Shapes of infinite conformally balanced trees

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Abstract

Numerical experiments by Werness, Lee and the third author suggested that dessin d'enfants associated to large trivalent trees approximate the developed deltoid introduced by Lee, Lyubich, Makarov and Mukherjee. In this paper, we confirm this conjecture. As a side product of our techniques, we give a new proof of a theorem of Bishop which says that "true trees are dense." We also exhibit a sequence of trees whose conformally natural shapes converge to the cauliflower, the Julia set of $z \mapsto z^2 + 1/4$.

1 Introduction

A finite tree \mathcal{T} in the plane is called a *conformally balanced tree* or a *true tree* if

(TT1) Every edge has the same harmonic measure as seen from infinity.

(TT2) Harmonic measures on the two sides of every edge are identical.

Conformally balanced trees are in one-to-one correspondence with Shabat polynomials: any conformally balanced tree is the pre-image of the segment [-1, 1] under an essentially unique polynomial p with critical values ± 1 . (The polynomial p(z) is uniquely determined up to multiplication by -1.)

We say that two trees T_1, T_2 in the plane are *equivalent* if there is an orientationpreserving homeomorphism of the plane which takes T_1 onto T_2 . It is well-known that every finite tree T in the plane is equivalent to a conformally balanced tree \mathcal{T} , which is unique up to affine transformations. A proof of these facts will be sketched in Section 4.1.

It is natural to ask if infinite trees also have a natural shape. In [10], the second and third authors developed the theory of Gehring trees and showed that the Aldous continuum random tree possesses a natural conformal structure. In this paper, we consider the *infinite trivalent tree* \mathcal{T} . To come up with a natural shape for \mathcal{T} , we truncate it at level n, form the conformally balanced tree \mathcal{T}_n and take $n \to \infty$.



Figure 1: The developed deltoid and some approximating true trees.

In order for the finite trees \mathcal{T}_n to converge, we need to normalize them in some way. Throughout the rest of the paper, we use the *hydrodynamic* normalization: we ask that each conformal map $\varphi_n : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{T}_n$ has the expansion $z \to z + O(1/z)$ near infinity. Our main theorem states:

Theorem 1.1. The trees \mathcal{T}_n converge in the Hausdorff topology to an infinite trivalent tree union a Jordan curve $\mathcal{T}_{\infty} \cup \partial \Omega$. The domain Ω enclosed by $\partial \Omega$ is the developed deltoid. The Shabat polynomials p_n converge to $F \circ R^{-1}$ where F is a modular function invariant under the $(3,3,\infty)$ triangle group and $R: \mathbb{H} \to \Omega$ is the Riemann map.

The developed deltoid will be defined in Section 1.1 below. We now give an intuitive explanation of the above theorem. As $n \to \infty$, the number of edges of \mathcal{T}_n tends to infinity. Since each edge has the same harmonic measure, it is natural to expect that the lengths of the edges tend to 0. However, a quick look at Figure 1 shows that something unexpected happens: as $n \to \infty$, the edges of \mathcal{T}_n converge to real-analytic arcs instead of shrinking.

Recall that harmonic measure is the probability distribution which describes where Brownian motion, starting at infinity, first hits \mathcal{T}_n . Evidently, to make up for the fact that the harmonic measures of individual edges tend to 0, edges of high generation need to screen edges of small generation. This intuitively explains why the branches of the tree appear to close up to form horoball-like regions: for Brownian motion to hit a long edge, it needs to pass through a very narrow gate. The proof of Theorem 1.1 involves providing a rigorous explanation for this phenomenon.

The choice of truncation is important: by considering other truncations of the infinite trivalent tree, one can obtain different limit sets. In fact, in Appendix A, we will show that any compact connected set in the plane can be approximated in the Hausdorff topology by conformally balancing finite truncations of the infinite trivalent tree, thereby giving another proof of a theorem of Bishop [3].

1.1 The developed deltoid

The deltoid $\triangle \subset \mathbb{C}$ is a remarkable domain in the plane bounded by a Jordan curve with three outward pointing cusps. This curve can be described as the curve traversed by a point on a circle of radius 1/3 as it rolls around in the interior of a circle of radius 1. Alternatively, one can describe the exterior of the deltoid $\triangle_e = \hat{\mathbb{C}} \setminus \overline{\triangle}$ as the image of $\mathbb{D}_e = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ under the conformal map $z \to z + \frac{1}{2z^2}$. The exterior of the deltoid is part of a somewhat mysterious family of domains called *quadrature domains*. Quadrature domains have several equivalent definitions such as possessing a *Schwarz reflection*, which is an anti-holomorphic function σ : $\Delta_e \to \mathbb{C}$ that is the identity on $\partial \Delta_e$. For other characterizations of quadrature domains, see [1, Lemma 2.3] or [9, Lemma 3.1].

By repeatedly reflecting the deltoid in its sides, one obtains the *developed deltoid*

$$\Omega = \bigcup_{k \ge 0} \sigma^{-k}(\triangle),$$

depicted in Figure 1 above. The developed deltoid was first studied by S-Y. Lee, M. Lyubich, N. G. Makarov and S. Mukherjee [8], who showed that it fuses Fuchsian dynamics with anti-holomorphic dynamics:

Theorem 1.2. (i) The boundary of the developed deltoid $\partial\Omega$ is the unique Jordan curve that realizes the mating of the group Γ generated by the reflections in the sides of an ideal triangle and $z \to \overline{z}^2$.

(ii) The developed deltoid Ω is a John domain. In particular, $\partial \Omega$ is conformally removable.

We now define the terms in the above theorem: A bounded domain $\Omega \subset \mathbb{C}$ is a *John domain* if there is a distinguished point $z_0 \in \Omega$, called the John center, and a constant C > 0 so that every point $z \in \Omega$ can be joined to z_0 by a rectifiable curve $\gamma(t)$ so that

$$\operatorname{dist}(\gamma(t), \partial \Omega) \ge C |\gamma(t) - z|$$

for all t. A set $E \subset \hat{\mathbb{C}}$ is conformally removable if every conformal map $h : \hat{\mathbb{C}} \setminus E \to \hat{\mathbb{C}} \setminus F$ which extends continuously to the Riemann sphere is a Möbius transformation. By [6, Corollary 1], boundaries of John domains are conformally removable.

Dynamics on \mathbb{D}_e . In the exterior of the unit disk, we consider the dynamical system $z \to \overline{z}^2$.

Dynamics on \mathbb{D} . Let $\triangle_{\text{hyp}} \subset \mathbb{D}$ be the ideal triangle in the unit disk with vertices at $1, \omega, \omega^2$, where $\omega = e^{2\pi i/3}$ is a third root of unity. Consider the group

 $\Gamma = \langle R_{s_1}, R_{s_2}, R_{s_3} \rangle \subset \operatorname{Aut}(\mathbb{D})$ generated by the reflections in the sides s_1, s_2, s_3 of Δ_{hyp} . The images

$$\{\gamma(\triangle_{hyp}): \gamma \in \Gamma\}$$

tessellate the unit disk. The Markov map $\rho : \mathbb{D} \setminus \triangle_{\text{hyp}} \to \mathbb{D}$ is defined as R_{s_1} on the (hyperbolic) half-plane cut off by s_1 , R_{s_2} on the half-plane cut off by s_2 and R_{s_3} on the half-plane cut off by s_3 .

What it means to be a mating. A Jordan curve $\partial\Omega$ is a mating of $z \to \overline{z}^2$ and Γ if there exist conformal maps $\varphi : \mathbb{D} \to \Omega, \ \psi : \mathbb{D}_e \to \Omega_e$ that glue the dynamical systems together, i.e. $\varphi \circ \rho \circ \varphi^{-1} = \psi \circ \overline{z}^2 \circ \psi^{-1}$ on $\partial\Omega$. In particular, this implies that

$$\sigma(z) = \begin{cases} \psi \circ \overline{z}^2 \circ \psi^{-1}, & z \in \overline{\Omega_e} \\ \varphi \circ \rho \circ \varphi^{-1}, & z \in \overline{\Omega} \setminus \varphi(\triangle_{\text{hyp}}) \end{cases}$$

is a Schwarz reflection for $\hat{\mathbb{C}} \setminus \varphi(\Delta_{hyp})$, and hence $\hat{\mathbb{C}} \setminus \varphi(\Delta_{hyp})$ is a quadrature domain.

1.2 Strategy of proof

The proof of Theorem 1.1 proceeds in four steps:

Step 1. We first show that any subsequential limit of the true trees \mathcal{T}_n in the Hausdorff topology is homeomorphic to an infinite trivalent tree union a Jordan curve $\mathcal{T}_{\infty} \cup \partial \Omega$, with $\mathcal{T}_{\infty} \subset \Omega$. Among our key tools are estimates for the diameters of edges by means of conformal modulus estimates of certain curve families. A notable difference to the setting of random trees is that in the truncated trivalent tree, the diameters of a fixed edge do not shrink to zero as $n \to \infty$. This step will be carried out in Section 5.

Step 2. We then show that any subsequential limit $\partial\Omega$ realizes the mating of $z \to \overline{z}^2$ and the Fuchsian reflection group Γ from Theorem 1.2. The main effort is to identify the Farey structure inside Ω and the $z \to \overline{z}^2$ structure outside Ω . This step will be carried out in Section 6.

At this point, to show the uniqueness of the subsequential limit, one can appeal to Theorem 1.2 by Lee, Lyubich, Makarov and Mukerjee which implies that the developed deltoid is conformally removable. We prefer not to rely on Theorem 1.2 because the proof given in [8] uses the dynamical nature of the developed deltoid in a crucial way. If one slightly changes the tree, then the resulting object will no longer carry an anti-conformal dynamical system.

Step 3. To show that the limit of the \mathcal{T}_n does not depend on the subsequence, we prove "partial conformal removability." Partial conformal removability is a much less stringent property than full conformal removability and it is easier to check. In essence, it asks that if $h : \hat{\mathbb{C}} \setminus E \to \hat{\mathbb{C}} \setminus F$ is a conformal map (which extends continuously to the Riemann sphere) onto the complement of a set F which has roughly the same geometry as E, then h is a Möbius transformation. This step will be carried out in Sections 5.4 and 6.4.

Step 4. Finally in Section 6.5, we will show that the Shabat polynomials converge and identify the limit as a modular function invariant under the $(3, 3, \infty)$ triangle group.

1.3 Acknowledgements

In 2014, Brent Werness (oral communication) proposed to study the natural shape of the infinite trivalent tree and posed the question if the diameters of the edges of the truncated trivalent tree \mathcal{T}_n tend to zero as n tends to infinity. During a visit of Seung-Yeop Lee to Seattle in 2015, Brent, Seung-Yeop and the third author performed computer experiments and observed the similarity between the trees and the developed deltoid, leading to the conjecture regarding their convergence. We are grateful to Brent and Seung-Yeop for our discussions and their contributions.

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2 Preliminaries

In this section, we gather a number of useful facts that will be used throughout this paper, dealing with moduli of curve families, convergence of Riemann maps, relative harmonic measure and weak conformal removability.

2.1 Moduli of annuli and rectangles

It is well known that any doubly-connected domain $\mathbf{A} \subset \mathbb{C}$ can be mapped onto a round annulus $A(0; r, R) = \{z : r < |z| < R\}$. The number Mod $\mathbf{A} := \frac{1}{2\pi} \log \frac{R}{r}$ is called the *modulus* of \mathbf{A} . Two doubly-connected domains are conformally equivalent if and only if their moduli coincide.

A metric $\rho(z)$ is a non-negative measurable function defined on a domain $\Omega \subset \mathbb{C}$. One can use $\rho(z)$ to measure lengths of rectifiable curves

$$\ell_{\rho}(\gamma) = \int_{\gamma} \rho(z) |dz|$$

and compute areas of shapes, for instance the total area of ρ is given by

$$A(\rho) = \int_{\Omega} \rho(z)^2 |dz|^2$$

The metric ρ is said to be *admissible* for a family of rectifiable curves Γ contained in Ω if the ρ -length of every curve $\gamma \in \Gamma$ is at least 1. The *modulus* of the curve family Γ is defined as

$$\operatorname{Mod} \Gamma := \inf_{\rho} A(\rho),$$

where the infimum is taken over all admissible metrics ρ . If one finds a conformal metric ρ such that $\ell_{\rho}(\gamma) \geq L$ for any $\gamma \in \Gamma$, then Mod $\Gamma \leq A(\rho)/L^2$.

The modulus of a doubly-connected domain is a special case of the above construction: Mod **A** is equal to the modulus of the family of curves $\Gamma_{\circlearrowright}$ that separate the two boundary components, while 1/ Mod **A** is equal to modulus of the family Γ_{\uparrow} of curves that connect the opposite boundary components of **A**. Thus one uses $\Gamma_{\circlearrowright}$ to give upper bounds for Mod **A** while one uses Γ_{\uparrow} go give lower bounds for Mod **A**.

We will frequently use the following two simple rules for modulus, which easily follow from the definitions, e.g. see [2, Chapter 4] or [4, Chapter IV.3]:

- 1. (Monotonicity rule) If $\mathbf{A}_1 \subset \mathbf{A}$ is an essential doubly-connected subdomain, so that $\Gamma_{\circlearrowright}(\mathbf{A}_1) \subset \Gamma_{\circlearrowright}(\mathbf{A})$, then $\operatorname{Mod} \mathbf{A}_1 \leq \operatorname{Mod} \mathbf{A}$.
- 2. (Parallel rule) If a doubly-connected domain $\mathbf{A} = \mathbf{A}_1 \cup \mathbf{A}_2$ can be represented as a union of two essential doubly-connected domains, then

$$\operatorname{Mod} \mathbf{A}_1 + \operatorname{Mod} \mathbf{A}_2 \leq \operatorname{Mod} \mathbf{A}_2$$

We will also use the following standard estimates which essentially go back to Loewner and express the fact that \mathbb{C} is a Loewner space, e.g. see [5, Theorem 8.2]:

Lemma 2.1. Let Ω be a simply-connected domain in the plane.

(a) Suppose F is a compact connected set contained in Ω . If $Mod(\Omega \setminus F) \ge m$ is bounded from below, then

$$\operatorname{dist}(\partial\Omega, F) \ge c \operatorname{diam} F,$$

for some c > 0 which depends only on m > 0. Furthermore, $c \to \infty$ as $m \to \infty$. Conversely, if dist $(\partial \Omega, F) \ge c$ diam F, then $Mod(\Omega \setminus F) \ge m(c)$.

(b) Suppose $E \subset F$ are two compact connected sets contained in Ω . If

$$m_1 \leq \operatorname{Mod}(\Omega \setminus F) \leq \operatorname{Mod}(\Omega \setminus E) \leq m_2$$

then diam $E \simeq \text{diam } F$. In fact, there exists a constant $C = C(m_1, m_2) > 1$ so that $F \subset B(e, C \cdot \text{diam } E)$ for any point $e \in E$, where B(x, r) denotes the ball of radius r centered at x.

A conformal rectangle \mathbf{R} is a simply connected domain with four marked prime ends z_1, z_2, z_3, z_4 . In this paper, all conformal rectangles will be *marked*, i.e. equipped with a distinguished pair of opposite sides. The Schwarz-Cristoffel formula provides a conformal map from \mathbf{R} onto a geometric rectangle $[0, m] \times [0, 1]$. If one insists that the marked sides of \mathbf{R} are mapped onto the vertical sides of $[0, m] \times [0, 1]$, then the number $m \in (0, \infty)$ is determined uniquely. The number $m := \text{Mod } \mathbf{R}$ is known as the modulus of \mathbf{R} and is equal to the modulus of the curve family Γ_{\uparrow} which separates the distinguished pair of opposite sides.

For further properties of conformal modulus, we refer the reader to [4, Chapter 4] and [12, Chapter 2].

2.2 On convergence of Jordan domains

We say that a sequence of Jordan curves γ_n converges *strongly* to a Jordan curve γ , if there exists a sequence of homeomorphism $h_n : \partial \mathbb{D} \to \gamma_n$ which converges to a homeomorphism $h : \partial \mathbb{D} \to \gamma$. One can similarly define strong convergence for Jordan arcs. The following lemma provides useful intuition, although we will only use the equivalence of (1) and (3) in this paper.

Lemma 2.2. Suppose γ_n is a sequence of Jordan curves which separate 0 and ∞ in $\hat{\mathbb{C}}$. Assume that the γ_n converge in the Hausdorff topology to another Jordan curve γ that separates 0 and ∞ in $\hat{\mathbb{C}}$. For each $n = 1, 2, ..., let \varphi_n : \mathbb{D} \to \text{Interior}(\gamma_n)$ be the conformal map with $\varphi_n(0) = 0$ and $\varphi'_n(0) > 0$. Similarly, let $\varphi : \mathbb{D} \to \text{Interior}(\gamma)$ be the conformal map with $\varphi(0) = 0$ and $\varphi'(0) > 0$. The following statements are equivalent:

- (1) The curves γ_n converge strongly to γ .
- (2) The curves γ_n converge to γ without backtracking.
- (3) The conformal maps $\varphi_n \to \varphi$ converge uniformly on the closed unit disk $\overline{\mathbb{D}}$.

We now explain the no backtracking condition. As is standard in complex analysis, we orient the Jordan curves γ_n and γ counter-clockwise. We say that γ_n converges to γ with *backtracking* if after passing to a subsequence, there exist two sets of arcs $\alpha_n^1, \alpha_n^2 : [0, 1] \to \gamma_n$ that have the same orientation as γ_n and converge in the Hausdorff topology to the same arc $\alpha = [a, b] \subset \gamma$ so that

$$\begin{aligned} \alpha_n^1(0) &\to a, \qquad \alpha_n^1(1) \to b, \\ \alpha_n^2(0) &\to b, \qquad \alpha_n^2(1) \to a. \end{aligned}$$

In other words, the arc α_n^1 passes by α in one direction, while α_n^2 passes by α in the other direction.

The equivalence of (1) and (2) is elementary and is left as an exercise to the reader. For $(1) \Rightarrow (3)$, we refer the reader to [14, Theorem 2.11]. The direction $(3) \Rightarrow (1)$ is trivial.

2.3 Relative harmonic measure

Suppose that U is a Jordan domain and $p \in \partial U$. While it does not make sense to talk about the harmonic measure of an arc $I \subset \partial U$ as viewed from p, one can talk about the *relative harmonic measure* of two arcs $I, J \subset \partial U$ that do not contain p:

$$\omega_{U,p}(I,J) = \lim_{z \to p} \frac{\omega_{U,z}(I)}{\omega_{U,z}(J)}.$$

From the definition, it is clear that relative harmonic measure is a conformal invariant: if $\varphi : U \to U'$ is a conformal map onto another Jordan domain, then $\omega_{U',p'}(I',J') = \omega_{U,p}(I,J)$, where $p' = \varphi(p)$, $I' = \varphi(I)$ and $J' = \varphi(J)$.

Example. When $U = \mathbb{H}$, $p = \infty$ and J = [0, 1], the relative harmonic measure $\omega_{\mathbb{H},\infty}(\cdot, [0, 1])$ is just Lebesgue measure on the real line.

The following lemma says that the quantity $\omega_{U,p}(I, J)$ varies continuously provided that p stays away from $I \cup J$:

Lemma 2.3. If a sequence of Jordan quadruples (U_n, p_n, I_n, J_n) converges strongly to a Jordan quadruple (U, p, I, J), then

$$\omega_{U_n,p_n}(I_n,J_n) = \lim_{n \to \infty} \omega_{U,p}(I,J).$$

Proof. According to the definition of strong convergence in Section 2.2, there exist parameterizations of ∂U_n which converge to a parameterization of ∂U . Lemma 2.2 implies that a sequence of conformal maps $R_n : \mathbb{D} \to U_n$ converges to a conformal map $R : \mathbb{D} \to U$ uniformly on the closed unit disk. The lemma now follows from the conformal invariance of relative harmonic measure.

2.4 Weak conformal removability

Lemma 2.4. Suppose X and X' are two compact sets in the complex plane and

$$\varphi: \hat{\mathbb{C}} \setminus X \to \hat{\mathbb{C}} \setminus X'$$

is a conformal map that extends continuously to a homeomorphism of the sphere. Assume that there is a countable exceptional set $E \subset X$ and a countable collection of closed subsets s_1, s_2, \ldots of X, called shadows, such that every point in $X \setminus E$ belongs to infinitely many sets s_i . If

$$\sum_{i=1}^{\infty} \operatorname{diam}^2 s_i < \infty, \qquad \sum_{i=1}^{\infty} \operatorname{diam}^2 \varphi(s_i) < \infty, \tag{2.1}$$

then φ is a Möbius transformation.

The term shadow is inspired by the following construction: let $\Omega \subset \mathbb{C}$ be a Jordan domain in the plane and z_0 be a point in Ω . Given a set $K \subset \Omega \setminus \{z_0\}$, the shadow s(K) is the union of the endpoints of hyperbolic geodesic rays emanating from z_0 that pass through K.

In a beautiful work, P. Jones and S. Smirnov [6] showed that when the sets s_i are shadows cast by Carleson boxes, then the first condition $\sum_{i=1}^{\infty} \operatorname{diam}^2 s_i < \infty$ is already sufficient for φ to extend conformally to the Riemann sphere, the second assumption $\sum_{i=1}^{\infty} \operatorname{diam}^2 \varphi(s_i) < \infty$ is not needed.

The lemma above can be viewed as a slight variation of [6, Proposition 1] and the proof below follows the argument in [6] very closely. On one hand, the assumptions of Lemma 2.4 are more general since we allow the s_i to be arbitrary closed sets; however, we get a weaker conclusion because we also impose a restriction on the images of the shadows.

Proof. For convenience, we write $s'_i = \varphi(s_i)$. Since φ is a homeomorphism, all but countably many points in X' are covered by infinitely many shadows s'_i . By condition (2.1), X and X' have 2-dimensional Lebesgue measure 0.

Call a direction $v \mod i$ for almost every line ℓ pointing in the direction of v, the set $\varphi(\ell \cap X)$ has linear Lebesgue measure 0. One says that φ is absolutely continuous on lines (ACL) if the directions parallel to the coordinate axes are good. It is well known that if X has 2-dimensional Lebesgue measure zero and $\varphi \in W_{\text{loc}}^{1,2}(\mathbb{C} \setminus X)$ is ACL, then $\varphi \in W_{\text{loc}}^{1,2}(\mathbb{C})$. Weyl's lemma then guarantees that φ is conformal on the Riemann sphere, and therefore, a Möbius transformation. Below, we will show that every direction is good.

Instead of showing that a set has zero 1-dimensional Lebesgue measure m_1 , we may instead show that it has zero 1-dimensional content m_1^{∞} . The definition of 1dimensional content is similar to that of 1-dimensional measure, but allows covers by balls of arbitrary size. Therefore, the lemma reduces to showing that for almost every line ℓ parallel to a given direction v, the 1-dimensional content of $\varphi(\ell \cap X)$ is 0.

Since E is countable, almost every line ℓ parallel to v misses E. For such a line,

$$m_1^{\infty}(\varphi(\ell \cap X)) \le \sum_{s_i \cap \ell \neq \emptyset, \, i > N} \operatorname{diam} s'_i. \tag{2.2}$$

The last equation holds for any $N \ge 1$ since any point in $X \setminus E$ is contained in infinitely many shadows, which allows us to avoid putting the first N - 1 shadows in the cover. In other words,

$$\sum_{s_i \cap \ell \neq \emptyset} \operatorname{diam} s'_i < \infty \quad \Longrightarrow \quad m_1^{\infty}(\varphi(\ell \cap X)) = 0.$$
(2.3)

As

$$\begin{split} \int_{\ell||v} \left\{ \sum_{s_i \cap \ell \neq \emptyset} \operatorname{diam} s'_i \right\} d\ell &\leq \sum_{i=1}^{\infty} \operatorname{diam} s_i \cdot \operatorname{diam} s'_i \\ &\leq \frac{1}{2} \left(\sum_{i=1}^{\infty} \operatorname{diam}^2 s_i + \operatorname{diam}^2 s'_i \right) \\ &< \infty, \end{split}$$

the integrand must be finite for a.e. ℓ parallel to v. In the first inequality above, we used that the summand diam s'_i participates in a set of lines of linear measure at most diam s_i . This completes the proof.

3 The Farey tesellation

Let $\triangle_{\text{hyp}} \subset \mathbb{D}$ be the ideal triangle in the unit disk with vertices 1, $\omega = e^{2\pi i/3}$ and $\overline{\omega} = e^{4\pi i/3}$. Repeatedly reflecting \triangle_{hyp} in its sides, one obtains a tessellation of the unit disk by ideal triangles. The dual graph (which joins centers of the triangles by hyperbolic geodesics) is called the *Farey tree*. We designate the center of \triangle_{hyp} as the root vertex. Throughout this paper, we view the Farey tree as an explicit embedded tree in the plane, depicted in Figure 2 below.



Figure 2: The Farey tesselation and Farey tree

The Farey tree partitions the unit disk into regions which we call *Farey horoballs*. We will usually index Farey horoballs by the point $p \in \partial \mathbb{D}$ where they touch the unit circle. We refer to p as the *cusp* of H_p . If H_p is not one of the three Farey horoballs that contain the root vertex, we can also label $H_p = H_v$ by the vertex v of the tree which is closest to the origin.

3.1 Diameters of triangles

Each non-root triangle \triangle can be labeled by a digit 1, 2, 3 followed by a finite sequence of *L*'s and *R*'s, which indicates the path one travels from \triangle_{hyp} to \triangle . For example, in the word

$$2\underbrace{L}_{k_1=1}\underbrace{R}_{k_2=1}\underbrace{LLL}_{k_3=3}\underbrace{RR}_{k_4=2}\underbrace{LL}_{k_5=2}\underbrace{R}_{k_6=1}\underbrace{LLLLL}_{k_7=5}\underbrace{RR}_{k_8=2},$$

the digit 2 indicates that we start by walking along the dual tree from the root vertex to its second child. After the first step, each vertex has two children and we have to decide whether to turn left or right. The options are indicated by 'L' and 'R' respectively.

Lemma 3.1. For a non-root triangle \triangle in the Farey tessellation,

$$\log \frac{1}{\operatorname{diam} \bigtriangleup} \asymp \sum_{i=1}^{m} \log(1+k_i).$$

Proof. It is easier and clearly equivalent to work in the upper half plane \mathbb{H} where \triangle_{hyp} has vertices 0, 1 and ∞ and in the first step, we walk down. Let

$$\triangle_0 = \triangle_{\text{hyp}}, \ \triangle_1 = (0, 1/2, 1), \ \triangle_2, \ \dots, \ \triangle_n = \triangle$$

be the sequence of triangles from \triangle_{hyp} to \triangle . Each triangle \triangle_j in this sequence has three vertices on the real axis $a_j < b_j < c_j$. To estimate diam \triangle_j , we keep track of the ratio

$$r(\Delta_j) := \frac{b_j - a_j}{c_j - a_j},$$

which measures the distortion of the triangle Δ_j . Each time we do an right turn after a left turn or vice versa, the ratio is "reset" to a value in [1/3, 2/3]. After a series of k consecutive left turns, $r \approx 1/k$, while after a series of k consecutive right turns, $1 - r \approx 1/k$.

After making k left or right turns in a row, the diameter goes down by a factor of roughly k + 1: for $1 \le k \le k_{j+1}$,

$$\log \frac{1}{\operatorname{diam} \triangle_{k_1+k_2+\dots+k_j+k}} - \log \frac{1}{\operatorname{diam} \triangle_{k_1+k_2+\dots+k_j+1}} \asymp \log(k+1).$$

When we make a right turn after a series of k_j left turns (or a left turn after a series of k_j left turns), the diameter goes down by a factor of $k_j + 1$, i.e.

$$\log \frac{1}{\operatorname{diam} \triangle_{k_1+k_2+\dots+k_j+1}} - \log \frac{1}{\operatorname{diam} \triangle_{k_1+k_2+\dots+k_j}} \asymp \log(k_j+1).$$

The above equations give the desired bound for diam \triangle .

3.2 Topology of the Farey tree

We write $d_{\mathcal{F}}(\cdot, \cdot)$ for the combinatorial distance between two vertices in the Farey tree. By a *branch* of the Farey tree, we mean an infinite sequence of vertices

 $[v_0, v_1, v_2, v_3, \ldots]$ with $d_{\mathcal{F}}(v_m, v_{\text{root}}) = m$. We can also label a branch of \mathcal{F} by a digit 1, 2, 3 followed by an infinite sequence of L's and R's which encodes the directions from the root vertex.

Lemma 3.2. (i) Any branch $\mathbf{v} = [v_0, v_1, v_2, v_3, ...]$ has a well-defined limit point: $p_{\mathbf{v}} = \lim_{m \to \infty} v_m$ exists and belongs to the unit circle.

(ii) The set of limit points of all branches of the Farey tree is the whole unit circle.

(iii) Two distinct branches $\mathbf{v} = [v_0, v_1, v_2, v_3, ...]$ and $\mathbf{w} = [w_0, w_1, w_2, w_3, ...]$ have the same limit point on the unit circle if and only if $\mathbf{v} = XLR^{\infty}$ and $\mathbf{w} = XRL^{\infty}$ or vice versa. In other words, \mathbf{v} and \mathbf{w} follow the same initial word X and then \mathbf{v} takes a right turn and infinitely many left turns, while \mathbf{w} takes a left turn and infinitely many right turns.

3.3 Shadows in the Farey tree

We denote the subtree which consists of $v \in \mathcal{F}$ and its descendants by $\mathcal{F}(v)$. To a non-root vertex $v \in \mathcal{F}$, we associate the shadow

$$s_v = \overline{\mathcal{F}(vLR) \cup \mathcal{F}(vRL)} \cap \partial \mathbb{D}.$$

It is not difficult to see that every point on the unit circle, which is not a cusp of one of the horoballs, is contained in infinitely many shadows s_v .

Lemma 3.3. For a non-root vertex $v \in \mathcal{F}$, we have

diam
$$s_v \simeq \operatorname{diam} H_v \simeq 1 - |v|$$
.

There exists a constant c > 0, independent of v, so that the Farey horoball H_v contains a Euclidean ball of radius c(1 - |v|).

Remark. At first glance, it may seem more natural to define s_v as $\overline{\mathcal{F}(v)} \cap \partial \mathbb{D}$. However, if the path from the root vertex to v ends on a lot of left or right turns, then diam $\mathcal{F}(v)$ will be a lot larger than 1 - |v|.

4 Background on true trees

In this section, we discuss the relation between true trees and Shabat polynomials, i.e. polynomials with critical values ± 1 . We then describe the local geometry of true trees whose vertices have bounded valence. Finally, we define the notions of shortcuts and obstacles, which will help in providing moduli estimates to control the global geometry of true trees.

4.1 True trees and Shabat polynomials

Let $T \subset \mathbb{C}$ be a finite tree in the plane. To find a conformally balanced tree \mathcal{T} with the same combinatorics as T, we label the sides of edges of T in counter-clockwise order: $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_{2N}}$. For each half-edge $\vec{e_i}$, we form an equilateral triangle $\Delta(\vec{e_i}, \infty)$ with unit-length sides, one of which is labelled $\vec{e_i}$ and the vertex opposite this side is labelled ∞ .

We first glue these equilateral triangles together to form a 2N-gon \mathbf{D}_{2N} with sides labeled counter-clockwise by \vec{e}_i , and central vertex labeled ∞ . We then glue \vec{e}_i with \vec{e}_j whenever \vec{e}_i, \vec{e}_j are opposite sides of the same edge $e \in T$, to obtain a topological sphere $X_T = \mathbf{D}_{2N}/\sim$ which has a flat structure except at the cone points located at the vertices of the triangles. Uniformizing $X_T \cong \hat{\mathbb{C}}$ so that the central vertex of \mathbf{D}_{2N} is placed at $\infty \in \hat{\mathbb{C}}$ produces the desired tree $\mathcal{T} \subset \hat{\mathbb{C}}$.

Remark. The conformally balanced tree \mathcal{T} resulting from the above construction is uniquely defined up to an affine transformation $z \to Az + B$. To obtain a unique tree, we will usually normalize \mathcal{T} so that the conformal map $\varphi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{T}$ has the expansion $z \to z + O(1/z)$ near infinity.

Given a conformally balanced tree \mathcal{T} , we construct a polynomial p(z) with critical values ± 1 such that $\mathcal{T} = p^{-1}([-1,1])$. For this purpose, we colour each triangle $\triangle(\vec{e}_i, \infty) \subset \hat{\mathbb{C}} \setminus \mathcal{T}$ black or white, so that adjacent triangles have opposite colours. On each black triangle $\triangle(\vec{e}_i, \infty)$, we define p(z) to be the conformal map onto the upper half-plane \mathbb{H} which takes $\vec{e}_i \to [-1, 1]$ and $\infty \to \infty$. Similarly, on each white triangle $\triangle(\vec{e}_i, \infty)$, we define p to be the conformal map onto the lower half-plane \mathbb{L} which takes $\vec{e}_i \to [-1, 1]$ and $\infty \to \infty$. Since \mathcal{T} is a true tree, p extends to a continuous function on the Riemann sphere. As \mathcal{T} is made up of real-analytic arcs, p is meromorphic on the Riemann sphere, and hence a rational function. As the only pole of p is at infinity, it is a polynomial. Finally, since p is N : 1 at infinity, p is a polynomial of degree N. From the construction, it is readily seen that p has critical values ± 1 and $\mathcal{T} = p^{-1}([-1, 1])$.

To define the Shabat polynomial uniquely, we need to specify which vertices are sent to +1 and which vertices are sent to -1. A different choice would correspond to multiplying p(z) by -1. If \mathcal{T} has a distinguished vertex v_{root} , then it is natural to choose the Shabat polynomial so that $p(v_{\text{root}}) = 1$.

4.2 Trees of bounded valence

We now present some general results on true trees whose vertices have bounded valence. We write $d_{\mathcal{T}}(v, w)$ for the graph distance between v and w.

Lemma 4.1. Let $d \ge 2$ be an integer. Suppose $e = \overline{v_1 v_2}$ is an edge in a true tree \mathcal{T} with deg $v_1 \le d$ and deg $v_2 \le d$. There is a simply connected neighbourhood $U \supset e$ with $Mod(U \setminus e) \ge m(d)$ such that only edges adjacent to e can intersect U.

Lemma 4.2. Fix an integer $d \ge 2$. Let v be a vertex of a conformally balanced tree \mathcal{T} . If the degrees of all vertices in $\{w : d_{\mathcal{T}}(v, w) \le 1\}$ are $\le d$, then the diameters of the edges $\overline{vv_i}$ emanating from v are comparable (with the comparison constant depending on d).

The proofs use the concept of a *star* of a vertex in a true tree. For a vertex v of \mathcal{T} , we define \star_v as the union of the triangles $\triangle(\vec{e}, \infty)$ that contain v. We enumerate the $2 \deg v$ triangles in \star_v counter-clockwise: $\triangle_1, \triangle_2, \ldots, \triangle_{2 \deg v}$.

Now, decompose the unit disk \mathbb{D} into $2 \deg v$ sectors $\sigma_1, \sigma_2, \ldots, \sigma_{2 \deg v}$ using $2 \deg v$ equally-spaced radial rays. For each $i = 1, 2, \ldots, 2 \deg v$, let ψ_i be the conformal map from σ_i to Δ_i which takes vertices to vertices, with 0 mapping to v. By Carathéodory's theorem, ψ_i extends to a homeomorphism of the closed regions. Since \mathcal{T} is a true tree, the maps ψ_i agree on the radial boundaries of the sectors σ_i and glue together to form a continuous map ψ_v on the unit disk. As the union of finitely many radial rays is conformally removable, ψ_v defines a conformal map from \mathbb{D} to \star_v .

On an edge $e_i = \overline{vv_i}$ of \mathcal{T} emanating from v, we mark the points a_i, b_i such that the segments $\overline{va_i}, \overline{a_ib_i}, \overline{b_iv_i}$ have equal length in the equilateral triangle model of $\triangle(\vec{e_i}, \infty)$, i.e. in D_{2N}/\sim . Note that the points a_i, b_i do not depend on which one of the two sides of e_i is used.

Applying Koebe's distortion theorem to ψ_v tells us that the diameters of the $2 \deg v$ segments

$$\left\{\overline{va_i}, \ \overline{a_ib_i} : i = 1, 2, \dots, \deg(v)\right\}$$

are comparable. By considering stars centered at the neighbouring vertices v_i , we see that

$$\{\overline{a_i b_i}, \overline{b_i v_i} : i = 1, 2, \dots, \deg(v)\}.$$

are also comparable. Putting these estimates together proves Lemma 4.2.

Lemma 4.1 follows from Lemma 2.1 (a) after applying Koebe's distortion theorem to ψ_{v_1} and ψ_{v_2} . Similar reasoning shows:

Lemma 4.3. Suppose $\{\mathcal{T}_n\}_{n=0}^{\infty}$ is an infinite sequence of hydrodynamically-normalized conformally balanced trees whose vertices have uniformly bounded degrees. Any subsequential Hausdorff limit of a sequence of edges $e^{(n)} \subset \mathcal{T}_n$ is either a point or a real-analytic arc. In the latter case, the convergence is in the strong topology.

Proof. Suppose the edge $e^{(n)}$ connects the vertices $v_1^{(n)}$ and $v_2^{(n)}$. As above, we mark the points $a^{(n)}$ and $b^{(n)}$ which trisect the edge $e^{(n)}$. We pass to a subsequence so that the maps $\psi_{v_1}^{(n)}$ and $\psi_{v_2}^{(n)}$ converge uniformly on compact subsets of the unit disk.

If the limiting maps $\psi_{v_1} = \lim_{n \to \infty} \psi_{v_1}^{(n)}$ and $\psi_{v_2} = \lim_{n \to \infty} \psi_{v_2}^{(n)}$ are constant, then the edges $e^{(n)}$ collapse to a point. Otherwise, the limiting edge $e = \lim e^{(n)}$ is covered by two compatible real-analytic arcs $\overline{v_1 b} = \lim_{n \to \infty} \overline{v_1^{(n)} b^{(n)}}$ and $\overline{av_2} = \lim_{n \to \infty} \overline{a^{(n)} v_2^{(n)}}$, and the conformal maps provide the requisite uniformly converging parametrizations.

4.3 Shortcuts and obstacles

Let \mathcal{T} be a conformally balanced tree in the plane, normalized so that the Riemann map $\varphi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{T}$ satisfies $\varphi(z) = z + O(1/z)$ as $z \to \infty$.

To control the geometry of \mathcal{T} , we estimate conformal moduli of various path families Γ contained in doubly-connected domains $\mathbf{A} \subset \mathbb{C}$. An instructive example is the family of closed curves surrounding an edge of the tree, which will be discussed in detail in Section 4.4.

Since we will only estimate moduli in the setting of finite balanced trees, we will not have to worry about the possibility that the area of \mathcal{T} might be positive.

By conformal invariance, we may estimate the modulus in any of the three conformally equivalent models $\hat{\mathbb{C}} \setminus \mathcal{T}$, \mathbf{D}_{2N}/\sim or $(\hat{\mathbb{C}} \setminus \mathbb{D})/\sim$. In the latter model, the equivalence relation on $\partial \mathbb{D}$ is given by the identifications of φ and the family $\varphi^{-1}(\Gamma)$ consists of sets $\varphi^{-1}(\gamma)$ that may be disconnected: if a curve $\gamma \in \Gamma$ crosses an edge $e \in \mathcal{T}$, then $\varphi^{-1}(\gamma)$ enters one side of $\varphi^{-1}(e)$, teleports through the identification provided by φ , and exits on the other side of $\varphi^{-1}(e)$.

We will construct admissible metrics of the form

$$\rho = \alpha_0 \Big(\rho_0 + \sum_{e \in \mathcal{T}} \alpha_e \rho_e \Big), \tag{4.1}$$

where the *background metric* $\rho_0 = \mathbf{1}_{\varphi^{-1}(\mathbf{A})}$ serves the purpose of controlling the length of curves γ that do not intersect \mathcal{T} , while the *obstacles* ρ_e have the purpose of penalizing teleportation so that shortcuts are not worthwhile. The constant α_0 is chosen so that curves that do not intersect \mathcal{T} have length ≥ 1 under $\alpha_0 \rho_0$.

We build the obstacles ρ_e so that they assign length ≥ 1 to all curves γ that intersect e (and are not confined to the union of the triangles that are incident to e). It is easiest to describe the construction in D_{2N}/\sim , which is a surface composed of 2N equilateral triangles $\Delta(\vec{e}_i, \infty)$ of side length 1: namely, we define ρ_e as three times the characteristic function of the 1/3-neighborhood of e in the flat metric, i.e.

$$\rho_e = 3 \times \mathbf{1}_{B_{1/3}(e)},$$

where $B_{1/3}(e)$ is the set of points of distance at most 1/3 from e.

We denote the conformal transport of this metric to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ again by ρ_e . Since any point $z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ can be in the support of at most $D = \max_{v \in \mathcal{T}} \deg(v)$ obstacles, it can be in at most D + 1 of the sets $\operatorname{supp} \rho_0 \cup {\operatorname{supp} \rho_e}$, and the area of ρ can be estimated by

$$A(\rho) = \alpha_0^2 \int_{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}} \left(\rho_0(z) + \sum_{e \in \mathcal{T}} \alpha_e \rho_e(z) \right)^2 |dz|^2$$

$$\leq (D+1)^2 \alpha_0^2 \int_{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}} \left(\rho_0(z)^2 + \sum_{e \in \mathcal{T}} \alpha_e^2 \cdot \rho_e(z)^2 \right) |dz|^2$$

$$\lesssim \alpha_0^2 \left(A(\rho_0) + \sum_{e \in \mathcal{T}} \alpha_e^2 \right), \qquad (4.2)$$

where in the second step, we used the elementary inequality $(\sum_{i=1}^{n} x_i)^2 \leq n(\sum_{i=1}^{n} x_i^2)$, and in the third step, we absorbed $(D+1)^2$ into the implicit constant.

For an edge e in \mathcal{T} , we denote by $\mathcal{T}(e) \subset \mathcal{T}$ the subtree consisting of the edge e and its descendants (as measured from the root vertex). It is easy to see that $S(e) = \varphi^{-1}(\mathcal{T}(e))$ is an arc in the unit circle $\partial \mathbb{D}$. We define the *outer shortcut* of e as the Euclidean length of S(e):

$$s(e) = \text{length}(S(e)) = 2\pi \cdot \omega_{\hat{\mathbb{C}} \setminus \mathcal{T},\infty}(S(e)).$$

We define $\mathcal{T}^{-}(e) = \mathcal{T}(e) \setminus e$ as the union of all the descendants of e. Naturally, we define the *inner shortcut* of e as

$$s^{-}(e) = \operatorname{length}(S^{-}(e)) = 2\pi \cdot \omega_{\hat{\mathbb{C}} \setminus \mathcal{T},\infty}(S^{-}(e)),$$

where $S^{-}(e) = \varphi^{-1}(\mathcal{T}^{-}(e))$. Unless *e* is a boundary edge, the difference between the outer and inner shortcuts $s(e) - s^{-}(e) = \pi/N$ is not significant.

4.4 A lower bound for the diameters of edges

Let \mathcal{T} be a true tree and $\Omega_2 \subset \mathbb{C}$ be the simply-connected domain bounded by the equipotential curve $\varphi(\{z : |z| = 2\})$. The hydrodynamic normalization of the conformal map φ implies that diam $\mathcal{T} \geq c_0 > 0$ is bounded from below by a universal constant (the sharp value $c_0 = 2$ is irrelevant for our purpose). In view of Lemma 2.1, to give a lower bound for the diameter of an edge e_0 in \mathcal{T} , it is enough to give an upper bound for the modulus of the family of curves $\Gamma_{\bigcirc}(\mathbf{A})$ that separate the boundary components of $\mathbf{A} = \Omega_2 \setminus e_0$. Denoting $A(0; 1, 2) = \{z : 1 < |z| < 2\}$, we will now show that the metric

$$\rho = \frac{1}{s^{-}(e_0)} \left(\mathbf{1}_{A(0;1,2)} + \sum_{e \in \mathcal{T}^{-}(e_0)} s(e) \rho_e \right)$$

is admissible for $\varphi^{-1}(\Gamma_{\circlearrowright}(\mathbf{A}))$, where the summation is over the descendants of e_0 .

Consider a curve $\gamma \in \Gamma_{\bigcirc}(\mathbf{A})$. If we pull γ back by φ^{-1} , we get a path in the annulus A(0; 1, 2) which may teleport from $x \in \partial \mathbb{D}$ to $y \in \partial \mathbb{D}$ if $\varphi(x) = \varphi(y) \in \mathcal{T} \setminus e_0$. Below, we denote the radial projection on the unit circle by $\pi_{\mathrm{rad}}(z) = z/|z|$. If γ does not pass through any edge in $\mathcal{T}^{-}(e_0)$, then $\pi_{\mathrm{rad}}(\varphi^{-1}(\gamma))$ contains $S^{-}(e_0)$ and the metric $\rho_0 = \mathbf{1}_{A(0;1,2)}$ assigns length $\geq s^{-}(e_0)$ to $\varphi^{-1}(\gamma)$. In general, the inclusion

$$S^{-}(e_0) \subset \pi_{\mathrm{rad}}(\varphi^{-1}(\gamma)) \cup \bigcup_{\substack{e \in \mathcal{T}^{-}(e_0)\\ \gamma \cap e \neq \emptyset}} S(e)$$

implies that

$$\int_{\varphi^{-1}(\gamma)} \left(\mathbf{1}_{A(0;1,2)} + \sum_{e \in \mathcal{T}^{-}(e_0)} s(e) \rho_e \right) |dz| \ge s^{-}(e_0),$$

thereby establishing the admissibility of ρ . Combined with (4.2), this shows the upper bound

$$M(\Gamma_{\circlearrowright}(A)) \leq A(\rho) \leq \frac{1}{s^{-}(e_{0})^{2}} \left[1 + \sum_{e \in \mathcal{T}^{-}(e_{0})} s(e)^{2} \right]$$

where the implicit constant depends on D, the maximum over the degrees of vertices in \mathcal{T} . As a consequence of the above estimate and Lemma 4.3, we obtain the following theorem:

Theorem 4.4. Let $\{\mathcal{T}_n\}_{n=0}^{\infty}$ be an increasing sequence of hydrodynamically-normalized conformally balanced trees whose vertices have uniformly bounded degrees. Suppose the sums $S_n = \sum_{e \in \mathcal{T}_n} s(e)^2$ are uniformly bounded. If $e_0^{(n)} \subset \mathcal{T}_n$ is a sequence of edges with $\inf s(e_0^{(n)}) > 0$, then any subsequential Hausdorff limit of the $e_0^{(n)}$ is a realanalytic arc. Furthermore, the edges $e_0^{(n)}$ converge without backtracking (see Section 2.2 for the definition). We now apply the above theorem to the sequence of the finite truncations $\{\mathcal{T}_n\}$ of the infinite trivalent tree. Inspection shows that for an edge $e \in \mathcal{T}_n$,

$$s(e) \approx 2^{-d_{\mathcal{T}_n}(v_{\operatorname{root}},e)}.$$

As the number of edges $v \in \mathcal{T}_n$ with $d_{\mathcal{T}_n}(v_{\text{root}}, e) = m$ is $\approx 2^m$, the sums

$$S_n = \sum_{e \in \mathcal{T}_n} s(e)^2$$

are uniformly bounded in $n = 1, 2, \ldots$ Fix an edge $e \subset \mathcal{T}$ in the infinite tree. For $n > \operatorname{dist}(v_{\operatorname{root}}, e)$, let $e^{(n)}$ denote the corresponding edge in \mathcal{T}_n . Since

$$s^{-}(e^{(n)}) \simeq 2^{-d_{\mathcal{T}_{\infty}}(v_{\text{root}},e)}, \qquad n > \text{dist}(v_{\text{root}},e) + 1,$$

the theorem above implies that the diameters of the edges $e^{(n)}$ are bounded from below.

On the other hand, if \mathcal{T}_n is a random conformally balanced tree with n edges, chosen uniformly among all of them, then one can show that

$$E[S_n] = \sum_{e \in \mathcal{T}_n} E[s(e)^2] \to \infty,$$

as $n \to \infty$, suggesting that the diameters of the edges tend to zero. This is indeed the case: it is proved in [10] that with high probability, the diameter of every edge is at most $1/n^{\alpha}$ for some $\alpha > 0$.

5 Structure of a subsequential limit

Let \mathcal{T}_n be the hydrodynamically-normalized conformally balanced trivalent tree of depth n. In this section, we show that any subsequential limit of the \mathcal{T}_n has the right topological type:

Theorem 5.1. For any subsequential Hausdorff limit of the \mathcal{T}_n , one can find a homeomorphism of the plane which takes it onto the Farey tree \mathcal{F} (defined in Section 3) union the unit circle $\partial \mathbb{D}$.

Remark. Theorem 5.1 holds in the slightly more general setting of Appendix A, with the same proof, where \mathcal{T}'_1 is an arbitrary finite trivalent tree and \mathcal{T}'_{n+1} is obtained from \mathcal{T}'_n by adding two edges to each boundary vertex.

We first pass to a subsequence so that every edge in the infinite trivalent tree has a limit along this sequence. In the previous section, we saw that the limit of each edge is a real-analytic arc. We write \mathcal{T}_{∞} for the union of the Hausdorff limits of the individual edges. We pass to a further subsequence so that the finite trees \mathcal{T}_n also possess a Hausdorff limit, which we denote by $\mathcal{T}_{\infty} \sqcup \Lambda$. We refer to Λ as the *limit set*.

The proof of Theorem 5.1 is based on a number of moduli estimates, which control the geometry of the finite trees \mathcal{T}_n . With the help of these moduli estimates, we prove the following assertions:

- (SL1) \mathcal{T}_{∞} is dense in the Hausdorff limit of the finite trees \mathcal{T}_n .
- (SL2) For any branch $[v_0, v_1, v_2, v_3, \dots]$ of \mathcal{T}_{∞} with $d_{\mathcal{T}_{\infty}}(v_m, v_{\text{root}}) = m$, $\lim_{m \to \infty} v_m$ exists.
- (SL3) Given two branches $[v_0, v_1, v_2, v_3, ...]$, $[w_0, w_1, w_2, w_3, ...]$, $\lim_{m \to \infty} v_m = \lim_{m \to \infty} w_m$ if and only if the limits of the corresponding branches in the Farey tree are the same.

We then show the following two topological assertions:

- (SL4) The limit set Λ is a Jordan curve $\partial \Omega$ which encloses \mathcal{T}_{∞} .
- (SL5) There is a natural correspondence between the complementary regions of $\mathcal{T}_{\infty} \cup \partial \Omega$ and $\mathcal{F} \cup \partial \mathbb{D}$.

Assuming the facts above, the proof of Theorem 5.1 runs as follows:

Proof of Theorem 5.1. Let h be a homeomorphism of \mathcal{T}_{∞} onto the Farey tree \mathcal{F} , which takes vertices to the corresponding vertices. In view of Lemma 3.2, the properties (SL1), (SL2) and (SL3) imply that h extends to a homeomorphism of the closures: $\mathcal{T}_{\infty} \cup \partial \Omega$ and $\mathcal{F} \cup \partial \mathbb{D}$. Since the complementary regions are Jordan domains, by (SL4) and (SL5), we can extend h to a homeomorphism of the plane. \Box

5.1 Shrinking of diameters

Given a vertex $v \in \mathcal{T}_n$, we denote the subtree which consists of v and its descendants by $\mathcal{T}_n(v)$. To prove (SL1) and (SL2), we show:

Lemma 5.2. (i) For every $\varepsilon > 0$, there exists $d_0(\varepsilon) > 0$ sufficiently large so that

diam $\mathcal{T}_n(v) < \varepsilon$,

for any $n \ge 1$ and $v \in \mathcal{T}_n$ with $d_{\mathcal{T}_n}(v_{\text{root}}, v) \ge d_0(\varepsilon)$.

(ii) For every $\varepsilon > 0$, there exists $k_0(\varepsilon) > 0$ sufficiently large so that

diam $\mathcal{T}_n(vLR^k) \cup \mathcal{T}_n(vRL^k) < \varepsilon$,

for any $n \ge 1$, $v \in \mathcal{T}_n$ and $k \ge k_0(\varepsilon)$.

As in the case of the Farey tree $\mathcal{F} \subset \mathbb{D}$, the diameter of $\mathcal{T}_n(v)$ depends on the nature of the word representing v. If the path joining v_{root} to v switches between left and right turns regularly, then the diameters of $\mathcal{T}_n(v)$ decrease exponentially quickly. On the other hand, if the word for v has long sequences of consecutive L's and R's, then the diameters of $\mathcal{T}_n(v)$ shrink at a polynomial rate. This dichotomy is reflected in the two types of estimates below.

5.1.1 Hyperbolic decay

At an interior vertex $v \in \mathcal{T}_n$, the domain $\hat{\mathbb{C}} \setminus \mathcal{T}_n$ has three prime ends. Assuming that $v \neq v_{\text{root}}$ is not the root vertex, we can name the three prime ends as left, right and middle. The *left* prime end lies between $\overline{v_{\text{parent}}v}$ and $\overline{vv_L}$, while the *right* prime end lies between $\overline{v_{\text{parent}}v}$ and $\overline{vv_L}$, while the *right* prime end lies between $\overline{v_{\text{parent}}v}$ and $\overline{vv_L}$ and $\overline{vv_R}$. Naturally, the *middle* prime end lies between $\overline{vv_L}$ and $\overline{vv_R}$.

Let $\gamma(v)$ denote the hyperbolic geodesic in $\hat{\mathbb{C}} \setminus \mathcal{T}_n$ which joins the left and right prime ends at v and V(v) be the domain enclosed by $\gamma(v)$, see Figure 3. With this definition, a vertex w is contained in V(v) if and only if w is represented by a word which begins with v. Moreover, if v_2 is a descendant of v_1 , then $V(v_2) \subset V(v_1)$.

Lemma 5.3. Suppose v is an interior vertex of \mathcal{T}_n , other than the root vertex. Then, Mod $V(v) \setminus V(vLR) \approx 1$ and Mod $V(v) \setminus V(vRL) \approx 1$.



Figure 3: To a non-root vertex $v \in \mathcal{T}_n$, we associate the domain V(v), bounded by the curve $\gamma(v)$.

It is enough to examine the modulus of $\mathbf{A} = V(v) \setminus V(vLR)$ as the situation for Mod $V(v) \setminus V(vRL)$ is entirely symmetric. To prove the lemma, we need to give uniform upper bounds for the moduli of the curve families $\Gamma_{\bigcirc}(\mathbf{A})$ and $\Gamma_{\uparrow}(\mathbf{A})$, which are independent of n and $v \in \mathcal{T}_n$.

To deal with the first curve family, we simply note that every $\gamma \in \Gamma_{\bigcirc}(\mathbf{A})$ intersects at least one of the two edges $\overline{vv_L}$ and $\overline{v_L v_{LR}}$ so that the sum of the two obstacles $\rho = \rho_{\overline{vv_L}} + \rho_{\overline{v_L v_{LR}}}$ is an admissible metric of area $A(\rho) = O(1)$.

To deal with the second curve family, by conformal invariance, we may give an upper bound for the modulus of the curve family $\varphi^{-1}(\Gamma_{\uparrow}(\mathbf{A}))$ in $\varphi^{-1}(\mathbf{A}) \subset \hat{\mathbb{C}} \setminus \mathbb{D}$ which allows teleportation, as we did before in Section 4.4. Cutting \mathbf{A} along the tree, we obtain a conformal rectangle $\mathbf{R} = \mathbf{A} \setminus \mathcal{T}_n$ whose vertices are the prime ends where $\gamma(v)$ and $\gamma(vLR)$ meet \mathcal{T}_n . Its pre-image $\hat{\mathbf{R}} = \varphi^{-1}(\mathbf{R}) \subset \hat{\mathbb{C}} \setminus \mathbb{D}$ is a conformal rectangle whose vertices are the points where the geodesics $\varphi^{-1}(\gamma(v))$ and $\varphi^{-1}(\gamma(vLR))$ meet the unit circle. We label the vertices z_1, z_2, z_3, z_4 in counterclockwise order such that z_1 corresponds to the right prime end of v. Due to the "left-right" turn between v and vLR, the distances between the points $z_i, 1 \leq i \leq 4$, are comparable:

$$|z_1 - z_2| \asymp |z_2 - z_3| \asymp |z_3 - z_4| \asymp 2^{-d}, \qquad d = d_{\mathcal{T}_n}(v_{\text{root}}, v).$$

Hence, the background metric $\rho_0 = \mathbf{1}_{\varphi^{-1}(\mathbf{A})}$ assigns length $\gtrsim 2^{-d}$ to every curve in

 $\varphi^{-1}(\Gamma_{\uparrow}(\mathbf{A}))$ that does not teleport.

Arguing as in Section 4.4 shows that the metric

$$\rho = C_0 2^d \left(\rho_0 + \sum_{e \in V(v)} s(e) \rho_e \right)$$
(5.1)

is admissible if C_0 is sufficiently large (independent of n and v). More precisely, while the set $\varphi^{-1}(\gamma)$ may be disconnected,

$$\sigma = \varphi^{-1}(\gamma) \cup \bigcup_{e \cap \gamma \neq \emptyset} S(e)$$

is connected and intersects both geodesics $\varphi^{-1}(\gamma(v))$ and $\varphi^{-1}(\gamma(vLR))$. Inspection shows that the integral $\int_{\sigma} \rho_0(z) |dz|$ computes the Euclidean length of $\sigma \setminus \partial \mathbb{D}$, whereas $\int_{\sigma} (\sum_{e \in V(v)} s(e) \rho_e) |dz|$ is bounded below by the Euclidean length of $\sigma \cap \partial \mathbb{D}$. As a result,

$$\int_{\sigma} \left\{ \rho_0 + \sum_{e \in V(v)} s(e) \rho_e \right\} |dz|$$

is greater or equal to the Euclidean distance between the geodesics $\varphi^{-1}(\gamma(v))$ and $\varphi^{-1}(\gamma(vLR))$, which is comparable to 2^{-d} . Consequently, the factor $C_0 2^d$ in (5.1) makes the metric ρ admissible.

From $s(e) \approx 2^{-d_{\tau_n}(v_{\text{root}},e)}$, it is clear that $\sum_{e \in V(v)} s(e)^2 \leq 2^{-2d}$. The area bound $A(\rho) = O(1)$ now follows from (4.2). Putting the above information together shows the desired modulus bound.

5.1.2 Parabolic decay

We continue to assume that $v \in \mathcal{T}_n$ is an interior vertex, other than the root vertex. For each $0 \leq j \leq n - 1 - d_{\mathcal{T}_n}(v_{\text{root}}, v)$, we connect the vertices vLR^j and vRL^j by two hyperbolic geodesics $\alpha_j, \beta_j \subset \hat{\mathbb{C}} \setminus \mathcal{T}_n$, with the *inner geodesic* α_j joining

$$(vLR^j)_{\text{right}}$$
 with $(vRL^j)_{\text{left}}$

and the outer geodesic β_j joining

$$(vLR^j)_{\text{left}}$$
 with $(vRL^j)_{\text{right}}$.



Figure 4: The region $W_j = W_j(v)$ is associated to a non-root vertex $v \in \mathcal{T}_n$ and an integer $j \ge 0$. It is bounded by the hyperbolic geodesics α_j and β_j .

We then define $W_j = W_j(v)$ as the simply-connected domain bounded by the Jordan curve $\alpha_j \cup \beta_j$. See Figure 4.

Lemma 5.4. Suppose v is an interior vertex of \mathcal{T}_n , other than the root vertex. Then,

Mod $W_0(v) \setminus W_k(v) \simeq \log(1+k)$,

for any $1 \leq k \leq n - 1 - d_{\mathcal{T}_n}(v_{\text{root}}, v)$.

Since the annulus $V(v) \setminus V(vLR^k) \supset W_0 \setminus W_k$, its modulus is strictly larger. In particular, the lemma implies that $\operatorname{Mod} V(v) \setminus V(vLR^k) \gtrsim \log(1+k)$.

Proof. For brevity, we write $\mathbf{A} = W_0 \setminus W_k$. To show the upper bound for Mod \mathbf{A} , we need to estimate the modulus of the family of curves Γ_{\bigcirc} which separate the boundary components of \mathbf{A} . The tree \mathcal{T}_n splits \mathbf{A} into two conformal rectangles \mathbf{R}_{α} and \mathbf{R}_{β} , with $\partial \mathbf{R}_{\alpha} \supset \alpha_0 \cup \alpha_k$ and $\partial \mathbf{R}_{\beta} \supset \beta_0 \cup \beta_k$. Since a curve in $\Gamma_{\bigcirc}(\mathbf{A})$ contains a crossing that joins the \mathcal{T}_n -sides of \mathbf{R}_{α} , Mod $\Gamma_{\bigcirc}(\mathbf{A}) \leq \text{Mod } \mathbf{R}_{\alpha}$. The latter modulus may be computed in the exterior unit disk: Mod $\mathbf{R}_{\alpha} = \text{Mod } \varphi^{-1}(\mathbf{R}_{\alpha}) \asymp \log(1+k)$ as desired.

We now turn to the lower bound. For this purpose, we decompose \mathbf{A} into a union of shells:

$$\mathbf{A} = \bigcup_{j=1}^{k} \mathbf{A}_{j} = \bigcup_{j=1}^{k} W_{j-1} \setminus W_{j}$$

By the parallel rule, it is enough to show that $\operatorname{Mod} \Gamma_{\uparrow}(\mathbf{A}_j) \leq j$, for each $j = 1, 2, \ldots, k$. As usual, we estimate the modulus of the family

$$\varphi^{-1}(\Gamma_{\uparrow}(\mathbf{A}_j)) \subset \varphi^{-1}(\mathbf{A}_j) \subset \hat{\mathbb{C}} \setminus \mathbb{D}.$$

The pre-image $\varphi^{-1}(\mathbf{A}_j) = \hat{\mathbf{R}}_{\beta,j} \cup \hat{\mathbf{R}}_{\alpha,j}$ consists of two conformal rectangles in $\hat{\mathbb{C}} \setminus \mathbb{D}$, with $\hat{\mathbf{R}}_{\alpha,j}$ bounded by $\hat{\alpha}_{j-1}, \hat{\alpha}_j$ and the unit circle, and $\hat{\mathbf{R}}_{\beta,j}$ bounded by $\hat{\beta}_{j-1}, \hat{\beta}_j$ and the unit circle.

Let $\rho_{\alpha,0}(z)$ be the extremal metric on the conformal rectangle $\mathbf{R}_{\alpha,j}$ for the family of curves contained in $\hat{\mathbf{R}}_{\alpha,j}$ that connect $\hat{\alpha}_{j-1}$ and $\hat{\alpha}_j$. It is easy to see that $A(\rho_{\alpha,0}) \approx$ j + 1. As in the proof of Lemma 5.3, there is a metric $\rho_{\beta,0}$ of the form (5.1) with $A(\rho_{\beta,0}) \approx 1$ which assigns length ≥ 1 to every curve γ in $\hat{\mathbf{R}}_{\beta,j}$ that connects $\hat{\beta}_{j-1}$ and $\hat{\beta}_j$ with or without teleportation. More precisely, since the four marked endpoints of $\hat{\beta}_{j-1}$ and $\hat{\beta}_j$ have mutually comparable distances $\approx 2^{-d\tau_n(v_{\text{root}},v)-j}$, the reasoning in the proof of Lemma 5.3 shows that the metric

$$\rho_{\beta,0} = C_0 2^{d_{\mathcal{T}_n}(v_{\text{root}},v)+j} \left(\mathbf{1}_{\hat{\mathbf{R}}_{\beta,j}} + \sum_{e \in V(v_{LR^{j-1}}) \cup V(v_{RL^{j-1}})} s(e) \rho_e \right)$$
(5.2)

is admissible if C_0 is sufficiently large.

A path in $\varphi^{-1}(\Gamma_{\uparrow}(\mathbf{A}_{j}))$ connects $\hat{\alpha}_{j-1} \cup \hat{\beta}_{j-1}$ with $\hat{\alpha}_{j} \cup \hat{\beta}_{j}$, where one is allowed to take shortcuts by teleporting from $x \in \partial \mathbb{D}$ to $y \in \partial \mathbb{D}$ if $\varphi(x) = \varphi(y) \in \mathcal{T}_{n}$. Such a path is either contained in $\hat{\mathbf{R}}_{\alpha,j}$, or contained in $\hat{\mathbf{R}}_{\beta,j}$, or intersects one of the two edges $e_{1} = [vLR^{j-1}, vLR^{j}]$ and $e_{2} = [vRL^{j-1}, vRL^{j}]$. To obtain a metric admissible for $\varphi^{-1}(\Gamma_{\uparrow}(\mathbf{A}_{j}))$, we modify $\rho_{0}(z) = \rho_{\alpha,0}(z) + \rho_{\beta,0}(z)$ by adding two obstacles along the edges e_{1} and e_{2} which make it impractical for a path to teleport from $\hat{\mathbf{R}}_{\beta,j}$ to $\hat{\mathbf{R}}_{\alpha,j}$:

$$\rho = (\rho_{\alpha,0} + \rho_{\beta,0}) + \rho_{e_1} + \rho_{e_2}.$$

As each obstacle has area O(1), the area $A(\rho) \simeq j + 1$, which gives the desired modulus bound.

5.1.3 Conclusion

We are now ready to show Lemma 5.2:

Proof of Lemma 5.2. Let $v \in \mathcal{T}_n$ be an interior vertex, other than the root vertex. We have seen that $\mathcal{T}_n(v) \subset V(v)$. Let

$$[v_{\text{root}}, v] = [v_0 = v_{\text{root}}, v_1, v_2, v_3, \dots, v_m = v]$$

be the path in \mathcal{T}_n joining v_{root} to v. In view of the hydrodynamic normalization, $V(v_1) \subset \Omega_2 \subset B(0,8)$ is contained in a ball of fixed size. Consequently, to prove (i), it is enough to show that $\operatorname{Mod} V(v_1) \setminus V(v)$ is large when $d_{\mathcal{T}_n}(v_{\text{root}}, v)$ is large.

There are two cases to consider. If the path $[v_{\text{root}}, v]$ frequently switches between left and right turns, then $\operatorname{Mod} V(v_1) \setminus V(v)$ will be large by Lemma 5.3 and the parallel rule. If we turn left many times or turn right many times without switching, then $\operatorname{Mod} V(v_1) \setminus V(v)$ will be large by Lemma 5.4. By considering the cases where the number of switches is $\geq \sqrt{m}$ or $\leq \sqrt{m}$, one gets the quantitative estimate

$$\operatorname{Mod} V(v_1) \setminus V(v) \gtrsim \log m$$

By Lemma 2.1(a), diam $V(v) \to 0$ uniformly in n as $m = d_{\mathcal{T}_n}(v_{\text{root}}, v) \to \infty$, which shows (i). To prove (ii), we note that

$$\mathcal{T}_n(vLR^k) \cup \mathcal{T}_n(vLR^k) \subset W_k(v) \subset V(v)$$

and appeal to Lemmas 5.4 and 2.1(a).

Remark. In conjunction with Lemma 3.1, a more careful reading of the above proof shows the estimate

$$\log \frac{1}{\operatorname{diam} \mathcal{T}_n(v)} \lesssim \log \frac{1}{\operatorname{diam} \mathcal{F}_n(v)}$$

With a little more work, one can show that the two quantities are comparable, but since we will not need this fact, we will not give the proof.

5.2 The limit set is a Jordan curve

Our next objective is to show (SL3) and (SL4). For two boundary vertices $v_1, v_2 \in \mathcal{T}_n$, we denote by $d_{\omega}(v_1, v_2)$ the harmonic measure as seen from infinity of the shorter arc on the unit circle with endpoints $\varphi^{-1}(v_1)$ and $\varphi^{-1}(v_2)$. The following lemma says that if the harmonic measure between two boundary vertices v_1, v_2 is small, then the Euclidean distance $|v_1 - v_2|$ is also small:

Lemma 5.5. For any $\varepsilon > 0$, there exists an $\eta > 0$, such that if $v_1, v_2 \in \mathcal{T}_n$ are two boundary vertices for which $d_{\omega}(v_1, v_2) < \eta$, then the Euclidean distance $|v_1 - v_2| < \varepsilon$.

Before explaining the proof, we examine an analogous situation: Consider the dyadic tree whose vertices are dyadic intervals in [0, 1], with the root vertex being the whole interval [0, 1], and a vertex corresponding to a dyadic interval I is connected to its two dyadic children I_L and I_R by edges. For two points $x_1, x_2 \in [0, 1]$, consider the minimal dyadic interval I containing them. There are two scenarios when $|x_1 - x_2|$ is small: either I is a short interval or there exists a large integer $k \geq 1$ so that $x_1 \in I_{LR^k}$ and $x_2 \in I_{RL^k}$ (or vice versa). Note that in the first case, $|I| \asymp |x_1 - x_2|$, while in the second case, $|I_{LR^k}| = |I_{RL^k}| \asymp |x_1 - x_2|$.

Proof. Let v be the last common ancestor of v_1 and v_2 so that $v_1, v_2 \in \mathcal{T}_n(v)$ and $v_1 = vLX, v_2 = vRY$ (or vice versa) for some sequences X, Y. Suppose that $d_{\omega}(v_1, v_2)$ is small. There are two non-mutually exclusive possibilities:

- 1. v is a vertex of high generation (i.e. far away from the root),
- 2. $v_1 \in \mathcal{T}_n(vLR^k)$ and $v_2 \in \mathcal{T}_n(vRL^k)$, where $k \geq 1$ is large integer.

In the first case, $\mathcal{T}_n(v) \subset V(v)$ has small diameter by Lemma 5.2(i). In the second case, $\mathcal{T}_n(vLR^k) \cup \mathcal{T}_n(vRL^k) \subset W_k(v)$ has small diameter by Lemma 5.2(ii). Since v_1, v_2 are contained in the above regions, in both cases, $|v_1 - v_2|$ is small.

Remark. We enumerate the boundary vertices of \mathcal{T}_n in the order that they appear as one walks counter-clockwise around \mathcal{T}_n . The above proof shows that for any $\varepsilon > 0$, when $n \ge n_0(\varepsilon)$ is sufficiently large, hyperbolic geodesics connecting consecutive boundary vertices have diameter less than ε . Indeed, any such geodesic is contained in regions of the form V(v) and $W_k(v)$, one of which is guaranteed to have small diameter.

We now show the converse to Lemma 5.5, namely, if the harmonic measure between two boundary vertices in \mathcal{T}_n is bounded below, then so is their Euclidean distance: **Lemma 5.6.** For any $\varepsilon > 0$, there exists an $\eta > 0$, such that if $v, w \in \mathcal{T}_n$ are two boundary vertices for which $d_{\omega}(v, w) > \eta$, then the Euclidean distance $|v - w| > \varepsilon$.

Proof. Let $[v_0 = v_{\text{root}}, v_1, v_2, v_3, \dots, v_n = v]$ be the path in \mathcal{T}_n joining v_{root} to v and $[w_0 = v_{\text{root}}, w_1, w_2, w_3, \dots, w_n = w]$ be the path joining v_{root} to w. The assumption implies that there exists an $n_0 = n_0(\eta) \ge 1$ sufficiently large so that the harmonic measure between $E = [v_{n_0}, v_{n_0+1}, \dots, v]$ and $F = [w_{n_0}, w_{n_0+1}, \dots, w]$ is at least $\eta/2$.

Recall that in Section 4.4, we showed that the diameters of E and F are bounded from below. To show that E and F are a definite distance apart, it is enough to give an upper bound for the modulus of the family of curves $\Gamma_{E\leftrightarrow F}$ that connect Eto F in $\Omega_2 = \varphi(\{z : |z| = 2\})$. By conformal invariance, we may instead estimate the modulus $\varphi^{-1}(\Gamma_{E\leftrightarrow F})$ in A(0; 1, 2) where teleportation is allowed between the pre-images of points in \mathcal{T}_n . An argument similar to the one in Section 4.4 shows that

$$\int_{\varphi^{-1}(\gamma)} \left(\mathbf{1}_{A(0;1,2)} + \sum_{e \in \mathcal{T}_n} s(e)\rho_e \right) |dz| \ge \eta/2, \qquad \gamma \in \Gamma_{E \leftrightarrow F},$$

i.e. $2/\eta$ times the integrand is an admissible metric ρ with $A(\rho) = O(1/\eta^2)$.

Comparing Lemmas 5.5 and 5.6 with the topological description of the Farey tree given in Lemma 3.2 shows (SL3). These lemmas also imply that $\Lambda = \partial \Omega$ is a Jordan curve: By joining consecutive boundary vertices of \mathcal{T}_n by hyperbolic geodesics, we obtain a sequence of Jordan curves Λ_n . If we parametrize these curves by the harmonic measure from infinity, then they converge uniformly to a continuous curve Λ by the remark following Lemma 5.5. Lemma 5.6 guarantees that this limit curve is simple, while Lemma 4.1 implies that it is disjoint from \mathcal{T}_{∞} . As \mathcal{T}_{∞} is a bounded set, it must be contained in the interior of Λ . This completes the verification of (SL4).

5.3 Formation of Ω -horoballs

We now turn to showing (SL5). Let $v_0 \neq v_{\text{root}}$ be a vertex of the infinite trivalent tree. For $j \geq 1$, set

$$v_i = v_0 L R^{j-1}$$
 and $v_{-i} = v_0 R L^{j-1}$.

By (SL3), the limits

$$\lim_{j \to +\infty} v_j \quad \text{and} \quad \lim_{j \to -\infty} v_j$$

exist and are equal. We refer to their common value p as a *cusp* or *parabolic point*. By Lemma 4.1, $p \notin \mathcal{T}_{\infty}$. The union of the edges

$$\bigcup_{j=-\infty}^{\infty} \overline{v_j v_{j+1}} \subset \mathcal{T}_{\infty}$$

together with p, defines a Jordan curve. We denote the region bounded by this curve as Ω_p . At the root vertex, one can similarly construct three Jordan domains $\Omega_{p_1}, \Omega_{p_2}, \Omega_{p_3}$. We refer to the regions $\{\Omega_{p_i}\}$ as Ω -horoballs.

Lemma 5.7. The regions $\{\Omega_{p_i}\}$ enumerate the bounded components of $\mathbb{C} \setminus \lim \mathcal{T}_n$.

Proof. We approximate the regions Ω_{p_i} by Jordan domains $\Omega_{p_i}^{(n)}$ constructed using the finite approximating trees \mathcal{T}_n as follows: Each finite tree \mathcal{T}_n contains only finitely many corresponding vertices $\{v_j\}_{j=-m}^m$, where $m = n - d(v_{\text{root}}, v_0)$. The union of the edges $\bigcup_{j=-m}^{m-1} \overline{v_j v_{j+1}} \subset \mathcal{T}_n$ is a Jordan arc. To form $\partial \Omega_{p_i}^{(n)}$, we close this Jordan arc with the hyperbolic geodesic $\alpha_{p_i}^{(n)} \subset \hat{\mathbb{C}} \setminus \mathcal{T}_n$ that connects the leaves $v_{-m}, v_m \in \mathcal{T}_n$.

In view of Lemma 5.4, diam $\alpha_{p_i}^{(n)} \to 0$ and $\Omega_{p_i} = \lim \Omega_{p_i}^{(n)}$. Since $\Omega_{p_i}^{(n)}$ is disjoint from the tree \mathcal{T}_n , the regions $\Omega_{p_i} = \lim \Omega_{p_i}^{(n)}$ are indeed bounded components of the complement $\mathbb{C} \setminus \lim \mathcal{T}_n$.

Can there be any more complementary components? If O is any connected component of $\mathbb{C} \setminus \lim \mathcal{T}_n$, then $\partial O \subset \mathcal{T}_\infty \cup \Lambda$. If ∂O intersects one of the edges of \mathcal{T}_∞ , then O is one of the four Ω -horoballs that form a neighborhood of this edge. If ∂O does not intersect \mathcal{T}_∞ , then $\partial O \subset \Lambda$, and since Λ is a Jordan curve, O must be the unbounded component of $\mathbb{C} \setminus \Lambda$.

Remark. For future reference, we note that the convergence of

$$\partial\Omega_{p_i}^{(n)} = \alpha_{p_i}^{(n)} \cup \bigcup_{j=-m}^{m-1} e_j^{(n)} \to \partial\Omega_{p_i}, \quad \text{as } n \to \infty,$$

takes place in the strong topology, where $e_j^{(n)} = \overline{v_j^{(n)}v_{j+1}^{(n)}}$ and $m = n - d(v_{\text{root}}, v_0)$ as before. Indeed, the strong convergence of the individual edges was established in

Lemma 4.3, while by Lemma 5.2, the diameters of

$$\alpha_{p_i}^{(n)} \cup \bigcup_{j \le -N} e_j^{(n)} \cup \bigcup_{j \ge N} e_j^{(n)}$$

are uniformly small when N and n are large.

Having established Properties (SL1)–(SL5), the proof of Theorem 5.1 is complete.

5.4 An estimate related to the uniqueness of the limit

For a non-root vertex $v \in \mathcal{T}$, we define the shadow $s_v \subset \partial\Omega$ as the arc of $\partial\Omega$ of smaller diameter which joins $vLRL^{\infty} = \lim_{m\to\infty} vLRL^m$ and $vRLR^{\infty} = \lim_{m\to\infty} vRLR^m$. A brief inspection of the homeomorphic picture of the Farey tree $\mathcal{F} \subset \mathbb{D}$ shows that any point on $\partial\Omega$ that is not a cusp of an Ω -horoball is contained in infinitely many shadows. In Section 6.4, we will use the following estimate in conjunction with Lemma 2.4 to show that the Hausdorff limit of the true trees \mathcal{T}_n is unique:

Lemma 5.8. Recall that $V(v) \subset \mathbb{C}$ is the largest domain bounded by a hyperbolic geodesic in \mathcal{T}_n that joins two of the three prime ends at v, depicted in Figure 3. The sums

$$\sum_{v \in \mathcal{T}_n, v \neq v_{\text{root}}} \left\{ \operatorname{diam}^2 V(vRL) + \operatorname{diam}^2 V(vLR) \right\}$$
(5.3)

are uniformly bounded above, independent of n.

Since s_v is contained in the Hausdorff limit as $n \to \infty$ of $V(vRL) \cup V(vLR)$, the above lemma implies that

$$\sum_{v \in \mathcal{T}_{\infty}, v \neq v_{\text{root}}} \operatorname{diam}^2 s_v < \infty.$$

In particular, $\partial \Omega$ has zero area.

Proof. For a hyperbolic geodesic $\hat{\gamma} \subset \{z \in \mathbb{C} : 1 < |z| < 2\} \subset \hat{\mathbb{C}} \setminus \mathbb{D}$, let $z_{\hat{\gamma}}$ be the Euclidean midpoint of $\hat{\gamma}$ and $B_{\hat{\gamma}} \subset \hat{\mathbb{C}} \setminus \mathbb{D}$ be the ball of hyperbolic radius 1/10 centered at $\frac{1+|z_{\hat{\gamma}}|}{2} \cdot z_{\hat{\gamma}}$. In view of the restriction on $\hat{\gamma}$, the ball $B_{\hat{\gamma}}$ is contained in the bounded domain enclosed by $\hat{\gamma}$ and the unit circle.

Recall that $\Omega_2 \subset \mathbb{C}$ is the domain bounded by the Jordan curve $\varphi(\{|z|=2\})$ where φ is the conformal map from the exterior unit disk to the exterior of the developed deltoid, in hydrodynamic normalization.

Similarly, to a hyperbolic geodesic $\gamma \subset \Omega_2 \subset \hat{\mathbb{C}} \setminus \mathcal{T}_n$, we can associate the topological disk $B_{\gamma} := \varphi(B_{\varphi^{-1}\gamma})$. By Koebe's distortion theorem, B_{γ} is approximately round in the sense that its area is comparable to its diameter squared.

We apply the above construction to the geodesics $\gamma(v) = \partial V(v) \subset \Omega_2$ from Section 5.1, where v ranges over interior vertices of \mathcal{T}_n , other than the root vertex. From the construction, it is clear that $B_{\gamma(v)} \subset V(v)$.

To prove the estimate (5.3), it is enough to show that

$$\operatorname{diam} V(vLR) \asymp \operatorname{diam} B_{\gamma(vLR)},\tag{5.4}$$

as the topological disks $B_{\gamma(vLR)}$ are disjoint and are contained in a bounded set. In view of Lemma 2.1, to prove (5.4), we may show the following two moduli estimates:

- 1. Mod $V(v) \setminus V(vLR)$ is bounded below.
- 2. Mod $V(v) \setminus B_{\gamma(vLR)}$ is bounded above.

The first estimate was already established in Lemma 5.3. The second estimate is automatic from Koebe's distortion theorem. $\hfill \Box$

Remark. One can show that the set s_v in the lemma above and the Jones-Smirnov shadow (defined in Section 2.4) of the closed ball of hyperbolic radius 1 centered at v with respect to $v_{\text{root}} \in \Omega$ have comparable diameters.

6 Convergence

In this section, we show that the Hausdorff limit $\mathcal{T}_{\infty} \cup \partial \Omega$ of the finite trees \mathcal{T}_n is unique. The main step is to prove that any subsequential Hausdorff limit realizes the mating of $z \to \overline{z}^2$, acting on the exterior unit disk \mathbb{D}_e , and the Markov map $z \to \rho(z)$ associated to the reflection group of an ideal triangle Δ_{hyp} , acting on the unit disk \mathbb{D} . (The Markov map has been defined in Section 1.1). The proof involves identifying the Farey structure inside Ω and the $z \to \overline{z}^2$ structure outside Ω .

6.1 Farey horoballs

Recall from Section 3 that the Farey tree \mathcal{F} partitions the unit disk \mathbb{D} into regions called Farey horoballs H_{p_i} , which are indexed by the point where they touch the unit circle. We label the vertices on ∂H_{p_i} in counter-clockwise order by $v^j(H_{p_i})$, for $j \in \mathbb{Z}$, with $v^0(H_{p_i})$ being the vertex with the smallest combinatorial distance to v_{root} . Since the Farey tree is invariant under the group generated by reflections in the sides of Δ_{hyp} , Farey horoballs enjoy the following two properties:

(F1) Any two edges $e_1, e_2 \subset \partial H_{p_i}$ have the same relative harmonic measure as viewed from p_i , i.e.

$$\omega_{H_{p_i},p_i}(e_1,e_2) = 1.$$

(F2) If an edge e belongs to two neighbouring Farey horoballs H_{p_i} and H_{p_j} , then the relative harmonic measures are the same from both sides:

$$\omega_{H_{p_i},p_i}(I,e) = \omega_{H_{p_j},p_j}(I,e), \qquad I \subseteq e.$$

6.2 Interior Structure of Ω

In Section 5, we saw that any subsequential Hausdorff limit $\mathcal{T}_{\infty} \cup \partial \Omega$ of the \mathcal{T}_n is ambiently homeomorphic to the union of the Farey tree \mathcal{F} and the unit circle $\partial \mathbb{D}$. Recall that the connected components of $\Omega \setminus \mathcal{T}_{\infty}$ are called Ω -horoballs. As with Farey horoballs, we index Ω -horoballs by the point where they touch $\partial \Omega$ and label the vertices on $\partial \Omega_{p_i}$ in counter-clockwise order by $v^j(\Omega_{p_i})$, for $j \in \mathbb{Z}$, with $v^0(H_{p_i})$ being the vertex with the smallest combinatorial distance to v_{root} .

Remark. With an eye towards Appendix A, we note that the notion of Ω -horoballs as well as the Lemmas and proofs of this section apply to the slightly more general setting described in the remark after Theorem 5.1.

Lemma 6.1. The Ω -horoballs also enjoy the properties (F1) and (F2).

Proof. Since the arguments are very similar, we only present the proof of the second property and leave the proof of the first property to the reader.

We approximate Ω_{p_i} by Jordan domains $\Omega_{p_i}^{(n)}$ as in the proof of Lemma 5.7. Pick an arbitrary point $p_i^{(n)} \in \alpha_{p_i}^{(n)}$. As the diameters of $\alpha_{p_i}^{(n)}$ tend to 0, the points $p_i^{(n)} \to p_i$. Suppose two neighbouring Ω -horoballs Ω_{p_i} and Ω_{p_j} meet along an edge e. Given an arc $I \subset e$, we can approximate it in the Hausdorff topology by arcs $I_n \subset e^{(n)} \subset \mathcal{T}_n$. By the remark after Lemma 5.7, $\Omega_{p_i}^{(n)}$ strongly converges to Ω_{p_i} as $n \to \infty$. Hence Lemma 2.3 shows that $\omega_{\Omega_{p_i},p_i}(I,e) = \lim \omega_{\Omega_{p_i}^{(n)},p_i^{(n)}}(I_n,e^{(n)})$. Consequently, to verify (F2), it is enough to show that

$$\omega_{\Omega_{p_i}^{(n)}, p_i^{(n)}}(I_n, e^{(n)}) \sim \omega_{\Omega_{p_j}^{(n)}, p_j^{(n)}}(I_n, e^{(n)}), \quad \text{as } n \to \infty.$$
(6.1)

An intuitive albeit somewhat informal explanation of (6.1) is as follows: Run Brownian motion from ∞ until it hits \mathcal{T}_n . If it is to hit the arc $I_n \subset e^{(n)}$ from the side of Ω_{p_i} , denoted by $I_n | \Omega_{p_i}^{(n)}$, then it must pass through the gate $\alpha_{p_i}^{(n)}$. Since the diameter of the gate $\alpha_{p_i}^{(n)}$ is very small,

$$\omega_{\Omega_{p_i}^{(n)},p_i^{(n)}}(I_n,e^{(n)}) \sim \frac{\omega_{\widehat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(I_n \mid \Omega_{p_i}^{(n)})}{\omega_{\widehat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(e^{(n)} \mid \Omega_{p_i}^{(n)})} = \frac{\omega_{\widehat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(I_n \mid \Omega_{p_j}^{(n)})}{\omega_{\widehat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(e^{(n)} \mid \Omega_{p_j}^{(n)})} \sim \omega_{\Omega_{p_j}^{(n)},p_j^{(n)}}(I_n,e^{(n)}).$$

The equality in the middle reflects the fact that the harmonic measures on the two sides of every edge e in a true tree are identical.

To rigorously justify these asymptotic equalities, notice that the pre-images of the approximate Ω -horoballs $\Omega_{p_i}^{(n)}$ and $\Omega_{p_j}^{(n)}$ under the hydrodynamically-normalized Riemann maps $\varphi_n : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus \mathcal{T}_n$ are of the form

$$\varphi_n^{-1}(\Omega_{p_i}^{(n)}) = \mathbb{D}_e \cap B_n,$$

where the B_n are small disks with centers near $\partial \mathbb{D}_e$ and $\varphi_n^{-1}(p_i^{(n)}) \in \partial B_n$. By the conformal invariance of relative harmonic measure, we have

$$\omega_{\Omega_{p_i}^{(n)}, p_i^{(n)}}(I_n, e^{(n)}) = \omega_{\mathbb{H}, \infty} \big(f_n(\varphi_n^{-1}(I_n)), [0, 1] \big) = \frac{\operatorname{length}(f_n(\varphi_n^{-1}(I_n)))}{\operatorname{length}([0, 1])}.$$

As the modulus of the annulus $B_n \setminus \varphi_n^{-1}(e^{(n)})$ tends to infinity, the Koebe distortion theorem implies that the above quantity is

$$\sim \frac{\operatorname{length}(\varphi_n^{-1}(I_n))}{\operatorname{length}(\varphi_n^{-1}(e^{(n)}))} = \frac{\omega_{\hat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(I_n \mid \Omega_{p_i}^{(n)})}{\omega_{\hat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(e^{(n)} \mid \Omega_{p_i}^{(n)})},$$

which is what we wanted to show.

Since $\partial \mathbb{D} \cup \mathcal{F}$ and $\partial \Omega \cup \mathcal{T}$ are ambiently homeomorphic, one has a correspondence between the bounded complementary components of $\partial \mathbb{D} \cup \mathcal{F}$ (Farey horoballs) and those of $\partial \Omega \cup \mathcal{T}$ (Ω -horoballs). For each pair of corresponding complementary regions, form the conformal mapping $\varphi_i : H_i \to \Omega_i$ which takes

$$p(H_i) \to p(\Omega_i), \quad v^0(H_i) \to v^0(\Omega_i), \quad v^1(H_i) \to v^1(\Omega_i).$$

(As Farey horoballs and Ω -horoballs are Jordan domains, by Carathéodory's theorem, the maps φ_i extend to homeomorphisms between the closures.)

Since both Farey and Ω -horoballs satisfy the property (F1), φ_i maps $v^j(H_i) \rightarrow v^j(\Omega_i)$ for any $j \in \mathbb{Z}$. Additionally, as both Farey and Ω -horoballs satisfy the property (F2), if two Farey horoballs H_i and H_j share a common edge e, then $\varphi_i|_e = \varphi_j|_e$. Consequently, the mappings $\varphi_i : H_i \rightarrow \Omega_i$ glue together to form a homeomorphism $\varphi : \mathbb{D} \rightarrow \Omega$, which maps \mathcal{F} onto \mathcal{T} . Since the edges of \mathcal{F} are analytic arcs and the homeomorphism φ is conformal on $\mathbb{D} \setminus \mathcal{F}$, φ extends analytically across the open edges. As the vertices are isolated points, they are removable singularities. Summarizing the above discussion, we have proved the following lemma:

Lemma 6.2 (Interior structure lemma). The mappings $\varphi_i : H_i \to \Omega_i$ glue up to form a conformal mapping $\varphi : \mathbb{D} \to \Omega$, which maps \mathcal{F} onto \mathcal{T} .

6.3 Exterior Structure of Ω

By definition, the harmonic measure $\omega_{\hat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}$ is supported on \mathcal{T}_n . From Koebe's 1/4 theorem, we know that the true trees $\mathcal{T}_n \subset B(0,8)$ are contained in a fixed compact set, so that any subsequential weak-* limit ω of the $\omega_{\hat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}$ is a probability measure supported on the Hausdorff limit $\mathcal{T}_\infty \cup \partial\Omega$. As the harmonic measure of any individual edge tends to zero, the support of the limiting measure ω is contained in $\partial\Omega$. Finally, since $\partial\Omega$ is uniformly perfect, being a Jordan curve, $\omega = \omega_{\Omega_e,\infty}$ is the harmonic measure on $\partial\Omega$ as seen from infinity.

Consider the map $f(z) = \overline{z}^2$ acting on the unit circle. It has fixed points at 1, $\omega = e^{2\pi i/3}$ and $\omega^2 = e^{4\pi i/3}$, which divide the circle into three equal arcs. We call this partition Π_0 . For k = 1, 2, ..., the partition $\Pi_k = f^{-k}(\Pi_0)$ divides the circle in $3 \cdot 2^k$ equal arcs. We now define an analogous sequence of partitions of $\partial \Omega$. We define the *order* of an Ω -horoball Ω_p as

ord
$$\Omega_p = \min_{v \in \partial \Omega_p} d_{\mathcal{T}_{\infty}}(v_{\text{root}}, v)$$
.

There are three Ω -horoballs of order 0, which contain v_{root} . Inspection shows that for $k \geq 1$, there are $3 \cdot 2^{k-1} \Omega$ -horoballs of order k and thus

$$3 + 3 + 6 + \dots + 3 \cdot 2^{k-1} = 3 \cdot 2^k$$

 Ω -horoballs of order at most k. For $k = 1, 2, \ldots$, we define Ψ_k as the partition of $\partial \Omega$ into $3 \cdot 2^k$ arcs by the cusps of order $\leq k$, i.e. the points where Ω -horoballs of order $\leq k$ meet $\partial \Omega$. The harmonic measures of each arc in Ψ_k are equal since every arc in Ψ_k subtends approximately the same number of edges of \mathcal{T}_n .

We now give a more precise explanation using the notation from the proof of Lemma 5.7: Let $\alpha = [p, p']$ be a counter-clockwise arc in $\partial\Omega$ bounded by two cusps. For $n \geq \max(\operatorname{ord} p, \operatorname{ord} p')$, we define $\mathcal{T}_n(\alpha) \subset \mathcal{T}_n$ as the part of the tree one encounters while traversing \mathcal{T}_n counter-clockwise from $v_0(p)R^{n-\deg v_0(p)}$ to $v_0(p')L^{n-\deg v_0(p)}$. From the construction, it is clear that the trees $\mathcal{T}_n(\alpha)$ accumulate onto the closed arc $\alpha \subset \partial\Omega$ while $\mathcal{T}_n \setminus \mathcal{T}_n(\alpha)$ accumulate onto $\overline{\partial\Omega \setminus \alpha} \subset \partial\Omega$. By the weak-* convergence of the harmonic measures, $\omega_{\hat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(\mathcal{T}_n(\alpha)) \to \omega_{\Omega_{e,\infty}}(\alpha)$. Inspection shows that when $\alpha = [p, p']$ is an arc in Ψ_k , the subtree $\mathcal{T}_n(\alpha)$ contains $2^{n-k} + O(n)$ edges. Since \mathcal{T}_n is a true tree, the harmonic measure of $\mathcal{T}_n(\alpha)$ is proportional to the number of edges in $\mathcal{T}_n(\alpha)$:

$$\omega_{\hat{\mathbb{C}}\setminus\mathcal{T}_n,\infty}(\mathcal{T}_n(\alpha)) = 1/(3\cdot 2^k) + o(1), \quad \text{as } n \to \infty.$$

Taking the limit as $n \to \infty$, we see that $\omega_{\Omega_{e,\infty}}(\alpha) = 1/(3 \cdot 2^k)$ as desired. We have therefore proved:

Lemma 6.3 (Exterior structure lemma). There is a conformal mapping

$$\psi: (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}, \infty) \to (\widehat{\mathbb{C}} \setminus \overline{\Omega}, \infty)$$

which takes Π_k to Ψ_k for any $k \ge 0$.

6.4 Uniqueness of the Hausdorff limit

The interior and exterior structure lemmas (Lemmas 6.2 and 6.3) show that any subsequential limit $\partial\Omega$ realizes the mating of $z \to \overline{z}^2$ and the Markov map $\rho(z)$ of the reflection group of an ideal triangle (see Section 1.1): Indeed, φ maps the cusps of the Farey horoballs of order $\leq k$ to the corresponding cusps of the Ω -horoballs by Lemma 6.2, and these are mapped to the endpoints of the partition Π_k by ψ^{-1} as stated in Lemma 6.3. Inspection shows that the welding homeomorphism $h = \psi^{-1} \circ \varphi$ conjugates the action of $\rho: \Psi_k \to \Psi_k$ to that of $\overline{z}^2: \Pi_k \to \Pi_k$, i.e. $h \circ \rho = \overline{z}^2 \circ h$. As $\bigcup_{k=0}^{\infty} \Pi_k$ and $\bigcup_{k=0}^{\infty} \Sigma_k$ are dense in the unit circle, we conclude that h conjugates ρ to $z \to \overline{z}^2$ on $\partial\mathbb{D}$. This allow us to glue the maps

$$\psi \circ \overline{z}^2 \circ \psi^{-1} : \overline{\Omega_e} \to \overline{\Omega_e} \quad \text{and} \quad \varphi \circ \rho \circ \varphi^{-1} : \overline{\Omega} \setminus \varphi(\triangle_{\text{hyp}}) \to \overline{\Omega} \setminus \varphi(\triangle_{\text{hyp}})$$

on $\partial\Omega$ to form the Schwarz reflection $\sigma : \hat{\mathbb{C}} \setminus \varphi(\Delta_{hyp}) \to \hat{\mathbb{C}}$, thereby verifying the definition of the mating.

The structure lemmas also show that Hausdorff limit of the \mathcal{T}_n is unique. Indeed, if $\mathcal{T}'_{\infty} \cup \partial \Omega'$ was another subsequential limit of \mathcal{T}_n , in addition to $\mathcal{T}_{\infty} \cup \partial \Omega$, we could conformally map each complementary region in $\hat{\mathbb{C}} \setminus (\mathcal{T}_{\infty} \cup \partial \Omega)$ to the corresponding complementary region in $\hat{\mathbb{C}} \setminus (\mathcal{T}'_{\infty} \cup \partial \Omega')$. Lemmas 6.2 and 6.3 guarantee that these conformal mappings patch together to form a continuous self-map of the sphere $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is conformal on $\hat{\mathbb{C}} \setminus (\mathcal{T}_{\infty} \cup \partial \Omega)$. The tree \mathcal{T}_{∞} is conformally removable as it is locally a finite union of real analytic arcs. Thus, h is conformal on $\hat{\mathbb{C}} \setminus \partial \Omega$. By Lemmas 2.4 and 5.8, h is a Möbius transformation.

6.5 Convergence of the Shabat polynomials

We subdivide each Ω -horoball Ω_p into triangles $\Delta(\vec{e}_i, p)$ by connecting the vertices of \mathcal{T}_{∞} on $\partial \Omega_p$ to p by hyperbolic geodesics of Ω_p . We colour the triangles $\Delta(\vec{e}_i, p) \subset \Omega$ black and white, so that adjacent triangles have opposite colours.

We conformally map each black triangle $\triangle(\vec{e}_i, p)$ onto the upper half-plane \mathbb{H} so that $\vec{e}_i \rightarrow [-1, 1], p_j \rightarrow \infty$ and each white triangle $\triangle(\vec{e}_i, p)$ onto the lower half-plane \mathbb{L} so that $\vec{e}_i \rightarrow [-1, 1], p \rightarrow \infty$. Properties (F1) and (F2) from Section 2.3 guarantee that these conformal maps glue together to form a holomorphic function h defined on

 Ω . By choosing the colouring scheme appropriately, we can ensure that $h(v_{\text{root}}) = 1$ rather than -1.

From the description of the Shabat polynomials p_n for the true trees \mathcal{T}_n given in Section 4.1, it is not difficult to see that $p_n \to h$, uniformly on compact subsets of Ω : By the strong convergence $\partial \Omega_p^{(n)} \to \partial \Omega_p$ as $n \to \infty$, discussed in remark after Lemma 5.7, it follows that the edges of \mathcal{T}_n converge to the corresponding edges of \mathcal{T}_{∞} . Additionally, the pieces of the hyperbolic geodesics in $\hat{\mathbb{C}} \setminus \mathcal{T}_n$ connecting the vertices of \mathcal{T}_n to ∞ , which lie in $\Omega_p^{(n)}$, converge to the hyperbolic geodesics connecting the vertices of \mathcal{T}_{∞} to p in Ω_p . As a result, the triangles $\Delta(\vec{e_i}, \infty) \subset \hat{\mathbb{C}}$, defined in Section 4.1, converge to the corresponding triangles $\Delta(\vec{e}_i, p) \subset \Omega_p$ in the Carathéodory topology. As p_n and h are conformal maps from these triangles to the upper or lower half-planes, this tells us that $p_n \to h$ uniformly on compact subsets of any triangle $\triangle(\vec{e}_i, p) \subset \Omega$. Applying similar reasoning to a pair of triangles that have a common edge shows that $p_n \to h$ uniformly on compact subsets of the union of these two triangles. Consequently, $p_n \to h$ uniformly on compact subsets of Ω away from the vertices of \mathcal{T}_{∞} . Finally, by examining the behaviour of the maps p_n and h on the stars \star_v , which were defined in Section 4.2, we conclude that $p \to h$ in a neighbourhood of each vertex $v \in \mathcal{T}_{\infty}$.

If R is the Riemann map from $\mathbb{H} \to \Omega$, then $F = h \circ R$ is a function on the upper half-plane which is invariant under a subgroup $\Gamma' < PSL(2,\mathbb{Z})$ of index 2. A fundamental domain for Γ' is depicted in Figure 5.

Alternatively, Γ' is the index 2 subgroup of orientation-preserving elements of the group generated by reflections in the sides of a $(3, 3, \infty)$ triangle, i.e. a hyperbolic triangle with angles $2\pi/3$, $2\pi/3$ and 0. Let $X_{\Gamma'}$ be the Riemann surface obtained by compactifying the quotient $(\mathbb{H} - S)/\Gamma'$, where $S = \Gamma'(e^{\pi i/3}) \cup \Gamma'(1 + e^{\pi i/3}) \subset \mathbb{H}$ is the set of points in the upper half-plane that have a non-trivial stabilizer under the action of Γ' . Inspection shows that compactification adds three points to the quotient, corresponding to the vertices of a $(3, 3, \infty)$ triangle, and the resulting Riemann surface $X_{\Gamma'}$ is a Riemann sphere.

Since F is invariant under Γ' , it descends to a function on $X_{\Gamma'}$, which we denote by f. From the description of F in terms of Riemann maps on triangles, it is clear that f is injective on $X_{\Gamma'}$ and the image of f is all of $\hat{\mathbb{C}}$. In other words, f is a



Figure 5: One can construct a fundamental domain for Γ' from two copies of the fundamental domain for $PSL(2,\mathbb{Z})$. The function h sends the blue part of the fundamental domain to the upper half-plane and the orange part to the lower half-plane.

conformal map from $X_{\Gamma'}$ to $\hat{\mathbb{C}}$.

Lemma 6.4. The modular function F in Theorem 1.1 is a Hauptmodul for Γ' , which means that F generates the field of meromophic functions on the upper halfplane invariant under Γ' .

Proof. Equivalently, the lemma states that f generates the field of meromophic functions on $X_{\Gamma'}$. This is straightforward as any Möbius transformation generates the field of meromorphic functions on the Riemann sphere.

Remark. (i) The Hauptmodul is unique up to post-composing F with a Möbius transformation.

(ii) The Hauptmodul for the $(2, 3, \infty)$ triangle group or $PSL(2, \mathbb{Z})$ is the Klein *j*-invariant. Consequently, one may think of *F* as an analogue of the *j*-invariant for the $(3, 3, \infty)$ triangle group.

A Trivalent true trees are dense

In this appendix, we show that one can approximate any connected compact set K in the plane in the Hausdorff topology by finite trivalent true trees, thereby giving another proof of Bishop's theorem [3]:

Theorem A.1 (Bishop). For any compact connected set $K \subset \mathbb{C}$ and $\varepsilon > 0$, there is a conformally balanced tree \mathcal{T} such that the Hausdorff distance $d_{\mathrm{H}}(K, \mathcal{T}) < \varepsilon$.



Figure 6: Unbalanced truncations of the infinite trivalent tree.

A.1 Generalized developed deltoids

Start with a finite trivalent tree \mathcal{T}'_1 , for instance with the tree on the left side of Figure 6 which consists of five edges. In each step, add two edges to each boundary vertex. This gives us a sequence of true trees $\{\mathcal{T}'_n\}_{n=1}^{\infty}$. The arguments presented in this paper (see the remark after Theorem 5.1) show that the finite trees \mathcal{T}'_n converge in the Hausdorff topology to an infinite trivalent tree union a Jordan curve: $\mathcal{T}'_{\infty} \cup \partial \Omega'$. We refer to the family of Jordan domains Ω' arising in this way as generalized developed deltoids. We will prove the following strengthening of Bishop's theorem:

Theorem A.2. For any compact connected set $K \subset \mathbb{C}$ and $\varepsilon > 0$, one can choose the starting tree \mathcal{T}'_1 so that the Hausdorff distance

$$d_{\mathrm{H}}(L \circ \mathcal{T}'_n, K) < \varepsilon,$$

for any $n \ge 1$ sufficiently large and some linear mapping L(z) = az + b in Aut \mathbb{C} .

The strategy of our proof is as follows: We first reduce the theorem to the problem of approximating a given Jordan curve γ (chosen to approximate K) by a generalized developed deltoid $\partial \Omega'$. Next, we observe that every Jordan curve γ can be approximated by the image $f(\partial\Omega)$ of the developed deltoid under a quasiconformal map that is conformal on Ω . Finally, to show that any such curve $\partial\tilde{\Omega} = f(\partial\Omega)$ can be approximated by a generalized developed deltoid $\partial\Omega'$, we consider the relations between the welding homeomorphisms h of $\partial\Omega$, h' of $\partial\Omega'$ and \tilde{h} of $f(\partial\Omega)$. A key observation is that $h' \circ h^{-1}$ is piecewise-linear and that such homeomorphisms are dense in the set of all quasisymmetric homeomorphisms in the appropriate topology, so in particular, they can approximate $\tilde{h} \circ h^{-1}$. Summarizing, our chain of approximations is $K \approx \gamma \approx f(\partial\Omega) \approx \partial\Omega'$.

A.2 Reductions



Figure 7: Approximating the unit circle and the unit disk in the Hausdorff topology by thin Jordan domains.

Reduction 1. To prove the theorem, it is enough to show that up to a linear rescaling, any Jordan curve γ is the Hausdorff limit of generalized developed deltoids.

To explain the reduction, we first approximate the compact connected set $K \subset \mathbb{C}$ in the Hausdorff topology by Jordan curves γ_k that are (1/k)-thin, i.e. any point in the domain Γ_k enclosed by γ_k lies within 1/k of γ_k . This ensures that as $k \to \infty$, both the curves γ_k and the closed domains $\overline{\Gamma_k}$ converge in the Hausdorff topology to K. See Figure 7 above for examples.

We then approximate each γ_k by a generalized developed deltoid $L_k \circ \partial \Omega'_k$ within Hausdorff distance 1/k. As the curves $L_k \circ \partial \Omega'_k$ are (2/k)-thin, both $L_k \circ \partial \Omega'_k$ and $L_k \circ \overline{\Omega'_k}$ converge to K. Therefore, if $\mathcal{T}'_{k,l}$, $l = 1, 2, \ldots$ is a sequence of finite trees associated to a generalized developed deltoid Ω'_k , then $d_{\rm H}(L_k \circ \mathcal{T}'_{k,l}, \gamma_k) \leq 4/k$ for all sufficiently large $l \geq l_0(k)$. A diagonal argument produces a sequence of finite trees which converges to K.

Reduction 2. It is enough to show that up to a linear rescaling, the image of the boundary of the developed deltoid $\partial\Omega$ under a quasiconformal map $f: \mathbb{C} \to \mathbb{C}$, which is conformal on Ω , is the Hausdorff limit of a sequence of generalized developed deltoids.

The reduction will be explained in Section A.4. It relies on the following lemma, which says that one can approximate Jordan curves by quasiconformal images of a fixed Jordan curve:

Lemma A.3. Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain. For any Jordan curve γ and $\varepsilon > 0$, one can find a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$, which is conformal on Ω and takes $\partial\Omega$ onto a Jordan curve $\partial\tilde{\Omega}$ for which the Hausdorff distance $d_{\mathrm{H}}(\partial\tilde{\Omega}, \gamma) < \varepsilon$.

Proof. Fix a smooth curve γ_{ε} surrounding γ such that $d_{\rm H}(\gamma, \gamma_{\varepsilon}) < \varepsilon$. Similarly, approximate Ω from the outside by the smoothly bounded domains

$$\Omega_{\delta} = \psi(\{|z| > 1 + \delta\}), \qquad \delta > 0,$$

where ψ is the conformal map from \mathbb{D}_e to Ω_e as before. Let f_{δ} be the conformal map from Ω_{δ} onto the region Γ_{ε} enclosed by γ_{ε} , normalized to map a point $z_0 \in \Omega$ to a point $w_0 \in \Gamma_{\varepsilon}$. Since Ω_{δ} converges to Ω in the Carathéodory topology, as $\delta \to 0$, the inverse maps f_{δ}^{-1} converge uniformly on compact to a conformal map $f^{-1}: \Gamma_{\varepsilon} \to \Omega$. For small $\delta > 0$, the curve $f_{\delta}^{-1}(\gamma)$ will be compactly contained in Ω , and so $f_{\delta}(\partial \Omega)$ will be contained in the doubly-connected region bounded by γ and γ_{ε} . Consequently, any smooth extension of $f_{\delta}|_{\overline{\Omega}}$ to the sphere, fixing ∞ , will fulfill the conclusion of the lemma.

A.3 Trivalent tree weldings

In this section, we discuss conformal weldings of generalized developed deltoids and show that they can approximate every quasisymmetric homeomorphism. Recall that any non-root vertex in the trees \mathcal{T}_n and \mathcal{T}_∞ can be labeled by a digit 1, 2, 3 followed by a sequence of left and right turns. In order to label the vertices of \mathcal{T}'_n and \mathcal{T}'_∞ in a similar fashion, we designate a vertex in \mathcal{T}'_1 as the root vertex and select one of the adjacent vertices as the vertex labeled 1.

Let $\varphi : (\mathbb{D}, 0, 1) \to (\Omega, v_{\text{root}}, p_{\Omega})$ and $\psi : (\mathbb{D}_e, \infty, 1) \to (\Omega_e, \infty, p_{\Omega})$ be conformal mappings to the interior and exterior of the developed deltoid respectively, where

$$p_{\Omega} = \lim_{k \to \infty} v_{1R^k} = \lim_{k \to \infty} v_{3L^k}$$

is one of the three cusps of the developed deltoid of order 0. The composition $h = \psi^{-1} \circ \varphi : \partial \mathbb{D} \to \partial \mathbb{D}$ defines a homeomorphism of the unit circle, which is called the *welding homeomorphism* of $(\partial \Omega, v_{\text{root}}, \infty, p_{\Omega})$. Form the analogous mappings φ', ψ' and h' for $(\partial \Omega', v'_{\text{root}}, \infty, p'_{\Omega})$.

Inspection shows that the weldings h and h' are related by a piecewise linear homeomorphism F of the unit circle: $h' = F \circ h$. For instance, in the example depicted in Figure 6, to describe F, we divide the unit circle into three equal arcs and map these onto arcs of lengths $\pi, \pi/2, \pi/2$ respectively, which corresponds to the fact that one third of the tree has the same number of edges as the other two thirds. Let $\text{TPL}_1 = \{F = h' \circ h^{-1} : \mathcal{T}'_1 \text{ finite, trivalent}\}$ denote the collection of piecewise linear homeomorphisms of the unit circle that arise in this way as \mathcal{T}'_1 ranges over all finite trivalent trees.

Lemma A.4. For any quasisymmetric homeomorphism of the unit circle $F \in QS_1$ which fixes $1 \in \partial \mathbb{D}$, there is a sequence of homeomorphisms $F_k \in TPL_1$ which converge to F uniformly on the unit circle and whose quasisymmetry constants are uniformly bounded. (In fact, one may choose the homeomorphisms F_k so that their quasisymmetry constants are comparable to the quasisymmetry constant of F.)

Proof. Recall the partition Π_n of S^1 into $3 \cdot 2^n$ equal arcs from Section 6.3. Since F is a quasisymmetry, there is a bound K (comparable to the quasisymmetry constant of F) such that

$$\frac{1}{K} \le \frac{|F(\alpha)|}{|F(\alpha')|} \le K$$

for all $n \ge 1$ and all pairs of adjacent arcs $\alpha, \alpha' \in \Pi_n$, and this set of inequalities characterizes quasisymmetry. For a given $\varepsilon = 1/k > 0$, we will construct a trivalent tree \mathcal{T}'_1 whose associated piecewise linear homeomorphism F_k satisfies

$$\left|\frac{|F(\alpha)|}{|F_k(\alpha)|} - 1\right| \le \varepsilon, \qquad \alpha \in \Pi_k, \tag{A.1}$$

and

$$\frac{1}{C(K)} \le \frac{|F_k(\alpha)|}{|F_k(\alpha')|} \le C(K), \tag{A.2}$$

for all pairs of adjacent arcs $\alpha, \alpha' \in \Pi_n$ with n > k. The quasisymmetry of Fand (A.1) ensure the quasisymmetry of F_k on large scales, while (A.2) guarantees quasisymmetry on small scales. Furthermore, (A.1) implies the convergence of $F_k \to$ F or equivalently of $F_k^{-1} \circ F$ to the identity.

As usual, let \mathcal{T}_k be the regular trivalent tree of depth k. Each leaf vertex $v \in \mathcal{T}_k$ corresponds to an arc $\alpha(v) \in \Pi_{k-1}$. We approximate the stretch factors $\lambda(v) = |F(\alpha)|/|\alpha|$ by choosing (possibly very large) positive integers n(v) such that

$$\left|\frac{n(v)}{n(v')} - \frac{\lambda(v)}{\lambda(v')}\right| \le \varepsilon,\tag{A.3}$$

for any pair of leaf vertices $v, v' \in \mathcal{T}_k$. We construct the tree \mathcal{T}'_1 by attaching $\tilde{n}(v) = 2 n(v) - 2$ vertices to each leaf vertex $v \in \mathcal{T}_k$, in such a way that v has n(v) leaf descendants in \mathcal{T}'_1 , all at distance ℓ or $\ell + 1$ from v, where $2^{\ell} \leq \tilde{n}(v) < 2^{\ell+1}$. (The relation between n(v) and $\tilde{n}(v)$ comes from trivalence.) In the example depicted Figure 6 above, for k = 1, the three stretch factors are 1/2, 1/4, 1/4, and the choice n(v) = 4, 2, 2, i.e. $\tilde{n}(v) = 6, 2, 2$, satisfies (A.3) exactly, i.e. with $\varepsilon = 0$.

Recall that the sequence of trees $\{\mathcal{T}'_n\}_{n=1}^{\infty}$ is built inductively from \mathcal{T}'_1 by attaching two edges to each boundary vertex at each step, while the Hausdorff limit of the \mathcal{T}'_n consists of an infinite tree \mathcal{T}'_{∞} together with a Jordan curve $\partial\Omega'$. By construction, each vertex $v \in \mathcal{T}'_n$ at distance k from the root has $2^n \cdot n(v)$ leaf descendants in \mathcal{T}'_n , and hence a total of $2^{n+1} \cdot n(v) - 2$ of descendants in \mathcal{T}'_n . An argument analogous to the one in the proof of Lemma 6.3 shows

$$\frac{\omega_{\Omega'_{e,\infty}}(\psi'(\alpha(v)))}{\omega_{\Omega'_{e,\infty}}(\psi'(\alpha(v')))} = \frac{n(v)}{n(v')}$$

for two vertices v, v' with $d(v_{\text{root}}, v) = d(v_{\text{root}}, v') = k$. Together with (A.3), the above identity implies (A.1).

Finally, (A.2) follows from the fact that the stretch factors associated to adjacent leaf vertices $v, v' \in \mathcal{T}_k$ are comparable. More precisely, let n > k and $\alpha = \alpha(v_n), \alpha' = \alpha(v'_n) \in \Pi_n$ be adjacent arcs, corresponding to vertices $v_n, v'_n \in \mathcal{T}'_\infty$ a distance n-1away from the root. If v_n, v'_n are children of the same leaf vertex $v \in \mathcal{T}_k$, then

$$\frac{1}{2} \le \frac{|F_k(\alpha)|}{|F_k(\alpha')|} \le 2.$$

If instead v_n, v'_n are children of adjacent leaf vertices $v, v' \in \mathcal{T}_k$, then

$$\frac{1}{2}\frac{n(v)}{n(v')} \le \frac{|F_k(\alpha)|}{|F_k(\alpha')|} \le 2\frac{n(v)}{n(v')},$$

as desired. The proof is complete.

A.4 Conclusion of the proof

Suppose $\tilde{\Omega} = f(\Omega)$ is the image of the developed deltoid under a quasiconformal mapping of the plane that is conformal on Ω . As in Section A.3, we denote the welding map associated to the developed deltoid by $h = \psi^{-1} \circ \varphi$, where $\varphi : (\mathbb{D}, 0, 1) \to (\Omega, v_{\text{root}}, p_{\Omega})$ and $\psi : (\mathbb{D}_e, \infty, 1) \to (\Omega_e, \infty, p_{\Omega})$ are the normalized conformal mappings to the interior and exterior of the developed deltoid respectively. Similarly, we denote the welding homeomorphism of $\partial \tilde{\Omega}$ by $\tilde{h} = \tilde{\psi}^{-1} \circ \tilde{\varphi}$, where $\tilde{\varphi} : (\mathbb{D}, 0, 1) \to (f(\Omega), f(v_{\text{root}}), f(p_{\Omega}))$ and $\tilde{\psi} : (\mathbb{D}_e, \infty, 1) \to (f(\Omega_e), \infty, f(p_{\Omega}))$ are the normalized conformal maps to the interior and exterior regions bounded by $\partial \tilde{\Omega}$. Since f is conformal on Ω , we have $\tilde{\varphi} = f \circ \varphi$ on \mathbb{D} . As

$$\hat{F} = \tilde{\psi}^{-1} \circ f \circ \psi$$

is a quasiconformal automorphism of \mathbb{D}_e , it extends to a quasisymmetric homeomorphism of the unit circle $\partial \mathbb{D}$ that fixes 1, which we denote by F. Consequently, on $\partial \mathbb{D}$, we have

$$\tilde{h} = \tilde{\psi}^{-1} \circ \tilde{\varphi} = \tilde{\psi}^{-1} \circ f \circ \varphi = F \circ h.$$

Conversely, given a quasisymmetric homeomorphism $F \in QS_1(\partial \mathbb{D})$, let \hat{F} be an arbitrary quasiconformal extension to the exterior unit disk which fixes the point

at infinity and $\psi_*\mu_{\hat{F}}$ be the pushforward of the Beltrami coefficient of \hat{F} under the conformal map $\psi: \mathbb{D}_e \to \Omega_e$. Inspection shows that any solution $f_{\hat{\mu}}: \mathbb{C} \to \mathbb{C}$ of the Beltrami equation with dilatation $\hat{\mu} = \psi_*\mu_{\hat{F}}$ in Ω_e and $\hat{\mu} = 0$ on Ω maps $\partial\Omega$ to a Jordan curve with welding $F \circ h$.

Let $F_k \in \text{TPL}_1$ be the piecewise-linear approximations of F given by Lemma A.4. With help of the Ahlfors-Beurling extension, it is not hard to construct quasiconformal extensions \hat{F}_k of F_k and \hat{F} of F to the exterior unit disk, such that the Beltrami coefficients $\mu_{\hat{F}_k}$ converge to the Beltrami coefficient $\mu_{\hat{F}}$ uniformly on compact subsets of \mathbb{D}_e and have uniformly bounded dilatation. (Alternatively, using an argument similar to the one in [16], one can construct a quasiconformal self-map of the exterior unit disk \hat{F}_k which agrees with F_k on the unit circle and with \hat{F} on $\hat{\mathbb{C}} \setminus B(0, r_k)$, for some radius r_k slightly larger than 1. In this case, $\mu_{\hat{F}_k} = \mu_{\hat{F}}$ outside the disc of radius r_k .)

Consequently, if one normalizes the solutions $f_k = f_{\hat{\mu}_k}$ of the Beltrami equation (with dilatations $\hat{\mu}_k = \psi_* \mu_{\hat{F}_k}$ in Ω_e and $\hat{\mu}_k = 0$ on Ω) such that

$$f_k(v_{\text{root}}) = f(v_{\text{root}}), \qquad f_k(p_\Omega) = f(p_\Omega), \qquad f_k(\infty) = f(\infty) = \infty,$$

then the curves $f_k(\partial \Omega)$ converge in the Hausdorff sense to $\partial \tilde{\Omega} = f(\partial \Omega)$. From the discussion above, it follows that for any $k = 1, 2, \ldots$, the four-tuple

$$(f_k(\partial\Omega), f_k(v_{\text{root}}), \infty, f_k(p_\Omega))$$

has welding homeomorphism $F_k \circ h$. In light of Reduction 2 from Section A.2, to complete the proof of Theorem A.2, it remains to explain that the curves $f_k(\partial\Omega)$ are linear rescalings of the generalized developed deltoids $\partial\Omega'_k$ with weldings $F_k \circ h$. This crucially relies on the conformal removability of the boundary of the developed deltoid, proved in [8]. Namely, the conformal removability of $\partial\Omega$ implies that the Jordan curve with welding $F_k \circ h$ is uniquely determined up to a Möbius transformation (and in fact, up to a linear map, given our normalization at infinity).

B True tree approximation of cauliflower

In this appendix, we describe a sequence of true trees whose limit set is the cauliflower, the Julia set of $f(z) = z^2 + 1/4$.



Figure 8: A sequence of planar trees given by an inductive construction.

Let T_1 be a planar tree which consists of a root vertex v_{root} and four edges

$$\overline{v_{\text{root}}v_{\uparrow}}, \quad \overline{v_{\text{root}}v_{\rightarrow}}, \quad \overline{v_{\text{root}}v_{\downarrow}}, \quad \overline{v_{\text{root}}v_{\leftarrow}},$$

labeled counter-clockwise. The arrows indicate the positions of the vertices relative to the root vertex. For instance, the vertex v_{\uparrow} is located above v_{root} . We colour the edges $\overline{v_{\text{root}}v_{\uparrow}}$, $\overline{v_{\text{root}}v_{\downarrow}}$ blue and $\overline{v_{\text{root}}v_{\leftarrow}}$, $\overline{v_{\text{root}}v_{\rightarrow}}$ red. To obtain T_{n+1} from T_n , we attach additional edges at each leaf vertex:

- If a leaf edge is red, we attach another red edge at the leaf vertex.
- If a leaf edge is blue, we attach three edges, coloured blue-red-blue in counterclockwise order.

The trees T_1 and T_2 are depicted on Figure 8. From this description, it is easy to see that T_n is made out of

$$4 + 8 + 16 + \dots + 2^{n+1} = 2^{n+2} - 4$$

edges, with the same number of red and blue edges.



Figure 9: A sequence of true trees which approximates $\mathcal{J}(z^2 + 1/4)$.

Let \mathcal{T}_n be the hydrodynamically-normalized true tree representative of T_n . Note that the colouring is only used to describe the combinatorics of T_n , it plays no role in how the true tree \mathcal{T}_n is constructed from T_n .

In order to state a result analoguous to Theorem 1.1, we recall two definitions from complex dynamics. Let f be a rational map acting on the Riemann sphere. The grand orbit of a point $z \in \hat{\mathbb{C}}$ is the set of points which can be obtained from zby means of forward and backward iteration. In other words, the grand orbit of zconsists of points $w \in \hat{\mathbb{C}}$ such that $f^{\circ m}(w) = f^{\circ n}(z)$ for some $m, n \geq 0$.

It is well-known that if p is a parabolic fixed point of f and Ω is one of the components of the immediate parabolic basin of attraction of p, then the quotient $\Omega/(z \sim f(z))$ is a cylinder, and so is conformally equivalent to $\mathbb{C}/(w \sim w + 1)$. Lifting the conformal equivalence of the cylinders, we obtain a holomorphic map $\psi: \Omega \to \mathbb{C}$ known as the *Fatou coordinate* which satisfies

$$\psi(f(z)) = \psi(z) + 1.$$
 (B.1)

From the construction, it is clear that the Fatou coordinate is determined uniquely up to an additive constant. We refer the reader to [13, Section 10] for details. **Theorem B.1.** The trees \mathcal{T}_n converge in the Hausdorff topoology to an infinite tree union a Jordan curve $\mathcal{T} \cup \partial \Omega$. The Jordan curve $\partial \Omega$ is the Julia set of $z^2 + 1/4$, while the set of vertices of \mathcal{T} is the grand orbit of the critical point 0 of $f(z) = z^2 + 1/4$. Let $\psi : \Omega \to \mathbb{C}$ be the Fatou coordinate at the parabolic fixed point $1/2 \in \mathcal{J}(f)$, with $\psi(0) = 0$. The Shabat polynomials $p_n(z)$ of \mathcal{T}_n , with $p_n(0) = 1$, converge uniformly on compact subsets of Ω to $\cos(\pi \cdot \psi(z))$.

The proof of the above theorem is similar to that of Theorem 1.1, but the moduli estimates are more cumbersome. Rather than presenting a full proof, we provide a few key insights to explain why Theorem B.1 holds.

B.1 Tile decomposition

As shown on the right side of Figure 9, the repeated pre-images of the line segment [0, 1/2) separate Ω , the interior of the filled Julia set of $f(z) = z^2 + 1/4$, into a countable collection of *tiles*. The union of these curves constitutes a tree, which we refer to as the *skeleton of the cauliflower*, whose vertices are points in the grand orbit of the critical point 0. We designate the critical point 0 as the root vertex. Note that [0, 1/2) is not a single edge but the union of countably many edges:

$$[0, 1/2) = [0, f(0)] \cup [f(0), f^{\circ 2}(0)] \cup [f^{\circ 2}(0), f^{\circ 3}(0)] \cup \dots$$

We label the tiles as $\Omega_{p,L}$ or $\Omega_{p,R}$, where p ranges over the cusps in $\partial\Omega$. Below, we write X for one of the symbols L, R. We define the *bi-tile* Ω_p as the union of $\Omega_{p,L}, \Omega_{p,R}$ and the arc in Ω that makes up the common boundary of the two tiles.

Under iteration, any tile is eventually mapped onto $\Omega_{1/2,L}$ or $\Omega_{1/2,R}$. The tiles $\Omega_{1/2,L}$ and $\Omega_{1/2,R}$ are invariant under f, and f restricts to a conformal automorphism on both $\Omega_{1/2,L}$ and $\Omega_{1/2,R}$. We record the following two properties of Ω , which come from the dynamics of f and the symmetry of Ω with respect to the real axis:

(CT1) If $\Omega_{p,X}$ is a tile, then each edge in $\partial \Omega_{p,X}$ has the same relative harmonic measure as viewed from p, i.e. if $e_1, e_2 \subset \Omega_{p,X}$, then

$$\lim_{z \to p, z \in \Omega_{p,X}} \frac{\omega_z(e_1)}{\omega_z(e_2)} = 1.$$

(CT2) If e is an edge that belongs to two neighbouring tiles $\Omega_{p,X}$ and $\Omega_{q,Y}$, then the relative harmonic measures are the same from both sides. This means that for any measurable subset $E \subset e$,

$$\lim_{z \to p, z \in \Omega_{p,X}} \frac{\omega_z(E)}{\omega_z(e)} = \lim_{z \to q, z \in \Omega_{q,Y}} \frac{\omega_z(E)}{\omega_z(e)}$$

We may further decompose each tile $\Omega_{p,X} \subset \Omega$ into countably many triangles $\triangle(e, p, X)$ by connecting the vertices in $\partial\Omega_{p,X}$ to the cusp $p \in \partial\Omega_{p,X}$ by hyperbolic geodesics in $\Omega_{p,X}$. Given that f maps tiles to tiles, vertices to vertices and cusps to cusps, it also maps triangles to triangles. (Since f maps the tile $\Omega_{p,X}$ conformally onto $\Omega_{f(p),X}$, it carries hyperbolic geodesics connecting p to the vertices on $\partial\Omega_{p,X}$ onto geodesics connecting f(p) to the vertices on $\partial\Omega_{f(p),X}$.)

We colour the triangles $\triangle(e, p, X) \subset \Omega$ black and white, so that

$$\Delta = \Delta \left(\overline{v_{\text{root}} f(v_{\text{root}})}, 1/2, R \right) \subset \Omega_{1/2,R} = \Omega_{1/2} \cap \mathbb{H}$$

is white and adjacent triangles have different colours. Inspection shows that f sends triangles to triangles of opposite colour. Reflecting Δ in the real line, we get a triangle $\overline{\Delta} \subset \Omega_{1/2,L} = \Omega_{1/2} \cap \mathbb{L}$. The union $\Delta \cup \overline{v_{\text{root}}f(v_{\text{root}})} \cup \overline{\Delta}$ constitutes a fundamental domain for the action of f on Ω .

B.2 Identifying the limit of the Shabat polynomials

Once the convergence of the finite true trees \mathcal{T}_n to the skeleton of the cauliflower has been established, the following lemma can be used to identify $\cos(\pi\psi(z))$ as the limit of the Shabat polynomials:

Lemma B.2. The map $z \to \cos(\pi\psi(z))$ takes each triangle $\triangle(e, p, X) \subset \Omega$ conformally onto the upper half-plane or the lower half-plane, with black triangles mapping onto the upper half-plane \mathbb{H} and white triangles mapping onto the lower half-plane \mathbb{L} . Furthermore, $\cos(\pi\psi(z))$ takes edges to [-1, 1], cusps to infinity and v_{root} to 1.

Proof. Step 1. Mapping properties of $\cos(\pi z)$. The lines $\{y = 0\}$ and $\{x = n : n \in \mathbb{Z}\}$ partition the complex plane into vertical half-strips $\{S_{n,\pm}\}$ of width 1. These may

be coloured black and white so that adjacent half-strips have opposite colours, with

$$\mathcal{S}_{0,+} = \{ z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, 0 < \operatorname{Im} z < \infty \}$$

being white. Inspection shows that $z \to \cos(\pi z)$ takes each black half-strip conformally onto the upper half-plane and each white half-strip conformally onto the lower half-plane. Furthermore, the horizontal side of each $S_{n,\pm}$ is mapped to the interval [-1, 1], while the vertical sides are mapped to the intervals $(-\infty, -1]$ and $[1, \infty)$.

Step 2. Mapping properties of the Fatou coordinate. Recall from the statement of Theorem B.1 that we normalize the Fatou coordinate ψ so that $\psi(v_{\text{root}}) = 0$. It is not difficult to show that ψ maps the bi-tile $\Omega_{1/2}$ conformally onto $\mathbb{C} \setminus (-\infty, 0]$. By symmetry considerations, ψ maps $\Omega_{1/2,R}$ conformally to \mathbb{H} and $\Omega_{1/2,L}$ conformally to \mathbb{L} . Since $\psi : \Omega_{1/2,R} \to \mathbb{H}$ takes 1/2 to infinity, it carries geodesics in $\Omega_{1/2,R}$ emanating from 1/2 onto vertical rays in \mathbb{H} . In view of the normalization on $\psi(v_{\text{root}})$, ψ maps the geodesic connecting 1/2 to v_{root} in $\Omega_{1/2,R}$ to $\{iy: 0 < y < \infty\}$. By the functional equation (B.1), ψ sends geodesics connecting 1/2 to the vertices on $\partial\Omega_{1/2,R}$ to the vertical rays $\{n + iy: 0 < y < \infty\}$ with $n \in \mathbb{Z}$. Consequently, ψ maps the triangles $\Delta(e, 1/2, R)$ conformally onto vertical half-strips $S_{n,\pm}$ of the same colour. By symmetry, the same is true for the triangles $\Delta(e, 1/2, L)$. By invariance, the same applies to any triangle $\Delta(e, p, X)$ which makes up Ω .

The lemma follows after composing the mappings from Steps 1 and 2. \Box

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