The geometry of the Weil-Petersson metric in complex dynamics

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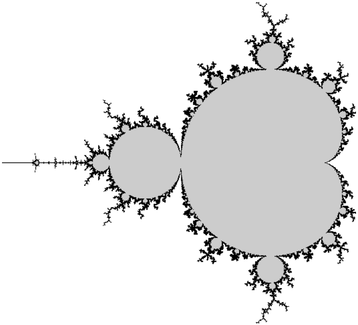
The Main Cardioid $\subset$ Mandelbrot Set



Conjecture: The Weil-Petersson metric is incomplete and its completion attaches the geometrically finite parameters.

Blaschke products

Let $\quad \mathcal{B}_d = \left\{ \begin{array}{c} \text{Blaschke products of degree } d \\ \text{with an attracting fixed point} \end{array} \right\} \Big/ \operatorname{Aut} \mathbb{D}$

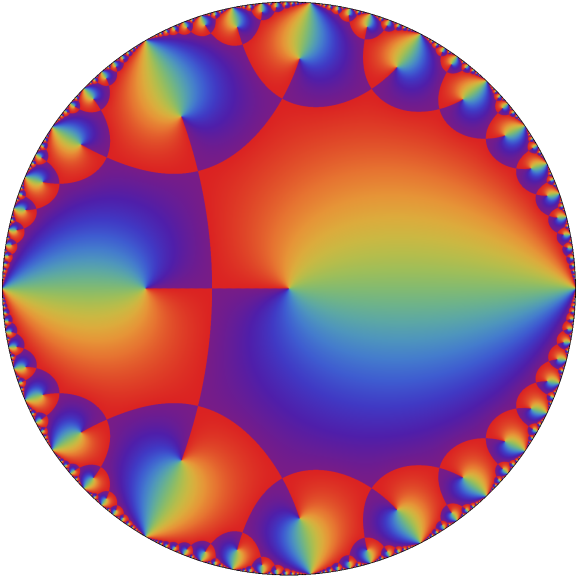
e.g $\mathcal{B}_2 \cong \mathbb{D}$:

$$a \in \mathbb{D} : \qquad z \to f_a(z) = z \cdot \frac{z+a}{1+\bar{a}z}.$$

All these maps are q.s. conjugate to each other on S^1

and except for for the special map $z \to z^2$, are q.c. conjugate on the entire disk.

$a = 0.5$



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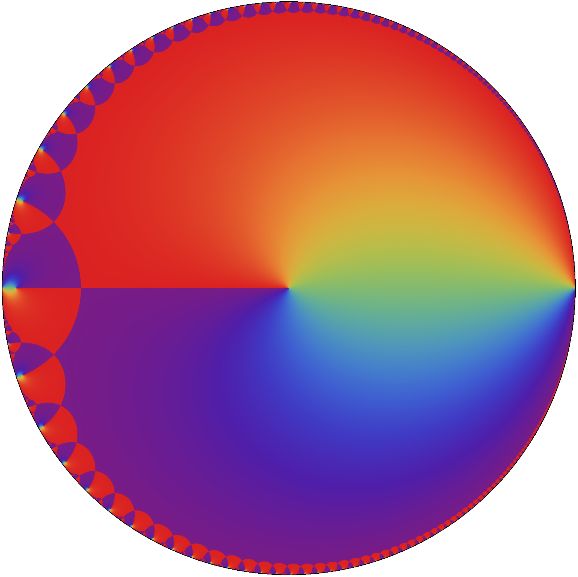


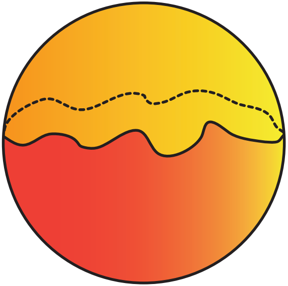
Mating

Let $f_{\mathbf{a}}, f_{\mathbf{b}}$ be Blaschke products.

Exists a rational map $f_{\mathbf{a},\mathbf{b}}$ and a Jordan curve γ s.t

- $f_{\mathbf{a},\mathbf{b}}|_{\Omega_-} \cong f_{\mathbf{a}}$,
- $f_{\mathbf{a},\mathbf{b}}|_{\Omega_+} \cong f_{\overline{\mathbf{b}}}$.

$f_{\mathbf{a},\mathbf{b}}, \gamma$ change continuously with $\mathbf{a}, \mathbf{b}$.



$$\left(\text{In degree 2,} \qquad f_{a,b} = z \cdot \frac{z+a}{1+\overline{b}z}\right)$$

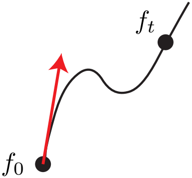
McMullen's paper on thermodynamics

Let $f_{\mathbf{a}(t)}$ be a curve in $\mathcal{B}_d$. Can form $f_{\mathbf{a}(0),\mathbf{a}(t)}$.

The function $t \to \mathrm{H.\,dim}\, \gamma_{0,t}$ satisfies:

$$\mathrm{H.\,dim}\, \gamma_{0,0} = 1.$$

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{H.\,dim}\, \gamma_{0,t} = 0.$$



Definition (McMullen).

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{H.\,dim}\, \gamma_{0,t} =: \|\dot{f}_{\mathbf{a}(t)}\|^2_{\mathrm{WP}}.$$

McMullen's paper on thermodynamics (ctd)

Let H_t denote the conformal conjugacy from $\mathbb{D}$ to $\Omega_-(f_{0,t})$. The initial map H_0 is the identity. Let

$$v = \frac{d}{dt}\bigg|_{t=0} H_t$$

be the holomorphic vector field of the deformation.

McMullen showed that

$$\|\dot{f}_{\mathbf{a}(t)}\|^2_{\mathrm{WP}} = \frac{4}{3} \cdot \lim_{r \to 1} \int_{|z|=r} \left| \frac{v'''}{\rho^2}(z) \right|^2 \frac{d\theta}{2\pi}.$$

Example: Weil-Petersson metric at z^2

Lacunary series $\quad v' \sim z + z^2 + z^4 + z^8 + \dots$

Can evaluate integral average explicitly due to orthogonality

$$\frac{1}{2\pi} \int_{S^1} z^k \overline{z^l} d\theta = \delta_{kl}.$$

Obtain Ruelle's formula

$$\mathrm{H.\,dim}\, J(z^2 + c) \sim 1 + \frac{|c|^2}{16 \log 2} + O(|c|^3).$$

Beltrami Coefficients

For an o.p. homeomorphism $w : \mathbb{C} \to \mathbb{C}$, we can compute its dilatation

$$\mu(w) = \frac{\overline{\partial} w}{\partial w}.$$

- If $\|\mu\|_\infty < 1$, we say w is quasiconformal.
- Conversely, given μ with $\|\mu\|_\infty < 1$, there exists a q.c. map w^μ with dilatation μ.

Dynamics: Given $f \in \mathrm{Rat}_d$ and $\mu \in M(\mathbb{D})^f$, can construct new rational maps by:

$$f^{t\mu}(z) = w^{t\mu} \circ f \circ (w^{t\mu})^{-1}.$$

Upper bounds on quadratic differentials

Suppose μ is supported on the exterior unit disk, $\|\mu\|_\infty \leq 1$. Then,

$$v'''(z) = -\frac{6}{\pi} \int_{|\zeta|>1} \frac{\mu(\zeta)}{(\zeta - z)^4} \cdot |d\zeta|^2.$$

Theorem:

$$\limsup_{r \to 1^-} \int_{|z|=r} \left| \frac{v'''}{\rho^2}(z) \right|^2 \frac{d\theta}{2\pi} \lesssim \limsup_{R \to 1^+} \left| \operatorname{supp} \mu \cap S_R \right|$$

where S_R is the circle $\{z : |z| = R\}$.

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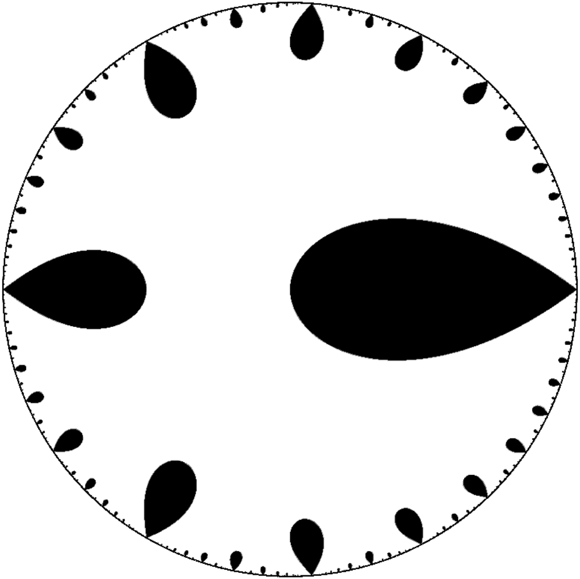

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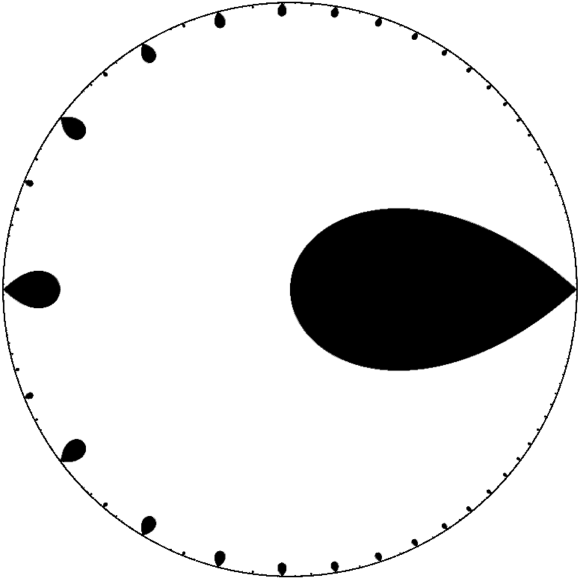

Incompleteness with a precise rate of decay

"Petal counting hypothesis" As $a \to e(p/q)$ radially, the WP metric is proportional to the petal count.

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Renewal theory:

Given a point $z \in \mathbb{D}$, let $\mathcal{N}(z, R)$ be the number of w satisfying $f^{\circ k}(w) = z$, for some $k \geq 0$, that lie in $B_{\text{hyp}}(0, R)$. Then,

$$\mathcal{N}(z, R) \sim \frac{1}{2} \cdot \frac{\log |1/z|}{h(f_a)} \cdot e^R \qquad \text{as } R \to \infty$$

where $h(f_a) = \int_{S^1} \log |f'(z)| \cdot \frac{d\theta}{2\pi}$ is the entropy of Lebesgue measure.

Incompleteness with a precise rate of decay (cont.)

If $\lim_{r\to 1^-} \int_{|z|=r} |v'''/\rho^2|^2 d\theta$ was proportional to the number of petals, then it would be asymptotically $\sim C_{p/q} \cdot \dfrac{|da|}{(1-|a|)^{3/4}}$.

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WARNING!

We might have correlations

$$\left| \sum_{P \neq Q} \frac{v_P'''}{\rho^2} \cdot \overline{\frac{v_Q'''}{\rho^2}} \right|.$$

Schwarz lemma: The petals are separated in the hyperbolic metric. Indeed, $d_{\mathbb{D}}(P, Q) \geq d_{\mathbb{D}}(P_1, P_2) \gtrsim d_{\mathbb{D}}(0, a)$.

Decay of Correlations

Fact: if $d_{\mathbb{D}}(z, \operatorname{supp} \mu^+) > R$, then $|v'''/\rho^2| \lesssim e^{-R}$.

Triangle inequality: For any $z \in \mathbb{D}$,

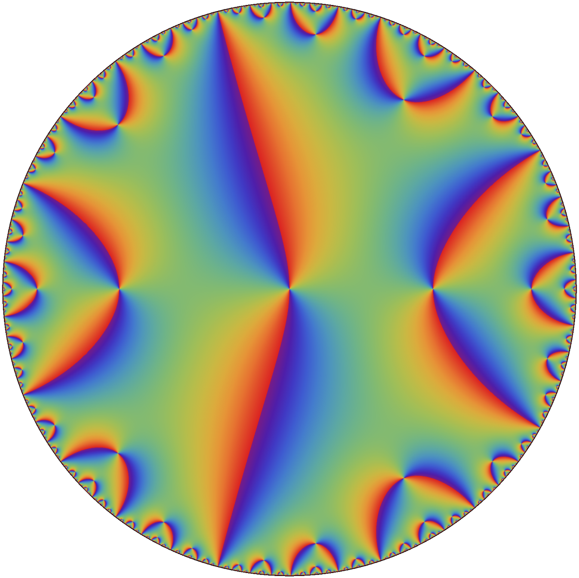
$$C(z) \leq \left| \sum_{P \neq Q} \frac{v_P'''}{\rho^2}(z) \cdot \frac{\overline{v_Q'''}}{\rho^2}(z) \right| \lesssim e^{-R_1} \cdot e^{-R_2} = e^{-R}.$$

As $e^{-d_{\mathbb{D}}(0,a)} \asymp 1 - |a|$, correlations decay like $\asymp 1 - |a|$.

REMARK!

This is neligible to the diagonal term $\sim \sqrt{1 - |a|}$.

$a \to -1$



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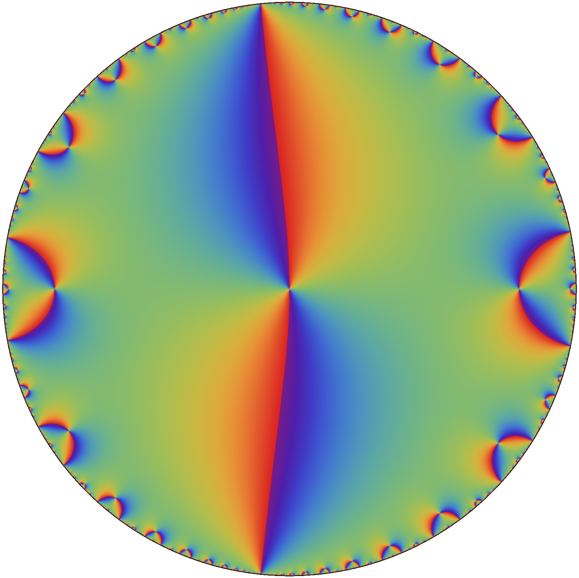
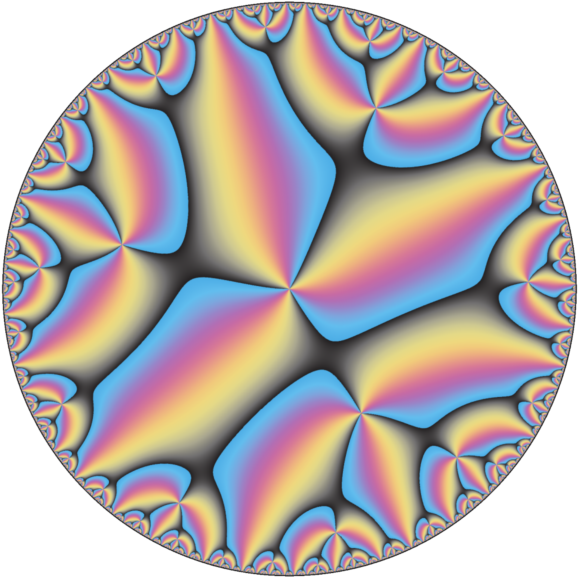


$a \to e(1/3)$



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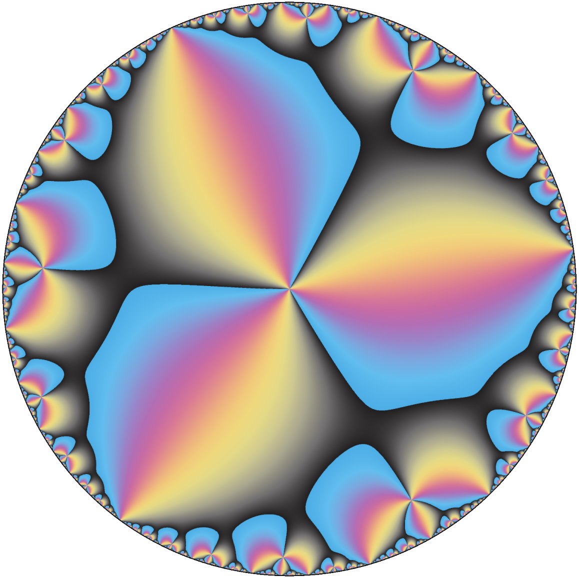

$a \to 1$ horocyclically

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Rescaling Limits

"Critically centered versions" $\quad \tilde{f}_a = m_{c,0} \circ f_a \circ m_{0,c}$

$a \to 1$ radially:

$$\tilde{f}_a \to \frac{z^2 + 1/3}{1 + 1/3z^2}.$$

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In $\mathbb{H}$, this is just $w \to w - 1/w$.

$a \to 1$ along a horocycle:

$$\tilde{f}_a \to w - 1/w + T$$

with $T > 0$ (clockwise) and $T < 0$ (counter-clockwise).

Rescaling Limits (ctd)

Amazingly, if $a \to e(p/q)$ along a horocycle, then $\tilde{f}_a^{\circ q}$ converges to the same class of maps, i.e

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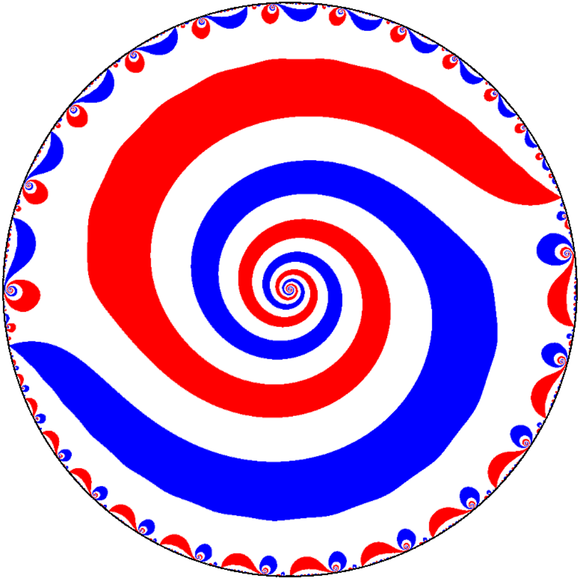
Lavaurs-Epstein boundary:

The WP metric is asymptotically periodic along horocycles

"Lavaurs phase"

We attach a punctured disk to every cusp with the same analytic and metric structure that models the limiting behaviour along horocycles.

$a \to -1$ horocyclically



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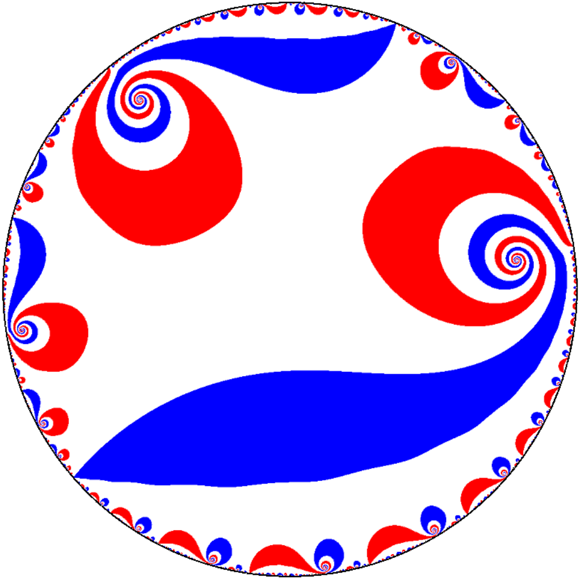
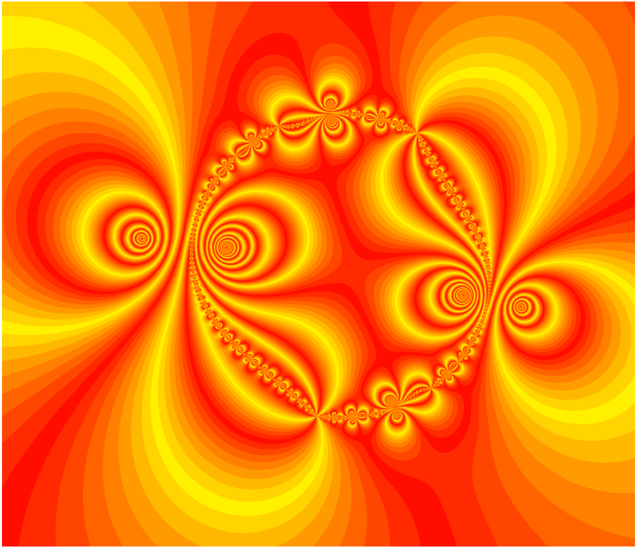
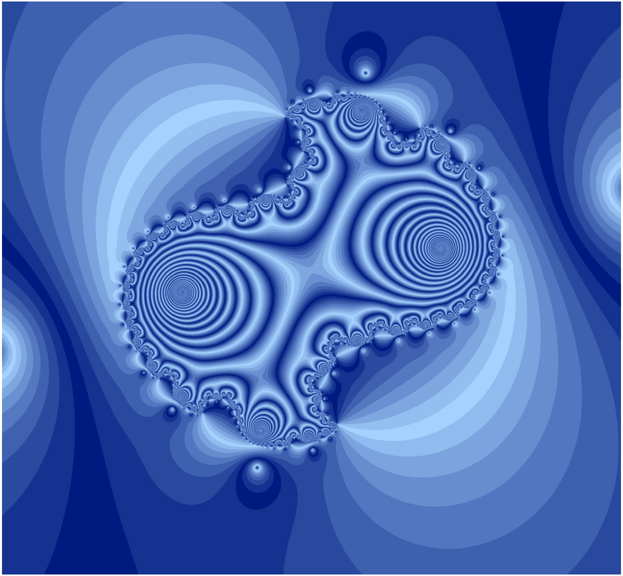


A quasi-Blaschke product – Horizontal direction



A quasi-Blaschke product – Vertical direction



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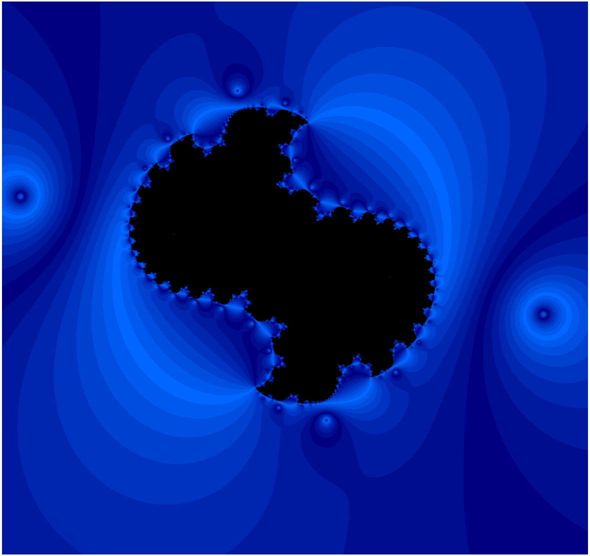

Beyond degree 2: Spinning in $\mathcal{B}_3$