Stable convergence of inner functions

Oleg Ivrii

January 9, 2020

Abstract

Let $\mathscr{J}$ be the set of inner functions whose derivative lies in the Nevanlinna class. In this paper, we discuss a natural topology on $\mathscr{J}$ where $F_n \to F$ if the critical structures of F_n converge to the critical structure of F. We show that this occurs precisely when the critical structures of the F_n are uniformly concentrated on Korenblum stars. The proof uses Liouville's correspondence between holomorphic self-maps of the unit disk and solutions of the Gauss curvature equation. Building on the works of Korenblum and Roberts, we show that this topology also governs the behaviour of invariant subspaces of a weighted Bergman space which are generated by a single inner function.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. An *inner function* is a holomorphic self-map of the unit disk such that for almost every $\theta \in [0, 2\pi)$, the radial limit $\lim_{r\to 1} F(re^{i\theta})$ exists and has absolute value 1. Let Inn denote the set of all inner functions and $\mathscr{J} \subset \text{Inn}$ be the subset consisting of inner functions which satisfy

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F'(re^{i\theta})| d\theta < \infty, \tag{1.1}$$

that is, with F' in the Nevanlinna class. The work of Ahern and Clark [1] implies that if $F \in \mathscr{J}$, then F' belongs to the Smirnov class and hence admits an "inner-outer" decomposition $F' = \operatorname{Inn} F' \cdot \operatorname{Out} F'$. Intuitively, $\operatorname{Inn} F' = BS$ describes the "critical structure" of the map F – the Blaschke factor records the locations of the critical points of F in the unit disk, while the singular inner factor describes the "boundary critical structure." In [10], the author proved the following theorem, answering a question posed in [6]:

Theorem 1.1. *Let $\mathscr{J}$ be the set of inner functions whose derivative lies in the Nevanlinna class. The natural map*

$$F \to \operatorname{Inn}(F') \quad : \quad \mathscr{J} / \operatorname{Aut}(\mathbb{D}) \to \operatorname{Inn} / \mathbb{S}^1$$

is injective. The image consists of all inner functions of the form BS_μ where B is a Blaschke product and S_μ is the singular factor associated to a measure μ whose support is contained in a countable union of Beurling-Carleson sets.

The above theorem says that an inner function $F \in \mathscr{J}$ is uniquely determined up to a post-composition with a holomorphic automorphism of the disk by its critical structure and describes all possible critical structures of inner functions. We need to quotient Inn by the group of rotations since the inner part is determined up to a unimodular constant. To help remember this, note that *Frostman shifts or post-compositions with elements of* $\operatorname{Aut}(\mathbb{D})$ *do not change the critical set of a function while rotations do not change the zero set.*

By definition, a *Beurling-Carleson set* $E \subset \mathbb{S}^1$ is a closed subset of the unit circle of zero Lebesgue measure whose complement is a union of arcs $\bigcup_k I_k$ with

$$\|E\|_{\mathcal{BC}} = \sum |I_k| \log \frac{1}{|I_k|} < \infty.$$

We say that $E \in \mathcal{BC}(N)$ if $\|E\|_{\mathcal{BC}} \le N$. We denote the collection of all Beurling-Carleson sets by $\mathcal{BC}$.

We will also need the notion of a *Korenblum star* which is the union of Stolz angles emanating from a Beurling-Carleson set $E \subset \mathbb{S}^1$:

$$K_E = B(0, 1/\sqrt{2}) \cup \bigl\{z \in \overline{\mathbb{D}} : 1 - |z| \ge \operatorname{dist}(\hat{z}, E)\bigr\}.$$

Here, $\hat{z} = z/|z|$ while dist denotes the Euclidean distance. With the above definition, $K_E \subset \overline{\mathbb{D}}$ is a closed set. We say that the Korenblum star has *entropy* or *norm* $\|E\|_{\mathcal{BC}}$.

We endow $\mathscr{J}$ with the topology of *stable convergence* where $F_n \to F$ if the F_n converge uniformly on compact subsets of the disk to F and the Nevanlinna splitting is preserved in the limit: $\operatorname{Inn} F'_n \to \operatorname{Inn} F'$, $\operatorname{Out} F'_n \to \operatorname{Out} F'$. Loosely speaking, our main result says that this occurs if and only if the critical structures of F_n are "uniformly concentrated" on Korenblum stars. A precise statement will be given later in the introduction. As observed in [10], in general, some part of the critical structure may disappear in the limit:

$$\operatorname{Inn} F'(z) \ge \limsup_{n\to\infty} \operatorname{Inn} F'_n(z), \qquad z \in \mathbb{D}. \tag{1.2}$$

Examples. (i) If F_n is a finite Blaschke product of degree $n+1$ which has a critical point at $1-1/n$ of multiplicity n, and is normalized so that $F_n(0)=0$, $F'_n(0)>0$, then the F_n stably converge to the unique inner function F_{δ_1} with critical structure $S_{\delta_1} = \exp\bigl(\frac{z+1}{z-1}\bigr)$. More generally, if F_n has n critical points (of multiplicity one) at $c_k = (1-1/n)e^{ik\theta_n}$, $k = 1, 2, \dots, n$, and $n\theta_n \log \frac{1}{\theta_n} \to 0$, then the F_n still stably converge to F_{δ_1}.

(ii) If $n\theta_n \log \frac{1}{\theta_n} \to \infty$ but $n\theta_n \to 0$, then the F_n converge to the identity even though the critical structures $\operatorname{Inn} F'_n \to S_{\delta_1}$.

(iii) For any $0 < c < 1$, one can choose θ_n appropriately so that the F_n converge to $F_{c\delta_1}$, the unique inner function with critical structure $S_{c\delta_1}$. In this case, $n\theta_n \log \frac{1}{\theta_n}$ must be bounded away from 0 and ∞.

We will examine this example more thoroughly in Section 4.4.

1.1 The Korenblum topology

A simple "normal families" argument [8, Lemma 7.6] shows that $\mathcal{BC}(N)$ is compact in the Hausdorff topology. For convenience of the reader, we give a brief sketch of the argument. Given a sequence of sets $\{E_n\} \subset \mathcal{BC}(N)$, let $I_n^{(1)}$ denote the longest complementary arc in $\mathbb{S}^1 \setminus E_n$ (in case of a tie, we choose $I_n^{(1)}$ to be one of the longest arcs). We pass to a subsequence so that the $I_n^{(1)}$ converge to a limit $I^{(1)}$. Since there is a definite lower bound for the length $\bigl|I_n^{(1)}\bigr|$, these arcs cannot shrink to a point. We then pass to a further subsequence along which the second longest arcs $I_n^{(2)} \to I^{(2)}$ converge. Continuing in this way, and diagonalizing, we obtain a subsequence of

$\{E_n\}$ which converges to a set $E \in \mathcal{BC}(N)$. Note that if E is a finite set, this process would terminate in finitely many steps. The above argument gives the inequality

$$\|E\|_{\mathcal{BC}} \le \liminf_{n\to\infty} \|E_n\|_{\mathcal{BC}}. \tag{1.3}$$

However, (1.3) could be a strict inequality if a definite amount of entropy gets trapped in smaller and smaller sets. To state this phenomenon precisely, we define the *local entropy* of a Beurling-Carleson $E \subset \mathbb{S}^1$ with *threshold* $\eta > 0$ as

$$\|E\|_{\mathcal{BC}_\eta} = \sum_{|I|<\eta} |I| \log \frac{1}{|I|},$$

where we sum over the connected components of $\mathbb{S}^1 \setminus E$ whose length is less than η. Then, (1.3) is a strict inequality if and only if $\liminf_{n\to\infty} \|E_n\|_{\mathcal{BC}_\eta} > c > 0$ is bounded below by a constant independent of η.

Topology on Beurling-Carleson sets

We define the *Korenblum topology* on $\mathcal{BC}$ by specifying that $E_n \to E$ if E_n converges to E in the Hausdorff sense and $\|E\|_{\mathcal{BC}} = \lim_{n\to\infty} \|E_n\|_{\mathcal{BC}}$. In this case, we say the sequence of sets $\{E_n\}$ is *concentrating* (the terminology is inspired by the work of Marcus and Ponce [15]).

Topology on measures on the unit circle

Let $M_{\mathcal{BC}(N)}(\mathbb{S}^1)$ denote the collection of finite positive measures that are supported on a Beurling-Carleson set of norm $\le N$ and $M_{\mathcal{BC}}(\mathbb{S}^1)$ denote the collection of measures supported on a countable union of Beurling-Carleson sets. Roughly speaking, a (weakly-convergent) sequence of measures $\mu_n \to \mu$ converges in the *Korenblum topology* on $M_{\mathcal{BC}}(\mathbb{S}^1)$ if up to arbitrarily small error, $\operatorname{supp} \mu_n \subseteq E_n$ for a concentrating sequence of Beurling-Carleson sets E_n. More precisely, we require that for any $\varepsilon > 0$, there exists an $N > 0$ and a "dominated" sequence $\nu_n \to \nu$ such that for all n sufficiently large,

(i) $0 \le \nu_n \le \mu_n$,

(ii) $\nu_n \in M_{\mathcal{BC}(N)}(\mathbb{S}^1)$,

(iii) $\operatorname{supp} \nu_n \to \operatorname{supp} \nu$ in $\mathcal{BC}$,

(iv) $(\mu_n - \nu_n)(\mathbb{S}^1) < \varepsilon$ and $(\mu - \nu)(\mathbb{S}^1) < \varepsilon$.

Diffuse sequences

We call a (weakly-convergent) sequence of measures $\mu_n \to \mu$ *diffuse* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any threshold $\eta > 0$, we have

$$\|E\|_{\mathcal{BC}_\eta} < \delta \quad \implies \quad \mu_n(E) < \varepsilon, \qquad n \geq n_0(\delta, \varepsilon, \eta), \quad E \in \mathcal{BC}\,.$$

It is not difficult to see that a sequence is concentrating if and only if it does not dominate any diffuse sequence with non-zero limit. This allows one to decompose any weakly convergent sequence $\mu_n \to \mu$ into concentrating and diffuse components, that is, to write $\mu_n = \nu_n + \tau_n$ with $\nu_n \to \nu$ concentrating and $\tau_n \to \tau$ diffuse. Even though there are infinitely many choices for the sequences $\{\nu_n\}$ and $\{\tau_n\}$, the limits ν and τ are uniquely determined by $\{\mu_n\}$. We leave the verification to the reader.

Topology on the closed unit disk

We say that a finite positive measure μ on the closed unit disk belongs to $M_{\mathcal{BC}(N)}(\overline{\mathbb{D}})$ if its support is contained in a Korenblum star of norm $\leq N$, while $\mu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ if it is supported on a countable union of Korenblum stars. Since any compact subset of the unit disk is contained in some Korenblum star, this is the same as asking that the restriction $\mu|_{\mathbb{S}^1} \in M_{\mathcal{BC}}(\mathbb{S}^1)$.

We define the *Korenblum topology* on $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ by specifying that a sequence of measures $\mu_n \to \mu$ converges if it does so weakly, and up to arbitrarily small error, $\operatorname{supp} \mu_n \subseteq K_{E_n}$ for a concentrating sequence of Beurling-Carleson sets E_n. Similarly, we say that a sequence of measures $\mu_n \to \mu$ in $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ is *diffuse* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any threshold $\eta > 0$, we have

$$\|E\|_{\mathcal{BC}_\eta} < \delta \quad \implies \quad \mu_n(K_E) < \varepsilon, \qquad n \geq n_0(\delta, \varepsilon, \eta), \quad E \in \mathcal{BC}\,.$$

1.2 Two embeddings of inner functions

To an inner function I, we associate the measure

$$\mu(I) = \sum (1 - |a_i|)\delta_{a_i} + \sigma(I) \in M(\overline{\mathbb{D}}), \tag{1.4}$$

where the sum ranges over the zeros of I (counted with multiplicity) and $\sigma(I)$ is the singular measure on the unit circle associated with the singular factor of I. This gives an embedding $\mathrm{Inn} / \mathbb{S}^1 \to M(\overline{\mathbb{D}})$. We say that the measure μ records the *zero structure* of I and write $I_\mu := I$. Clearly, the function I_μ is uniquely determined up to a rotation.

We can also embed $\mathscr{J} / \mathrm{Aut}(\mathbb{D}) \to M_{\mathcal{BC}}(\overline{\mathbb{D}})$ by taking $F \to \mu(\mathrm{Inn}\, F')$. This embedding records the *critical structure* of F. We use the symbol F_μ to denote an inner function with $\mathrm{Inn}\, F'_\mu = I_\mu$ and $F_\mu(0) = 0$ (again, such a function is unique up to a rotation).

We can now state our main result:

Theorem 1.2. *The embedding* $\mathscr{J} / \mathrm{Aut}(\mathbb{D}) \to M_{\mathcal{BC}}(\overline{\mathbb{D}})$ *is a homeomorphism onto its image when* $\mathscr{J} / \mathrm{Aut}(\mathbb{D})$ *is equipped with the topology of stable convergence and* $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ *is equipped with the Korenblum topology.*

1.3 Connections with the Gauss curvature equation

We now give an alternative (and slightly more general) perspective of our main theorem in terms of conformal metrics and nonlinear differential equations. Given a conformal pseudometric $\lambda(z)|dz|$ on the unit disk, its *Gaussian curvature* is given by the expression

$$k_\lambda = -\frac{\Delta \log \lambda}{\lambda^2},$$

where the Laplacian is taken in the sense of distributions. An easy computation shows that the Poincaré metric $\lambda_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}$ has constant curvature -4.

The importance of Gaussian curvature to complex analysis comes from Gauss Theorema Egregium [14, Theorem 2.5] which says that curvature is a conformal invariant: if $F \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ is a holomorphic self-map of the unit disk, then

$$\lambda_F := F^* \lambda_{\mathbb{D}} = \frac{|F'|}{1 - |F|^2}$$

has curvature -4 on $\mathbb{D} \setminus \mathrm{crit}(F)$ where $\mathrm{crit}(F)$ is the critical set of F. On the critical set, $\lambda_F = 0$ while its curvature has δ-masses. Its logarithm $u_F = \log \lambda_F$ satisfies the *Gauss curvature equation*

$$\Delta u = 4e^{2u} + 2\pi\tilde{\nu}, \tag{1.5}$$

where $\tilde{\nu} = \sum_{c \in \mathrm{crit}(F)} \delta_c$ is an integral sum of point masses. A theorem of Liouville [14, Theorem 5.1] states that any solution of Gauss curvature equation with integral singularities arises in this way. In other words, the correspondence $F \to u_F$ is a bijection between

$$\mathrm{Hol}(\mathbb{D}, \mathbb{D}) \,/\, \mathrm{Aut}(\mathbb{D}) \quad \Longleftrightarrow \quad \big\{\text{solutions of (1.5) with } \tilde{\nu} \text{ integral}\big\}.$$

In principle, Liouville's theorem allows one to translate questions about holomorphic self-maps of the disk to problems in PDE and vice versa. In practice, however, it is difficult to find questions that are simultaneously interesting in both settings.

Ahlfors showed that $u_{\mathbb{D}} = \log \lambda_{\mathbb{D}}$ is the *maximal* solution of (1.5) in the sense that it dominates every solution of (1.5) pointwise with any $\tilde{\nu} \geq 0$. It turns out that the question of describing inner functions with derivative in the Nevanlinna class is related to studying the Gauss curvature equation with *nearly-maximal* boundary values

$$\begin{cases} \Delta u = 4e^{2u} + 2\pi\tilde{\nu}, & \text{in } \mathbb{D}, \\ u_{\mathbb{D}} - u = \mu, & \text{on } \mathbb{S}^1, \end{cases} \tag{1.6}$$

where we allow $\tilde{\nu} \in M(\mathbb{D})$ to be any positive measure on the unit disk which satisfies the *Blaschke condition*

$$\int_{\mathbb{D}} (1 - |z|) d\tilde{\nu}(z) < \infty, \tag{1.7}$$

and $\mu \in M(\mathbb{S}^1)$ to be any finite positive measure on the unit circle. We say that u has *singularity* $2\pi\tilde{\nu}$ and *deficiency* μ.

The first equality in (1.6) is understood weakly in the sense of distributions: we require $u(z)$ and $e^{2u(z)}$ to be in $L^1_{\mathrm{loc}}(\mathbb{D})$, and ask that for any test function $\phi \in C_c^\infty(\mathbb{D})$, compactly supported in the disk,

$$\int_{\mathbb{D}} u\Delta\phi \, |dz|^2 = \int_{\mathbb{D}} 4e^{2u}\Delta\phi \, |dz|^2 + 2\pi \int_{\mathbb{D}} \phi d\tilde{\nu}, \tag{1.8}$$

while the second equality expresses the fact that the measures $(u_{\mathbb{D}} - u)(d\theta/2\pi)|_{\{|z|=r\}}$ converge weakly to μ as $r \to 1$. As we shall see, it is natural to combine μ and $\tilde{\nu}$ into a single measure:

$$\omega(z) = \mu(z) + \nu(z) := \mu(z) + \tilde{\nu}(z)(1 - |z|) \in M(\overline{\mathbb{D}}). \tag{1.9}$$

Theorem 1.3. *Given a measure $\omega = \mu + \nu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, the equation (1.6) admits a unique solution, which we denote $u_{\mu,\nu}$ or u_ω. The solution u_ω is decreasing in ω, that is, $u_{\omega_1} > u_{\omega_2}$ if $\omega_1 < \omega_2$. If $\omega \notin M_{\mathcal{BC}}(\overline{\mathbb{D}})$, then no solution exists.*

We endow the space of solutions of (1.6) with the *stable topology* where $u_{\omega_n} \to u_\omega$ if $\omega_n \to \omega$ and $u_{\omega_n} \to u_\omega$ weakly. Similar to what happens for inner functions in $\mathscr{J}$, for general sequences of measures $\omega_n \to \omega$, some part of the "mass" can disappear in the limit, see Lemma 2.8. In the setting of nearly-maximal solutions of the Gauss curvature equation, our main theorem states:

Theorem 1.4. *The stable topology on the space of solutions with nearly maximal boundary values corresponds to the Korenblum topology on $M_{\mathcal{BC}}(\overline{\mathbb{D}})$.*

In fact, Theorem 1.2 is the restriction of Theorem 1.4 to integral measures (we say that a measure $\omega = \mu + \nu \in M(\overline{\mathbb{D}})$ is *integral* if $\tilde{\nu}$ is an integral sum of δ-masses, while μ can be anything). The connection comes from [10, Lemma 3.3] which says that if $F_\omega \in \mathscr{J}$ is an inner function with critical structure ω, then

$$u_\omega \;=\; \log \lambda_{F_\omega} \;=\; \log \frac{|F_\omega'|}{1 - |F_\omega|^2}.$$

We spend a moment to check that the embedding $F \to u_F$ is a homeomorphism onto its image. This follows from the following lemma:

Lemma 1.5. *A sequence of functions $\{F_n\} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \,/\, \mathrm{Aut}(\mathbb{D})$ converges to F uniformly on compact subsets if and only if $u_{F_n} \to u_F$ weakly on the disk.*

Proof. Suppose the F_n converge uniformly on compact subsets of the disk to a map $F \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$. Since $F_n(0)$ cannot escape to the unit circle (otherwise F would be constant), we can normalize the F_n and the limiting map F to fix the origin.

As $\log \frac{1}{1-|F_{\omega_n}|^2}$ remain uniformly bounded on compact sets, they converge in $L^1_{\text{loc}}(\mathbb{D})$ to $\log \frac{1}{1-|F_\omega|^2}$. To see that $\log|F_n'| \to \log|F'|$ also converge in $L^1_{\text{loc}}(\mathbb{D})$, note that the critical sets of the F_n converge to the critical set of F_n, and the singularity at each critical point is integrable. The result follows since $L^1_{\text{loc}}(\mathbb{D})$ convergence implies weak convergence.

Conversely, suppose that $u_{F_n} \to u_F$ weakly. We normalize the F_n so they fix the origin. Let G be a subsequential limit of the F_n. By the direct implication, $u_F = u_G$. Liouville's theorem tells us that $F = G$ up to post-composition with an automorphism of the disk. □

1.4 Invariant subspaces of Bergman space

For a fixed $\alpha > -1$ and $1 \le p < \infty$, consider the weighted Bergman space $A^p_\alpha(\mathbb{D})$ which consists of holomorphic functions on the unit disk satisfying the norm boundedness condition

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p \cdot (1-|z|)^\alpha |dz|^2 \right)^{1/p} < \infty. \tag{1.10}$$

For a function $f \in A^p_\alpha$, let $[f]$ denote the (closed) z-invariant subspace generated by f, that is the closure of the set $\{p(z)f(z)\}$, where $p(z)$ ranges over polynomials. In the book [19, p. 34], Nikol'skii equipped subspaces of $A^p_\alpha(\mathbb{D})$ with the *strong topology* where $X_n \to X$ if any $x \in X$ can be obtained as a limit of a converging sequence of $x_n \in X_n$ and vice versa.

We focus our attention on a small but important subclass of invariant subspaces which are generated by a single inner function (here, we mean a usual Hardy-inner function rather than a Bergman-inner function). Following [7], we refer to such subspaces as of *κ-Beurling-type*. According to a classical theorem of Korenblum [11] and Roberts [22], the equality $[BS_{\mu_1}] = [BS_{\mu_2}]$ holds if and only if $\mu_1 - \mu_2$ does not charge Beurling-Carleson sets. Comparing with Theorem 1.1, we see that the subspaces of κ-Beurling-type are in bijection with elements of $\mathscr{J} / \operatorname{Aut}(\mathbb{D})$. We show that this bijection is a homeomorphism:

Theorem 1.6. *For any $\alpha > -1$ and $1 \le p < \infty$, the strong topology on subspaces of κ-Beurling-type agrees with the Korenblum topology on $M_{\mathcal{BC}}(\overline{\mathbb{D}})$.*

In the work [13], Kraus proved that the critical sets of Blaschke products coincide with zero sets of functions in A^2_1. It is therefore plausible that inner functions modulo Frostman shifts are in bijection with the collection of z-invariant subspaces of A^2_1 satisfying the index one property $\dim(E \ominus zE) = 1$. The work of Shimorin [23] on approximate spectral synthesis is likely to be of use here.

2 The Gauss curvature equation

We say that u is a (weak) *solution* of Gauss curvature equation

$$\Delta u = 4e^{2u} + 2\pi\tilde{\nu}, \qquad \tilde{\nu} \ge 0, \tag{2.1}$$

if for any non-negative function $\phi \in C_c^\infty(\mathbb{D})$,

$$\int_{\mathbb{D}} u\Delta\phi \, |dz|^2 = \int_{\mathbb{D}} 4e^{2u}\phi \, |dz|^2 + 2\pi \int_{\mathbb{D}} \phi d\tilde{\nu}. \tag{2.2}$$

By analogy with subharmonic functions, we say that u is a (weak) *subsolution* if one has $\ge$ in (2.2) while the word *supersolution* indicates the sign $\le$.

Theorem 2.1 (Perron method)**.** *Suppose u is a function on the unit disk which is a subsolution of the Gauss curvature equation (2.1) with free boundary, where $\tilde{\nu} \ge 0$ is a locally finite measure on the unit disk. There exists a unique minimal solution $\Lambda^{\tilde{\nu}}[u]$ which exceeds u. If $\overline{u}$ is a supersolution with $\overline{u} \ge u$ then $\overline{u} \ge \Lambda^{\tilde{\nu}}[u]$.*

Theorem 2.2. *Given a finite measure $\tilde{\nu} \ge 0$ on the unit disk and a measurable function $h : \mathbb{S}^1 \to \mathbb{R}$ that is bounded above, the Gauss curvature equation*

$$\begin{cases} \Delta u = 4e^{2u} + 2\pi\tilde{\nu}, & \text{in } \mathbb{D}, \\ u = h, & \text{on } \mathbb{S}^1, \end{cases} \tag{2.3}$$

admits a unique solution. If u_1 and u_2 are two solutions with $h_1 \le h_2$ and $\tilde{\nu}_1 \ge \tilde{\nu}_2$ then $u_1 \le u_2$ on $\mathbb{D}$.

The boundary data h in (2.3) is interpreted in terms of weak limits of measures: we require that $h\,d\theta$ is the weak limit of $u\,d\theta|_{\{|z|=r\}}$ as $r \to 1$. The uniqueness and monotonicity statements of Theorem 2.2 can be easily deduced from Kato's inequality [21, Proposition 6.9] which states that *if* $u \in L^1_{\text{loc}}$ *and* $\Delta u \geq f$ *in the sense of distributions with* $f \in L^1_{\text{loc}}$, *then* $\Delta u^+ \geq f \cdot \chi_{u>0}$. As usual, $u^+ = \max(u, 0)$ denotes the positive part of u.

Proof of Theorem 2.2: uniqueness and monotonicity. Since $\tilde{\nu}_1 \geq \tilde{\nu}_2$,

$$\Delta(u_1 - u_2) \geq 4e^{2u_1} - 4e^{2u_2}$$

in the sense of distributions. By Kato's inequality,

$$\Delta(u_1 - u_2)^+ \geq (4e^{2u_1} - 4e^{2u_2}) \cdot \chi_{\{u_1 > u_2\}} \geq 0$$

is a subharmonic function. However, the inequality $h_1 \leq h_2$ implies that $(u_1 - u_2)^+$ has zero boundary values. The maximal principle shows that $(u_1 - u_2)^+ \leq 0$ or $u_1 \leq u_2$. The same argument also proves uniqueness. □

In order to not interrupt the presentation, we defer the existence statement in Theorem 2.2 to Appendix B and instead explain how to derive Theorem 2.1 from Theorem 2.2.

Suppose u is a subsolution of (2.1). For $0 < r < 1$, we use the symbol $\Lambda_r^{\tilde{\nu}}[u]$ to denote the unique solution of (2.1) on $\mathbb{D}_r = \{z : |z| < r\}$ which agrees with u on $\partial\mathbb{D}_r$. (The function u is bounded above on $\partial\mathbb{D}_r$ since it is subharmonic on the disk.) It may alternatively be described as the minimal solution which dominates u on $\mathbb{D}_r$. With this definition, $\Lambda_r^{\tilde{\nu}}[u]$ does not depend on $\tilde{\nu}|_{\mathbb{D}\setminus\mathbb{D}_r}$.

Proof of Theorem 2.1. As $r \to 1$, the $\Lambda_r^{\tilde{\nu}}[u]$ form an increasing family of solutions (defined on an increasing family of domains) which are bounded above by $u_{\mathbb{D}}$, and therefore they converge to a solution, see Lemma 2.3 below. From the construction, it is clear that $\Lambda_r^{\tilde{\nu}}[u] = \lim_{r\to 1} \Lambda^{\tilde{\nu}}[u]$ is the Perron hull we seek.

Suppose that $\overline{u} \geq u$ is a dominating supersolution. To show that $\overline{u} \geq \Lambda^{\tilde{\nu}}[u]$, it suffices to show $\overline{u} \geq \Lambda_r^{\tilde{\nu}}[u]$ on $\mathbb{D}_r$ for any $0 < r < 1$. Consider the difference $v = \Lambda_r^{\tilde{\nu}}[u] - \overline{u}$. Since $\Delta v \geq 4e^{2\Lambda_r^{\tilde{\nu}}[u]} - 4e^{2\overline{u}}$, by Kato's inequality, we have

$$\Delta v^+ \geq (4e^{2\Lambda_r^{\tilde{\nu}}[u]} - 4e^{2\overline{u}}) \cdot \chi_{\{\Lambda_r^{\tilde{\nu}}[u] > \overline{u}\}} \geq 0.$$

Hence, v^+ is a subharmonic function on $\mathbb{D}_r$ with zero boundary values. The maximal principle shows that $v^+ \leq 0$ in $\mathbb{D}_r$ and hence must be identically 0. The proof is complete. $\square$

Lemma 2.3. *Suppose $\{u_n\}$ is a sequence of solutions of (2.1) with measures $\{\tilde{\nu}_n\}$. If $u_n \to u$ and $\tilde{\nu}_n \to \tilde{\nu}$ weakly on the unit disk, then u is a solution of (2.1) with measure $\tilde{\nu}$.*

Proof. Since the $u_n \leq u_{\mathbb{D}}$ are locally uniformly bounded above, the exponentials e^{2u_n} converge weakly to e^{2u}. It is now a simple matter to examine the definition of a weak solution (2.2) and apply the dominated convergence theorem. $\square$

Corollary 2.4. *Suppose $\{u_n\}$ is a sequence of subsolutions of (2.1) with measures $\{\tilde{\nu}_n\}$. If $u_n \to u$ and $\tilde{\nu}_n \to \tilde{\nu}$ weakly on the unit disk, then for any $0 < r < 1$,*

$$\liminf_{n\to\infty} \Lambda_r^{\tilde{\nu}_n}[u_n] \geq \Lambda_r^{\tilde{\nu}}[u].$$

The same statement also holds with Λ in place of Λ_r.

2.1 Generalized Blaschke products

We say that a measure $\tilde{\nu}$ on the unit disk satisfies the *Blaschke condition* if

$$\int_{\mathbb{D}} (1 - |a|) d\tilde{\nu}(a) < \infty. \tag{2.4}$$

In this case, $\nu(a) := (1-|a|)\tilde{\nu}(a)$ is a finite measure. It will be convenient to use both symbols ν and $\tilde{\nu}$. We define the *generalized Blaschke product* with zero structure ν as

$$B_\nu(z) = \exp\left(\int_{\mathbb{D}} \log \frac{z-a}{1-\bar{a}z}\, d\tilde{\nu}(a) \right), \tag{2.5}$$

cf. (1.4). While $B_\nu(z)$ may not be a single-valued function on the unit disk, its absolute value and hence zero set are well-defined. Multiplying B_ν by a singular inner function S_μ, we obtain the *generalized inner function* $I_\omega = B_\nu S_\mu$ with zero structure $\omega = \mu + \nu$. Note that $\log 1/|S_\mu|$ is just the Poisson extension of μ, while $\log \frac{1}{|B_\nu|}$ is the Green integral of $2\pi\tilde{\nu}$ (it has zero boundary values on the unit circle and $\Delta \log \frac{1}{|B_\nu|} = 2\pi\tilde{\nu}$, for more information, see Appendix B).

The following lemma is well known:

Lemma 2.5. (i) *For* $\nu \in M(\mathbb{D})$, *the measures* $(\log 1/|B_\nu|)(d\theta/2\pi)\big|_{\{|z|=r\}}$ *tend weakly to the zero measure as* $r \to 1$.

(ii) *If* $\mu \in M(\mathbb{S}^1)$ *is a singular measure, then* $(\log 1/|S_\mu|)(d\theta/2\pi)\big|_{\{|z|=r\}} \to \mu$ *as* $r \to 1$.

We will also need:

Lemma 2.6. *Suppose measures* $\omega_n \in M(\overline{\mathbb{D}})$ *converge weakly to* ω. *Then,*

$$\log \frac{1}{|I_{\omega_n}|} \to \log \frac{1}{|I_\omega|}$$

weakly on the unit disk in the sense of distributions.

We leave the proof as an exercise for the reader.

2.2 Nearly-maximal solutions

We now prove Theorem 1.3 which identifies the space of nearly-maximal solutions of the Gauss curvature equation with $M_{\mathcal{BC}}(\overline{\mathbb{D}})$. The heavy-lifting has been done in [10] where Theorem 1.3 was proved in the case when $\tilde{\nu} = 0$. Here, we explain the extension to general measures $\tilde{\nu} \geq 0$ satisfying the Blaschke condition (2.4).

Proof of Theorem 1.3. Observe that $u_{\mathbb{D}} - \log \frac{1}{|I_\omega|}$ is a subsolution of $\Delta u = 4e^{2u} + 2\pi\tilde{\nu}$. We claim that if u_ω is a nearly-maximal solution of the Gauss curvature equation with data $\omega = \mu + \nu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, then

$$u_\omega = \Lambda^{\tilde{\nu}}\left[u_{\mathbb{D}} - \log \frac{1}{|I_\omega|}\right]. \tag{2.6}$$

Since (2.6) gives an explicit formula for u_ω, the nearly-maximal solution with data ω is unique. In view of the monotonicity properties of Λ, the fundamental identity (2.6) also shows that u_ω is decreasing in ω.

Consider the function

$$h = u_{\mathbb{D}} - u_\omega - \log \frac{1}{|I_\omega|}.$$

Since $\Delta h = 4e^{2u_{\mathbb{D}}} - 4e^{2u_\omega} \geq 0$, h is subharmonic. By Lemma 2.5, as $r \to 1$, $h(re^{i\theta})d\theta$ tends weakly to the zero measure on the unit circle, so that h is negative in the unit disk. By the definition of the Perron hull,

$$u_\omega \;\geq\; u_* \;:=\; \Lambda^{\tilde{\nu}}\left[u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}\right] \;\geq\; u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}.$$

Some rearranging gives

$$u_{\mathbb{D}} - u_\omega \;\leq\; u_{\mathbb{D}} - u_* \;\leq\; \log\frac{1}{|I_\omega|}.$$

Taking the weak limit as $r \to 1$ shows that u_* has deficiency μ on the unit circle, that is, $(u_{\mathbb{D}} - u_*)(d\theta/2\pi)|_{\{|z|=r\}} \to \mu$ weakly as $r \to 1$. Since u_* has singularity $2\pi\tilde{\nu}$, it is also a nearly-maximal solution of the Gauss curvature equation with data ω. To see that $u_* = u_\omega$, notice that the difference $u_\omega - u_*$ is a non-negative subharmonic function which tends to the zero measure on the unit circle (and therefore must vanish identically). This proves the claim.

Let u_μ be the nearly-maximal solution of the Gauss curvature equation $\Delta u = 4e^{2u}$ with deficiency $\mu \in M_{\mathcal{BC}}(\mathbb{S}^1)$. The existence of u_μ was proved in [10] using the connection with complex analysis provided by the Liouville correspondence. For any Blaschke measure $\tilde{\nu} \geq 0$ on the unit disk, the Perron method finds the least solution of $\Delta u = 4e^{2u} + 2\pi\tilde{\nu}$ satisfying $u_\mu \;\geq\; u \;\geq\; u_\mu - \log\frac{1}{|B_\nu|}$. By Lemma 2.5, u has the correct boundary behaviour in order to solve (1.6), thereby proving the existence of $u_{\mu,\nu}$.

Conversely, suppose that $\mu \notin M_{\mathcal{BC}}(\mathbb{D})$. It was proved in [10] that u_μ does not exist in this case. To show that $u_{\mu,\nu}$ does not exist for any $\nu \in M(\mathbb{D})$, we argue by contradiction: we use the existence of $u_{\mu,\nu}$ to construct u_μ. To this end, we notice that $\Lambda^0(u_{\mu,\nu})$ is a solution of the Gauss curvature $\Delta u = e^{2u}$ which is squeezed between $u_{\mu,\nu} \leq \Lambda^0(u_{\mu,\nu}) \leq u_{\mu,\nu} + \log\frac{1}{|B_\nu|}$, and so must be u_μ by Lemma 2.5. □

In the proof above, we saw the importance of the formula (2.6). In the next lemma, we give a slight generalization of this identity:

Lemma 2.7. *Given two measures $\omega_i = \mu_i + \nu_i \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, $i = 1, 2$, we have*

$$u_{\omega_1+\omega_2} = \Lambda^{\tilde{\nu}_1+\tilde{\nu}_2}\left[u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}\right], \tag{2.7}$$

$$= \Lambda^{\tilde{\nu}_1+\tilde{\nu}_2}\left[\Lambda^{\tilde{\nu}_1}\left[u_{\mathbb{D}} - \log\frac{1}{|I_{\omega_1}|}\right] - \log\frac{1}{|I_{\omega_2}|}\right]. \tag{2.8}$$

Proof. Not surprisingly, the proof of (2.7) is similar to that of (2.6). Since the quantity on the right side (2.7) is a solution of the Gauss curvature equation with "singularity" $2\pi(\tilde{\nu}_1+\tilde{\nu}_2)$, we simply need to check that it has the correct "deficiency" on the unit circle. To see this, we observe that it is squeezed by quantities with deficiency $\mu_1 + \mu_2$:

$$u_{\omega_1+\omega_2} \geq \Lambda^{\tilde{\nu}_1+\tilde{\nu}_2}\left[u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}\right] \geq u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}.$$

We leave it to the reader to justify the first inequality by checking that $u_{\omega_1+\omega_2} \geq u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}$ using the argument from the proof of (2.6). To obtain (2.8), one only needs to substitute (2.6) into (2.7). □

2.3 Sequences of nearly-maximal solutions

We now present three useful lemmas concerning sequences of nearly-maximal solutions. The first lemma says that mass can only disappear in the limit, cf. (1.2):

Lemma 2.8. *Suppose ω_n is a sequence of measures in $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ which converges weakly to ω. If the nearly-maximal solutions u_{ω_n} also converge weakly, then* $\lim u_{\omega_n} = u_{\omega^*}$ *for some $0 \leq \omega^* \leq \omega$.*

Proof. According to (2.6), $u_{\omega_n} \geq u_{\mathbb{D}} - \log\frac{1}{|I_{\omega_n}|}$. Taking weak limits gives $u_{\omega^*} \geq u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}$. Since $u_\omega = \Lambda^{\tilde{\nu}}\big[u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}\big]$, $u_{\omega^*} \geq u_\omega$ as desired. □

The second lemma says that modification by a sequence with zero limit does not change the limiting solution:

Lemma 2.9. *In the setting of Lemma 2.8, if $\omega_n' \to 0$ weakly, then* $\lim u_{\omega_n+\omega_n'} = \lim u_{\omega_n}$.

Proof. By Lemma 2.7, $u_{\omega_n} \geq u_{\omega_n+\omega'_n} \geq u_{\omega_n} - \log \frac{1}{|I_{\omega'_n}|}$. The statement follows from the weak convergence $\log \frac{1}{|I_{\omega'_n}|} \to 0$. □

The third and last lemma says that any sequence dominated by a stable sequence is also stable:

Lemma 2.10. *Suppose $\omega_n \to \omega$ and $\omega'_n \to \omega'$ are two weakly convergent sequences of measures in $M_{\mathcal{BC}}(\overline{\mathbb{D}})$. If $u_{\omega_n+\omega'_n} \to u_{\omega+\omega'}$ then $u_{\omega_n} \to u_{\omega'}$.*

Proof. Passing to a subsequence, we may assume that u_{ω_n} converges weakly to a nearly-maximal solution u_{ω^*} with $\omega^* \leq \omega$. By Lemma 2.7,

$$(u_{\mathbb{D}} - u_{\omega_n+\omega'_n}) - (u_{\mathbb{D}} - u_{\omega_n}) \;=\; u_{\omega_n} - u_{\omega_n+\omega'_n} \;\leq\; \log \frac{1}{|I_{\omega'_n}|}.$$

Taking $n \to \infty$ and examining the boundary data, we arrive at $(\omega + \omega') - \omega^* \leq \omega'$ which implies that $\omega^* = \omega$. □

3 Concentrating sequences

In this section, we study concentrating sequences of inner functions. We show:

Theorem 3.1. *Suppose $\{F_{\omega_n}\}$ is a sequence of inner functions. If the measures $\omega_n \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ converge to ω in the Korenblum topology, then the F_{ω_n} converge uniformly on compact subsets to F_ω.*

Actually, we give two proofs of the above theorem. The first proof uses hyperbolic geometry to estimate the derivative of a Blaschke product whose critical structure is supported on a Korenblum star. The second proof uses PDE techniques and applies to arbitrary sequences of nearly-maximal solutions.

For a Beurling-Carleson set $E \subset \mathbb{S}^1$ and parameters $\alpha \geq 1$, $0 < \theta \leq 1$, we define the *generalized Korenblum star of order α* as

$$K_E^\alpha(\theta) = B(0, 1/\sqrt{2}) \cup \big\{ z \in \overline{\mathbb{D}} : 1 - |z| \geq \theta \cdot \operatorname{dist}(\hat{z}, E)^\alpha \big\}. \tag{3.1}$$

If $\alpha = 1$ and $\theta = 1$, the above definition reduces to the one given earlier: $K_E = K_E^1(1)$. By default we take $\theta = 1$, i.e. we write $K_E^\alpha = K_E^\alpha(1)$.

Lemma 3.2. *Suppose $F \in \mathscr{J}$ with $F(0) = 0$ whose "critical structure" $\mu(\operatorname{Inn} F')$ is supported on a Korenblum star K_E of order 1 and "critical mass" $\mu(\operatorname{Inn} F')(\overline{\mathbb{D}}) < M$. Then,*

$$\frac{1 - |F(z)|}{1 - |z|} \le C(M) \cdot \operatorname{dist}(z, E)^{-4}, \qquad z \in \mathbb{D} \setminus K_E^4, \tag{3.2}$$

where dist *denotes Euclidean distance.*

Under the assumptions of the above lemma, we have:

Corollary 3.3. *The set $\{z \in \mathbb{D} : |F(z)| < 1/2\}$ is supported on a higher-order Korenblum star $K_E^4(\theta)$ where θ is a parameter which depends on M. In particular, by Schwarz reflection, F extends to a bounded analytic function on $\mathbb{C} \setminus \mathbf{r}(K_E^4(\theta))$ where $\mathbf{r}(z) = 1/\overline{z}$ denotes the reflection in the unit circle.*

Corollary 3.4. *For a point $\zeta \in \mathbb{S}^1$ on the unit circle,*

$$|F'(\zeta)| \le C(M) \cdot \operatorname{dist}(\zeta, E)^{-4}.$$

In particular, if I is a connected component of $\mathbb{S}^1 \setminus E$, then

$$\int_I \log |F_n'| d\theta \lesssim \int_I \log \frac{1}{\operatorname{dist}(x, \partial I)} d\theta \asymp |I| \log \frac{1}{|I|}.$$

Proof of Theorem 3.1. Special case. We first prove the theorem in the special case when each measure ω_n is supported on a Korenblum star K_{E_n} and the sets $E_n \to E$ converge in $\mathcal{BC}$.

To simplify the notation, let us write F_n instead of F_{ω_n}. By a normal families argument and Corollary 3.3, we may assume that $F_n \to F$ converge locally uniformly on $\mathbb{C} \setminus \mathbf{r}(K_E^4(\theta))$, where the domains of definition $\mathbb{C} \setminus \mathbf{r}(K_{E_n}^4(\theta))$ are changing but converge to $\mathbb{C} \setminus \mathbf{r}(K_E^4(\theta))$. In this case, the derivatives $F_n' \to F'$ also converge locally uniformly.

According to [10, Section 4], to show that the sequence $\{F_n\}$ is stable, it is enough to check that the outer factors converge at the origin:

$$\lim_{n \to \infty} \int_{\mathbb{S}^1} \log |F_n'| d\theta = \int_{\mathbb{S}^1} \log |F'| d\theta.$$

The proof will be complete if we can argue that the functions $\log |F_n'|$ are uniformly integrable on the unit circle. This means that for any $\varepsilon > 0$, there exists a $\delta > 0$ so that $\int_A \log |F_n'| d\theta < \varepsilon$ for any n, whenever $A \subset \mathbb{S}^1$ is a measurable set with $m(A) < \delta$. This estimate is provided by Corollary 3.4 above and the definition of a concentrating sequence of Beurling-Carleson sets.

General case. According to the definition, $\omega_n \to \omega$ in the Korenblum topology if for any $\varepsilon > 0$, one can find a concentrating sequence $\omega_n^N \to \omega^N$ in $M_{\mathcal{BC}(N)}(\overline{\mathbb{D}})$ with $0 \le \omega_n^N \le \omega_n$, $(\omega_n - \omega_n^N)(\overline{\mathbb{D}}) < \varepsilon$ and $(\omega - \omega^N)(\overline{\mathbb{D}}) < \varepsilon$. Lemma 2.7 tells us that

$$\log \frac{1}{|I_{\omega_n - \omega_n^N}|} \ \ge\ u_{\omega_n^N} - u_{\omega_n} \ \ge\ 0, \qquad n = 1, 2, \dots$$

and

$$\log \frac{1}{|I_{\omega - \omega^N}|} \ \ge\ u_{\omega^N} - u_{\omega} \ \ge\ 0.$$

By the triangle inequality,

$$|u_\omega - u_{\omega_n}| \le \log \frac{1}{|I_{\omega - \omega^N}|} + |u_{\omega^N} - u_{\omega_n^N}| + \log \frac{1}{|I_{\omega_n - \omega_n^N}|}.$$

From the special case of the theorem, we know that $u_{\omega_n^N}$ converge weakly to u_{ω^N}. Lemma 2.6 implies that $u_{\omega_n} \to u_\omega$ weakly. Finally, by Lemma 1.5, this is equivalent to the uniform convergence of $F_{\omega_n} \to F_\omega$ on compact subsets of the unit disk. □

3.1 Blaschke products as approximate isometries

To prove Lemma 3.2, we use the following principle: *away from the critical points, an inner function is close to a hyperbolic isometry.* Our discussion is inspired by the work of McMullen [18, Section 10] which deals with finite Blaschke products of fixed degree. Here, we require "degree independent" estimates. To this end, given an inner function $F(z)$, we consider the quantity

$$\gamma_F(z) = \log \frac{1}{|\operatorname{Inn} F'(z)|} \tag{3.3}$$

which measures how much F deviates from a Möbius transformation near z. The quantity γ_F satisfies the Möbius invariance relation

$$\gamma_{M_1 \circ F \circ M_2}(z) = \gamma_F(M_2(z)), \qquad M_1, M_2 \in \operatorname{Aut}(\mathbb{D}), \tag{3.4}$$

which follows from the identity $\operatorname{Inn}\big[(M_1 \circ F \circ M_2)'\big] = \operatorname{Inn} F' \circ M_2$. Let $G(z,w) = \log\left|\frac{1-\overline{w}z}{z-w}\right|$ denote the Green's function of the unit disk. If the singular measure $\sigma(F')$ is trivial (e.g. if F is a finite Blaschke product), the above definition reduces to

$$\gamma_F(z) = \sum_{c \in \operatorname{crit}(F)} G(z,c). \tag{3.5}$$

For two points $x, y \in \mathbb{D}$, we write $d_{\mathbb{D}}(x,y)$ for the hyperbolic distance and denote the segment of the hyperbolic geodesic that joins x and y by $[x,y]$. There is a convenient way to estimate hyperbolic distance. Let $z \in [x,y]$ be the point closest to the origin. If $z = x$ or $z = y$, then $d_{\mathbb{D}}(x,y) = d_{\mathbb{D}}(|x|,|y|) + \mathcal{O}(1)$ is essentially the "vertical distance" from x to y. If z lies strictly between x and y, then $d_{\mathbb{D}}(x,y) = d_{\mathbb{D}}(|x|,|z|) + d_{\mathbb{D}}(|z|,|y|) + \mathcal{O}(1)$.

Lemma 3.5 (cf. Proposition 10.9 of [18])**.** *Suppose $F \in \mathscr{J}$. At a point $z \in \mathbb{D}$ which is not a critical point of F, the 2-jet $\big(F(z), F'(z), F''(z)\big)$ of F matches the 2-jet of a hyperbolic isometry with an error of $\mathcal{O}(\gamma_F(z))$.*

Proof. By Möbius invariance (3.3), it suffices to consider the case when $z = F(z) = 0$ and $F'(0) > 0$. Set $\delta = \gamma_F(0)$. To prove the lemma, we need to show that $1 - F'(0) = |F''(0)| = \mathcal{O}(\delta)$. Since $1 - t \le \log(1/t)$ for $0 \le t \le 1$, $1 - |(\operatorname{Inn} F')(0)| \le \delta$. Applying [10, Lemma 2.3] gives the desired estimate for the first derivative:

$$F'(0) \;=\; \lambda_F(0) \;\ge\; |(\operatorname{Inn} F')(0)| \cdot \lambda_{\mathbb{D}}(0) \;\ge\; 1 - \delta.$$

By the Schwarz lemma applied to $F(z)/z$, we have $d_{\mathbb{D}}\big(F(z)/z, F'(0)\big) = \mathcal{O}(1)$ for $z \in B(0, 1/2)$. Taking note of the location of $F'(0) \in \mathbb{D}$, this estimate can be written as $|F(z) - z| = \mathcal{O}(\delta)$ for $z \in B(0,1/2)$. Cauchy's integral formula now gives the estimate for the second derivative. □

Corollary 3.6 (cf. Theorem 10.11 and Corollary 10.7 of [18])**.** *Suppose $F(z)$ is a finite Blaschke product and $[z_1, z_2]$ is a segment of a hyperbolic geodesic. If for each $z \in [z_1, z_2]$, $\gamma_F(z) < \delta$ is sufficiently small, then $F(z_1) \ne F(z_2)$. In fact, for any $\varepsilon > 0$, we can choose $\delta(\varepsilon) > 0$ small enough to guarantee that*

$$(1-\varepsilon)\cdot d_{\mathbb{D}}(z_1,z_2) \;\le\; d_{\mathbb{D}}\big(F(z_1), F(z_2)\big) \;\le\; d_{\mathbb{D}}(z_1, z_2). \tag{3.6}$$

Sketch of proof. If we choose $\delta > 0$ small enough, then $F|_{[z_1,z_2]}$ is so close to an isometry that the geodesic curvature of its image is nearly 0. But a path in hyperbolic space with geodesic curvature less than 1 (the curvature of a horocycle) cannot cross itself, so $F(z_1) \neq F(z_2)$. Similar reasoning gives the second statement. □

Remark. If $\gamma_F(z)$ decays exponentially along $[z_1, z_2]$, i.e. satisfies a bound of the form $\gamma_F(z) < M \exp\bigl(-d_{\mathbb{D}}(z, z_1)\bigr)$, for some $M > 0$, then McMullen's argument gives the stronger conclusion

$$d_{\mathbb{D}}(F(z_1), F(z_2)) = d_{\mathbb{D}}(z_1, z_2) + \mathcal{O}(1).$$

See the proof of [18, Theorem 10.11].

Lemma 3.7. *Suppose I is an inner function whose zero structure $\mu(I)$ is contained in a Korenblum star K_E and its zero mass $\mu(I)(\overline{\mathbb{D}}) < M$. Then, $|I(z)| > c(M) > 0$ is bounded from below on $\mathbb{D} \setminus K_E^2$. More precisely,*

$$\log \frac{1}{|I(z)|} \lesssim M \exp\bigl(-d_{\mathbb{D}}(z, K_E^2)\bigr), \qquad z \in \mathbb{D} \setminus K_E^2.$$

For a point $z \in \mathbb{D}$ and an integer $n \geq 1$ such that $z \notin K_E^n$, let z_n denote the unique point of intersection of $[0, z]$ and ∂K_E^n. Elementary hyperbolic geometry shows that for $w \in K_E^n$, $d_{\mathbb{D}}(z, w) = d_{\mathbb{D}}(z, z_n) + d_{\mathbb{D}}(z_n, w) - \mathcal{O}(1)$. In particular, $d_{\mathbb{D}}(z, \partial K_E^n) = d_{\mathbb{D}}(z, z_n) - \mathcal{O}(1)$.

Proof. We may assume that I is a finite Blaschke product as the general case follows by approximating I by finite Blaschke products whose zero sets are contained in K_E.

Let a be a zero of I. By the triangle inequality, the hyperbolic distance

$$\begin{aligned} d_{\mathbb{D}}(z, a) &= d_{\mathbb{D}}(z, z_1) + d_{\mathbb{D}}(z_1, a) - \mathcal{O}(1), \\ &\geq d_{\mathbb{D}}(z, z_1) + d_{\mathbb{D}}(0, a) - d_{\mathbb{D}}(0, z_1) - \mathcal{O}(1), \\ &\geq d_{\mathbb{D}}(z, z_2) + d_{\mathbb{D}}(0, a) - \mathcal{O}(1). \end{aligned}$$

In other words, the Green's function

$$G(z, a) \lesssim G(0, a) \exp\bigl(-d_{\mathbb{D}}(z, z_2)\bigr)$$

decays exponentially quickly in the hyperbolic distance $d_{\mathbb{D}}(z,z_2)$. For $a \in B(0,1/2)$, we have the "trivial" estimate $G(z,a) \asymp G(z,0) \lesssim \exp\bigl(-d_{\mathbb{D}}(z,z_2)\bigr)$. Combining the two inequalities, we get

$$G(z,a) \lesssim G^*(0,a)\exp\bigl(-d_{\mathbb{D}}(z,z_2)\bigr)$$

where $G^*(z,w) := \min\bigl(G(z,w),1\bigr)$ is the truncated Green's function. Summing over the zeros of I gives

$$\log\frac{1}{|I(z)|} = \sum_{a\in \mathrm{zeros}(I)} G(z,a) \lesssim M\exp\bigl(-d_{\mathbb{D}}(z,z_2)\bigr) \asymp M\exp\bigl(-d_{\mathbb{D}}(z,K_E^2)\bigr),$$

where in the second step we made use of $\sum_{a\in\mathrm{zeros}(I)} G^*(0,a) \asymp \mu(I)(\overline{\mathbb{D}}) \le M$. This proves the lemma. □

Corollary 3.8. *Suppose F is an inner function which satisfies the hypotheses of Lemma 3.2. For $z \in \mathbb{D}\setminus K_E^2$, the characteristic $\gamma_F(z) \lesssim M\exp\bigl(-d_{\mathbb{D}}(z,K_E^2)\bigr)$.*

With these preparations, we can now prove Lemma 3.2:

Proof of Lemma 3.2. Suppose $z \in \mathbb{D}\setminus K_E^4$. Divide $[0,z]$ into two parts: $[0,z_2]$ and $[z_2,z]$. By the Schwarz lemma,

$$d_{\mathbb{D}}(F(0),F(z_2)) \le d_{\mathbb{D}}(0,z_2).$$

However, since F restricted to $[z_2,z]$ is close to a hyperbolic isometry,

$$d_{\mathbb{D}}(F(z_2),F(z)) \ge d_{\mathbb{D}}(z_2,z) - \mathcal{O}(1). \tag{3.7}$$

The triangle inequality gives

$$d_{\mathbb{D}}(F(0),F(z)) \ge d_{\mathbb{D}}(z_4,z) - \mathcal{O}(1),$$

which is equivalent to (3.2). □

3.2 Concentrating sequences of solutions

We now prove the generalization of Theorem 3.1 for concentrating sequences of nearly-maximal solutions:

Theorem 3.9. *Suppose the measures $\omega_n \to \omega$ converge in the Korenblum topology. Then, the associated nearly-maximal solutions of the Gauss curvature equation $u_{\omega_n} \to u_\omega$ converge weakly on the unit disk.*

The proof of Theorem 3.9 rests on three simple observations:

Lemma 3.10. *If $E \subset \mathbb{S}^1$ is a Beurling-Carleson set and $\alpha \geq 1$ then*

$$\int_{K_E^\alpha} \frac{|dz|^2}{1-|z|} \asymp \alpha \cdot \|E\|_{\mathcal{BC}}. \tag{3.8}$$

If the sets $E_n \to E$ in $\mathcal{BC}$, then

$$\int_{K_{E_n}^\alpha} \frac{|dz|^2}{1-|z|} \to \int_{K_E^\alpha} \frac{|dz|^2}{1-|z|}. \tag{3.9}$$

We leave the verification to the reader.

Lemma 3.11. *Suppose u is a nearly-maximal solution of the Gauss curvature equation and $\omega \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$. Then, $u = u_\omega$ if and only if*

$$(u_{\mathbb{D}} - u)(z) = \log \frac{1}{|I_\omega(z)|} - \frac{1}{2\pi} \int_{\mathbb{D}} J_u(z,w)\,|dw|^2. \tag{3.10}$$

where

$$J_u(z,w) = 4\Big(e^{2u_{\mathbb{D}}(w)} - e^{2u(w)}\Big)\, G(z,w).$$

The lemma follows after applying the Poisson-Jensen formula for subharmonic functions on $\mathbb{D}_r$ and taking $r \to 1$.

Lemma 3.12. *Suppose the functions $h_n : \mathbb{D} \to \mathbb{R}$ converge pointwise to h and are locally uniformly bounded. Then, they converge weakly to h in the sense of distributions, that is, for any test function $\phi \in C_c^\infty(\mathbb{D})$, $\int_{\mathbb{D}} h_n \phi\,|dz|^2 \to \int_{\mathbb{D}} h\phi\,|dz|^2$.*

This is immediate from the dominated convergence theorem.

Proof of Theorem 3.9. To prove the theorem, we show that if a sequence of solutions $u_n = u_{\omega_n}$ converges weakly to a solution u, then $u = u_\omega$ where ω is the weak limit of the ω_n. The reduction described in the proof of Theorem 3.1 allows us to assume that each measure ω_n is supported on a Korenblum star K_{E_n} with $E_n \to E$ in $\mathcal{BC}$.

By Lemma 3.11, for each $n = 1, 2, \dots$, we know that

$$u_{\mathbb{D}}(z) - u_n(z) - \log \frac{1}{|I_{\omega_n}(z)|} = -\frac{1}{2\pi} \int_{\mathbb{D}} J_{u_n}(z, w)\, |dw|^2. \tag{3.11}$$

We claim that if we take the weak limit of (3.11) as $n \to \infty$, we will end up with

$$u_{\mathbb{D}}(z) - u(z) - \log \frac{1}{|I_{\omega}(z)|} = -\frac{1}{2\pi} \int_{\mathbb{D}} J_{u}(z, w)\, |dw|^2, \tag{3.12}$$

which would mean that $u = u_\omega$.

By assumption, the u_n converge weakly to u, while by Lemma 2.6, $\log \frac{1}{|I_{\omega_n}(z)|}$ converge weakly to $\log \frac{1}{|I_{\omega}(z)|}$. It remains to show that

$$\frac{1}{2\pi} \int_{\mathbb{D}} J_{u_n}(z, w)\, |dw|^2 \to \frac{1}{2\pi} \int_{\mathbb{D}} J_{u}(z, w)\, |dw|^2 \tag{3.13}$$

also converge weakly. For this purpose, we will use the following bounds on the integrands $J_n(z, w) = J_{u_n}(z, w)$:

- If $d_{\mathbb{D}}(w, z) \le 1$, we use the bound $J_n(z, w) \le C_1(z) \cdot G(z, w)$. Note that the singularity of the Green's function is integrable.

- For $w \in K^2_{E_n}$ with $d_{\mathbb{D}}(w, z) > 1$, we use the coarse estimate

$$J_n(z, w) \le e^{2u_{\mathbb{D}}}(w)\, G(z, w) \le C_2(z) \cdot \frac{1}{1 - |w|}.$$

- For $w \in \mathbb{D} \setminus K^2_{E_n}$ with $d_{\mathbb{D}}(w, z) > 1$, we use the fine estimate

$$\begin{aligned} J_n(z, w) &\le C_2(z) \cdot \frac{1}{1 - |w|} \cdot \bigl(1 - e^{-2(u_{\mathbb{D}} - u_n)}(w)\bigr) \\ &\le C_3(z) \cdot \frac{1}{1 - |w|} \cdot \log \frac{1}{|I_{\omega_n}(w)|} \\ &\le M C_3(z) \cdot \frac{1}{1 - |w|} \cdot \exp\bigl(-d_{\mathbb{D}}(w, K^2_{E_n})\bigr), \end{aligned}$$

where $M = \sup_{n\geq 1} \omega_n(\overline{\mathbb{D}})$. The second inequality follows from (2.6) while the third inequality is provided by Lemma 3.7 (which also holds for generalized Blaschke products with the same proof).

In view of the "area convergence" (3.9), the second estimate on J_n and the weak convergence of $u_n \to u$ show

$$\frac{1}{2\pi}\int_{K^2_{E_n}} J_{u_n}(z,w)\,|dw|^2 \to \frac{1}{2\pi}\int_{K^2_E} J_u(z,w)\,|dw|^2.$$

For $0 < \theta_1 < \theta_2 \leq 1$, let $K^2_E(\theta_1, \theta_2)$ denote $K^2_E(\theta_2) \setminus K^2_E(\theta_1)$. A similar argument implies that

$$\frac{1}{2\pi}\int_{K^2_{E_n}(e^{-(k+1)},e^{-k})} J_{u_n}(z,w)\,|dw|^2 \to \frac{1}{2\pi}\int_{K^2_E(e^{-(k+1)},e^{-k})} J_u(z,w)\,|dw|^2,$$

for any $k \geq 0$. However, by the third estimate on J_n, these integrals decay exponentially in k, which proves the pointwise convergence in (3.13).

Since $C_1(z), C_2(z), C_3(z)$ can be taken to be continuous in $z \in \mathbb{D}$, the functions $z \to \frac{1}{2\pi}\int_{\mathbb{D}} J_{u_n}(z,w)\,|dw|^2$ are locally uniformly bounded, which allows us to use Lemma 3.12 to upgrade pointwise convergence to weak convergence. This completes the proof. □

4 Diffuse sequences

In this section, we show the converse to Theorem 3.9, thereby completing the proof of Theorem 1.4:

Theorem 4.1. *If a weakly-convergent sequence of measures $\mu_n \to \mu$ does not converge in the Korenblum topology, then the associated nearly-maximal solutions u_{μ_n} do not converge weakly to u_μ.*

Combining the above theorem with Lemma 2.8, we see that if a sequence of measures $\{\mu_n\}$ is not concentrating, then some mass is lost in the limit. One may be inclined to believe that if the sequence $\{\mu_n\}$ is diffuse (has no concentrating component), then $u_{\mu_n} \to u_{\mathbb{D}}$, however, in Section 4.4 we will give a counterexample.

In order to guarantee that $u_{\mu_n} \to u_{\mathbb{D}}$, we need to assume a stronger condition on the sequence $\{\mu_n\}$. We say that $\{\mu_n\}$ is *totally diffuse* if for any $N > 0$,

$$\sup_{E \in \mathcal{BC}(N)} \mu_n(K_E) \to 0, \qquad \text{as } n \to \infty.$$

Theorem 4.2. *For any totally diffuse sequence of measures $\{\mu_n\} \subset M_{\mathcal{BC}}(\overline{\mathbb{D}})$ whose masses $\mu_n(\overline{\mathbb{D}})$ are uniformly bounded above, the associated nearly-maximal solutions of the Gauss curvature equation u_{μ_n} converge weakly to $u_{\mathbb{D}}$.*

4.1 Roberts decompositions

Our main tool is a variant of the Roberts decomposition [22] for measures on the closed unit disk. The decomposition depends on two parameters: a real number $c > 0$ and an integer $j_0 \geq 2$. Set $n_j := 2^{2^{j+j_0}}$ and $r_j := 1 - 10/n_j$. The factor "10" may appear somewhat artificial but it will save us some headache later on (any constant larger than 2π will do).

For an arc $I \subset \mathbb{S}^1$ of the unit circle, we write

$$\square_{I,r,R} := \{z \in \overline{\mathbb{D}} : z/|z| \in I,\, r \leq |z| \leq R\},$$

with the convention that we include the left edge into $\square_{I,r,R}$ but not the right edge. We write $\square_{\mathbb{S}^1,r,R}$ for the annulus $\{z \in \overline{\mathbb{D}} : r \leq |z| \leq R\}$.

Theorem 4.3. *Given a finite measure $\mu \in M(\overline{\mathbb{D}})$ on the closed unit disk, one can write it as*

$$\mu \;=\; \overline{\mu} + \nu_{\text{cone}} \;=\; (\mu_2 + \mu_3 + \mu_4 + \dots) + \nu_{\text{cone}} \tag{4.1}$$

where each measure μ_j, $j \geq 2$, satisfies

$$\operatorname{supp} \mu_j \subset \square_{\mathbb{S}^1, r_{j-1}, 1} \tag{4.2}$$

and

$$\mu_j(\square_{I,r_{j-1},1}) \leq 2(c/n_j)\log n_j, \qquad \forall I \subset \mathbb{S}^1, \quad |I| = 2\pi/n_j; \tag{4.3}$$

while the cone measure ν_{cone} is supported on a Korenblum star $K_{E_{\text{cone}}}$ of norm $\|E_{\text{cone}}\|_{\mathcal{BC}} \leq N\big(c, j_0, \mu(\overline{\mathbb{D}})\big)$.

It will be important for us that the measure μ admits infinitely many decompositions with different parameters c and j_0.

Proof. We obtain the decomposition by means of an algorithm which sorts out the mass of μ into various components. For each $j = 2, 3, \dots$, we consider a partition P_j of the unit circle into n_j equal arcs. Since n_j divides n_{j+1}, each next partition can be chosen to be a refinement of the previous one.

As Step 1 of our algorithm, we move $\mu|_{B(0,r_1)}$ into ν_{cone}. (We remove this mass from μ.)

In Step j, $j = 2, 3, \dots$, we consider all intervals in the partition P_j. Define an interval to be *light* if $\mu(\square_{I,0,1}) \le (c/n_j) \log n_j$ and *heavy* otherwise. We do one of the following three operations:

L. If I is light, we move the mass $\mu|_{\square_{I,0,1}}$ into μ_j.

H1. If I is heavy, we look at the box $\square_{I,r_{j-1},r_j}$. If $\mu(\square_{I,r_{j-1},r_j}) \ge (c/n_j) \log n_j$, we move $\mu|_{\square_{I,r_{j-1},r_j}}$ into ν_{cone}.

H2. If $\mu(\square_{I,r_{j-1},r_j}) < (c/n_j) \log n_j$, we move $\mu|_{\square_{I,r_{j-1},r_j}}$ to μ_j. We also move some mass from $\mu|_{\square_{I,r_j,1}}$ to μ_j so that $\mu_j(\square_{I,0,1}) = (c/n_j) \log n_j$.

After we followed the above instructions for $j = 2, 3, \dots$, it is possible that the measure μ has not been exhausted completely: some "residual" mass may remain on the unit circle. We move this remaining mass to ν_{cone}.

From the construction, it is clear that the conditions (4.2) and (4.3) are satisfied. The factor of 2 in (4.3) is due to the fact that any interval $I \subset \mathbb{S}^1$ of length $2\pi/n_j$ is contained in the union of two adjacent intervals from the partition P_j.

Let Λ be the collection of light intervals (of any generation) which are maximal with respect to inclusion. Define $E_{\text{cone}} := \mathbb{S}^1 \setminus \bigcup_{I \in \Lambda} \operatorname{Int} I$ as the complement of the interiors of these intervals. Since the measure $\nu_{\text{cone}}|_{\mathbb{S}^1}$ is supported on the set of points which lie in heavy intervals at every stage, $\operatorname{supp} \nu_{\text{cone}}|_{\mathbb{S}^1} \subset K_{E_{\text{cone}}}$. Observe that if I is an interval of generation j, then the box $\square_{I,r_{j-1},r_j}$ is contained in the union of two Stolz angles emanating from the endpoints of I. If I is heavy, these endpoints are contained in E_{cone}, from which we see that $\operatorname{supp} \nu_{\text{cone}}|_{\mathbb{D}}$ is also contained in the Korenblum star $K_{E_{\text{cone}}}$.

To check that E_{cone} is a Beurling-Carleson set, we follow the computation from Roberts [22]. The relation $\log n_{j+1} = 2 \log n_j$ shows

$$\sum_{I \in \Lambda} |I| \log \frac{1}{|I|} \le \sum_{I \in P_2} |I| \log \frac{1}{|I|} + 2 \sum_{\text{heavy}} |J| \log \frac{1}{|J|} \lesssim 2^{j_0} + \frac{\mu(\overline{\mathbb{D}})}{c}, \tag{4.4}$$

where we have used the fact that a maximal light interval of generation $j \ge 3$ is contained in a heavy interval of the previous generation. □

Remark. The above proof shows that if we take the threshold $\eta = \pi/n_2 = \pi/2^{2^{2+j_0}}$, then the local entropy

$$\|E_{\text{cone}}\|_{\mathcal{BC}_\eta} \lesssim \frac{\mu(\overline{\mathbb{D}})}{c},$$

can be made arbitrarily small by asking for $c > 0$ to be large. Crucially, this estimate is independent of j_0. Hence, if $\mu_n \to \mu$ is a diffuse sequence of measures, then for sufficiently large n, most of the mass of μ_n falls into the series $\overline{\mu}_n = \mu_2 + \mu_3 + \dots$.

4.2 Estimating nearly-maximal solutions

For $0 < r \le 1$, $C > 0$, let $u_{r,C}$ denote the unique solution of $\Delta u = 4e^{2u}$ defined on $\mathbb{D}_r$ with constant boundary values $u|_{\partial \mathbb{D}_r} \equiv C$. It is easy to write down the solution explicitly: $u_{r,C}(z) = \log \frac{L}{1-|Lz|^2}$ where $L > 0$ is chosen so that $\log \frac{L}{1-(Lr)^2} = C$. Our current objective is to show the following theorem:

Theorem 4.4. *Suppose $\mu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ is a finite measure on the closed unit disk which can be expressed as a countable sum*

$$\mu = \mu_2 + \mu_3 + \mu_4 + \dots,$$

where each piece satisfies (4.2) and (4.3) and $c_* > 0$ is sufficiently small so that (4.5) below holds. Then, $u_\mu \ge u_{r_0,(4/5)u_{\mathbb{D}}}$ on $\mathbb{D}_{r_0}$.

Since $u_{r_0,(4/5)u_{\mathbb{D}}} \to u_{\mathbb{D}}$ uniformly on compact subsets of the unit disk as $j_0 \to \infty$, the above theorem tells us that if $j_0 \ge 1$ is large, then u_μ is close to the maximal solution $u_{\mathbb{D}}$. As the proof of Theorem 4.4 is similar to that of [10, Theorem 1.10], we only give a sketch the argument and refer the reader to [10, Section 6] for the details. We will need the following lemma:

Lemma 4.5. *Suppose $\mu \in M(\overline{\mathbb{D}})$ is a finite measure on the closed unit disk which satisfies* $\operatorname{supp}\mu \subset \square_{\mathbb{S}^1,1-1/n,1}$ *and*

$$\mu(\square_{I,1-1/n,1}) \le 2c \cdot |I| \log \frac{1}{|I|}, \qquad \forall I \subset \mathbb{S}^1, \quad |I| = 2\pi/n.$$

Then,

$$|I_\mu(z)| > \frac{1}{(1-|z|^2)^{c'}}, \qquad |z| < 1 - 2/n,$$

for some $c' \asymp c$.

The lemma is well known when $\operatorname{supp}\mu \subseteq \mathbb{S}^1$, e.g. see [22, Lemma 2.2]. The same proof applies to the general case. We choose c_* in Theorem 4.4 so that $c' < 1/10$, i.e.

$$|I_{\mu_j}(z)| > \frac{1}{(1-|z|^2)^{1/10}}, \qquad \text{for } z \in \mathbb{D}_{r_{j-2}}. \tag{4.5}$$

Sketch of proof of Theorem 4.4. To simplify notation, let us write $\Lambda_r := \Lambda^0_r$. As $u_\mu = \lim_{j\to\infty} u_{\mu_2+\mu_3+\mu_4+\cdots+\mu_j}$, it suffices to prove the theorem when $\mu = \mu_2+\mu_3+\mu_4+\cdots+\mu_j$ is a finite sum. By construction, the function

$$\tilde u \;:=\; \Lambda_{r_0}\left[\ldots \Lambda_{r_{j-3}}\left[\Lambda_{r_{j-2}}\left[u_{\mathbb{D}} - \log\frac{1}{|I_{\mu_j}|}\right] - \frac{1}{|I_{\mu_{j-1}}|}\right]\cdots - \log\frac{1}{|I_{\mu_2}|}\right] \tag{4.6}$$

solves the Gauss curvature equation $\Delta u = 4e^{2u}$ on the disk $\mathbb{D}_{r_0}$. By the monotonicity properties of Λ and the repeated use of Lemma 2.7,

$$\begin{aligned}
\tilde u \;&\le\; \Lambda^{\tilde\mu_2+\tilde\mu_3+\cdots+\tilde\mu_j}\left[\ldots \Lambda^{\tilde\mu_{j-1}+\tilde\mu_j}\left[\Lambda^{\tilde\mu_j}\left[u_{\mathbb{D}} - \log\frac{1}{|I_{\mu_j}|}\right] - \frac{1}{|I_{\mu_{j-1}}|}\right]\cdots - \log\frac{1}{|I_{\mu_2}|}\right], \\
&=\; \Lambda^{\tilde\mu_2+\tilde\mu_3+\cdots+\tilde\mu_j}\left[u_{\mathbb{D}} - \log\frac{1}{|I_{\mu_2+\mu_3+\cdots+\mu_j}|}\right], \\
&= u_\mu
\end{aligned}$$

on $\mathbb{D}_{r_0}$, where we made use of the fact that $\operatorname{supp}\mu_j \cap \mathbb{D}_{r_{j-2}} = \emptyset$. To show that $\tilde u > u_{r_0, u_{\mathbb{D}}-\log 2}$ on $\mathbb{D}_{r_0}$, we estimate $\tilde u$ by recursively unwinding the definition (4.6):

0. We begin with $u_{\mathbb{D}} - \log 2$.

1. We subtract $\log \frac{1}{|I_{\mu_j}|}$. By the estimate (4.5),

$$u_{\mathbb{D}} - \log 2 - \log \frac{1}{|I_{\mu_j}|} \geq (4/5)u_{\mathbb{D}}, \qquad \text{on } \partial\mathbb{D}_{r_{j-2}}.$$

2. We form the solution $u_{r_{j-2},(4/5)u_{\mathbb{D}}}$. By the computation in [10, Section 6], $u_{r_{j-2},(4/5)u_{\mathbb{D}}} > u_{\mathbb{D}} - \log 2$ on $\partial\mathbb{D}_{r_{j-3}}$.

Repeating this process gives the desired estimate. □

4.3 Diffuse sequences lose mass

We now prove Theorems 4.1 and 4.2:

Proof of Theorem 4.1. We split $\mu_n = \nu_n + \tau_n$ so that $\nu_n \to \nu$ is concentrating and $\tau_n \to \tau$ is diffuse. Since the sequence $\{\mu_n\}$ does not converge in the Korenblum topology, the diffuse part is non-trivial, i.e. $\tau \neq 0$. Form the Roberts decompositions

$$\mu_n = \overline{\mu}_{n,j_0} + \nu_{n,j_0,\text{cone}} = (\mu_{n,j_0,2} + \mu_{n,j_0,3} + \mu_{n,j_0,4} + \dots) + \nu_{n,j_0,\text{cone}}.$$

By the remark following the proof of Theorem 4.3, when the threshold $\eta(j_0) = \pi/n_2 = \pi/2^{2^{2+j_0}}$, the local entropy

$$\|E_{n,j_0,\text{cone}}\|_{\mathcal{BC}_{\eta(j_0)}} \lesssim \frac{\mu_n(\overline{\mathbb{D}})}{c}. \tag{4.7}$$

By the definition of a diffuse sequence, we can choose $c > 0$ so that for any given $j_0 \geq 2$, $\mu_{n,j_0}(K_{E_{n,j_0,\text{cone}}}) < \tau_n(\overline{\mathbb{D}})/2$ for all sufficiently large $n \geq n_0(j_0)$, in which case, a definite chunk of τ_n will fall into $\overline{\mu}_n$. Diagonalizing, we obtain a sequence $j_0(n) \to \infty$ with $\liminf_{n\to\infty} \overline{\mu}_{n,j_0(n)}(\mathbb{D}) \geq \tau(\overline{\mathbb{D}})/2$. By Theorem 4.4, $u_{(c_*/c)\overline{\mu}_n} \to u_{\mathbb{D}}$. Since $(c_*/c)\overline{\mu}_n$ have definite mass, Lemma 2.10 prevents u_{μ_n} from converging to u_μ. The proof is complete. □

Proof of Theorem 4.2. For each $n = 1, 2, \dots$, consider the Roberts decomposition

$$\mu_n = \overline{\mu}_n + \nu_{n,\text{cone}} = (\mu_{n,2} + \mu_{n,3} + \mu_{n,4} + \dots) + \nu_{n,\text{cone}}$$

with $c = c_*$ being the constant from Theorem 4.4. Since the sequence $\{\mu_n\}$ is totally diffuse, for any *fixed* $j_0 \geq 2$, $\nu_{n,j_0,\text{cone}}(\overline{\mathbb{D}}) \to 0$ as $n \to \infty$. Diagonalizing allows us to choose $j_0(n) \to \infty$ so that $\nu_{n,j_0(n),\text{cone}}(\overline{\mathbb{D}}) \to 0$ as $n \to \infty$. By Theorem 4.4, $u_{\overline{\mu}_{n,j_0(n)}} \to u_{\mathbb{D}}$. Lemma 2.9 guarantees that $u_{\mu_n} \to u_{\mathbb{D}}$ as well. □

4.4 An instructive example

Given $M > 0$, consider the sequence of probability measures $\mu_{n,M} = (1/n)\sum_{k=1}^n \delta_{e^{ik\theta_n}}$, where θ_n is chosen so that $n\theta_n \log \frac{1}{\theta_n} = M$. We show:

Lemma 4.6. *For $M > 0$ sufficiently large, the nearly-maximal solutions $u_{n,M} = u_{\mu_{n,M}}$ converge to $u_{\mathbb{D}}$. For $M > 0$ sufficiently small, the $u_{n,M}$ do not converge to $u_{\mathbb{D}}$.*

Proof. We assume that n is sufficiently large so that $n\theta_n < \pi$. Since any arc $I \subset \mathbb{S}^1$ of length $\theta_n/2$ contains at most one of the points $\{e^{ik\theta_n}\}_{k=1}^n$,

$$\mu_{n,M}(I) \leq 1/n = (1/M) \cdot \theta_n \log \frac{1}{\theta_n} \leq (3/M) \cdot |I| \log \frac{1}{|I|}.$$

For the first assertion, it is enough to request that $3/M < c_*$ where c_* is the constant from Theorem 4.4. If the second assertion were false, a diagonalization argument would produce a sequence $u_{n_j,M_j} \to u_{\mathbb{D}}$ with $n_j \to \infty$ and $M_j \to 0$. But this diagonal sequence is concentrating, so by Theorem 3.9, its limit should be u_{δ_1}, which is a contradiction. □

Corollary 4.7. *There exists a $0 < k < 1$ and a sequence of measures $\mu_n \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ such that $u_{\mu_n} \not\to u_{\mathbb{D}}$ but $u_{k\cdot\mu_n} \to u_{\mathbb{D}}$.*

Proof. By the second statement of Lemma 4.6, we can choose $M > 0$ so that the $u_{\mu_{n,M}}$ do not converge to $u_{\mathbb{D}}$. If $k = (M/3)c_*$, then for any arc $I \subset \mathbb{S}^1$ of length $\theta_n/2$, $k \cdot \mu_{n,M}(I) \leq c_*|I| \log \frac{1}{|I|}$, which implies that $u_{k\cdot\mu_{n,M}}$ tends to $u_{\mathbb{D}}$ as $n \to \infty$. □

Remark. Actually, for *any* $0 < k < 1$, there exists a sequence of measures $\mu_n \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ such that $u_{\mu_n} \not\to u_{\mathbb{D}}$ but $u_{k\cdot\mu_n} \to u_{\mathbb{D}}$. To see this, one simply needs to scale the sequence from Corollary 4.7 by an appropriate constant.

5 Invariant subspaces of Bergman spaces

For a fixed $\alpha > -1$ and $1 \le p < \infty$, consider the weighted Bergman space $A^p_\alpha(\mathbb{D})$ of holomorphic functions satisfying the norm boundedness condition (1.10). Let $\{I_n\}$ be a sequence of inner functions which converge uniformly on compact subsets of the disk to an inner function I. Assume that the measures $\mu(I_n)$ and $\mu(I)$ are in $M_{\mathcal{BC}}(\overline{\mathbb{D}})$. Let $[I_n] \subset A^p_\alpha$ be the z-invariant subspace generated by I_n. In this section, we prove Theorem 1.6 which says that $\lim_{n\to\infty}[I_n] = [I]$ if and only if the measures $\mu(I_n)$ converge to $\mu(I)$ in the Korenblum topology. Note that the inclusion

$$[I] \subseteq \liminf_{n\to\infty} \, [I_n] \tag{5.1}$$

is automatic since $\liminf_{n\to\infty} [I_n]$ is an invariant subspace which contains I.

5.1 Concentrating sequences: special case

Theorem 5.1. *Suppose $I_n \to I$ is a sequence of inner functions which converges uniformly on compact subsets of the unit disk. If the zero structure of I_n belongs to a Korenblum star K_{E_n} and the $E_n \to E$ converge in $\mathcal{BC}$ then $[I_n] \to [I]$.*

The proof of the above theorem is based on the arguments of Korenblum [12]. For a Beurling-Carleson set E, one can construct an outer function $\Phi_E(z) \in C^\infty(\overline{\mathbb{D}})$ which vanishes precisely on E and does so to infinite order. Examining the construction in [8, Proposition 7.11], we may assume that Φ_E enjoys two extra properties:

1. The function $\Phi_E(z)$ varies continuously with the Beurling-Carleson set E, in the sense that $\Phi_{E_n} \to \Phi_E$ uniformly on compact subsets of the disk if $E_n \to E$ and $\|E_n\|_{\mathcal{BC}} \to \|E\|_{\mathcal{BC}}$.

2. For each $N \ge 0$,
$$|\Phi_E(z)| \cdot \operatorname{dist}(z, E)^{-N} \le C_E(N)$$
is bounded by a constant which depends continuously on E. It is convenient to take $C_E(0) = 1$ so that $|\Phi_E(z)| \le 1$ on the disk.

A brief sketch of the construction will be provided in Appendix C. The central idea in Korenblum's vision is the following division principle:

Theorem 5.2 (Korenblum's division principle)**.** *Suppose I is an inner function with zero structure* $\operatorname{supp}\mu(I) \subset K_E$ *and $f \in [I]$. For any $\delta > 0$,*

$$f^\delta(z) := (\Phi_E^\delta/I)f(z) \in A^p_\alpha, \tag{5.2}$$

with the norm estimate $\|f^\delta\|_{A^p_\alpha} \le C_E\|f\|_{A^p_\alpha}$ where $C_E = C_E\big(\delta, \mu(I)(\overline{\mathbb{D}})\big)$ depends continuously on E, δ and $\mu(I)(\overline{\mathbb{D}})$.

Assuming Theorem 5.2, the proof of Theorem 5.1 runs as follows:

Proof of Theorem 5.1. Suppose that a sequence of functions $f_n \in [I_n]$ converges to f in A^p_α. Norm convergence implies that the f_n converge to f uniformly on compact subsets of $\mathbb{D}$. By Korenblum's division principle, for a fixed $\delta > 0$, the functions $g_n = (\Phi_n^\delta/I_n) \cdot f_n(z)$ have bounded A^p_α norms and converge uniformly on compact subsets to

$$g = (\Phi^\delta/I) \cdot f(z).$$

Fatou's lemma implies that $g \in A^p_\alpha$ and therefore $\Phi^\delta \cdot f = Ig \in [I]$. Taking $\delta \to 0$ shows that $f \in [I]$ and therefore $[I] \supseteq \limsup_{n\to\infty} [I_n]$. By (5.1), the other inclusion is automatic. □

Since the exact statement of Theorem 5.2 is not present in Korenblum's work [12], we give a proof below.

Proof of Korenblum's division principle (Theorem 5.2). We first consider the case when I is a finite Blaschke product and E is a finite set. Afterwards, we will deduce the general case by a limiting argument. If I is a finite Blaschke product, it is clear that $f^\delta \in A^p_\alpha$. We need to give a uniform estimate on its norm.

Recall that K_E^2 denotes the generalized Korenblum star of order 2, see (3.1) for the definition. According to Lemma 3.7, $|1/I(z)| \le C\big(\mu(I)(\overline{\mathbb{D}})\big)$ is uniformly bounded on $\mathbb{D} \setminus K_E^2$ so that $|f^\delta(z)| \le C|f(z)|$ there.

To estimate f^δ on K_E^2, we examine its values on the boundary ∂K_E^2. It is well known that a function in Bergman space does not grow too rapidly:

$$|f(z)| \le C_2 \|f\|_{A^p_\alpha} (1-|z|)^{-\beta}, \qquad z \in \mathbb{D}, \tag{5.3}$$

for some $\beta = \beta(p,\alpha) > 0$. However, the C^∞ decay of the outer function Φ^δ cancels out this grows rate on ∂K_E^2 and we end up with

$$|f^\delta(z)| \le C_3 \|f\|_{A^p_\alpha}, \qquad z \in \partial K_E^2.$$

Since $f^\delta \in A^p_\alpha$, we can use the Phragmén-Lindelöf principle to conclude that this bound extends to the interior of K_E^2. Putting the above estimates together completes the proof when I is a finite Blaschke product.

For the general case, we approximate I uniformly on compact subsets by finite Blaschke products I_n whose zeros are contained in K_E. Using the semicontinuity property (5.1), we may then approximate $f \in [I]$ by $f_n \in [I_n]$ in the A^p_α-norm. By the finite case of the lemma, $f_n^\delta = (\Phi_E^\delta / I_n) f_n(z) \in A^p_\alpha$ with $\|f_n^\delta\|_{A^p_\alpha}$ bounded above. By Fatou's lemma, $\|f^\delta\|_{A^p_\alpha} \le \liminf_{n\to\infty} \|f_n^\delta\|_{A^p_\alpha}$ as desired. □

5.2 Concentrating sequences: general case

Suppose I is an inner function with $\mu(I) \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ and $I^N \to I$ is an approximating sequence of inner functions such that $\mu(I^N) \le \mu(I)$ is supported on a Korenblum star of norm $\le N$. We claim that $\bigcap_{N=1}^\infty [I^N] = [I]$. The "$\supseteq$" inclusion is trivial. For the converse, note that if $f \in \bigcap_{N=1}^\infty [I^N]$ then $f(I/I_N) \in [I]$ for any N. Since $[I]$ is a closed subspace, $f \in [I]$, which proves the claim.

Theorem 5.3. *Suppose $I_n \to I$ is a sequence of inner functions which converges uniformly on compact subsets of the disk. If the associated measures $\mu(I_n) \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ converge in the Korenblum topology, then $[I_n] \to [I]$.*

Proof. By the definition of the Korenblum topology, there exist "approximations" $I_n^N \to I^N$ supported on Korenblum stars of norm $\le N$. By Theorem 5.1,

$$\limsup_{n\to\infty} [I_n] \subseteq \limsup_{n\to\infty} [I_n^N] = [I^N] \to_{N\to\infty} [I].$$

The other inclusion follows from (5.1). □

5.3 Diffuse sequences

To complete the proof of Theorem 1.6, we need to show:

Theorem 5.4. *If $I_{\mu_n} \to I_\mu$ is a sequence of inner functions whose zero structures $\mu(I_n) \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ do not converge in the Korenblum topology to $\mu(I) \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, then the invariant subspaces $[I_{\mu_n}]$ do not converge to $[I_\mu]$.*

We will also show:

Theorem 5.5. *Suppose $I_n \to I$ is a convergent sequence of inner functions such that the associated measures $\mu(I_n)$ are totally diffuse. Then, $[I_n] \to [1]$.*

The proofs of Theorems 5.4 and 5.4 are similar to their counterparts in Section 4, except that the estimates on solutions of the Gauss curvature equation are replaced with the use of corona theorem. We begin by describing an analogue of Theorem 4.4 in this setting. Let $\overline{\mu} = \mu_2 + \mu_3 + \mu_4 + \dots$ be a measure on the closed unit disk where the individual pieces satisfy (4.2) and (4.3). In [22], Roberts explained how to estimate $d(1, [I_{\overline{\mu}}])$, the distance of the constant function 1 to the invariant subspace generated by $I_{\overline{\mu}}$ in A^p_α. To be honest, Roberts only considered singular inner functions (in which case, the measures μ_j are supported on the unit circle), but his argument extends to general inner functions almost verbatim. We give only a brief outline of his argument and leave it to the interested reader to fill in the details.

Lemma 5.6 (cf. Lemma 2.3 of [22])**.** *Fix $\beta > 0$ so that $\|z^n\|_{A^p_\alpha} \le n^{-\beta}$ for $n \ge 2$. Suppose I is an inner function which enjoys the estimate*

$$|I(z)| \ge n^{-\gamma}, \qquad |z| \le 1 - 1/n. \tag{5.4}$$

If $0 < \gamma < (\beta/3)K$ where K is the constant from the corona theorem and $n \ge N(\gamma)$ is sufficiently large, then there exists a function $g \in H^\infty(\mathbb{D})$ with

$$\|g\|_\infty \le n^{\beta/3}, \qquad \|1 - gI\|_{A^p_\alpha} \le n^{-2\beta/3}. \tag{5.5}$$

Roberts introduced the characteristic $D[\{n_1, n_2, \dots, n_k\}]$ which is defined recursively by $D[\emptyset] = 0$ and $D[\{n_1, n_2, \dots, n_k\}] = n_1^{\beta/3} D[\{n_2, n_3, \dots, n_k\}] + n_1^{-2\beta/3}$. In view of monotonicity, this definition naturally extends to infinite sequences. He noticed that if $j_0(\beta) \geq 1$ is large, then the sequence of integers $n_j = 2^{2^{j+j_0}}$ in the Roberts decomposition (Theorem 4.3) is sufficiently sparse to ensure that $D[\{n_j\}]$ is small.

Theorem 5.7 (cf. Lemma 2.4 of [22])**.** *Let $0 < \gamma < (\beta/3)K$ as in Lemma 5.6. Suppose $I_0, I_1, \ldots, I_{k-1}$ are inner functions such that*

$$|I_j(z)| \geq n_j^{-\gamma}, \qquad |z| \leq 1 - 1/n_j, \qquad j = 0, 1, \ldots, k-1. \tag{5.6}$$

Assume that $\min(n_0, n_1, \ldots, n_{k-1}) \geq N(\gamma)$*. If $I = \prod_{j=0}^{k-1} I_j$ then $d(1, [I]) \leq D[\{n_j\}]$.*

By Lemma 4.5, if the parameter $c > 0$ of the Roberts series $\overline{\mu} = \mu_2 + \mu_3 + \mu_4 + \ldots$ is sufficiently small, then $I_j = I_{\mu_{j+2}}$ verifies the condition (5.6) with $0 < \gamma < (\beta/3)K$, which allows us to apply the above theorem to estimate $d(1, [I_{\overline{\mu}}])$. In order to be able to use the strategy outlined in Section 4.3 to prove Theorem 5.4, it is enough to establish an analogue of Lemma 2.10 in the current setting:

Lemma 5.8. *Suppose $I_{\tau_n} \to I_\tau$ and $I_{\nu_n} \to I_\nu$ are two sequences of inner functions with $\tau \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$. If $[I_{\nu_n+\tau_n}]$ converge to $[I_{\nu+\tau}]$, then $[I_{\tau_n}]$ converge to $[I_\tau]$.*

Proof. Clearly, $\liminf_{n\to\infty}[I_{\tau_n}] \supseteq [I_\tau]$. For the reverse inclusion, note that any $f \in \limsup_{n\to\infty}[I_{\tau_n}]$ can be approximated by elements of the form $I_{\tau_n}p_n$ in A^p_α. Since $I_{\nu_n+\tau_n}p_n \to I_\nu f$ in A^p_α, the assumption of the lemma tells us that $I_\nu f \in [I_{\nu+\tau}]$. Using Korenblum's division principle, it is easy to see that $f \in [I_\tau]$ as desired. □

Similarly, to show Theorem 5.5, it is enough to prove the analogue of Lemma 2.9 for invariant subspaces of A^p_α:

Lemma 5.9. *Suppose $I_{\mu_n} \to I_\mu$ is a convergent sequence of inner functions. Assume that the invariant subspaces $[I_{\mu_n}]$ converge to $[I_{\mu^*}]$. If ν_n is a sequence of measures converging to 0, then $[I_{\mu_n+\nu_n}]$ also converge to $[I_{\mu^*}]$.*

Proof. Clearly, $\limsup_{n\to\infty}[I_{\mu_n+\nu_n}] \subseteq [I_{\mu^*}]$. For the reverse inclusion, we approximate a function $f \in [I_{\mu^*}]$ by functions $f_n \in [I_{\mu_n}]$ in A^p_α and notice that $f_n I_{\nu_n} \in [I_{\mu_n+\nu_n}]$ also converge to f in A^p_α. □

Remark. In the special case of the weighted Bergman space A^2_1, we can give an alternative proof of Theorem 5.5. By the Korenblum-Roberts theorem, we may assume that $\mu(I_n) \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$. For each I_n, we may form an inner function F_n with $F_n(0) = 0$, $F_n'(0) > 0$ and $\operatorname{Inn} F_n' = I_n$. According to Theorem 4.2, $F_n \to z$ uniformly on compact subsets. However, the bound $\|F_n\|_{H^\infty} \le 1$ implies that $\|F_n\|_{H^2} \le \|z\|_{H^2}$ which forces $F_n \to z$ to converge in the H^2-norm. The Littlewood-Paley formula

$$\|F_n\|_{H^2} = \frac{1}{\pi}\int_{\mathbb{D}} |F_n'|^2 \log\frac{1}{|z|^2}\,|dz|^2 \asymp \|F_n'\|_{A^2_1}$$

then shows that $F_n' \to 1$ in the A^2_1-norm. Since $F_n' \in [I_n]$,

$$\liminf_{n\to\infty}\,[I_n] \supset \liminf_{n\to\infty}\,[F_n'] \supset [1] = A^2_1.$$

A Entropy of universal covering maps

Let m be the Lebesgue measure on the unit circle, normalized to have unit mass. It is well known that if F is an inner function with $F(0) = 0$, then m is F-invariant, i.e. $m(E) = m(F^{-1}(E))$ for any measurable set $E \subset \mathbb{S}^1$. In the work [5], M. Craizer showed that if $F \in \mathscr{J}$, then the integral

$$\int_{|z|=1} \log|F'(z)|dm$$

has the dynamical interpretation as the measure-theoretic entropy of m. It is therefore of interest to compute it in special cases. For finite Blaschke products, one may easily compute the entropy using Jensen's formula:

Theorem A.1. *Suppose F is a finite Blaschke product with $F(0) = 0$ and $F'(0) \ne 0$. We have*

$$\frac{1}{2\pi}\int_{|z|=1} \log|F'(z)|d\theta \;=\; \sum_{\mathrm{crit}(F)} \log\frac{1}{|c_i|} - \sum_{\mathrm{zeros}(F)\setminus\{0\}} \log\frac{1}{|z_i|}, \tag{A.1}$$

In this appendix, we discuss a complementary example:

Theorem A.2 (Pommerenke)**.** *Let P be a relatively closed subset of the unit disk not containing 0. Let $\mathcal{U}_P : \mathbb{D} \to \mathbb{D} \setminus P$ be the universal covering map, normalized so that $\mathcal{U}_P(0) = 0$ and $\mathcal{U}_P'(0) > 0$. Then $\mathcal{U}_P \in \mathscr{J}$ if and only if P is a Blaschke sequence, in which case*

$$\frac{1}{2\pi}\int_{|z|=1} \log|\mathcal{U}_P'(z)|d\theta \,=\, \sum_{p_i \in P} \log\frac{1}{|p_i|} - \sum_{\mathrm{zeros}(F)\setminus\{0\}} \log\frac{1}{|z_i|}. \tag{A.2}$$

A theorem of Frostman says that $\mathcal{U}_P$ is an inner function if and only if the set P has logarithmic capacity 0, see [4, Chapter 2.8]. In particular, $\mathcal{U}_P$ is inner if P is countable. For brevity, we will write $F = \mathcal{U}_P$. While Pommerenke did not explicitly state (A.2), in the work [20], he proved the equivalent statement

$$\operatorname{Inn} F'(z) \,=\, \prod_{i=1}^{k} F_{p_i}(z) \,=\, \prod_{i=1}^{k} \frac{F(z)-p_i}{1-\overline{p_i}F(z)}, \tag{A.3}$$

so we feel that it is appropriate to name the above theorem after him. Actually, Pommerenke worked in the significantly greater generality of Green's functions for Fuchsian groups of Widom type, so this is only a special case of his result. Below, we give a direct proof of Theorem A.2 based on stable approximation by finite Blaschke products.

A.1 Preliminaries

We first recall a well known property of Nevanlinna averages:

Lemma A.3. *If $f \in \mathcal{N}$ is a function in the Nevanlinna class and is not identically 0, then*

$$\frac{1}{2\pi}\int_{|z|=1} \log|f(z)|d\theta - \lim_{r\to 1}\left\{\frac{1}{2\pi}\int_{|z|=r} \log|f(z)|d\theta\right\} = \sigma(f)(\mathbb{S}^1). \tag{A.4}$$

See [10, Section 3] for a proof. For $x \in \mathbb{D}$, let $F_x = T_x \circ F$ denote the *Frostman shift* of F with respect to x, where $T_x(z) = \frac{z-x}{1-\bar{x}z}$. Frostman showed that if x avoids an exceptional set $\mathscr{E}$ of capacity zero, then F_x is a Blaschke product, in which case $\sigma(F_x) = 0$. We will also need:

Lemma A.4. *Let F be an inner function with $F(0) = 0$. For any $x \in \mathbb{D} \setminus \{0\}$,*

$$\log \frac{1}{|x|} = \sum_{F(y)=x} \log \frac{1}{|y|} + \sigma(F_x)(\mathbb{S}^1). \tag{A.5}$$

Proof. Taking $f = F_x$ in Lemma A.3 gives

$$0 = \lim_{r\to 1} \frac{1}{2\pi} \int_{|z|=r} \log |F_x(z)| d\theta + \sigma(F_x)(\mathbb{S}^1).$$

The lemma follows after applying Jensen's formula and taking $r \to 1$. □

In the case when $F \in \mathscr{J}$, Ahern and Clark [1] observed that the exceptional set $\mathscr{E}$ of F is at most countable and that the singular masses of different Frostman shifts F_x are mutually singular. More precisely, they showed that the measure $\sigma(F_x)$ is supported on the set of points on the unit circle at which the radial limit of F is x. Since the singular inner function $\operatorname{Sing} F_x$ divides F_x', it must also divide its inner part $\operatorname{Inn} F_x' = \operatorname{Inn} F'$. This shows that

$$\sigma(F') \ge \sum_{x \in \mathscr{E}} \sigma(F_x). \tag{A.6}$$

In other words, $\operatorname{Inn} F'$ is divisible by the product $\prod_{x \in \mathscr{E}} \operatorname{Sing} F_x$.

A.2 Proof of Theorem A.2 when P is a finite set

We first prove Theorem A.2 when $P = \{p_1, p_2, \dots, p_k\}$ is a finite set. In the formula (A.1), one considers the sum $\sum_{\text{crit}} \log \frac{1}{|c_i|}$ over critical points. It appears that the identity (A.5) allows one to sum over the "critical values" $p_1, p_2, \dots, p_k$ instead. To make this rigorous, we will construct a special approximation $F_n \to F$ by finite Blaschke products with critical values sets $\{p_1, p_2, \dots, p_k\}$.

Assuming the existence of such an approximating sequence, the argument runs as follows: since the entropy can only decrease after taking limits [10, Theorem 4.2],

$$\begin{aligned} \frac{1}{2\pi} \int_{|z|=1} \log |F'(z)| d\theta &\le \liminf_{n\to\infty} \frac{1}{2\pi} \int_{|z|=1} \log |F_n'(z)| d\theta, \\ &\le \liminf_{n\to\infty} \biggl\{ \log |F_n'(0)| + \sum_{i=1}^{k} \sum_{F_n(q_i)=p_i} \log \frac{1}{|q_i|} \biggr\}, \end{aligned}$$

$$= \log |F'(0)| + \sum_{i=1}^{k} \log \frac{1}{|p_i|}.$$

However, by (A.6), the other direction is automatic:

$$\begin{aligned}
\frac{1}{2\pi} \int_{|z|=1} \log |F'(z)| d\theta &= \lim_{r \to 1} \frac{1}{2\pi} \int_{|z|=r} \log |F'(z)| d\theta + \sigma(F')(\mathbb{S}^1), \\
&\geq \log |F'(0)| + \sum_{i=1}^{k} \sigma(F_{p_i})(\mathbb{S}^1), \\
&= \log |F'(0)| + \sum_{i=1}^{k} \log \frac{1}{|p_i|}.
\end{aligned}$$

Logic dictates that the sequence $F_n \to F$ is stable and the formula (A.2) holds.

A.3 Construction of the approximating sequence

For the construction of the approximating sequence, we employ the gluing technique of Stephenson [24], also see the paper of Bishop [2]. For each puncture p_i, choose a real-analytic arc which joins p_i to a point on the unit circle, so that the arcs are disjoint and do not pass through the origin. Define a *tile* or *sheet* to be the shape $\mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i$. Let $\Gamma = \langle g_1, g_2, \ldots, g_k \rangle$ be the free group on k generators. Consider the countable collection $\{T_g\}_{g \in \Gamma}$ of tiles indexed by elements of Γ. We form a simply-connected Riemann surface S by gluing the lower side of γ_i in T_g to the upper side of γ_i in $T_{g_i g}$. The surface S comes equipped with a natural projection to the disk $\mathbb{D}$ which sends a point in a tile T_g to its representative in the model $\mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i$. We may uniformize $S \cong \mathbb{D}$ by taking 0 in the base tile T_e to 0. In this uniformizing coordinate, the projection F becomes a holomorphic self-map of the disk. Since all the slits have been glued up, F is an inner function, and a little thought shows that it is the universal covering map of $\mathbb{D} \setminus \{p_1, p_2, \ldots, p_k\}$.

We now give a slightly different description of the above construction. For this purpose, we need the notion of an ∞-*stack*: a countable collection of tiles $\{T_j\}_{j \in \mathbb{Z}}$, where the lower side of γ_i in T_j is identified with the upper side of γ_i in T_{j+1}. To highlight the dependence on the curve γ_i, we say that the ∞-stack is glued over γ_i.

Similarly, by an *n-stack*, we mean a set of n tiles with the above identifications made modulo n. Now, to construct S, we begin with the base tile $T_e \cong \mathbb{D} \setminus \cup_{i=1}^k \gamma_i$, and at each slit $\gamma_i \subset T_e$, we glue an ∞-stack (i.e. we add the tiles $\{T_j\}_{j \in \mathbb{Z} \setminus \{0\}}$ and treat T_e as T_0). We refer to the tiles that were just added as the tiles of generation 1. To each of the $k-1$ unglued slits in each tile of generation 1, we glue a further ∞-stack of tiles, which we call tiles of generation 2. Repeating this construction infinitely many times gives the Riemann surface S from before.

For the finite approximations, we slightly modify the above procedure. We begin with a base tile $T_e \cong \mathbb{D} \setminus \cup_{i=1}^k \gamma_i$ with k slits. At each of these k slits, we glue in an n-stack of sheets (sheets of generation 1). At each of the $k-1$ unresolved slits of sheet of generation 1, we glue in a further n-stack (sheets of generation 2). We repeat for n generations. Finally, at sheets of generation n, we resolve the slits by simply sowing their edges together. This gives us a Riemann surface S_n and a finite Blaschke product F_n with critical values $p_1, p_2, \ldots, p_k$.

Since the Riemann surfaces $S_n \to S$ converge in the Carathéodory topology, the maps $F_n \to F$ converge uniformly on compact sets. With the construction of the special approximating sequence, the proof of Theorem A.2 is complete (when the number of punctures is finite).

A.4 Proof of Theorem A.2 when P is infinite

We handle the infinite case by reducing it to the finite case. This is achieved by the following lemma:

Lemma A.5. *Suppose that $\mathcal{U}_P$ is an inner function. Then,*

$$\frac{1}{2\pi} \int_{|z|=1} \log |\mathcal{U}_P'(z)| d\theta \geq \frac{1}{2\pi} \int_{|z|=1} \log |\mathcal{U}_Q'(z)| d\theta, \tag{A.7}$$

for any $Q \subseteq P$.

Proof. Topological considerations allow us to factor $\mathcal{U}_P = \mathcal{U}_Q \circ h$, where h is a holomorphic map of the disk. The normalizations $\mathcal{U}_P(0) = \mathcal{U}_Q(0) = 0$ imply that $h(0) = 0$. Since $\mathcal{U}_P$ is inner, h must also be inner. The chain rule and the h-invariance of Lebesgue measure give

$$\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_P'(z)|d\theta = \frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_Q'(z)|d\theta + \frac{1}{2\pi}\int_{|z|=1}\log|h'(z)|d\theta$$

Since h is inner and $h(0) = 0$, $|h'(z)| \geq 1$ for $z \in \mathbb{S}^1$, see e.g. [17, Theorem 4.15]. Dropping second term gives (A.7). □

Proof of Theorem A.2 when P is infinite. The above lemma shows that if P is not a Blaschke sequence, then $\mathcal{U}_P$ cannot be an inner function of finite entropy. Conversely, if $P = \{p_1, p_2, \dots\}$ is a Blaschke sequence, then the integrals

$$\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_{P_k}'(z)|d\theta, \qquad P_k = \{p_1, p_2, \dots, p_k\},$$

are increasing in k and

$$\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_P'(z)|d\theta \geq \lim_{k\to\infty}\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_{P_k}'(z)|d\theta. \tag{A.8}$$

Since the entropy can only decrease in the limit [10, Theorem 4.2], we must have equality in (A.8). This completes the proof. □

B Existence of Perron hulls

We now prove the existence statement in Theorem 2.2. The proof is a standard application of Schauder's fixed point theorem. Our exposition is inspired by [14, Appendix].

Recall that $G(z,\zeta) = \log\left|\frac{1-z\overline{\zeta}}{z-\zeta}\right|$ denotes the Green's function of the unit disk. Below, we will make use of two properties of the Green's function:

1. If μ is a finite measure on the unit disk, then

   $$G_\mu(z) = \frac{1}{2\pi}\int_{\mathbb{D}} G(z,\zeta)d\mu$$

   solves the linear Dirichlet problem

   $$\begin{cases} \Delta u = -\mu, & \text{in } \mathbb{D}, \\ u = 0, & \text{on } \mathbb{S}^1, \end{cases} \tag{B.1}$$

   where the boundary condition is understood in terms of weak limits.

2. The function
$$z \;\to\; \int_{\mathbb{D}} G(z,\zeta)|d\zeta|^2$$
is uniformly bounded on $\mathbb{D}$ and tends to 0 as $|z| \to 1$.

Remark. One can check Property 2 by using the interpretation of the Green's function as the occupation density of Brownian motion. In fact, the quadratic scaling of Brownian motion gives the stronger estimate
$$\int_{\mathbb{D}} G(z,\zeta)|d\zeta|^2 \le C(1-|z|)^2.$$

Proof of Theorem 2.2: existence. Let P_h denote the harmonic extension of h to the unit disk. Since $h : \mathbb{S}^1 \to \mathbb{R}$ is bounded above by assumption, P_h is bounded above on the unit disk. Consider the closed convex set
$$\mathscr{K}_h \;=\; \Big\{ v \in L^1(\mathbb{D}, |dz|^2),\; v \le P_h \Big\} \;\subset\; L^1(\mathbb{D}, |dz|^2)$$
and the operator
$$(Tv)(z) = P_h(z) - \frac{1}{2\pi} \int_{\mathbb{D}} \big(4e^{2v(\zeta)}|d\zeta|^2 + 2\pi d\tilde{\nu}_\zeta\big) G(z,\zeta). \tag{B.2}$$
Since $\nu_\zeta = 4e^{2v(\zeta)}|d\zeta|^2 + 2\pi d\tilde{\nu}_\zeta$ is a finite measure, by Property 2, $G_\nu(z) \in L^1(\mathbb{D}, |dz|^2)$, which shows that T maps $\mathscr{K}_h$ into itself.

By Property 1, every function in the image of T has boundary data h. In other words, Tv is the unique solution of the linear Dirichlet problem
$$\begin{cases} \Delta u = 4e^{2v} + 2\pi\tilde{\nu}, & \text{in } \mathbb{D}, \\ u = h, & \text{on } \mathbb{S}^1. \end{cases} \tag{B.3}$$
In particular, $u \in \mathscr{K}_h$ is a fixed point of T if and only if u solves the Gauss curvature equation (2.3) with data $(2\pi\tilde{\nu}, h)$.

To see that the image $T(\mathscr{K}_h)$ is compact, note that by Property 2, the functions
$$\int_{\mathbb{D}} e^{2v(\zeta)} G(z,\zeta)|d\zeta|^2, \qquad v \in \mathscr{K}_h,$$
are uniformly continuous on the closed unit disk $\overline{\mathbb{D}}$, which allows one to extract uniform subsequential limits using Arzelà-Ascoli. Since we have verified the assumptions of Schauder's fixed point theorem, T has a fixed point $u \in \mathscr{K}_h$. The proof is complete. □

The reader who wishes to learn more about non-linear elliptic PDEs involving measures can consult [21, 16].

C Carleson's theorem on outer functions

We now briefly outline the construction of an outer function $\Phi_E \in C^\infty(\overline{\mathbb{D}})$ which vanishes on a Beurling-Carleson set E to infinite order. In the literature, this fact is known as Carleson's theorem, even though the original construction due to Carleson [3] only gave $\Phi_E \in C^N(\overline{\mathbb{D}})$, where $N \geq 1$ could be any positive integer. Here, we follow the exposition from [8, Proposition 7.11], although we slightly modify the construction to ensure that Φ_E varies continuously with E.

For a closed subset K of the unit circle, we denote the collection of open arcs that make up $\mathbb{S}^1 \setminus K$ by $\mathcal{I}(K)$. The construction begins by subdividing each interval $I_n \in \mathcal{I}(E)$ into countably many pieces $\{J_{n,k}\}_{k\in\mathbb{Z}}$ such that $J_{n,0}$ is the middle third interval in I_n and

$$|J_{n,k}| = \operatorname{dist}(E, J_{n,k}) = \frac{1}{3\cdot 2^{|k|}} \cdot |I_n|.$$

Inspection shows that $F = \mathbb{S}^1 \setminus \bigcup J_{n,k}$ is a Beurling-Carleson set and that the map $\Delta : \mathcal{BC} \to \mathcal{BC}$ which sends E to F is continuous, that is, if $E_n \to E$ and $\|E_n\|_{\mathcal{BC}} \to \|E\|_{\mathcal{BC}}$ then $F_n \to F$ and $\|F_n\|_{\mathcal{BC}} \to \|F\|_{\mathcal{BC}}$. It is not difficult to see that there exists a function $\lambda_F : \mathcal{I}(F) \to [1,\infty)$ which satisfies

$$\lambda_F(J) \to \infty, \qquad |J| \to 0, \tag{C.1}$$

and

$$\sum \lambda_F(J) \cdot |J| \log \frac{1}{|J|} < \infty. \tag{C.2}$$

With help of λ_F, we define

$$\Phi_E(z) = \exp\Bigg[-\sum_{J\in\mathcal{I}(F)} \frac{\lambda_F(J)\cdot |J|\log\frac{1}{|J|}\cdot e^{i\theta_J}}{a_J - z}\Bigg], \tag{C.3}$$

where $e^{i\theta_J}$ is the midpoint of J and $a_J = r_J e^{i\theta_J}$ is the point in $\mathbb{C} \setminus \mathbb{D}$ from which J is seen from a $60°$ angle (such a point exists since $|J| \leq 2\pi/3$). Since the real part of each term in the sum is negative, $|\Phi_E(z)| \le 1$ on the disk. The condition (C.2) ensures that $\Phi_E(z)$ is not identically 0.

We now show that Φ_E vanishes on E to infinite order, that is, $|\Phi_E(z)|\,\mathrm{dist}(z,E)^{-N}$ is bounded on the unit disk for any given $N \ge 1$. Actually, it is enough to show that $|\Phi_E(z)|\,\mathrm{dist}(z,E)^{-N}$ is bounded on the unit circle: to extend the bound to the unit disk we can apply the maximum-modulus principle to the functions $\Phi_E(z)(\zeta - z)^{-N}$ with $\zeta \in E$.

For $z \in \mathbb{S}^1 \setminus E$, let $J_z \in \mathcal{I}(F)$ denote an arc which contains z (if z is contained in two arcs by virtue of being an endpoint of both, we may choose J_z to be either one of these arcs). By the choice of a_J, $\frac{|J|e^{i\theta_J}}{a_J - z}$ is confined to a compact subset of $\{\mathrm{Re}\, z < 0\}$. It follows that

$$|\Phi_E(z)| \le \frac{1}{|J|^{c_1\cdot\lambda_F(J)}} \le \mathrm{dist}(z,E)^{c_2\cdot\lambda_F(J)}$$

where $c_1, c_2 > 0$ are universal constants. The condition (C.1) gives the required decay.

For an interval $J \in \mathcal{I}(F)$, we would like to define

$$\lambda_F(J) = \max\left\{1,\, \log\frac{1}{h_F(J)}\right\}$$

where

$$h_F(J) = \sum_{J' \in \mathcal{I}(F):\, |J'| \le |J|} |J'| \log\frac{1}{|J'|}.$$

With this definition, the sum (C.2) is finite and its tails converge to zero uniformly as

$$\sum_{J \in \mathcal{I}(F):\, e^{-(k+1)} \le h_F(J) \le e^{-k}} \lambda_F(J)\cdot |J| \log\frac{1}{|J|} \le (k+1)e^{-k}, \tag{C.4}$$

however, the $h_F(J), \lambda_F(J)$ will not depend continuously with respect to the Beurling-Carleson set F, because they are sensitive to small changes in the lengths of the intervals and the entropy of F.

To rectify this, we smoothen out the definitions of $h(J)$ and $\lambda(J)$, that is, we define

$$h_F(J) = \sum_{J' \in \mathcal{I}(F):\, |J'| < 2|J|} \psi\left(\frac{|J'|}{|J|}\right)\cdot |J'| \log\frac{1}{|J'|},$$

where $\psi : (0, \infty) \to [0, 1]$ is a smooth function such that $\psi(t) = 1$ for $t < 1$ and $\psi(t) = 0$ for $t > 2$, and

$$\lambda_F(J) = \phi\left(\log \frac{1}{|J|}\right),$$

where ϕ is an increasing smooth function which satisfies $\phi(t) = t$ for $t > 2$ and $\phi(t) = 1$ for $t < 1$.

We leave it to the reader to check that Φ_E enjoys the continuity properties described in Section 5.1.

References

[1] P. R. Ahern, D. N. Clark, *On inner functions with H_p-derivative*, Michigan Math. J. 21 (1974), no. 2, 115–127.

[2] C. J. Bishop, *An indestructible Blaschke product in the little Bloch space*, Publicacions Matemtiques 37 (1993), no. 1, 95–109.

[3] L. Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta Math. 87 (1952), 325–345.

[4] E. F. Collingwood, A. J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 56. Cambridge University Press, Cambridge, 1966.

[5] M. Craizer, *Entropy of inner functions*, Israel J. Math. 74 (1991), no. 2, 129–168.

[6] K. M. Dyakonov, *Inner functions and inner factors of their derivatives*, Integr. Equ. Oper. Theory 82 (2015), no. 2, 151–155.

[7] H. Hedenmalm, B. Korenblum, K. Zhu, *Beurling type Invariant Subspaces of the Bergman Spaces*, J. London Math. Soc. 53 (1996), no. 2, 601–614.

[8] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics 199, Springer-Verlag, 2000.

[9] M. Heins, *On a class of conformal metrics*, Nagoya Math. J. 21 (1962), 1–60.

[10] O. Ivrii, *Prescribing inner parts of derivatives of inner functions*, J. d'Analyse Math. 139 (2019), 495–519.

[11] B. Korenblum, *Cyclic elements in some spaces of analytic functions*, Bull. Amer. Math. Soc. 5 (1981), 317–318.

[12] B. Korenblum, *Outer Functions and Cyclic Elements in Bergman Spaces*, J. Funct. Anal. 151 (1993), no. 1, 104–118.

[13] D. Kraus, *Critical sets of bounded analytic functions, zero sets of Bergman spaces and nonpositive curvature*, Proc. London Math. Soc. 106 (2013), no. 4, 931–956.

[14] D. Kraus, O. Roth, *Conformal metrics*, Lecture Notes Ramanujan Math. Society, Lecture Notes Series 19 (2013), 41–83.

[15] M. Marcus, A. C. Ponce, *Reduced limits for nonlinear equations with measures*, J. Funct. Anal 258 (2010), no. 7, 2316–2372.

[16] M. Marcus, L. Véron, *Nonlinear second order elliptic equations involving measures*, De Gruyter Series in Nonlinear Analysis and Applications 21, De Gruyter, Berlin, 2014.

[17] J. Mashreghi, *Derivatives of Inner Functions*, Fields Institute Monographs, 2012.

[18] C. T. McMullen, *Ribbon $\mathbb{R}$-trees and holomorphic dynamics on the unit disk*, J. Topol. 2 (2009), 23–76.

[19] N. K. Nikol'skii, *Treatise on the Shift Operator*, Grundlehren der mathematischen Wissenschaften 273, Springer-Verlag, Berlin, 1986.

[20] C. Pommerenke, *On the Green's function of Fuchsian groups*, Ann. Acad. Sci. Fenn. Math. 2 (1976), 409–427.

[21] A. C. Ponce, *Elliptic PDEs, Measures and Capacities: From the Poisson Equation to Nonlinear Thomas–Fermi Problems*, EMS Tracts in Mathematics 23, Zürich, 2016.

[22] J. W. Roberts, *Cyclic inner functions in the Bergman spaces and weak outer functions in H^p, $0 < p < 1$*, Illinois J. Math. 29 (1985), 25–38.

[23] S. M. Shimorin, *Approximate spectral synthesis in the Bergman space*, Duke Math. J. 101 (2000), no. 1, 1–39.

[24] K. Stephenson, *Construction of an inner function in the little Bloch space*, Trans. Amer. Math. Soc. 308 (1988), no. 2, 713–720.