. Since a is close to 1 along a horocycle, multiplication by a looks like a rotation by $e^{i \arg a}$, but there is also a small radial contraction. A simple calculation reveals that when $x_n(y)$ is large,

$$\arg a \sim 2/x, \qquad 1 - |a| \sim 2y/x^2.$$

In particular, it takes $N_a := \lceil \frac{2\pi}{\arg a} \rceil$ steps to make one complete revolution, plus a little extra. In this time, the total contraction $|z_{N_a}|/|z_0| = |a|^{N_a} \sim \pi y \arg a$ is comparable to the rotation of a single step. This suggests an alternative tiling of $\mathbb{C}^*/(\cdot\, a)$ of the form Ce^{Λ} where $\Lambda = \langle \tau, N_a\tau - 2\pi i \rangle$, $\tau = \log a$. If one chooses the constant C so that z_0 is a corner of a tile, the sequence $\{z_i\}_{i\in\mathbb{Z}}$ describes the set of all corners. In Figure 5 below, we show a renormalized fundamental domain – a "parallelogram" with sides (z_0, az_0) and $(z_0, a^{N_a} z_0)$.

Scaling Λ by τ (i.e. switching to the Ca^{Λ^*} representation), one obtains the lattice $\Lambda^* = \langle 1, \tau^* \rangle = \langle 1, N_a - \frac{2\pi i}{\tau} \rangle$. When x_n is large, $\Lambda^* \approx \big\langle 1,\, i\pi y + \lceil \pi x_n \rceil - \pi x_n \big\rangle$. In particular, these computations show that the tori $\mathbb{C}^*/(\cdot\, a)$ converge in moduli space if the *renormalized excess rotation* $\lceil \pi x_n \rceil - \pi x_n$ converges modulo 1. The parameter y determines the height of the limiting quotient torus.

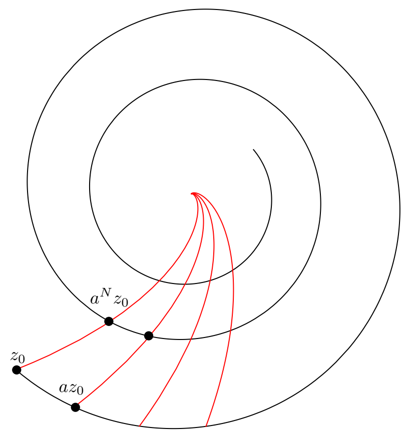

Figure 5: A renormalized fundamental domain.

7.2 Strong linearization principle

Suppose $\{f_n\} \subset \mathcal{B}_d$ is a sequence of Blaschke products with one critical cluster, with $a_n \to 1^{\circlearrowright}$ clockwise along a horocycle. One may suspect that in the ball $B(0, 1 - K_0\delta_c)$ with K_0 large, the iterate $f^{\circ j}$ acts approximately as multiplication by a^j. However, for general $1 \le j \le N_a$, the error between $f^{\circ j}(z)$ and $a^j z$ can be significant, for instance $|\log_a f^{\circ j}(z) - \log_a a^j z|$ need not be $\mathcal{O}(1)$. Nevertheless, when $n = N_a = \lceil 2\pi/\arg a \rceil$, $a^{N_a} z$ approximates $f^{\circ N_a}(z)$ fairly accurately. This crucially uses the horocyclic nature of a. In essence, one compares $\varphi(f^{\circ N_a}(z)) = a^{N_a} \cdot \varphi(z)$ with $\varphi(f(z)) = a \cdot \varphi(z)$ and uses $|a^{N_a} z - z| \asymp |az - z|$. However, these computations are best performed in the logarithmic coordinate, where the grid of Figure 5 becomes the lattice Λ_n^*. We defer the rigorous statement and proof to Section 8.

7.3 Canonical Cylinders

Our current objective is to prove the following theorem:

Theorem 7.1. *Suppose $\{f_n\} \subset \mathcal{B}_d$ is a sequence of Blaschke products with*

$$a_n = \mathtt{HtoD}_1(iy + x_n) \to 1, \quad x_n \to -\infty, \quad \tilde{f}_n \to g_T.$$

If x_n converge mod π^{-1} to $\sigma \in \mathbb{R}/\pi^{-1}\mathbb{Z}$, then $\tilde{f}_n^{\circ N_{a_n}} \to g_\sigma$.

The proof of Theorem 7.1 hinges on the main construction from the theory of parabolic implosion. We first build two fundamental domains for g_T: an "incoming" fundamental domain $U_-(g_T)$ and an "outgoing" fundamental domain $U_+(g_T)$. The incoming fundamental domain $U_-(g_T)$ has the property that any forward orbit of g_T passes through $\tilde{U}_-$ exactly once. Further, we use the sequence $\{\tilde{f}_n\}$ of approximating hyperbolic Blaschke products to construct a conformal map of half-cylinders $\tilde{\Delta} : U_-(g_T)/(g_T) \to U_+(g_T)/(g_T)$.

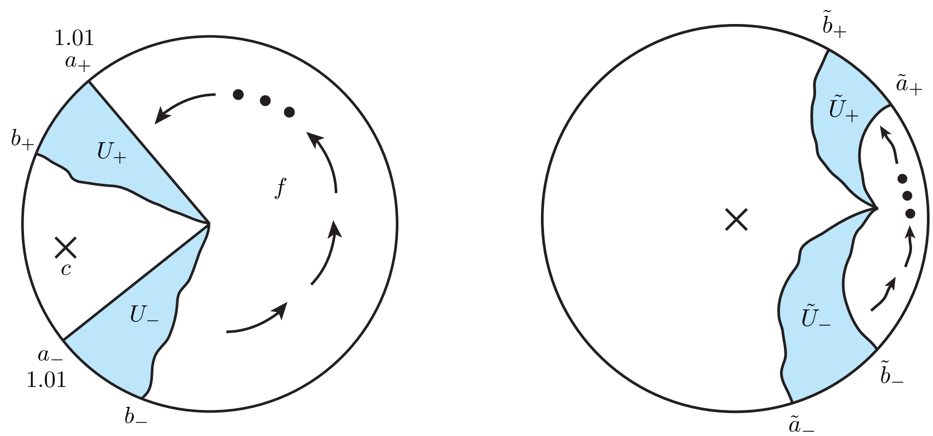

Figure 6: Canonical cylinders.

For each $n = 1, 2, \ldots$, we mark two points a_- and a_+ on the unit circle chosen so that $a_+, \widehat{c_1}, a_-$ ordered counter-clockwise and

$$|a_+ - \widehat{c_1}| = |a_- - \widehat{c_1}| = M \cdot \delta_c, \tag{7.1}$$

where $M > 0$ is sufficiently large to guarantee that f_n is injective on $\mathbb{S}^2 \setminus B(\widehat{c_1}, M\delta_c)$. By construction, $m_{c_1 \to 0}(a_\pm)$ converges as $n \to \infty$. Set $b_- := f_n(a_-)$ and $b_+ := f_n(a_+)$. We have thus constructed two intervals $[a_-, b_-]$ and $[a_+, b_+]$ on the unit circle. Let $U_-(f_n)$ be the region enclosed by the curves

$$[0, a), \quad [a_-, b_-], \quad f\big([0, a)\big)$$

and $C_-(f_n) := U_-/(f_n)$ denote the quotient half-cylinder. We define U_+ and $C_+(f_n)$ similarly. By construction, as $n \to \infty$, the rescaled domains $U_\pm(\tilde{f}_n)$ converge in the Carathéodory topology to limiting domains $U_\pm(\tilde{g}_T)$.

The dynamics of f_n determine a natural conformal equivalence between the half-cylinders $\Delta_n : C_-(f_n) \to C_+(f_n)$. Namely, given a point $z \in C_-(f_n)$, we lift it to $U_-(f_n)$, take the first iterate that lands in $U_+(f_n)$, and then project down to $C_+(f_n)$.

Rescaling by $m_{c\to 0}$ gives maps $\tilde{\Delta}_n : C_-(\tilde{f}_n) \to C_+(\tilde{f}_n)$. As $n \to \infty$, these half-cylinders converge in bulk $C_\pm(\tilde{f}_n) \to C_\pm(g_T)$. Since a map of half-cylinders is defined up to an additive constant in $\mathbb{R}/\mathbb{Z}$, in order for the maps $\tilde{\Delta}_n$ to converge, these constants need to become aligned. To pinpoint this constant, we examine the angular correspondence between $U_-(f_n)$ and $U_+(f_n)$ near the origin. The computation below uses the strong linearization principle which provides the necessary uniformity needed to make the argument rigorous.

Let θ_n be the counter-clockwise angle from α_+ to α_-. From the existence of an the rescaling limit, we know that $\theta_n / \arg a_n \to \theta^*$ converges as $n \to \infty$. It takes roughly

$$N'_{a_n} := \left\lceil \frac{2\pi - \theta_n}{\arg a_n} \right\rceil$$

steps to go from $U_-(f_n)$ to $U_+(f_n)$, at least near the origin. After N'_{a_n} steps, the renormalized excess rotation from α_- to α_+ is

$$\sigma'_n := \frac{N'_{a_n} \arg a_n - (2\pi - \theta_n)}{\arg a_n} \in \mathbb{R}/\mathbb{Z}. \tag{7.2}$$

It is of course more natural to consider the quantity

$$\sigma_n := \frac{N_{a_n} \arg a_n - 2\pi}{\arg a_n} \in \mathbb{R}/\mathbb{Z}, \tag{7.3}$$

which is more canonical since it does not depend on choices of a_- and a_+. However, $\sigma - \sigma' \equiv \theta^*$ modulo 1, and so σ and σ' carry the same information.

Summarizing, the use of the half-cylinder maps shows that $f_n^{\circ N'_{a_n}}$ converge if and only if the σ'_n converge modulo 1. This occurs precisely when the σ_n converge modulo 1, which is the same as asking for the convergence of the compactified quotient tori T_{f_n} in the moduli space $\mathcal{M}_1$. Since $N_{a_n} - N'_{a_n} = \mathcal{O}(1)$, the convergence of $f_n^{\circ N'_{a_n}}$ implies the convergence of $f_n^{\circ N_{a_n}}$. We refer to $g_\sigma = \lim_{n\to\infty} f_n^{\circ N_{a_n}}$ as a *Lavaurs map*.

7.4 Basic properties of Lavaurs maps

We briefly describe the basic properties of g_σ. From the construction, it is clear that g_σ is an endomorphism of the unit disk which extends analytically to the arc $m_{c\to 0}([a_-, b_-])$. Using the fact that f and g_σ commute, we see that g_σ extends analytically to to the complement of the Julia set $\mathcal{J}(g_T)$. Since the Julia set $\mathcal{J}(g_T)$ has measure 0, the boundary values of $\lim g_T$ exist a.e. on the unit circle and have absolute value 1, in other words, g_σ defines an inner function.

Furthermore, p is the Denjoy-Wolff point of g_σ. In fact, under iteration by g_σ, the forward orbit of any point $z \in \mathbb{D}$ is eventually contained in arbitrarily small horoballs resting on p (like for a doubly parabolic Blaschke product). To see this, first note that for any $K \geq K_0$ and $R > 0$, one can find $m \geq 1$ such that $f^{\circ m}(B_{\text{hyp}}(z_0, R)) \subset B(0, 1 - K\delta_c)$, for all $n \geq n_0$ sufficiently large. As $n \to \infty$,

$$m_{c \to 0}\big(B(0, 1 - K\delta_c)\big) \to \mathcal{H}_p,$$

tend to a horoball resting on p. These facts show that $\mathcal{H}_p$ is an absorbing region for g_σ, a furthermore,

$$g(z) \approx m_{c_1 \to 0} \circ (\cdot\, a^{N_a}) \circ m_{0 \to c_1}(z), \qquad z \in \mathcal{H}_p, \tag{7.4}$$

up to small error in the hyperbolic metric, cf. (5.4). Inspecting the dynamics of $(\cdot\, a^{N_a})$ shows that the horoball parametr is contracted. In fact, a more careful reflection shows the quotient $T^{\times}_{\mathcal{G}} = \Omega_{\mathcal{G}}/\mathcal{G}$ is a torus whose closure is isomorphic to $\mathbb{C} \setminus \Lambda^*$.

8 Logarithmic coordinate

The proof of the strong linearization principle uses the following lemma which one can consider as the *key estimate* of the theory of parabolic implosion:

Lemma 8.1 (e.g. see [S]). *Suppose Φ and v are holomorphic functions in a domain $\mathcal{U}$, with Φ injective in $\mathcal{U}$ and*

- $|v(w) - 1| < 1/4$ *for $w \in \mathcal{U}$,*
- $\Phi(w + v(w)) = \Phi(w) + 1$ *whenever $w, \Phi(w) \in \mathcal{U}$.*

There exist universal constants C_1, C_2 and R_0 such that

$$|\Phi'(w) - 1/v| \leq C_2/R, \tag{8.1}$$

whenever $\operatorname{dist}(w, \partial\mathcal{U}) \geq R > R_0$, *where* dist *denotes the Euclidean distance.*

This estimate will be applied in the logarithmic coordinate. We first note that by Lemma 2.5, we may choose $K_0 > 1$ sufficiently large so that in $B(0, 1 - K_0\delta_c)$,

(i) $f(z)$ is injective,

(ii) thus $\varphi(z) := \lim_{k\to\infty} f^{\circ k}(z)/a^k$ is also injective, being the uniform limit of univalent functions,

(iii) $|\log_a f(z) - \log_a(az)| < C/K$.

We are now ready to introduce the *logarithmic coordinate* $w(z) = \mathscr{E}^{-1}(z)$. Let $\mathcal{U}$ be the half-plane $\{w \in \mathbb{C} : |a^w| < 1\}$. By construction, it slopes downwards, i.e. is forward-invariant under the translation $w \to w + 1$. The map $\mathscr{E}$ is essentially a^w, although for convenience, we take $\mathscr{E}(w) := a^{w + w_0}$ where (a) the translation $w \to w + w_0$ fixes $\mathcal{U}$ and (b) one of the pre-images $w_1 \in \mathscr{E}^{-1}(c_1)$ lies on the y-axis.

By construction, multiplication by a becomes translation by 1. The functions f and $g = f_n^{\circ N_{a_n}}$ become

$$\mathbf{F}(w) := \mathscr{E}^{-1} \circ f \circ \mathscr{E}(w) = w + 1 + v(w), \tag{8.2}$$

$$\mathbf{G}(w) := \mathbf{F}^{\circ N_{a_n}}(w) - \frac{2\pi i}{\log a}. \tag{8.3}$$

Here, in the definition of $\mathbf{F}$, we use pick the branch $\mathscr{E}^{-1}$ so that $v(w) \to 0$ if $\operatorname{dist}(w, \partial\mathcal{U}) \to \infty$. Note that $\mathbf{F}$ and $\mathbf{G}$ are genuine holomorphic functions on $\mathcal{U}$. Next, we transfer the "grid map"

$$\Psi = \frac{\log \varphi(z)}{\log a} - \frac{\log \varphi(c_1)}{\log a} \qquad \Longrightarrow \qquad \mathbf{\Psi}(w) = \mathscr{E}^{-1} \circ \Psi \circ \mathscr{E}(w).$$

By construction,

$$\mathbf{\Psi}(\mathbf{F}(w)) = w + 1, \tag{8.4}$$

$$\mathbf{\Psi}(\mathbf{G}(w)) = w + \tau^*. \tag{8.5}$$

We caution the reader that $\mathbf{\Psi}$ is multivalued over $\mathcal{U}$; however, given any $\rho > 0$ arbitrarily large, $\mathbf{\Psi}_n$ is single-valued on $B_{\text{hyp}}(w_1, \rho) \subset \mathcal{U}$, provided $n \geq n(\rho)$ is sufficiently large. By the observation (iii) above, there exists R_0 sufficiently large so that $|v(w) - 1| < C_1/R$ on $\mathcal{U}_R$ for any $R \geq R_0$, where $\mathcal{U}_R := \{w \in \mathcal{U} : \operatorname{dist}(w, \partial\mathcal{U}) \geq R\}$. Lemma 8.1 gives $|\Phi'(z) - 1| < C/R$ for $z \in \mathcal{U}_R$. In particular, the hyperbolic distance

$$d_{\mathcal{U}}\big(\mathbf{G}(w),\, w + \tau^*\big) < C_2/R^2, \qquad w \in \mathcal{U}_R, \quad R > R_0, \quad n > n_0. \tag{8.6}$$

Applying the triangle inequality gives

$$d_{\mathcal{U}}\big(\mathbf{G}^{\circ k}(w),\, w + k\tau^*\big) < C_3/R, \quad \text{for any } k \geq 1. \tag{8.7}$$

Since

$$\mathbf{\Psi}(w) = \lim_{k\to\infty} (\mathbf{G}^{\circ k}(w) - k\tau^*), \tag{8.8}$$

we deduce that for any $\varepsilon > 0$, exists R_0 and n_0 sufficiently large so that

$$|\mathbf{\Psi}(w) - w| < \varepsilon, \qquad w \in \mathcal{U}_R, \quad R > R_0, \quad n > n_0. \tag{8.9}$$

This is the desired uniformity which makes the argument counting work.

Taking $n \to \infty$ (optional)

As $n \to \infty$, the domains $\mathcal{U}_n$ converge to $\mathbb{H}$. The theory of rescaling limits states the assumptions which guarantee that the limit of the maps $\mathbf{F}_n(w)$ exists. At first glance, this may look a little strange but if n is large, $\mathbf{F}_n(w)$ is nearly identical to the upper half-plane rescaling $\texttt{DtoH} \circ f_n \circ \texttt{HtoD}(w)$. The grid maps $\mathbf{\Psi}_n$ converge to the grid map for the decorated rescaling limit.

9 Lavaurs maps: $a \to e(p/q)$

We now briefly describe the rather minor modifications that need to be made when $a_n \to e(p/q)^{\circlearrowright}$ clockwise along a horocycle. Suppose $\{f_n\} \subset \mathcal{B}_d$ is a sequence of Blaschke products with

$$a_n = \mathtt{HtoD}_{p/q}(q^2 \cdot (iy + x_n)), \qquad x_n \to -\infty,$$

such that $\tilde{f}_n^{\circ q} \to g_T$ and $\sigma = \lim x_n \pmod{\pi^{-1}}$ exists. Our objective is to construct the Lavaurs map g_σ and show that it only depends on (g_T, σ) and not on p/q.

Renormalizing the quotient tori

We first explain how to renormalize the quotient tori $T_{a_n} = \mathbb{C}^*/(\cdot\, a_n)$. Fix an arbitrary point $z_0 \in \mathbb{C}^*$ and consider the sector

$$S_{z_0} = \{z \in \mathbb{C}^* : \arg z_0 < \arg z < \arg a_n^q \cdot z_0\}. \tag{9.1}$$

We define N_{a_n} as the first return time of z_0 to S_{z_0} with respect to the action of $(\cdot\, a_n)$, that is, N_{a_n} is the minimal positive integer k for which $a_n^k \cdot z_0 \in S$. From the definition,

$$N_{a_n} \approx \frac{2\pi}{q \cdot [\arg(a \cdot e(-p/q))]}.$$

Consider the lattices $\Lambda_n = \left\langle \log a_n^q, \log a_n^{N_{a_n}} \right\rangle$. Scaling Λ_n by $\log a_n^q$, we obtain the lattices

$$\Lambda_n^* = \langle 1, \tau \rangle, \qquad \tau = \frac{\log a_n^{N_{a_n}}}{\log a_n^q}.$$

With this scaling, the lattices converge, and $\Lambda^* = \lim \Lambda_n^*$ is independent of (p/q) – it only depends on y and $\lim x_n \pmod{\pi^{-1}}$.

Canonical cylinders

The construction of the canonical cylinders is nearly identical to the $a \to 1^{\circlearrowright}$ case: for each f_n, we select $a_\pm$ according to (7.1), but this time we take $b_\pm := f^{\circ q}(a_\pm)$. Let $U_-(f_n^{\circ q})$ be the region bounded by $\{[0, a_-], [a_-, b_-], f^{\circ q}([0, a_-])\}$ and $C_-(f_n^{\circ q})$ be the half-cylinder obtained by identifying the two radial sides of $U_-(f_n^{\circ q})$ by $f_n^{\circ q}$. The objects $U_+(f_n^{\circ q}), C_+(f_n^{\circ q})$ are defined similarly. From the construction, the rescaled versions $U_-(\tilde{f}_n^{\circ q})$ and $U_+(\tilde{f}_n^{\circ q})$ converge to the same $U_-(g_T)$ and $U_+(g_T)$ from the $a \to 1^{\circlearrowright}$ case.

Renormalized excess rotation

We define N'_n as the first hitting time of S_{a_+} where multiplication by a is started from a_-. Recall that N_n is the first hitting time of the sector S_{a_-}. From the construction, $N_{a_n} > N'_{a_n}$ and $N_{a_n} - N'_{a_n} = \mathcal{O}(1)$. In fact, if θ_n is the counter-clockwise angle from α_+ to α_- (where as before, $\alpha_\pm = [0, \alpha_\pm]$), from the existence of an the rescaling limit, we know that $\theta_n / \arg a_n^q \to \theta^*$ converges as $n \to \infty$, and this limit depends only on g (and crucially not on p/q). Thus, the renormalized excess rotation from α_- to α_+ is given by

$$\sigma'_n := \frac{\arg a_n^{N'_{a_n}} - (2\pi - \theta)}{\arg a_n^q} \in \mathbb{R}/\mathbb{Z}. \tag{9.2}$$

Like before, the *Lavaurs phase*

$$\sigma_n := \frac{\arg a_n^{N_{a_n}} - 2\pi}{\arg a_n^q} \in \mathbb{R}/\mathbb{Z} \tag{9.3}$$

is more natural since it does not depend on the arbitrary choices involved in the construction of the canonical cylinders. Furthermore, the Lavaurs phase σ_n is determined by the closed torus X – it is independent of p/q.

Transfer maps

To define $\Delta_n : C_-(f_n^{\circ q}) \to C_+(f_n^{\circ q})$, we follow the same recipe as before: given a point $z \in C_-(f_n)$, we lift it to $U_-(f_n^{\circ q})$, take the first iterate that lands in $U_+(f_n^{\circ q})$, and then project down to $C_+(f_n^{\circ q})$. The reader can check for $1 \le k \le N'_n$, $f^{\circ k}(U_-(f_n^{\circ q}))$ are disjoint from the sector enclosed by α_+ and α_- which contains the critical points of f_n, so that Δ_n is a conformal equivalence. (This time, the iterates do quite a bit of hopping, however, all the hops are injective.)

Since the space of half-cylinder maps coincides with the one from the $a \to 1^{\circlearrowleft}$ setting, the possible limits of $f_n^{\circ N'_n}$ are the maps g_σ from before. Since the strong linearization principle holds for N_n, it holds for N'_n as well. Therefore, σ'_n is exactly the additive constant that specifies the half-cylinder map Δ_n.

10 Mating decorated Blaschke products

11 Applications to the Weil-Petersson metric

TO BE WRITTEN

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