THE GEOMETRY OF THE WEIL-PETERSSON METRIC IN COMPLEX DYNAMICS

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ABSTRACT. In this work, we study an analogue of the Weil-Petersson metric on the space of Blaschke products of degree 2 proposed by McMullen. Via the Bers embedding, one may view the Weil-Petersson metric as a metric on the main cardioid of the Mandelbrot set. We prove that the metric completion attaches the geometrically finite parameters from the Euclidean boundary of the main cardioid and conjecture that this is the entire completion.

For the upper bound, we estimate the intersection of a circle $S_r = \{z : |z| = r\}$, $r \approx 1$, with an invariant subset $\mathcal{G} \subset \mathbb{D}$ called a half-flower garden, defined in this work. For the lower bound, we use gradients of multipliers of repelling periodic orbits on the unit circle. Finally, utilizing the convergence of Blaschke products to vector fields, we compute the rate at which the Weil-Petersson metric decays along radial degenerations.

CONTENTS

1. Introduction

1.1. **Basic notation.** We write $\mathbb{D}$ for the unit disk and S^1 for the unit circle. Let m denote the Lebesgue measure on S^1, normalized to have unit mass. Given two points $z_1, z_2 \in \mathbb{D}$, we denote the hyperbolic distance between them by $d_{\mathbb{D}}(z_1, z_2) = \inf \int_\gamma \rho$. We use the convention that the hyperbolic metric on the unit disk is $\rho(z)|dz| = \frac{2|dz|}{1-|z|^2}$, while the Kobayashi metric is $\frac{|dz|}{1-|z|^2}$. The hyperbolic geodesic connecting the two points is denoted by $[z_1, z_2]$. For $z \in \mathbb{C} \setminus \{0\}$, let $\hat{z} := z/|z|$. Let $\mathcal{B}_{p/q}(\eta) \subset \mathbb{D}$ be the horoball of Euclidean diameter η/q^2 which rests on $e(p/q) := e^{2\pi i(p/q)}$ and $\mathcal{H}_{p/q}(\eta) = \partial \mathcal{B}_{p/q}(\eta)$ be its boundary horocycle. To compare quantities, we use:

- $A \lesssim B$ means $A < \text{const} \cdot B$,
- $A \sim B$ means $A/B \to 1$,
- $A \asymp B$ means $C_1 \cdot B < A < C_2 \cdot B$ for some constants $C_1, C_2 > 0$,
- $A \approx_\epsilon B$ means $|A/B - 1| \lesssim \epsilon$.

1.2. **The traditional Weil-Petersson metric.** To set the stage, we recall the definition and basic properties of the Weil-Petersson metric on Teichmüller space. Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of marked Riemann surfaces of genus g with n punctures. For a Riemann surface $X \in \mathcal{T}_{g,n}$, consider the spaces

- $Q(X)$ of holomorphic quadratic differentials with $\int_X |q| < \infty$,
- $M(X)$ of measurable Beltrami coefficients satisfying $\|\mu\|_\infty < \infty$.

There is a natural pairing between quadratic differentials and Beltrami coefficients given by integration $\langle \mu, q \rangle = \int_X \mu q$. One has natural identifications

$$T_X^* \mathcal{T}_{g,n} \cong Q(X), \qquad T_X \mathcal{T}_{g,n} \cong M(X)/Q(X)^\perp.$$

We will discuss two natural metrics on Teichmüller space: the Teichmüller metric and the Weil-Petersson metric. On the cotangent space, the Teichmüller and Weil-Petersson norms are given by

$$\|q\|_T = \int_X |q|, \qquad \|q\|_{\mathrm{WP}}^2 = \int_X \rho^{-2}|q|^2.$$

The Teichmüller and Weil-Petersson lengths of tangent vectors are defined by duality, i.e. $\|\mu\|_T := \sup_{\|q\|_T=1} |\int_X \mu q|$ and $\|\mu\|_{\mathrm{WP}} := \sup_{\|q\|_{\mathrm{WP}}=1} |\int_X \mu q|$. From the definitions, it is clear that the Teichmüller and Weil-Petersson metrics are invariant under the mapping class group $\mathrm{Mod}_{g,n}$. However, unlike the Teichmüller metric, the Weil-Petersson metric is not complete.

For the Teichmüller space of a punctured torus $\mathcal{T}_{1,1} \cong \mathbb{H}$, the mapping class group is $\mathrm{Mod}_{1,1} \cong \mathrm{SL}(2, \mathbb{Z})$. Let us denote the Weil-Petersson metric on $\mathcal{T}_{1,1}$ by $\omega_T(z)|dz|$.

To describe the metric completion of $(\mathcal{T}_{1,1}, \omega_T)$, we introduce a system of disjoint horoballs. Let $B_{1/0}(\eta)$ denote the horoball $\{z : y \geq 1/\eta\}$ that rests on $\infty = 1/0$ and $B_{p/q}(\eta)$ denote the horoball of Euclidean diameter η/q^2 that rests on p/q. For a fixed $\eta \geq 0$, $\bigcup_{p/q \in \mathbb{Q} \cup \{\infty\}} B_{p/q}(\eta)$ is an SL$(2, \mathbb{Z})$-invariant collection of horoballs. When $\eta = 1$, the horoballs have disjoint interiors but many mutual tangencies. We denote the boundary horocycles by $H_{p/q}(\eta) := \partial B_{p/q}(\eta)$ and $H_{1/0}(\eta) := \partial B_{1/0}(\eta)$.

Consider $\mathbb{H}$ with the usual topology. Extend this topology to $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ by further requiring $\{B_{p/q}(\eta)\}_{\eta \geq 0}$ to be open sets for $p/q \in \mathbb{Q} \cup \{\infty\}$. Let us also consider a family of incomplete SL$(2, \mathbb{Z})$-invariant model metrics ρ_α on the upper half-plane: for $\alpha > 0$, let ρ_α be the unique SL$(2, \mathbb{Z})$-invariant metric which coincides with the hyperbolic metric $|dz|/y$ on $\mathbb{H} \setminus \bigcup_{p/q \in \mathbb{Q} \cup \{\infty\}} B_{p/q}(1)$ and is equal to $|dz|/y^{1+\alpha}$ on $B_{1/0}(1)$.

Lemma 1.1. *For $\alpha > 0$, the metric completion of $(\mathbb{H}, \rho_\alpha)$ is homeomorphic to $\mathbb{H}^*$.*

Sketch of proof. To see that the irrational points are infinitely far away in the ρ_α metric, notice that the horoballs $\{B_{p/q}(2)\}$ cover the upper half-plane, while by SL$(2, \mathbb{Z})$-invariance, the distance between $H_{p/q}(2)$ and $H_{p/q}(3)$ is bounded below in the ρ_α metric. Therefore, any path γ that tends to an irrational number must pass through infinitely many protective shells $B_{p/q}(3) \setminus B_{p/q}(2)$. In fact, this argument shows that an incomplete path γ is trapped within some horoball $B_{p/q}(3)$, from which it follows that it must eventually enter arbitrarily small horoballs. By the form of ρ_α in $B_{p/q}(1)$, it is easy to see that the completion attaches only one point to the cusp at p/q. $\square$

Theorem 1.1 (Wolpert). *The Weil-Petersson metric on $\mathcal{T}_{1,1}$ is comparable to $\rho_{1/2}$, i.e. $1/C \leq \omega_T/\rho_{1/2} \leq C$ for some $C > 1$.*

Corollary. *The metric completion of $(\mathcal{T}_{1,1}, \omega_T)$ is homeomorphic to $\mathbb{H}^*$.*

1.3. **Main results.** In this paper, we replace the study of Fuchsian groups with complex dynamical systems on the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Inspired by Sullivan's dictionary, we are interested in understanding the Weil-Petersson metric on the space

$$\mathcal{B}_2 = \left\{ \begin{array}{c} f : \mathbb{D} \to \mathbb{D} \text{ is a proper degree 2 map} \\ \text{with an attracting fixed point} \end{array} \right\} \Big/ \text{ conjugacy by Aut}(\mathbb{D}). \quad (1.1)$$

The multiplier at the attracting fixed point $a : f \to f'(p)$ gives a holomorphic isomorphism $\mathcal{B}_2 \cong \mathbb{D}$. By putting the attracting fixed point at the origin, we can parametrize $\mathcal{B}_2$ by

$$a \in \mathbb{D} : \qquad z \to f_a(z) = z \cdot \frac{z + a}{1 + \overline{a}z}. \quad (1.2)$$

All degree 2 Blaschke products are quasisymmetrically conjugate to each other on the unit circle, and except for the special map $z \to z^2$, they are quasiconformally conjugate on the entire disk. For this reason, it is somewhat simpler to work with $\mathcal{B}_2^\times := \mathcal{B}_2 \setminus \{z \to z^2\}$, the quasiconformal moduli space $\mathcal{M}(f)$ of a rational map described in [MS]. Given a Blaschke product $f \in \mathcal{B}_2^\times$, an f-invariant Beltrami coefficient on the unit disk $\mu \in M(\mathbb{D})^f$ defines a tangent vector in $T_f \mathcal{B}_2^\times$. Since an f-invariant Beltrami coefficient descends to a Beltrami coefficient on the quotient torus of the attracting fixed point, $M(\mathbb{D})^f \cong M(T_f)$. According to [MS], μ defines a trivial deformation in $\mathcal{B}_2^\times$ if and only if it defines a trivial deformation of $T_f \in \mathcal{T}_{1,1}$. In other words, one has a natural identification of tangent spaces $T_f \mathcal{B}_2^\times \cong T_{T_f} \mathcal{T}_{1,1}$ which shows that $\mathcal{T}_{1,1}$ is the universal cover of $\mathcal{B}_2^\times$.

To make the parallels with Teichmüller theory more explicit, we state our results on the universal cover. For this purpose, we pullback the Weil-Petersson metric on $\mathcal{B}_2^\times$ by $a(\tau) = e^{2\pi i \tau}$ to obtain a metric on $\mathcal{T}_{1,1} \cong \mathbb{H}$, which we also denote ω_B.

Conjecture A. The metric ω_B on $\mathcal{T}_{1,1} \cong \mathbb{H}$ is comparable to $\rho_{1/4}$ on $\{\tau : \operatorname{Im} \tau < 1\}$. In particular, the metric completion of $(\mathcal{T}_{1,1}, \omega_B)$ is homeomorphic to $\mathbb{H}^*$.

In this paper, we show that $1/4$ is the correct exponent in the conjecture above. More precisely, we show that:

Theorem 1.2. *The Weil-Petersson metric ω_B on $\mathcal{T}_{1,1} \cong \mathbb{H}$ satisfies:*

(a) $\omega_B \leq C \rho_{1/4}$.
(b) *There exists $C_{\mathrm{small}} > 0$ such that on $\bigcup_{p/q \in \mathbb{Q}} B_{p/q}(C_{\mathrm{small}})$, $\omega_B \geq (1/C)\rho_{1/4}$.*

Corollary. *The Weil-Petersson metric on $\mathcal{B}_2$ is incomplete. In fact, the Weil-Petersson length of each line segment $e(p/q) \cdot [1/2, 1)$ is finite.*

Corollary. *The space $\mathbb{H}^*$ naturally embeds into the completion of $(\mathcal{T}_{1,1}, \omega_B)$.*

Remark. Since the Weil-Petersson metric is a real-analytic metric on $\mathcal{B}_2$, the cusp at infinity in the $\mathbb{H}^*$-model is somewhat special:

$$w_B \sim C e^{-2\pi \operatorname{Im} \tau} |d\tau|, \qquad \text{as } \operatorname{Im} \tau \to \infty.$$

Along radial rays $a \to e(p/q)$, we have a more precise estimate:

Theorem 1.3. *Given a rational number $p/q \in \mathbb{Q}$, as $\tau = p/q + it \to p/q$ vertically, the ratio $\omega_B / \rho_{1/4} \to C_q$, where C_q is a positive constant independent of p.*

Conjecture B. We conjecture that C_q is a universal constant, independent of q.

In a forthcoming work [Ivr], we will show that the Weil-Petersson metric is asymptotically periodic if we approach $a \to e(p/q)$ along a horocycle. The proof combines ideas from the work of Epstein [E] on rescaling limits with parabolic implosion.

1.4. **Properties of the Weil-Petersson metric.** In this section, we give a definition of the Weil-Petersson metric on $\mathcal{B}_2^\times \subset \mathcal{B}_2$ in the form most useful for our later work. In Section 1.7, we will give equivalent definitions which work on the entire space $\mathcal{B}_2$. For example, the Weil-Petersson metric may be described as the second derivative of the Hausdorff dimension of one-parameter families of Julia sets.

It is convenient to put the Beltrami coefficient on the exterior unit disk. For a Beltrami coefficient $\mu \in M(\mathbb{D})$, we let μ^+ denote the reflection of μ in the unit circle:

$$\mu^+ = \begin{cases} 0 & \text{for } z \in \mathbb{D}, \\ (1/\overline{z})^* \mu & \text{for } z \in S^2 \setminus \mathbb{D}. \end{cases} \tag{1.3}$$

Suppose $X \in \mathcal{T}_{g,n}$ is a Riemann surface and $\mu \in M(X)$ is a Beltrami coefficient. If $X \cong \mathbb{D}/\Gamma$, we can consider μ as a Γ-invariant Beltrami coefficient on the unit disk. Let v be a solution of $\overline{\partial} v = \mu^+$. Since the set of all solutions is of the form $v + \mathrm{sl}(2, \mathbb{C})$, the third derivative v''' uniquely depends on μ^+. As v''' is an infinitesimal version of the Schwarzian derivative, it is naturally a quadratic differential. In [McM2], McMullen observed that

$$\frac{\|\mu\|_{\mathrm{WP}}^2}{4 \cdot \mathrm{Area}(X, \rho^2)} = \mathcal{I}[\mu] = \lim_{r \to 1^-} \frac{1}{2\pi} \int_{|z|=r} \left| \frac{v_{\mu^+}'''(z)}{\rho(z)^2} \right|^2 d\theta. \tag{1.4}$$

Similarly, given a Blaschke product $f \in \mathcal{B}_2^\times$, we can solve the equation $\overline{\partial} v = \mu^+$ for $\mu \in M(\mathbb{D})^f$. As above, a solution v of the equation $\overline{\partial} v = \mu^+$ is well-defined up to adding a holomorphic vector field in $\mathrm{sl}(2, \mathbb{C})$ so that v''' is uniquely defined. Following [McM2], we *define* the Weil-Petersson metric $\|\mu\|_{\mathrm{WP}}^2 := \mathcal{I}[\mu]$ provided that the limit exists. In Section 7, we will use renewal theory to establish the existence of this limit for any $\mu \in M(\mathbb{D})^f$, invariant under a degree 2 Blaschke product other than $z \to z^2$.
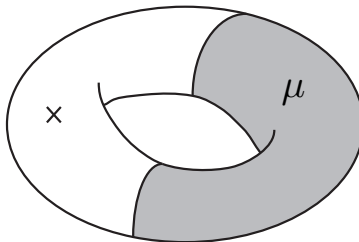

FIGURE 1. The support of the Beltrami coefficient takes up half of the quotient torus.

1.5. **A glimpse of incompleteness.** We now sketch the proof of the upper bound in Theorem 1.2. To establish the incompleteness of the Weil-Petersson metric, we consider "half-optimal" Beltrami coefficients $\mu_\lambda \cdot \chi_{\mathcal{G}(f_a)}$ which take up half of the quotient torus at the attracting fixed point, but are sparse near the unit circle.
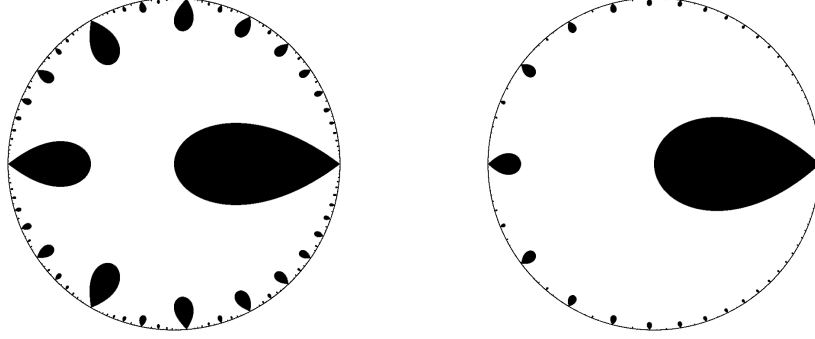

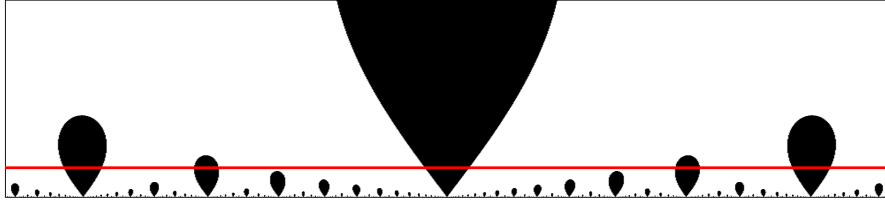FIGURE 2. Gardens $\mathcal{G}(f_a)$ for the Blaschke products with $a = 0.5$ and 0.8.

FIGURE 3. A blow-up of $\mathcal{G}(f_{0.5})$ near the boundary. A circle $\{z : |z| = r\}$ with r close to 1 meets $\mathcal{G}(f_{0.5})$ in small density.

The garden $\mathcal{G}(f_a) \subset \mathbb{D}$ is an invariant subset of the unit disk whose quotient $A = \mathcal{G}(f_a)/f_a \subset T_a$ is a certain annulus that takes up half of the Euclidean area of the quotient torus. To give an upper bound for the Weil-Petersson metric, we estimate the length of the intersection of $\mathcal{G}(f_a)$ with $S_r := \{z : |z| = r\}$. In general, one has the estimate

$$\left(\frac{\omega_B}{\rho_{\mathbb{D}^*}}\right)^2 \leq C \cdot \limsup_{r \to 1} |\mathcal{G}(f_a) \cap S_r|. \tag{1.5}$$

In order for this estimate to be efficient, we take A to be a collar neighbourhood of the shortest p/q-geodesic in the quotient torus $T_{f_a} \in \mathcal{T}_{1,1}$. For the Blaschke product f_a with parameter $a = e^{2\pi i \tau}$, $\tau \in H_{p/q}(\eta)$, we prove

$$\limsup_{r \to 1} |\mathcal{G}(f_a) \cap S_r| = O(\eta^{1/2}). \tag{1.6}$$

Combining (1.5) and (1.6), we see that $\omega_B \leq C\rho_{1/4}$ on $\{\tau : \operatorname{Im}\tau < 1\}$ as desired.

Remark. The trick of truncating the support of the Beltrami coefficient can be found in the proof of [McM1, Corollary 1.3]. See also [B].

1.6. **A glimpse of the convergence $\omega_B/\rho_{1/4} \to C_q$.** We now sketch the proof of Theorem 1.3. To understand the behaviour of the Weil-Petersson metric as $a \to e(p/q)$ radially, we study the convergence of Blaschke products to vector fields. For example, as $a \to 1$ along the real axis, we will see that even though the maps $f_a(z) = z \cdot \frac{z+a}{1+\bar{a}z}$ tend pointwise to the identity, their long-term dynamics tends to the flow of the holomorphic vector field $\kappa_1 = z \cdot \frac{z-1}{z+1} \cdot \frac{\partial}{\partial z}$. For the radial approach $a \to e(p/q)$, the maps $f_a(z) \to e(p/q)z$ converge pointwise to a rotation, and therefore the q-th iterates tend to the identity. We are thus led to extract a limiting vector field κ_q by considering limits of the high iterates of $f_a^{\circ q}$. It turns out that the vector field κ_q is a q-fold cover of the vector field κ_1. In particular, it is independent of p.
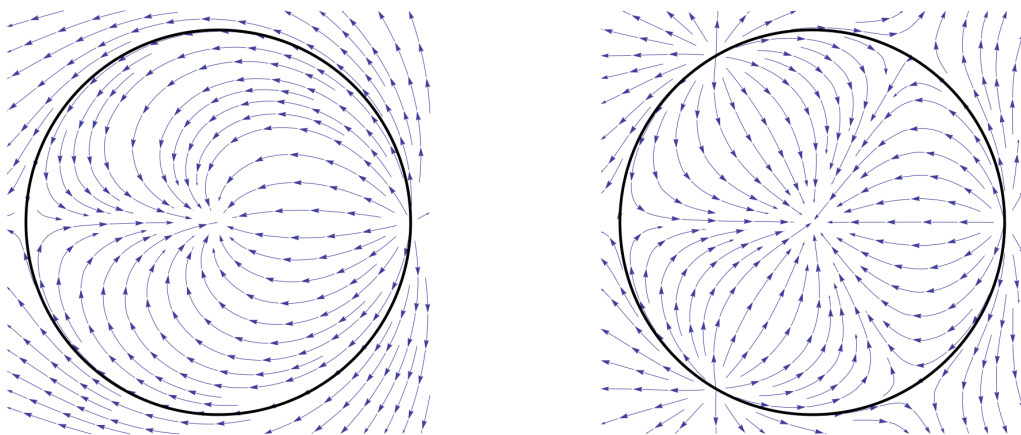
FIGURE 4. The vector fields κ_1 and κ_3.

From the convergence of Blaschke products to vector fields, it follows that the flowers that make up the gardens $\mathcal{G}(f_a)$ for $a \approx e(p/q)$ have nearly the same shape, up to affine scaling. Intuitively, for the integral average (1.4) to exist, when we replace $r = 1-\delta$ by $r = 1-\delta/2$ say, we expect to intersect twice as many flowers to replenish the integral, i.e. we expect the number of flowers to be inversely proportional in δ. This leads us to explore an orbit counting problem for Blaschke products. The decay rate of the Weil-Petersson metric is governed by the dependence of the flower count on the parameter variable a.

1.7. **Notes and references.** In this section, we describe the space of Blaschke products of higher degree and equivalent definitions of the Weil-Petersson metric.

Blaschke products of higher degree. More generally, we can consider the space $\mathcal{B}_d$ of marked Blaschke products of degree d which have an attracting fixed point modulo conformal conjugacy. By moving the attracting fixed point to the origin as before, one can parametrize $\mathcal{B}_d$ by

$$\{a_1, a_2, \ldots, a_{d-1}\} \in \mathbb{D}: \qquad z \to f_{\mathbf{a}}(z) = z \cdot \prod_{i=1}^{d-1} \frac{z + a_i}{1 + \overline{a_i} z}. \qquad (1.7)$$

Let $a := a_1 a_2 \cdots a_{d-1} = f'_{\mathbf{a}}(0)$ denote the multiplier of the attracting fixed point. It is because the maps are *marked* that we can distinguish the conformal conjugacy classes of $\mathbf{a} = \{a_1, a_2, \ldots, a_{d-1}\}$ and $\zeta \cdot \mathbf{a} = \{\zeta a_1, \zeta a_2, \ldots, \zeta a_{d-1}\}$. See [McM3] for more on markings.

Mating. It is a remarkable fact that given two Blaschke products $f_{\mathbf{a}}, f_{\mathbf{b}}$ of the same degree, one can find a rational map $f_{\mathbf{a},\mathbf{b}}(z)$ – the *mating* of $f_{\mathbf{a}}, f_{\mathbf{b}}$ – whose Julia set is a quasicircle $\mathcal{J}_{\mathbf{a},\mathbf{b}}$ which separates the Riemann sphere into two domains Ω_-, Ω_+ such that on one side $f_{\mathbf{a},\mathbf{b}}(z)$ is conformally conjugate to $f_{\mathbf{a}}$, and to $f_{\overline{\mathbf{b}}}$ on the other. The mating is unique up to conjugation by a Möbius transformation. One can prove the existence of a mating by quasiconformal surgery (see [Mil] for details). The mating $\mathcal{B}_d \times \overline{\mathcal{B}_d} \to \operatorname{Rat}_d$ varies holomorphically with parameters. A natural way to put a complex structure on $\mathcal{B}_d$ is via the *Bers embedding* $\mathcal{B}_d \to \mathscr{P}_d$ which takes a Blaschke product and mates it with z^d to obtain a polynomial of degree d. Here, the space $\mathscr{P}_d \cong \mathbb{C}^{d-1}$ is considered modulo affine conjugacy. The image of the Bers embedding is the generalized main cardioid in $\mathscr{P}_d$.

Question. For $d \geq 3$, what is the completion of $\mathcal{B}_d$ with respect to the Weil-Petersson metric? Are the additional points precisely the geometrically finite parameters on the boundary of the generalized main cardioid? What is the topology on $\overline{\mathcal{B}_d}$?

Remark. Wolpert showed that the metric completion of $(\mathcal{T}_{g,n}, \omega_T)$ is the augmented Teichmüller space $\overline{\mathcal{T}_{g,n}}$, the action of the mapping class group $\operatorname{Mod}_{g,n}$ extends isometrically to $(\overline{\mathcal{T}_{g,n}}, \omega_T)$ and the quotient $M_{g,n} = \overline{\mathcal{T}_{g,n}}/\operatorname{Mod}_{g,n}$ is the Deligne-Mumford compactification. See [Wol] for more information.

Equivalent definitions of the Weil-Petersson metric. Suppose $f \in \mathcal{B}_d$ and f_t, $t \in (-\epsilon, \epsilon)$ is a smooth path with $f_0 = f$, representing a tangent vector in $T_f \mathcal{B}_d$. Consider the vector field $v(z) := \frac{d}{dt}\big|_{t=0} H_{0,t}(z)$ where $H_{0,t} : \mathbb{D} \to \Omega_-(f_{0,t})$ is the conformal conjugacy between f_0 and $f_{0,t}$. If f is a Blaschke product other than $z \to z^d$, one can define $\|\dot{f}_t\|_{\mathrm{WP}}^2$ by the integral average (1.4), while if $f(z) = z^d$, one can use a more complicated integral average described in [McM2].

Remark. The definition of the Weil-Petersson metric via mating is slightly more general than the one via quasiconformal conjugacy given earlier because quasiconformal deformations do not exhaust the entire tangent space $T_f \mathcal{B}_d$ at the special parameters $f \in \mathcal{B}_d$ that have critical relations.

In [McM2], McMullen showed that

$$\|\dot{f}_t\|_{\mathrm{WP}}^2 = \frac{3}{4} \cdot \frac{\mathrm{Var}(\dot{\phi}, m)}{\int \log |\phi'| dm} = \frac{3}{4} \cdot \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{H.\,dim}\, \mathcal{J}_{0,t} \tag{1.8}$$

$$= -\frac{3}{16} \cdot \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{H.\,dim}(H_{t,t})_* m \tag{1.9}$$

where

$\mathcal{J}_{0,t}$ is the Julia set of $f_{0,t}$,

$H_{t,t} : S^1 \to S^1$ is the conjugacy between f_0 and f_t on the unit circle,

$(H_{t,t})_* m$ is the push-forward of the Lebesgue measure,

$\phi_t = \log |f'_{0,t}(H_{0,t}(z))|$,

$\int \log |\phi'| dm$ is the Lyapunov exponent,

$\mathrm{Var}(h, m) := \lim_{n \to \infty} \frac{1}{n} \int |S_n h(x)|^2 dm$ denotes the "asymptotic variance" in the context of dynamical systems.

Remark. Since $\mathcal{J}_{0,t}$ is a Jordan curve, $\mathrm{H.\,dim}\, \mathcal{J}_{0,t} \geq 1$, so $\frac{d}{dt}\big|_{t=0} \mathrm{H.\,dim}\, \mathcal{J}_{0,t} = 0$ and $\frac{d^2}{dt^2}\big|_{t=0} \mathrm{H.\,dim}\, \mathcal{J}_{0,t} \geq 0$. Similarly, since $(H_{t,t})_* m$ is a measure supported on the unit circle, $\mathrm{H.\,dim}(H_{t,t})_* m \leq 1$, $\frac{d}{dt}\big|_{t=0} \mathrm{H.\,dim}(H_{t,t})_* m = 0$ and $\frac{d^2}{dt^2}\big|_{t=0} \mathrm{H.\,dim}(H_{t,t})_* m \leq 0$.

1.8. **Relations with quasiconformal geometry.** The characterizations (1.8) and (1.9) of the Weil-Petersson metric are reflected in the duality between quasiconformal expansion and quasisymmetric compression:

Theorem 1.4 (Smirnov [S]). *For a k-quasiconformal map $f : S^2 \to S^2$,*

$$\mathrm{H.\,dim}\, f(S^1) \leq 1 + k^2.$$

Remark. If the dilatation $\mu(z) = \frac{\overline{\partial} f}{\partial f}$ is supported on the exterior unit disk, one has the stronger estimate $\mathrm{H.\,dim}\, f(S^1) \leq 1 + \tilde{k}^2$ where $k = \frac{2\tilde{k}}{1+\tilde{k}^2}$.

Theorem 1.5 (Prause, Smirnov [PrSm]). *For a k-quasiconformal map $f : S^2 \to S^2$, symmetric with respect to the unit circle, one has $\mathrm{H.\,dim}\, f_* m \geq 1 - k^2$.*

An application of Theorem 1.4 or Theorem 1.5 shows:

Corollary. *The Weil-Petersson metric on $\mathcal{B}_2$ is bounded above by $\sqrt{3/32} \cdot \rho_{\mathbb{D}}$.*

Proof. For a map $f_a \in \mathcal{B}_2$, the Bers embedding β_{f_a} gives a holomorphic motion of the exterior unit disk $H : \mathcal{B}_2 \times (S^2 \setminus \mathbb{D}) \to \mathbb{C}$ given by $H(b, z) := H_{b,a}(z)$. Note that the motion H is centered at a since $H(a, \cdot)$ is the identity. By the λ-lemma (e.g. see [AIM, Theorem 12.3.2]), one can extend H to a holomorphic motion $\tilde{H}$ of the Riemann sphere satisfying $\|\mu_{\tilde{H}(b,\cdot)}\|_\infty \leq \frac{b-a}{1-\overline{a}b}$. Observe that as $d_\mathbb{D}(b, a) \to 0$, $\frac{b-a}{1-\overline{a}b} \sim \frac{1}{2} \cdot d_\mathbb{D}(b, a)$. Since each map $\tilde{H}(b, \cdot)$ is conformal on $S^2 \setminus \mathbb{D}$, by the remark following Theorem 1.4, we have $\|\dot{f_t}\|_{\mathrm{WP}}^2 \leq \frac{1}{4} \cdot \frac{3}{8} \cdot \|\dot{f_t}\|_{\rho_\mathbb{D}}^2$ as desired. $\square$

Acknowledgements. I would like to express my deepest gratitude to Curtis T. McMullen for his time, energy and invaluable insights. I also want to thank Ilia Binder for many interesting conversations.

2. BACKGROUND IN ANALYSIS

In this section, we explain how to bound the integral (1.4) in terms of the density of the support of μ. We also discuss a version of Koebe's distortion theorem for maps that preserve the unit circle.

2.1. **Teichmüller theory in the disk.** For a Beltrami coefficient μ, let $v(z) = v_\mu(z)$ be a solution of the equation $\overline{\partial} v = \mu$. The following formula is well-known (e.g. see [IT, Theorem 4.37]):

$$v'''(z)dz^2 = \left(-\frac{6}{\pi} \int_\mathbb{C} \frac{\mu(\zeta)}{(\zeta - z)^4}|d\zeta|^2\right)dz^2 \tag{2.1}$$

for $z \notin \mathrm{supp}\,\mu$.

Lemma 2.1. *For a Beltrami coefficient μ and a Möbius transformation $\gamma \in \mathrm{Aut}(S^2)$, we have $v'''_{\gamma^*\mu}(z) = v'''_\mu(\gamma z) \cdot \gamma'(z)^2$ whenever $\gamma z \notin \mathrm{supp}\,\mu$. In particular, if μ is supported on the exterior of the unit disk and $\gamma \in \mathrm{Aut}(\mathbb{D})$, then*

$$\left|\frac{v'''_\mu}{\rho^2}(\gamma(z))\right| = \left|\frac{v'''_{\gamma^*\mu}}{\rho^2}(z)\right|, \qquad z \in \mathbb{D}. \tag{2.2}$$

Proof. The first statement follows from a change of variables and the identity

$$\frac{\gamma'(z_1)\gamma'(z_2)}{(\gamma(z_1) - \gamma(z_2))^2} = \frac{1}{(z_1 - z_2)^2}, \qquad z_1 \neq z_2 \in \mathbb{C}, \ \gamma \in \mathrm{Aut}(S^2), \tag{2.3}$$

while the second statement follows from the fact that $\gamma^*\rho = \rho$ for all $\gamma \in \mathrm{Aut}(\mathbb{D})$. $\square$

To obtain upper bounds for the Weil-Petersson metric, we will use the following estimate:

Theorem 2.1. *Suppose μ is a Beltrami coefficient which is supported on the exterior of the unit disk and has $\|\mu\|_\infty \leq 1$. Then,*

$$\limsup_{r \to 1^-} \frac{1}{2\pi} \int_{|z|=r} \left| \frac{v_\mu'''(z)}{\rho(z)^2} \right|^2 d\theta \;\leq\; \frac{9}{4} \cdot \|\mu\|_\infty^2 \cdot \limsup_{R \to 1^+} \frac{1}{2\pi} \big| \operatorname{supp} \mu \cap S_R \big|. \qquad (2.4)$$

Theorem 2.2. *Suppose μ is a Beltrami coefficient which is supported on the exterior of the unit disk and has $\|\mu\|_\infty \leq 1$. Let $\mu^- := \overline{(1/\overline{z})^*\mu}$ be its reflection in the unit circle. Then,*

(a) *$|(v'''/\rho^2)(z)| \leq 3/2 \cdot \|\mu\|_\infty$ for $z \in \mathbb{D}$.*
(b) *If $d_\mathbb{D}(z, \operatorname{supp}\mu^-) \geq R$ then $|(v'''/\rho^2)(z)| \lesssim e^{-R}$.*
(c) *v'''/ρ^2 is uniformly continuous in the hyperbolic metric.*

Proof. By the Möbius invariance of $|v_\mu'''/\rho^2|$, it suffices to prove these assertions at the origin. Clearly,

$$|v'''(0)| \leq \frac{6}{\pi} \int_{|\zeta|>1} \frac{1}{|\zeta|^4} \cdot |d\zeta|^2 \leq 12 \int_1^\infty \frac{dr}{r^3} = 6.$$

Hence $|v'''/\rho^2(0)| \leq \frac{3}{2}$. This proves (a). For (b), recall that $d_\mathbb{D}(0,z) = -\log(1-|z|) + O(1)$. Then,

$$|v'''(0)| \leq \frac{6}{\pi} \int_{1+Ce^{-R}>|\zeta|>1} \frac{1}{|\zeta|^4} \cdot |d\zeta|^2 \lesssim e^{-R}.$$

For (c), it suffices to observe that the kernel $\frac{1}{(\zeta-z)^4}$ is uniformly continuous at $z = 0$ for $\{\zeta : |\zeta| > 1\}$. $\qquad \square$

Proof of Theorem 2.1. Let $V_\mu(z) := \frac{6}{\pi} \int_{|\zeta|>1} \frac{|\mu(\zeta)|}{|\zeta-z|^4} \cdot |d\zeta|^2$. The calculation from part (a) of Theorem 2.2 shows that $|V_\mu/\rho^2| \leq 3/2 \cdot \|\mu\|_\infty$ has the same L^∞ bound. Set $\nu(\zeta) := \frac{1}{2\pi} \int |\mu(e^{i\theta}\zeta)| d\theta$. From Fubini's theorem, it is clear that

$$\int_{|z|=r} |V_\mu/\rho^2| d\theta = \int_{|z|=r} |V_\nu/\rho^2| d\theta, \qquad 0 < r < 1.$$

Since $\limsup_{|\zeta|\to 1^+} |\nu(\zeta)| \leq \|\mu\|_\infty \cdot \limsup_{R \to 1^+} \frac{1}{2\pi} \big| \operatorname{supp} \mu \cap S_R \big|$,

$$\limsup_{r \to 1^-} \frac{1}{2\pi} \int_{|z|=r} \left| \frac{V_\mu(z)}{\rho(z)^2} \right| d\theta \leq \frac{3}{2} \cdot \|\mu\|_\infty \cdot \limsup_{R \to 1^+} \frac{1}{2\pi} \big| \operatorname{supp} \mu \cap S_R \big|.$$

Equation (2.4) follows by multiplying the L^1 and L^∞ bounds. $\qquad \square$

2.2. A distortion theorem.

The classical version of Koebe's distortion theorem says that if $h : B(0,1) \to \mathbb{C}$ is univalent, then $|h'(z) - 1| \lesssim |z|$ for $|z| < 1/2$. We will mostly use a version of Koebe's distortion theorem for maps which preserve the real line or the unit circle:

Theorem 2.3. *Suppose* $h : B(0,1) \to \mathbb{C}$ *is a univalent function which satisfies* $h(0) = 0$, $h'(0) = 1$ *and takes real values on* $(-1,1)$. *For* $t < 1/2$, h *is nearly an isometry in the hyperbolic metric on* $B(0,t) \cap \mathbb{H}$, *i.e.* $h^*(|dz|/y) \approx_t (|dz|/y)$.

In particular, h distorts hyperbolic distance and area by a small amount:

Corollary. *For* $z_1, z_2 \in B(0,t) \cap \mathbb{H}$, $d_{\mathbb{H}}(z_1, z_2) = d_{\mathbb{H}}(h(z_1), h(z_2)) + O(t)$.

Corollary. *If* $\mathscr{B}$ *is a round ball contained in* $B(0,t) \cap \mathbb{H}$, *then*

$$\mathrm{Area}\left(\mathscr{B}, \frac{|dz|^2}{y^2}\right) \approx_t \mathrm{Area}\left(h(\mathscr{B}), \frac{|dz|^2}{y^2}\right).$$

Above, "$A \approx_t B$" denotes that $|A/B - 1| \lesssim t$. For a set $E \subset B(0,t)$, we call a set of the form $h(E)$ a *t-nearly-affine copy* of E.

Suppose μ is a Beltrami coefficient supported on the upper half-ball $B(0,1) \cap \mathbb{H}$. It is easy to see that for $z \in B(0,t) \cap \mathbb{H}$, $\left|(h^*\mu)(z) - \mu(h(z))\right| \lesssim t \cdot \|\mu\|_{\infty}$ where $h^*\mu = \mu(h(z)) \cdot \frac{\overline{h'(z)}}{h'(z)}$. In terms of quadratic differentials, we have:

Lemma 2.2. *On the lower half-ball* $B(0,t) \cap \overline{\mathbb{H}}$,

$$\left| \frac{v_{\mu}'''}{\rho^2}(h(z)) - \frac{v_{h^*\mu}'''}{\rho^2}(z) \right| \lesssim \phi_1(t) \cdot \|\mu\|_{\infty}, \qquad (2.5)$$

for some function $\phi_1(t)$ *satisfying* $\phi_1(t) \to 0^+$ *as* $t \to 0^+$.

Proof. Given $R, \epsilon > 0$, we can choose $t > 0$ sufficiently small to guarantee that

$$|h'(\zeta) - 1| < \epsilon \qquad \text{and} \qquad (z - \zeta) \approx_{\epsilon} (h(z) - h(\zeta))$$

for $z \in B(0,t) \cap \overline{\mathbb{H}}$ and $\zeta \in \mathscr{B} = \{w : d_{\mathbb{H}}(\overline{z}, w) < R\}$. Together with Theorem 2.3, these facts imply (2.5) with μ replaced by $\mu \chi_{h(\mathscr{B})}$. However, by part (b) of Theorem 2.2, the contributions of $\mu(1 - \chi_{h(\mathscr{B})})$ and $(h^*\mu)(1 - \chi_{\mathscr{B}})$ to $(v_{\mu}'''/\rho^2)(h(z))$ and $(v_{h^*\mu}'''/\rho^2)(z)$ respectively are exponentially small in R. $\qquad \square$

2.3. Applications to Blaschke products.

For a Blaschke product $f \in \mathcal{B}_d$, let $\delta_c := \min_{c \in \mathbb{D}}(1 - |c|)$ where c ranges over the critical points of f that lie inside the unit disk. By the Schwarz lemma, the post-critical set of $f : S^2 \to S^2$ is contained in the union of $B(0, 1 - \delta_c)$ and its reflection in the unit circle.

If $\zeta \in S^1$, the ball $B(\zeta, \delta_c)$ is disjoint from the post-critical set, and therefore all possible inverse branches f^{-n} are well-defined univalent functions on $B(\zeta, \delta_c)$. For $0 < t < 1/2$, let $U_t := \{z : 1 - t \cdot \delta_c \leq |z| < 1\}$. For Blaschke products, we have the following analogue of Lemma 2.2:

Lemma 2.3. *If μ is an invariant Beltrami coefficient supported on the exterior unit disk, and if the orbit $z \to f(z) \to \cdots \to f^{\circ n}(z)$ is contained in some U_t with $t < 1/2$ sufficiently small, then*

$$\left| \frac{v_\mu'''}{\rho^2}(f^{\circ n}(z)) \cdot f^{\circ n}(z)^2 - \frac{v_\mu'''}{\rho^2}(z) \cdot z^2 \right| \lesssim \phi_2(t) \cdot \|\mu\|_\infty, \tag{2.6}$$

for some function $\phi_2(t)$ satisfying $\phi_2(t) \to 0^+$ as $t \to 0^+$.

3. Blaschke Products

In this section, we give background information on Blaschke products. We discuss the quotient torus at the attracting fixed point and special repelling periodic orbits called "simple cycles" on the unit circle. In the next section, we will examine the interface between these two objects.

3.1. Attracting tori.

The dynamics of forward orbits of a Blaschke product

$$f_a(z) = z \cdot \frac{z + a}{1 + \bar{a}z} \tag{3.1}$$

is very simple: all points in the unit disk are attracted to the origin. In this paper, we mostly assume that the multiplier of the attracting fixed point $a = f'(0) \neq 0$. In this case, the linearizing coordinate $\varphi_a(z) := \lim_{n \to \infty} a^{-n} \cdot f_a^{\circ n}(z)$ conjugates f_a to multiplication by a, i.e.

$$\varphi_a : \mathbb{D} \to \mathbb{C}, \qquad \varphi_a(f_a(z)) = a \cdot \varphi_a(z). \tag{3.2}$$

It is well-known that (3.2) determines φ_a uniquely with the normalization $\varphi_a'(0) = 1$.

Let Ω denote the unit disk with the grand orbits of the attracting fixed and critical point removed. From the existence of the linearizing coordinate, it is easy to see that the quotient $\hat{\varphi}_a : \Omega \to T_a^\times := \Omega/(f_a)$ is a torus with one puncture. We denote the underlying closed torus by T_a. We will also consider the intermediate covering map $\pi_a : \mathbb{C}^* \to T_a \cong \mathbb{C}^*/(\cdot a)$ defined implicitly by $\hat{\varphi}_a = \pi_a \circ \varphi_a$.

Higher degree. For a Blaschke product $f_{\mathbf{a}} \in \mathcal{B}_d$ with $a = f_{\mathbf{a}}'(0) \neq 0$, the quotient torus $T_{\mathbf{a}}^\times$ has at most $(d-1)$ punctures but there could be less if there are critical relations. The reader may view the space $\mathcal{B}_d^\times \subset \mathcal{B}_d$ consisting of Blaschke products for which $T_{\mathbf{a}}^\times \in \mathcal{T}_{1,d-1}$ as a natural generalization of $\mathcal{B}_2^\times$.

3.2. Multipliers of simple cycles.

On the unit circle, a Blaschke product has many repelling periodic orbits or cycles. Since all Blaschke products of degree 2 are quasisymmetrically conjugate on the unit circle, we can label the periodic orbits of $f \in \mathcal{B}_2$ by the corresponding periodic orbits of $z \to z^2$.

A cycle is *simple* if f preserves its cyclic ordering. In this case, we say that $\langle \xi_1, \xi_2, \ldots, \xi_q \rangle$ has *rotation number* p/q if $f(\xi_i) = \xi_{i+p \pmod q}$. (For simple cycles, we prefer to index the points $\{\xi_i\} \subset S^1$ in counter-clockwise order, rather than by their dynamical order.)

Examples of cycles of degree 2 Blaschke products:

- $(1,2)/3$ has rotation number $1/2$,
- $(1,2,4)/7$ has rotation number $1/3$,
- $(1,2,3,4)/5$ is not simple.

In degree 2, for every fraction $p/q \in \mathbb{Q}/\mathbb{Z}$, there is a unique simple cycle of rotation number p/q. We denote its multiplier by $m_{p/q} := (f^{\circ q})'(\xi_1)$. Since Blaschke products preserve the unit circle, $m_{p/q}$ is a positive real number (greater than 1). It is sometimes more convenient to work with $L_{p/q} := \log(f^{\circ q})'(\xi_1)$ which is an analogue of the length of a closed geodesic of a hyperbolic Riemann surface.

4. Petals and Flowers

In this section, we give an overview of petals, flowers and gardens. As suggested by the terminology, gardens are made of flowers, and flowers are made of petals. We first give a general definition of a garden, but then we specify to "half-flower gardens" which will be used throughout this work.

In fact, for a Blaschke product $f_a \in \mathcal{B}_2^\times$, we will construct infinitely many half-flower gardens $\mathcal{G}_{[\gamma]}(f_a)$ – one for every outgoing homotopy class of simple closed curves $[\gamma] \in \pi_1(T_a, *)$. However, in practice, we use the garden $\mathcal{G}(f_a) := \mathcal{G}_{[\gamma]}(f_a)$ associated to the shortest geodesic γ in the flat metric on the torus. For parameters $a \in \mathcal{B}_{p/q}(C_{\mathrm{small}})$, the shortest curve γ is uniquely defined and has rotation number p/q. It is precisely for this choice of half-flower garden that the estimate (1.6) holds. For example, to study radial degenerations with $a \to 1$, we consider gardens where flowers have only one petal (see Figure 2), while for other parameters, it is more natural to use gardens where the flowers have more petals (see Figure 5 below).

4.1. **Curves on the quotient torus.** Inside the first homotopy group $\pi_1(T_a, *) \cong \mathbb{Z} \oplus \mathbb{Z}$, there is a canonical generator α which is represented by counter-clockwise loops $\hat\varphi_a(\{z : |z| = \epsilon\})$ with $\epsilon > 0$ sufficiently small. By a *neutral* curve, we mean a curve whose homotopy class in $\pi_1(T_a, *)$ is an integral power of α. All non-neutral curves can be classified as either *incoming* or *outgoing*, depending on their orientation: a curve $\gamma : \mathbb{R}/\mathbb{Z} \to T_a$ is *outgoing* if some (and hence every) lift $\gamma_i^* = \pi_a^{-1}\gamma_i$ in $\mathbb{C}^*$
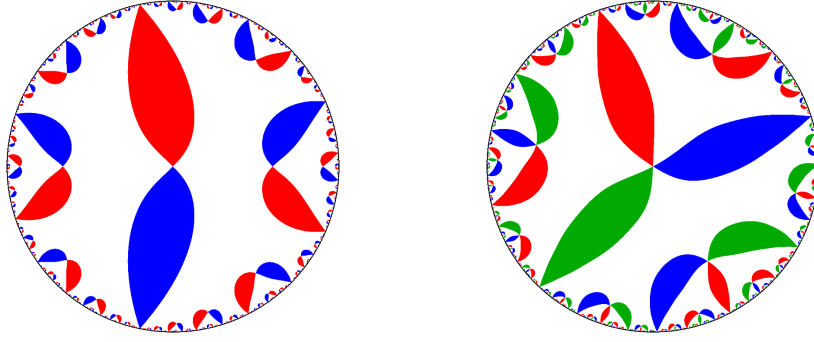
FIGURE 5. The gardens $\mathcal{G}_{1/2}(f_{-0.6})$ and $\mathcal{G}_{1/3}(f_{0.66 \cdot e^{2\pi i/3}})$.

satisfies

$$\gamma_i^*(t+1) = (1/a)^q \cdot \gamma_i^*(t) \qquad \text{for some } q \geq 1.$$

In other words, γ is outgoing if $\gamma_i^*(t) \to \infty$ as $t \to \infty$. A curve is *incoming* if the opposite holds, i.e. if instead $\gamma_i^*(t) \to 0$ as $t \to \infty$.

A complementary (outgoing) generator β is only canonically defined up to an integer multiple of α. In terms of the basis $\{\alpha, \beta\}$, we say that an outgoing curve homotopic to $(q-p)\alpha + p\beta$ has rotation number p/q. If we don't specify the choice of β, then p/q is only well-defined modulo 1.

4.2. **Lifting outgoing curves.** Suppose γ is a simple closed outgoing curve in $T_a^{\times}$ of rotation number p/q mod 1. It has q lifts to $\mathbb{C}^*$ under the projection $\pi_a : \mathbb{C}^* \to T_a$, which we denote $\gamma_1^*, \gamma_2^*, \ldots, \gamma_q^*$. The curves γ_i^* are "spirals" that join 0 to ∞. Each individual spiral is invariant under multiplication by a^q. We typically index the spirals so that multiplication by a sends γ_i^* to γ_{i+p}^*. Let $\tilde{\gamma}_i := \varphi_a^{-1}(\gamma_i^*)$ be (further) lifts in the unit disk emanating from the attracting fixed point.

Lemma 4.1. *Suppose γ is a simple closed outgoing curve in $T_a^{\times}$ of rotation number p/q. Then, $\tilde{\gamma}_i$ joins the attracting fixed point at the origin to a repelling periodic point $\xi_i \in S^1$ of rotation number number p/q.*

Proof. Pick a point z_1 on $\tilde{\gamma}_i$, and approximate $\tilde{\gamma}_i$ by the backwards orbit of $f^{\circ q}$: $z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_n \leftarrow \ldots$ By the Schwarz lemma, the backwards orbit is eventually contained in $U_{1/2} = \{z : 1 - \delta_c/2 \leq |z| < 1\}$, i.e. $z_n \in U_{1/2}$ for $n \geq N$. Since the Blaschke product is asymptotically affine, the hyperbolic distance $d_{\mathbb{D}}(z_n, z_{n+1})$ between successive points is bounded as it cannot substantially grow for $n \geq N$. The boundedness of the backward jumps forces the sequence $\{z_n\}$ to converge to a repelling periodic point ξ_i on the unit circle. The same argument shows that the hyperbolic length of the arc of $\tilde{\gamma}_i$ from z_n to z_{n+1} is bounded, and therefore $\tilde{\gamma}_i$ itself

must converge to ξ_i. Since $f(\tilde{\gamma}_i) = \tilde{\gamma}_{i+p}$, we have $f(\xi_i) = \xi_{i+p}$. Furthermore, since the lifts $\tilde{\gamma}_i \subset \mathbb{D}$ are disjoint, the points $\{\xi_i\}$ are arranged in counter-clockwise order which means that the repelling periodic orbit $\langle \xi_1, \xi_2, \ldots, \xi_q \rangle$ has rotation number p/q. $\qquad \square$

4.3. Definitions of petals and flowers.

An annulus $A \subset T_a^\times$ homotopy equivalent in $T_a^\times$ to an outgoing geodesic of rotation number p/q has q lifts in the unit disk emanating from the origin. We call these lifts *petals* and denote them $\mathcal{P}_i^A$, with $i = 1, 2, \ldots, q$. Each petal connects the attracting fixed point to a repelling periodic point. Naturally, the *flower* is defined as the union of the petals: $\mathcal{F} = \bigcup_{i=1}^{q} \mathcal{P}_i^A$. We refer to the attracting fixed point as the *A-point* of the flower and to the repelling periodic points as the *R-points*. By construction, flowers are forward-invariant regions. The *garden* is the totally-invariant region obtained by taking the union of all the repeated pre-images of the flower:

$$\mathcal{G} = \bigcup_{n=0}^{\infty} f_a^{-n}(\mathcal{F}).$$

We refer to the iterated pre-images of petals and flowers as *pre-petals* and *pre-flowers* respectively. In degree 2, a flower has two pre-images: itself and an *immediate pre-flower* which we denote $\mathcal{F}_*$ for convenience. Each pre-flower has two proper pre-images. We define the A and R points of pre-flowers as the pre-images of the A and R points of the flower. We typically label a pre-petal by its R-point and a pre-flower by its A-point.

4.4. Half-flower gardens.

We now construct the special gardens that will be used in this work. For this purpose, observe that an outgoing homotopy class $[\gamma] \in \pi_1(T_a, *)$ determines a foliation of the quotient torus T_a by parallel lines, which are closed geodesics in the flat metric on T_a. Explicitly, we can first foliate the punctured plane $\mathbb{C}^*$ by the logarithmic spirals

$$\gamma_\theta^* := \{e^{t \log a^q} \cdot e^{i\theta} : t \in [-\infty, \infty)\}, \qquad 0 \le \theta < 2\pi,$$

and then quotient out by $(\cdot a)$. The branch of $\log a^q$ is chosen so that $\pi_a(\gamma_\theta^*) \in [\gamma]$. Note that since each individual spiral is only invariant under $(\cdot a^q)$, a single line on the quotient torus T_a corresponds to q equally-spaced spirals in $\mathbb{C}^*$. Therefore, T_a is foliated by the parallel lines $\gamma_\theta := \pi_a(\gamma_\theta^*)$ with $0 \le \theta < 2\pi/q$.

For a Blaschke product $f_a \in \mathcal{B}_2^\times$, the quotient torus $T_a^\times$ has one puncture. Let $A^1 = T_a \setminus \gamma_{\theta_c}$ be the complement of the "singular line" that passes through this puncture. For $0 < \alpha \le 1$, let $A^\alpha \subset A^1$ be the middle round annulus with $\text{Area}(A^\alpha)/\text{Area}(A^1) = \alpha$. By the construction of Section 4.3, the annulus A^1 defines a system of petals $\mathcal{P}_i^1$,

$i = 1, 2, \ldots, q$, which we calls *whole petals*. Similarly, an α-petal $\mathcal{P}_i^\alpha$ is defined as a petal constructed using the annulus $A^\alpha \subset T_a^\times$. By default, we take $\alpha = 1/2$ and write $\mathcal{P}_i = \mathcal{P}_i^{1/2}$. We define the half-flower $\mathcal{F}$ as the union of all the half-petals.

Alternatively, one can describe whole petals and half-petals in terms of linearizing rays. A *linearizing ray*, or a *linearizing spiral* if $a \notin (0,1)$, is defined as the pre-image $\tilde{\gamma}_\theta := \varphi_a^{-1}(\gamma_\theta^*)$, $0 \le \theta \le 2\pi$ emanating from the attracting fixed point. If a whole petal $\mathcal{P}^1$ consists of linearizing rays with arguments in $(\theta_1, \theta_2) = (\frac{x-y}{2}, \frac{x+y}{2})$, then the associated α-petal $\mathcal{P}^\alpha$ is the union of the linearizing rays with arguments in $(\frac{x-\alpha y}{2}, \frac{x+\alpha y}{2})$.

Convention. In the rest of the paper, we use this system of flowers. When working with $a \approx e(p/q)$, we let $\mathcal{F} = \mathcal{F}_{p/q}$ denote the flower constructed from a foliation of the quotient torus by p/q-curves, arising from the choice of $\log a^q \approx \log 1 = 0$.

Higher degree. One can similarly define petals and flowers similarly for Blaschke products of degree $d \ge 3$: Call a line $\gamma_\theta \subset T_{\mathbf{a}}$ *regular* if it is contained in $T_{\mathbf{a}}^\times$ and *singular* if it passes through a puncture. The singular lines partition $T_{\mathbf{a}}$ into annuli, the lifts of which we call *whole petals*. The number of (p/q)-cycles of whole petals is at most $d - 1$, but there could be less if several critical points lie on a single line.

5. Quasiconformal Deformations

In this section, we describe the Teichmüller metric on $\mathcal{B}_2^\times$ and define the half-optimal Beltrami coefficients which are supported on the half-flower gardens from the previous section. We also discuss pinching deformations.

For a Beltrami coefficient μ with $\|\mu\|_\infty < 1$, let w_μ be the quasiconformal map fixing $0, 1, \infty$ whose dilatation is μ. Given a rational map $f(z) \in \mathrm{Rat}_d$, an invariant Beltrami coefficient $\mu \in M(S^2)^f$ defines a (possibly trivial) tangent vector in $T_f \mathrm{Rat}_d$ represented by the path $f_t = w_{t\mu} \circ f \circ (w_{t\mu})^{-1}$, $t \in (-\epsilon, \epsilon)$.

If $\mu \in M(\mathbb{D})$, one can also consider the symmetrized version w^μ which is the quasiconformal map that has dilatation μ on the unit disk and is symmetric with respect to inversion in the unit circle. For a Blaschke product $f \in \mathcal{B}_d$ and a Beltrami coefficient $\mu \in M(\mathbb{D})^f$, the symmetric deformation

$$f_t = w^{t\mu} \circ f \circ (w^{t\mu})^{-1}, \qquad t \in (-\epsilon, \epsilon),$$

defines a path in $\mathcal{B}_d$. Note that while we use symmetric deformations to move around the space $\mathcal{B}_d$, we use asymmetric deformations $w_{t\mu^+} \circ f \circ (w_{t\mu^+})^{-1}$ to compute the Weil-Petersson metric as the definition of $\|\mu\|_{\mathrm{WP}}$ involves $v(z) = \frac{d}{dt}\big|_{t=0} w_{t\mu^+}(z)$.

The formula for the variation of the multiplier of a fixed point of a rational map will play a fundamental role in this work:

Lemma 5.1 (e.g. Theorem 8.3 of [IT]). *Suppose $f_0(z)$ is a rational map with a fixed point at p_0 which is either attracting or repelling, and $\mu \in M(S^2)^{f_0}$. Then, $f_t = w_{t\mu} \circ f_0 \circ (w_{t\mu})^{-1}$ has a fixed point at $p_t = w_{t\mu}(p_0)$ and*

$$\left.\frac{d}{dt}\right|_{t=0} \log f_t'(p_t) = \pm\frac{1}{\pi} \cdot \int_{T_{p_0}} \frac{\mu(z)}{z^2} \cdot |dz|^2 \qquad (5.1)$$

where T_{p_0} is the quotient torus at p_0. The sign is "$+$" in the repelling case and "$-$" in the attracting case.

5.1. Teichmüller metric.

As noted in the introduction, $\mathcal{T}_{1,1}$ is the universal cover of $\mathcal{B}_2^\times$ since one has an identification of the tangent spaces $T_{f_a}\mathcal{B}_2^\times \cong T_{T_a}\mathcal{T}_{1,1}$. The Teichmüller metric on $\mathcal{B}_2^\times$ makes this correspondence a local isometry. More precisely, for a Beltrami coefficient $\mu \in M(\mathbb{D})^{f_a}$ representing a tangent vector in $T_{f_a}\mathcal{B}_2^\times$,

$$\|\mu\|_{T(\mathcal{B}_2^\times)} := \|(\hat{\varphi}_a)_*\mu\|_{T(\mathcal{T}_{1,1})}.$$

A well-known result of Royden says that the Teichmüller metric on $\mathcal{T}_{1,1}$ is equal to the Kobayashi metric; therefore, the same is true for the Teichmüller metric on $\mathcal{B}_2^\times \cong \mathbb{D}^*$. Explicitly, the Teichmüller metric on $\mathcal{B}_2^\times$ is $\frac{|da|}{|a| \log |a|^2}$.

Lemma 5.1 distinguishes a one-dimensional subspace of Beltrami coefficients in $M(\mathbb{D})^{f_a}$, namely ones of the form $\mu_\lambda = \varphi_a^*(\lambda \cdot (w/\overline{w}) \cdot (d\overline{w}/dw))$ with $\lambda \in \mathbb{C}$. We refer to these coefficients as *optimal* Beltrami coefficients. Here, "optimal" is short for "multiplier-optimal" which refers to the fact that μ_λ maximizes the absolute value of $(d/dt)|_{t=0} \log a_t$ out of all Beltrami coefficients with L^∞-norm $|\lambda|$.

For a tangent vector $\mathbf{v} \in T_{T_a^\times}\mathcal{T}_{1,1}$, the *Teichmüller coefficient* $\mu_\mathbf{v}$ associated to $\mathbf{v}$ is the unique Beltrami coefficient of minimal L^∞ norm which represents $\mathbf{v}$. It is well-known that Teichmüller coefficients have the form $\lambda \overline{q}/|q|$ with $q \in Q(T_a^\times)$, where $Q(T_a^\times)$ is the space of integrable holomorphic quadratic differentials on the punctured torus $T_a^\times$. In particular, $\|\mu_\mathbf{v}\|_T = \sup_{\|q\|_T = 1}\left|\int_{T_a^\times} \mu q\right| = \|\mu_\mathbf{v}\|_\infty$.

Since the quotient torus $T_a^\times$ associated to a degree 2 Blaschke product $f_a \in \mathcal{B}_2^\times$ has one puncture, $Q(T_a^\times)$ is one-dimensional. If we represent $T_a^\times \cong \mathbb{C}^*/(\cdot a)$, then $Q(T_a^\times)$ is spanned by $(\pi_a)_*(dw^2/w^2)$. Thus, in degree 2, the notions of Teichmüller coefficients and optimal coefficients agree.

Higher degree. For a Blaschke product $f_\mathbf{a} \in \mathcal{B}_d^\times$ of degree $d \geq 3$, the quotient torus has $d-1 \geq 2$ punctures, and so $Q(T_\mathbf{a}) \subsetneq Q(T_\mathbf{a}^\times)$. Therefore, optimal Beltrami coefficients represent only a complex 1-dimensional set of directions in $T_{T_\mathbf{a}^\times}\mathcal{T}_{1,d-1}$. In particular,

to understand the Weil-Petersson metric on spaces of Blaschke products of higher degree, one would need to study other deformations.

Given an optimal Beltrami coefficient μ_λ and a half-flower garden $\mathcal{G}(f_a)$, we define the *half-optimal Beltrami coefficient* as $\mu_\lambda \cdot \chi_\mathcal{G}$.

Lemma 5.2. *The half-optimal Beltrami coefficient $\mu \cdot \chi_\mathcal{G}$ is half as effective as the optimal Beltrami coefficient μ, i.e. the map $f_t(\mu \cdot \chi_\mathcal{G}) := w^{t\mu \cdot \chi_\mathcal{G}} \circ f_0 \circ (w^{t\mu \cdot \chi_\mathcal{G}})^{-1}$ is conformally conjugate to $f_{\tilde{t}}(\mu) := w^{\tilde{t}\mu} \circ f_0 \circ (w^{\tilde{t}\mu})^{-1}$ where $\tilde{t}$ is chosen so that $d_\mathbb{D}(0, t) = 2d_\mathbb{D}(0, \tilde{t})$.*

5.2. **Pinching deformations.** A closed torus $X = X_\tau = \mathbb{C}/\langle 1, \tau \rangle$, $\tau \in \mathbb{H}$, carries a natural flat metric which is unique up to scale. To study lengths of curves on X, we normalize the total area to be 1. Given a slope $p/q \in \mathbb{Q} \cup \{\infty\}$, let $\gamma_{p/q} \subset X$ denote the Euclidean geodesic obtained by projecting $(\tau - p/q) \cdot \mathbb{R}$ down to X. We define the *pinching deformation* (with respect to $\gamma_{p/q}$) as the geodesic in $\mathcal{T}_1 \cong \mathbb{H}$ which joins τ to p/q. We further define the *pinching coefficient* $\mu_{\text{pinch}} \in M(X)$ as the Teichmüller coefficient which represents the unit tangent vector in the direction of this geodesic. Intrinsically, the pinching deformation is "the most efficient deformation" that shrinks the Euclidean length of $\gamma_{p/q}$. More precisely, X_t is the marked Riemann surface with $d_T(X, X_t) = \frac{1}{2}\log\frac{t+1}{t-1}$ for which $L_{X_t}(\gamma)$ is minimal, where d_T is the Teichmüller distance in $\mathcal{T}_1$.

One can also define pinching deformations for annuli: given an annulus $A = A_0$, the pinching deformation $(A_t)_{t \geq 0}$ is the deformation for which the modulus of A_t grows as quickly as possible. For the annulus $A_{r,R} := \{z : r < |z| < R\}$, the pinching deformation is given by the Beltrami coefficients

$$t \cdot \mu_{\text{pinch}} = t \cdot (w/\overline{w}) \cdot (d\overline{w}/dw), \quad t \in [0, 1). \tag{5.2}$$

With these definitions, the operation of "pinching a torus X with respect to a Euclidean geodesic γ" is the same as "pinching the annulus $A = X \setminus \gamma$." Indeed, the modulus of $X_\tau \setminus \gamma_{p/q}$ is just

$$\text{mod}(X_\tau \setminus \gamma_{p/q}) \;=\; \frac{\text{Area}\,X_\tau}{L_{X_\tau}(\gamma_{p/q})^2} \;=\; \begin{cases} \frac{|\operatorname{Im}\tau|}{|q\tau - p|^2}, & \text{if } p/q \neq \infty, \\ |\operatorname{Im}\tau|, & \text{if } p/q = \infty. \end{cases} \tag{5.3}$$

The above formula appears in [McM4, Section 5], although McMullen normalizes the area of X_τ to be $|\operatorname{Im}\tau|$. The modulus of course is independent of the normalization.

6. Incompleteness: Special Case

In this section, we show that the Weil-Petersson metric on $\mathcal{B}_2$ is incomplete as we take $a \to 1$ along the real axis. As noted in the introduction, to show the estimate $\omega_B/\rho_{\mathbb{D}^*} \lesssim (1 - |a|)^{1/4}$ on $(1/2, 1]$, it suffices to prove:

Theorem 6.1. *For a Blaschke product $f_a \in \mathcal{B}_2$ with $a \in [1/2, 1)$, we have*

$$\limsup_{r \to 1} \ |\mathcal{G}(f_a) \cap S_r| = O\big(\sqrt{1 - |a|}\big). \tag{6.1}$$

We will deduce Theorem 6.1 from:

Theorem 6.2. *For a Blaschke product $f_a \in \mathcal{B}_2$ with $a \in [1/2, 1)$,*

   (a) *Every pre-petal lies within a bounded hyperbolic distance of a geodesic segment.*

   (b) *The hyperbolic distance between any two pre-petals exceeds $d_{\mathbb{D}}(0, a) - O(1)$.*

One curious feature of hyperbolic geometry is that *a horocycle connecting two points is exponentially longer than the geodesic.* Indeed, if $-x + iy, x + iy \in \mathbb{H}$, then the hyperbolic length of the horocycle joining them is $2(x/y)$ while the geodesic length is only $\int_\theta^{\pi - \theta} \frac{dt}{\sin t} = 2\log(\cot(\theta/2))$ where $\cot\theta = x/y$. As $\cot\theta \approx 1/\theta$ for θ small, this is approximately $2\log(2 \cdot x/y)$. With this in mind, we argue as follows:

Proof of Theorem 6.1. By part (a) of Theorem 6.2, the hyperbolic length of the intersection of S_r with any single pre-petal is $O(1)$. By part (b) of Theorem 6.2, whenever the circle S_r intersects a pre-petal, an arc of hyperbolic length $O\big(\sqrt{1 - |a|}\big)$ is disjoint from the other pre-petals. Therefore, only the $O\big(\sqrt{1 - |a|}\big)$-th part of S_r can be covered by pre-petals. $\qquad\square$

6.1. Quasi-geodesic property.

Lemma 6.1. *For $a \in [1/2, 1)$, the petal $\mathcal{P}(f_a)$ lies within a bounded hyperbolic neighbourhood of a geodesic ray.*

Proof. By symmetry, the linearizing ray $\tilde{\gamma}_0 = \varphi_a^{-1}((0, \infty))$ is the line segment $(0, 1)$ which happens to be a geodesic ray. We therefore need to show that the petal $\mathcal{P}(f_a) = \varphi_a^{-1}(\{\mathrm{Re}\, z > 0\})$ lies within a bounded hyperbolic neighbourhood of $\tilde{\gamma}_0$. Suppose $z \in \mathcal{P}(f_a)$ lies outside a small ball $B(0, \delta)$. Let F be the fundamental domain bounded by $\{\zeta : |\zeta| = \delta\}$ and its image under f_a. Under iteration, z eventually lands in F, e.g. $z_0 = f_a^{\circ N}(z) \in F$, with $\lim_{n \to \infty} \arg f^{\circ n}(z) \in (-\pi/2, \pi/2)$. On the other hand, the limiting argument of the critical point $\lim_{n \to \infty} \arg f^{\circ n}(c) = \pi$ since the forward orbit of the critical point is contained in the segment $(-1, 0)$. Therefore,

we can pick a point $x_0 \in \tilde{\gamma}_0$ for which $d_\Omega(z_0, x_0) = d_{T_a^\times}(\pi_a(z_0), \pi_a(x_0)) = O(1)$. Let $x = f^{-N}(x_0)$ be the N-th pre-image of x_0 along $\tilde{\gamma}_0$. Clearly,

$$d_\mathbb{D}(z, x) \leq d_\Omega(z, x) = d_{T_a^\times}(\pi_a(z_0), \pi_a(x_0)) = O(1). \tag{6.2}$$

This completes the proof. □

6.2. **The structure lemma.** To establish the quasi-geodesic property for pre-petals, we show the "structure lemma" which says that the pre-petals are nearly-affine copies of the immediate pre-petal, while $f : \mathcal{P}_{-1} \to \mathcal{P}$ is approximately the involution about the critical point, i.e. $f|_{\mathcal{P}_{-1}} \approx m_{0\to c} \circ (-z) \circ m_{c\to 0}$, where $m_{0\to c} = \frac{z+c}{1+\bar{c}z}$ and $m_{c\to 0} = \frac{z-c}{1-\bar{c}z}$. For a Blaschke product f, its *critically-centered version* is given by

$$\tilde{f} = m_{c\to 0} \circ f \circ m_{0\to c}.$$

Naturally, the petals and pre-petals of $\tilde{f}$ are defined as the images of petals and pre-petals of f under $m_{c\to 0}$.

Lemma 6.2 (Structure lemma)**.** *For $a \in [1/2, 1)$ on the real axis,*
   (i) *The critically-centered petal $\tilde{\mathcal{P}} \subset B\big(1, \text{const} \cdot \sqrt{1 - |a|}\big)$.*
   (ii) *The immediate pre-petal $\mathcal{P}_{-1} \subset B\big(-1, \text{const} \cdot (1 - |a|)\big)$.*

Proof. Part (i) follows from Lemma 6.1 as $m_{c\to 0}\big((0, 1)\big) = (-c, 1)$. To pin down the size and location of the immediate pre-petal, we use the fact that for a degree 2 Blaschke product, c is the hyperbolic midpoint of $[0, -a]$. This implies that in the critically-centered picture, the A-point of the petal is $m_{c\to 0}(0) = -c$ while the A-point of the immediate pre-petal is $m_{c\to 0}(-a) = c$. Therefore, by Koebe's distortion theorem, $\tilde{\mathcal{P}}_{-1} \subset B\big(-1, \text{const} \cdot \sqrt{1 - |a|}\big)$. Part (ii) follows by applying $m_{0\to c}$ to the last statement. □
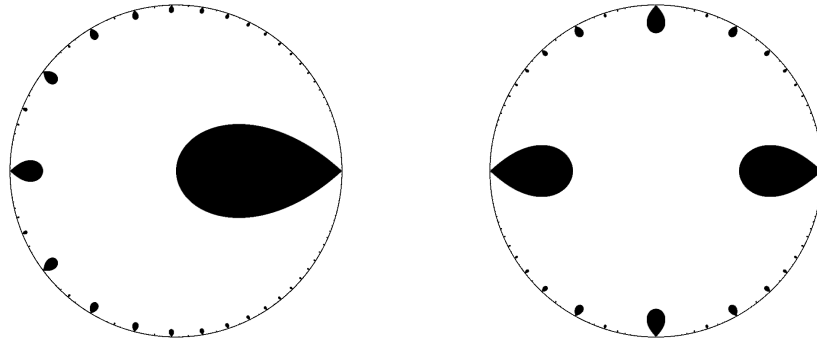

FIGURE 6. Half-petal families for the Blaschke products $f_{0.8}$ and $\tilde{f}_{0.8}$.

6.3. **Petal separation.** We can now prove that the petals are far apart:

Proof of part (b) *of Theorem 6.2.* Since the petal $\mathcal{P}$ is contained in a bounded hyperbolic neighbourhood of $(0, 1)$ and the immediate pre-petal $\mathcal{P}_{-1}$ is contained in a bounded hyperbolic neighbourhood of $(-1, -a)$, it follows that

$$d_{\mathbb{D}}(\mathcal{P}, \mathcal{P}_{-1}) = d_{\mathbb{D}}(0, -a) - O(1).$$

By the Schwarz lemma, given two pre-petals $\mathcal{P}_{\zeta_1}$ and $\mathcal{P}_{\zeta_2}$ with $f^{\circ n_1}(\zeta_1) = f^{\circ n_2}(\zeta_2) = 1$ and $n_1 \neq n_2$ (say $n_1 > n_2$),

$$d_{\mathbb{D}}(\mathcal{P}_{\zeta_1}, \mathcal{P}_{\zeta_2}) \geq d_{\mathbb{D}}\left(f^{\circ(n_1-1)}(\mathcal{P}_{\zeta_1}), f^{\circ(n_1-1)}(\mathcal{P}_{\zeta_1}) \right) \geq d_{\mathbb{D}}(\mathcal{P}_{-1}, \mathcal{P}_1).$$

To complete the proof, it suffices to show that pre-petals $\mathcal{P}_{\zeta_1}$ and $\mathcal{P}_{\zeta_2}$ are far apart in the case that they have a common parent, e.g. when $f(\zeta_1) = f(\zeta_2) = \zeta$. We prove this using a topological argument. Observe that -1 and 1 separate the unit circle in two arcs, each of which is mapped to $S^1 \setminus \{1\}$ by f_a. Therefore, any path in the unit disk connecting $\mathcal{P}_{\zeta_1}$ and $\mathcal{P}_{\zeta_2}$ must intersect the line segment $(-1, 1) \subset \overline{\mathcal{P}_1^1} \cup \overline{\mathcal{P}_{-1}^1}$.

However, we already know that the distance between $\mathcal{P}_{\zeta_i}$ to either $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ is greater than $d_{\mathbb{D}}(0, a) - O(1)$ which tells us that the hyperbolic $(\frac{1}{2} \cdot d_{\mathbb{D}}(0, a) - O(1))$-neighbourhood of $(-1, 1)$ is disjoint from $\mathcal{P}_{\zeta_1}$ and $\mathcal{P}_{\zeta_2}$. This completes the proof. □

7. Renewal Theory

In this section, we show that for a Blaschke product other than $z \to z^d$, the integral average (1.4) defining the Weil-Petersson metric converges. The proof is based on renewal theory, which is the study of the distribution of repeated pre-images of a point. In the context of hyperbolic dynamical systems, this has been developed by Lalley [La]. We apply his results to Blaschke products, thinking of them as maps from the unit circle to itself. Using an identity for the Green's function, we extend renewal theory to points inside the unit disk. Renewal theory will also be instrumental in giving bounds for the Weil-Petersson metric.

For a point x on the unit circle, let $n(x, R)$ denote the number of repeated pre-images y (i.e. $f^{\circ n}(y) = x$ for some $n \geq 0$) for which $\log |(f^{\circ n})'(y)| \leq R$. Also consider the probability measure $\mu_{x,R}$ on the unit circle which gives equal mass to each of the $n(x, R)$ pre-images. We show:

Theorem 7.1. *For a Blaschke product $f \in \mathcal{B}_d$ other than $z \to z^d$,*

$$n(x, R) \sim \frac{e^R}{\int \log |f'| dm} \qquad as \quad R \to \infty. \tag{7.1}$$

Furthermore, as $R \to \infty$, the measures $\mu_{x,R}$ tend weakly to the Lebesgue measure.

For a point $z \in \mathbb{D}$, let $\mathcal{N}(z, R)$ be the number of repeated pre-images of z that lie in the ball centered at the origin of hyperbolic radius R.

Theorem 7.2. *Under the assumptions of Theorem 7.1, we have*

$$\mathcal{N}(z, R) \sim \frac{1}{2} \cdot \log \frac{1}{|z|} \cdot \frac{e^R}{\int \log |f'| dm} \qquad as \quad R \to \infty. \qquad (7.2)$$

As before, when $R \to \infty$, the $\mathcal{N}(z, R)$ pre-images become equidistributed on the unit circle with respect to the Lebesgue measure.

7.1. Green's function.
Let $G(z) = \log \frac{1}{|z|}$ be the Green's function of the disk with a pole at the origin. It is uniquely characterized by three properties:

(i) $G(z)$ is harmonic on the punctured disk,
(ii) $G(z)$ tends to 0 as $|z| \to 1$,
(iii) $G(z) - \log \frac{1}{|z|}$ is harmonic near 0.

Lemma 7.1. *For a Blaschke product $f \in \mathcal{B}_d$, we have*

$$\sum_{f(w_i)=z} G(w_i) = G(z), \qquad z \in \mathbb{D}. \qquad (7.3)$$

To prove Lemma 7.1, it suffices to check that $\sum_{f(w_i)=z} G(w_i)$ also satisfies the three properties above. We leave the verification to the reader. From equation (7.3), it follows that the Lebesgue measure on the unit circle is invariant under f. Indeed, for a point $x \in S^1$, one can apply the lemma to $z = rx$ and take $r \to 1$ to obtain $\sum_{f(y)=x} |f(y)|^{-1} = 1$. (Alternatively, one can apply $\frac{\partial}{\partial z}$ to both sides of (7.3) to obtain the somewhat stronger statement $\sum_{f(w)=z} \frac{f(w)}{wf'(w)} = 1$.)

In fact, the Lebesgue measure is ergodic. The argument is quite simple (see [SS] or [Ha]); for the convenience of the reader, we reproduce it here: given an invariant set $E \subset S^1$, form the harmonic extension $u_E(z)$ of χ_E. Since $\chi_{f^{-1}E} = \chi_E \circ f$, u_E is a harmonic function in the disk which is invariant under f. But 0 is an attracting fixed point, so u_E must actually be constant, which forces E to have measure 0 or 1 as desired. From the ergodicity of Lebesgue measure, it follows that conjugacies of distinct Blaschke products are not absolutely continuous.

7.2. Weak mixing.
For the exceptional Blaschke product $z \to z^d$, the pre-images of a point $x \in S^1$ come in packets and so $n(x, R)$ is a step function. Explicitly,

$$n(x, R) = 1 + d + d^2 + \cdots + d^{\lfloor \log R / \log d \rfloor}.$$

While $n(x, R)$ has exponential growth, due to the lack of mixing, some values of R are special. All other Blaschke products satisfy the required mixing property and Theorem 7.1 follows from [La, Theorem 1 and formula (2.5)].

Sketch of proof of Theorem 7.1. In the language of thermodynamic formalism, we must check that the potential $\phi_f(x) = -\log|f'(x)|$ is non-lattice, i.e. that the cohomology equation $\phi - \psi = \gamma - \gamma \circ f$ does not admit solutions (ψ, γ) with $\psi(x)$ valued in a discrete subgroup of $\mathbb{R}$ and $\gamma(x)$ bounded. To the contrary, the existence of such a pair of functions would imply that the multiplier spectrum

$$\{\log(f^{\circ n})'(\xi) : f^{\circ n}(\xi) = \xi\}$$

is contained in a discrete subgroup of $\mathbb{R}$. Following the proof of [PP, Proposition 5.2], we see that there exists a function $w \in C^\alpha(\Sigma)$ satisfying

$$w(f(x)) = e^{ia\phi_f(x)}w(x), \qquad \text{for some } a \in \mathbb{R} \setminus \{0\}. \tag{7.4}$$

Here, $\Sigma = \{0, 1, \ldots, d-1\}^{\mathbb{N}}$ is the shift space which codes the dynamics of f on the unit circle. However, if we work directly on the unit circle and repeat the proof of [PP, Proposition 4.2], we obtain a function $w \in C^\alpha(S^1)$ satisfying (7.4). Since $w(x)$ is non-vanishing and has constant modulus, we can scale it by a constant if necessary so that $|w(x)| = 1$. By comparing the topological degrees of both sides of (7.4), we see that the topological degree of w is 0. In particular, w admits a continuous branch of logarithm.

If $w(x) = e^{iv(x)}$ then $v \circ f = a \cdot \phi_f + v + 2\pi k$ for some constant $k \in \mathbb{Z}$. Therefore, $\phi_f \sim 2\pi k/a$ is cohomologous to a constant. This tells us that the Lebesgue measure m must also be the measure of maximal entropy. However, the measure of the maximum entropy is a topological invariant, thus if we have a conjugacy h between z^d and $f(z)$, then the measure of the maximal entropy is $h_* m$. However, we know that the conjugacies of distinct Blaschke products are *not* absolutely continuous, therefore, we must have $f(z) = z^d$. $\square$

7.3. Computation of entropy.
Since the dimension of the unit circle is equal to 1, the entropy $h(f, m)$ of the Lebesgue measure coincides with the Lyapunov exponent $\frac{1}{2\pi} \int \log|f'(e^{i\theta})|d\theta$. We may compute the latter quantity using Jensen's formula:

Lemma 7.2. *If $a = f_{\mathbf{a}}'(0) \neq 0$, the entropy of the Lebesgue measure for the Blaschke product $f_{\mathbf{a}}(z)$ with critical points $\{c_i\}$ and zeros $\{z_i\}$ is given by*

$$\frac{1}{2\pi} \int \log|f_{\mathbf{a}}'(e^{i\theta})|d\theta = \sum_{cp} G(c_i) - G(a) = \sum_{cp} G(c_i) - \sum_{zeros} G(z_i). \tag{7.5}$$

In particular, for degree 2 Blaschke products, as a tends to the unit circle, the entropy $h(f_a, m) \sim 1 - |c| \sim \sqrt{2(1 - |a|)}$.

7.4. **Laminated area.** For a measurable set E in the unit disk, let $\hat{E}$ denote its *saturation* under taking pre-images, i.e. $\hat{E} = \{\zeta : f^{\circ n}(\zeta) \in E \text{ for some } n \geq 0\}$. For a saturated set $\hat{E}$, we define its *laminated area* as $\mathcal{A}(\hat{E}) = \lim_{r \to 1^-} \frac{1}{2\pi} |E \cap S_r|$ and say that "E subtends the $\mathcal{A}(\hat{E})$-th part of the lamination." By Koebe's distortion theorem (see Section 2.2), we have the following useful estimate:

Lemma 7.3. *Suppose E is a subset of $U_t := \{z : 1 - t \cdot \delta_c \leq |z| < 1\}$ with $t < 1/2$. If E is is disjoint from all of its pre-images, then*

$$\mathcal{A}(\hat{E}) \approx_t \frac{1}{2\pi \, h(f_a, m)} \int_E \frac{1}{1 - |z|} \cdot |dz|^2. \tag{7.6}$$

(The notation "$A \approx_\epsilon B$" means that $|A/B - 1| \lesssim \epsilon$.)

Proof. By breaking up the set E into little pieces, we may assume that $E \subset B(x, t)$ for some $x \in S^1$. We claim that $\int_E \frac{1}{1-|z|} \cdot |dz|^2 \approx_t \int_{f^{-n}(E)} \frac{1}{1-|z|} \cdot |dz|^2$, uniformly in $n \geq 0$. By Lemma 2.2, for each n-fold pre-image E_y of E, with $f^{\circ n}(y) = x$, we have

$$\int_{E_y} \frac{1}{1 - |z|} \cdot |dz|^2 \approx_t |(f^{\circ n})'(y)|^{-1} \cdot \int_E \frac{1}{1 - |z|} \cdot |dz|^2.$$

The claim follows in view of the the identity $\sum_{f^{\circ n}(y) = x} |(f^{\circ n})'(y)|^{-1} = 1$ (recall that the Lebesgue measure is invariant). Therefore, we may assume that $E \subset U_{t'}$ with $t' > 0$ arbitrarily small, i.e. we can pretend that f^{-1} is essentially affine.

By approximation, it suffices to consider the case when $E = \mathscr{R}$ is a "rectangle" of the form

$$\Big\{z : \ 1 - |z| \in \big(\delta, (1 + \epsilon_1)\delta\big), \ \arg z \in \big(\theta_0, \theta_0 + \epsilon_2\delta\big)\Big\}$$

with ϵ_1, ϵ_2 small. For k large, the circle $S_{1-\delta/k} = \{z : |z| = 1 - \delta/k\}$ intersects $\approx \epsilon_1 k/h$ pre-images of $\mathscr{R}$. As the hyperbolic length of $S_{1-\delta/k}$ is $\sim 2\pi k/\delta$ and each pre-image has "horizontal" hyperbolic length $\approx \epsilon_2$, the laminated area $\mathcal{A}(\hat{\mathscr{R}}) \approx \frac{\epsilon_1 \epsilon_2}{2\pi h} \cdot \delta$ as desired. $\qquad\square$

Recall from [McM2] that a continuous function $h : \mathbb{D} \to \mathbb{C}$ is *almost-invariant* if for any $\epsilon > 0$, there exists $r(\epsilon) < 1$, so that for any orbit $z \to f(z) \to \cdots \to f^{\circ n}(z)$ contained in $\{z : r \leq |z| < 1\}$, we have $|h(z) - h(f^{\circ n}(z))| < \epsilon$.

Theorem 7.3. *Suppose f is a Blaschke product other than $z \to z^d$, and h is an almost-invariant function. Then the limit $\lim_{r \to 1^-} \frac{1}{2\pi} \int_{|z|=r} h(z) d\theta$ exists.*

Proof. Let E be a backwards fundamental domain near the unit circle, e.g. take $E = f^{-1}(B(0, s)) \setminus B(0, s)$ with $s \approx 1$. Split E into many pieces on which h is approximately constant. Applying Lemma 7.3 to each piece and summing over the pieces, we see that as $r \to 1$, $\frac{1}{2\pi} \int_{|z|=r} h(z)d\theta$ oscillates by an arbitrarily small amount. Therefore, the limit exists. $\square$

Applying the above theorem with $h = |v'''/\rho^2|^2$, which is almost-invariant by Lemma 2.3, gives:

Corollary. *Given a Blaschke product $f \in \mathcal{B}_d$ other than $z \to z^d$, the limit in the definition of the Weil-Petersson metric (1.4) exists for every vector field v that is associated to a tangent vector $T_f \mathcal{B}_d$.*

8. Multipliers of Simple Cycles

In this section, we study the behaviour of repelling periodic orbits of degree 2 Blaschke products with small multipliers. Recall from Section 3 that $L_{p/q}$ denotes the logarithm of the multiplier of the unique cycle that has rotation number p/q. Given $\mu \in M(\mathbb{D})^{f_0}$ representing a vector in $T_{\mathcal{B}_2^\times} f_0$, let $\dot{L}_{p/q}[\mu] := (d/dt)|_{t=0} L_{p/q}(f_t)$ where we perturb f_0 using the symmetric deformation $f_t = w^{t\mu} \circ f \circ (w^{t\mu})^{-1}$, $t \in (-\epsilon, \epsilon)$.

Let $\mathcal{B}_{p/q}(\eta)$ be the horoball in the unit disk of Euclidean diameter η/q^2 which rests on $e(p/q) \in S^1$ and $\mathcal{H}_{p/q}(\eta) = \partial \mathcal{B}_{p/q}(\eta)$ be its boundary horocycle. We show:

Theorem 8.1. *There exists a constant $C_{\text{small}} > 0$ such that for a Blaschke product $f_a \in \mathcal{B}_2$ with $a \in \mathcal{H}_{p/q}(\eta)$ and $\eta < C_{\text{small}}$, we have:*

(i) *As $\eta \to 0^+$, $m_{p/q} - 1 \sim \eta/2$.*

(ii) *If $\gamma_{p/q} \subset T_a$ is the shortest curve in the quotient torus at the attracting fixed point (which necessarily has rotation number p/q) and $\mu_{pinch} \in M(\mathbb{D})^{f_a}$ is the associated pinching coefficient with $\|\mu_{\text{pinch}}\|_\infty = 1$, then*

$$|\dot{L}_{p/q}[\mu]/L_{p/q}| \asymp 1.$$

In other words, the gradient of $L_{p/q}$ is within a bounded factor of the maximal possible. We now make some useful definitions. Let $T_{p/q}$ denote the quotient torus associated to the repelling periodic orbit of rotation number p/q and $T_{p/q}^{\text{in}} \subset T_{p/q}$ be the half of the torus which is associated to points inside the unit disk. Let $P_{p/q}^1 \subset T_{p/q}^{\text{in}}$ be the footprint of $\mathcal{F}^1$ in $T_{p/q}^{\text{in}}$, i.e. the part of $T_{p/q}^{\text{in}}$ filled by $\mathcal{F}^1$. The footprint $P_{p/q}$ of $\mathcal{F} = \mathcal{F}^{1/2}$ is defined similarly. To prove Theorem 8.1, we need:

Lemma 8.1. *There exists $C_{\text{small}} > 0$ sufficiently small so that for $a \in \mathcal{B}_{p/q}(C_{\text{small}})$,*

(i) *The footprint $P_{p/q}^1$ of the whole petal contains a definite angle of opening at least $0.99\,\pi$.*

(ii) *The footprint $P_{p/q}$ of the half-petal is contained in a central angle of $0.51\,\pi$.*

In turn, Lemma 8.1 is proved by comparing the "petal correspondence" with the holomorphic index formula. The argument is essentially due to McMullen, see [McM4, Theorem 6.1]; however, we will spell out the details since we need slightly more information.

8.1. **Conformal modulus of an annulus.** We use the convention that the annulus $A_{r,R} := \{z : r < |z| < R\}$ has modulus $\frac{\log(R/r)}{2\pi}$, which is the extremal length of the curve family $\Gamma_{\uparrow}(A_{r,R})$ consisting of curves that join the two boundary components of $A_{r,R}$. We denote the dual curve family by $\Gamma_{\circlearrowleft}(A_{r,R})$, consisting of curves that separate the two boundary components. Then, $\lambda_{\Gamma_{\uparrow}(A)} \cdot \lambda_{\Gamma_{\circlearrowleft}(A)} = 1$. For background on extremal length and moduli of curve families, we refer the reader to [GM].

If $B \subset A$ is an essential sub-annulus of A, we say that B is *round* in A if the pair (A, B) is conformally equivalent to a pair of concentric round annuli $(A_{r,R}, A_{r',R'})$ with $A_{r',R'} \subset A_{r,R}$. Alternatively, B is round in A if the pinching deformations for A and B are compatible, i.e. if $\mu_{\text{pinch}}(B) = \mu_{\text{pinch}}(A)|_B$.

Lemma 8.2. *Suppose $S^* = \{e^{i\theta} \cdot e^{\mathbb{R}\log\alpha} : \theta_1 < \theta < \theta_2\} \subset \mathbb{C}^*$ where $|\alpha| > 1$ and a branch of $\log\alpha$ has been chosen. Then the annulus*

$$S^* \,/\, \{z \sim \alpha z\} \qquad \text{has modulus} \qquad (\theta_2 - \theta_1)\,\text{Re}\Big(\frac{1}{\log\alpha}\Big). \qquad (8.1)$$

Suppose $T^* \subset \mathbb{C}^*$ is a region bounded by two Jordan curves γ_1, γ_2 which are invariant under multiplication by α, with $|\alpha| > 1$. By analogy with (8.1), we define the *generalized angle* β between γ_1 and γ_2 by the formula $\text{mod}(T^* \,/\, \{z \sim \alpha z\}) = \beta\,\text{Re}\big(\frac{1}{\log\alpha}\big)$.

8.2. **Holomorphic index formula.** We now recall the statement of the holomorphic index formula. If $g(z)$ is a holomorphic map, the *index* of a fixed point ζ is defined as

$$I_\zeta := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - g(z)} \qquad (8.2)$$

where γ is any sufficiently small counter-clockwise loop around ζ. If the multiplier $\lambda = g'(\zeta)$ is not 1, this expression reduces to $\frac{1}{1-\lambda}$. By the residue theorem, one has:

Theorem 8.2 (Holomorphic Index Formula). *Suppose $R(z)$ is a rational function and $\{\zeta_i\}$ are its fixed points. Then, $\sum I_{\zeta_i} = 1$.*

For a Blaschke product $f \in \mathcal{B}_d$, the holomorphic index formula says that

$$\sum \frac{1}{r_i - 1} = \frac{1 - |a|^2}{|1 - a|^2} \tag{8.3}$$

where the sum ranges over the repelling fixed points on the unit circle, and $a = f'(0)$ is the multiplier of the attracting fixed point.

8.3. **Petal correspondence.** Since a whole petal joins the attracting fixed point to a repelling periodic point, it provides a conformal equivalence between the annuli $A^1 \subset T_a^\times$ and $P_{p/q}^1 \subset T_{p/q}$. As there are q whole petals at the attracting fixed point,

$$\frac{\beta}{\log m_{p/q}} = \operatorname{Re} \frac{1}{q} \cdot \frac{2\pi}{\log(1/a^q)} \tag{8.4}$$

where β is the generalized angle representing the modulus of mod $P_{p/q}^1$. Observe that the holomorphic index formula gives a lower bound on $m_{p/q}$:

$$\frac{1}{m_{p/q} - 1} \leq \frac{1}{q} \cdot \frac{1 - |a^q|^2}{|1 - a^q|^2}. \tag{8.5}$$

Proof of Lemma 8.1. Suppose $a \in \mathcal{H}_{p/q}(\eta)$. If $\eta > 0$ is small, then $a^q \in \mathcal{H}_1(\frac{\eta + \theta}{q})$ with $|\theta|$ small. On this horocycle, $\operatorname{Re} \frac{1}{\log(1/a^q)} \approx \frac{q}{\eta + \theta}$ while the Poisson kernel $\frac{1 - |a^q|^2}{|1 - a^q|^2} \approx \frac{2q}{\eta + \theta}$. Note that if $\eta > 0$ is small, equation (8.5) forces $m_{p/q}$ to be close to 1, which in turn ensures that the ratio $\frac{\log m_{p/q}}{m_{p/q} - 1}$ is close to 1. Comparing (8.4) and (8.5) like in [McM4], we deduce that β is close to π. By the standard modulus estimates (see Lemmas 8.3 and 8.4 below), it follows that the footprint $P_{p/q}^1$ must contain an angle of opening close to π. They also show that the footprint of the half-petal $P_{p/q}$ is contained in a central angle of opening slightly greater than $\pi/2$. □

With preparations complete, we can now prove Theorem 8.1:

Proof of Theorem 8.1. For (i), we plug $\beta \approx \pi$ into (8.4) to obtain

$$1/\log m_{p/q} \approx 2/\eta \quad \text{or} \quad m_{p/q} \approx 1 + \eta/2.$$

Part (ii) requires a bit more work. Since the footprint of the whole petal $P_{p/q}^1$ contains an angle of $> 0.51\pi$, it is easy to construct an invariant Beltrami coefficient which effectively deforms the quotient torus of the repelling periodic orbit. As $\mathcal{B}_2$ is one-dimensional, we see that for an optimal Beltrami coefficient μ, we must have either

$$|\dot{L}_{p/q}[\mu]/L_{p/q}| \asymp 1 \quad \text{or} \quad |\dot{L}_{p/q}[i\mu]/L_{p/q}| \asymp 1. \tag{8.6}$$

We need to show that the first alternative holds when $\mu = \mu_{\text{pinch}} \in M(\mathbb{D})^f$ is the optimal pinching coefficient built from the attracting torus. As the dynamics of

$f^{\circ q}$ is approximately linear near a repelling periodic point, $\mu = \mu_{\text{pinch}}$ descends to a Beltrami coefficient $\nu \in M(T_{p/q})$, with $\operatorname{supp}\nu \subset T^{\text{in}}_{p/q}$. Since $\mu|_{A^1}$ is the optimal pinching coefficient for A^1, $\nu|_{P^1_{p/q}}$ is the optimal pinching coefficient for the annulus $P^1_{p/q}$. By Lemma 8.1, when $\eta > 0$ is small, the footprint $P^1_{p/q}$ takes up most of $T^{\text{in}}_{p/q}$, and since $T^{\text{in}}_{p/q}$ is a round annulus in $T^{p/q}$, ν is approximately equal to the optimal pinching coefficient for $T_{p/q}$ on $T^{\text{in}}_{p/q}$.

When we consider deformations $f^{t\mu}$ in the Blaschke slice, we use the Beltrami coefficient $\mu + \mu^+$, which corresponds to $\nu + \nu^+ \in M(T_{p/q})$. We see that $\nu + \nu^+ \in M(T_{p/q})$ is approximately equal to the optimal pinching coefficient for $T_{p/q}$ (at least away from the trace of the unit circle in $T_{p/q}$). In other words, pinching T_a with respect to a p/q curve has nearly the same effect as pinching $T_{p/q}$ with respect to a $0/1$ curve. This gives $|\dot{L}_{p/q}[\mu]/L_{p/q}| \asymp 1$. $\qquad\square$

8.4. Standard modulus estimates.

For the convenience of the reader, we state the standard estimates for moduli of annuli that we have used in the proofs of Lemma 8.1 and Theorem 8.1.

Lemma 8.3. *Suppose $A = A_{r,R}$ and $B \subset A$ is an essential sub-annulus. For any $\epsilon > 0$, there exists $\delta > 0$ and $m_0 > 0$ such that if $\operatorname{mod} A > m_0$ and*

$$\operatorname{mod} B \geq (1 - \delta)\operatorname{mod} A,$$

then B contains the "middle" annulus of modulus $(1 - \epsilon)\operatorname{mod} A$.

Proof. We first prove an analogous statement with rectangles in place of annuli. Suppose $R = [0, m] \times [0, 1]$ is a rectangle of modulus $m \geq 4/\epsilon$, and $S = (ABCD)$ is a conformal sub-rectangle, with $(AB) \subset [0, m] \times \{1\}$ and $(CD) \subset [0, m] \times \{0\}$. We will show that if S does not contain the middle sub-rectangle of modulus $(1 - \epsilon)m$, then $\operatorname{mod} S \leq (1 - \epsilon/4)m$.

By symmetry, we may assume that S is missing a curve joining $z_1 = iy_0$ and $z_2 = (\epsilon/2)m + iy_1$. Note that $m = \lambda_{\Gamma_{\leftrightarrow}(R)}$ is the extremal length of the horizontal curve family. Giving an upper bound on the extremal length of $\Gamma_{\leftrightarrow}(S)$ is equivalent to finding a lower bound on the extremal length of the vertical curve family $\Gamma_{\updownarrow}(S)$. For this purpose, consider the metric

$$\rho = \begin{cases} \chi_S, & \operatorname{Re} z \geq (\epsilon/4)m, \\ 0, & \operatorname{Re} z < (\epsilon/4)m. \end{cases} \tag{8.7}$$

Observe the ρ-length of any curve in $\Gamma_{\updownarrow}(S)$ is at least 1, yet $\operatorname{Area}(\rho) \leq (1 - \epsilon/4)m$. Therefore, $\lambda_{\Gamma_{\updownarrow}(S)} > \frac{\lambda_{\Gamma_{\updownarrow}(R)}}{1 - \epsilon/4}$ as desired.

We can deduce the original statement with annuli from the special case when $(AB) = (CD) + i$ by representing the pair $B \subset A$ as $A = R/\{z \sim z + i\}$ and $B = S/\{z \sim z + i\}$. Indeed, $\mod A = m$ while $\mod B \geq \mod S$ can only increase since a path in $\Gamma_{\circlearrowleft}(B)$ contains a path in $\Gamma_{\updownarrow}(S)$. $\qquad\qquad\square$

Essentially the same argument shows that:

Lemma 8.4. *Suppose $A = A_{r,R}$ has modulus $\mod A > m_0$ and $B_1, B_2, B_3 \subset A$ are three essential disjoint annuli, with B_2 sandwiched between B_1 and B_3. For any $\epsilon > 0$, there exists $\delta > 0$ and $m_0 > 0$ such that if $\mod A > m_0$ and*

$$\mod B_2 \geq (1/2 - \delta) \mod A, \qquad \mod B_i \geq (1/4 - \delta) \mod A, \quad i = 1, 3,$$

then B_2 is contained within the "middle" annulus of modulus $(1/2 + \epsilon) \mod A$.

We leave the details to the reader.

9. Lower bounds for the Weil-Petersson metric

In this section, we explain how one can obtain lower bounds for the Weil-Petersson metric using the multipliers of repelling periodic orbits on the unit circle. We first consider the Fuchsian case and then handle the Blaschke case by approximation. Somewhat frustratingly, the approximation argument comes with a price: in the Blaschke case, to give a lower bound for the Weil-Petersson metric, we must insist that *the quotient torus of the repelling periodic orbit changes at a definite rate in the Teichmüller metric.* It is precisely this "minor" detail which prevents us from showing that the completion of the Weil-Petersson metric on $\mathcal{B}_2$ attaches precisely the points $e(p/q) \in S^1$ and forces us to restrict our attention to small horoballs. The difficulty is caused by the error term in Lemma 2.3. For details, see the proof of Theorem 9.1 below.

For instance, it is well-known that in Teichmüller space, the Weil-Petersson length of a curve $X : [0, 1] \to \mathcal{T}_g$ with $L_{X(0)}(\gamma) = L_1$ and $L_{X(1)}(\gamma) = L_2 > L_1$ is bounded below by a definite constant $C(g, L_1, L_2)$. As hinted above, we are unable to prove the analogous statement for the Weil-Petersson metric on $\mathcal{B}_2$ where we replace the "length of a hyperbolic geodesic" by "the logarithm of the multiplier of a periodic orbit." We note that in order to resolve Conjecture A from the introduction using the method described here, one would need to show:

Conjecture C. For any Blaschke product $f \in \mathcal{B}_2$, there exists a repelling periodic orbit $f^{\circ q}(\xi) = \xi$ with $(f^{\circ q})'(\xi) < M_2$ and $\mu \in M(\mathbb{D})^f$ of norm 1 for which $|\dot{L}_{0,t}(\xi)/L(\xi)| \asymp 1$, where we perturb $f = f_0$ asymmetrically with $f_{0,t} = w_{t\mu} \circ f \circ$

$(w_{t\mu})^{-1}$. In terms of symmetric deformations $f_{t,t} = w^{t\mu} \circ f \circ (w^{t\mu})^{-1}$, it suffices to check that either $|\dot{L}_{t,t}(\xi)/L(\xi)| \asymp 1$ or $|\dot{L}_{it,it}(\xi)/L(\xi)| \asymp 1$.

9.1. Lower bounds in Teichmüller space.

Consider a linear map $f(z) = \lambda z$ with $\lambda > 1$. Given a Beltrami coefficient $\mu \in M(\mathbb{H})^f$ supported on the upper half-plane, form the maps $f_t = w_{t\mu} \circ f_0 \circ (w_{t\mu})^{-1}$. Since we use the asymmetric deformations $w_{t\mu}$, the multipliers $\lambda_t = f_t'(w_{t\mu}(0))$ are not necessarily real. We view $v = (d/dt)|_{t=0} \, w_{t\mu}$ as a holomorphic vector field on the lower half-plane.

Let $\pi : \mathbb{C} \to \mathbb{C}/(\cdot \lambda)$ be the quotient map. The Beltrami coefficient μ descends to the quotient torus, which we also denote μ when there is no risk of confusion. Our goal is to give a lower bound for $|v'''/\rho^2|$ in terms of $\|\mu\|_{T(\mathcal{T}_1)} = |\dot{L}_0/(2L_0)|$ where $L_t = \log \lambda_t$ and $\dot{L}_t = (d/dt)|_{t=0} \log \lambda_t$. Suppose first that μ is a *radial* Beltrami coefficient of the form

$$\mu(z) = k(\theta) \cdot \frac{z}{\bar{z}} \cdot \frac{d\bar{z}}{dz}. \tag{9.1}$$

Lemma 9.1. *For the radial Beltrami coefficient μ given by (9.1),*

$$v(z) = \frac{d}{dt}\bigg|_{t=0} w_{t\mu}(z) = -\frac{1}{2\pi} \cdot z \log z \cdot \int_0^\pi k(\theta) d\theta, \qquad z \in \overline{\mathbb{H}}, \tag{9.2}$$

and therefore,

$$v'''(z) = \frac{1}{2\pi} \cdot \frac{1}{z^2} \cdot \int_0^\pi k(\theta) d\theta, \qquad z \in \overline{\mathbb{H}}. \tag{9.3}$$

Proof. We compute:

$$\begin{aligned}
v(z) &= \frac{1}{2\pi} \int_{\mathbb{H}} \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)} \cdot k(\theta) \cdot (\zeta/\bar{\zeta})|d\zeta|^2, \\
&= \frac{z}{2\pi} \int_0^\pi k(\theta) \int_0^\infty \frac{(z-1)e^{i\theta}}{(re^{i\theta}-1)(re^{i\theta}-z)} dr d\theta, \\
&= \frac{z}{2\pi} \int_0^\pi k(\theta) \int_0^\infty \left(\frac{1}{r - e^{-i\theta}} - \frac{1}{r - ze^{-i\theta}} \right) dr d\theta, \\
&= \frac{z}{2\pi} \int_0^\pi k(\theta) \cdot (-\log z) d\theta.
\end{aligned}$$

(Since we are working in $\mathbb{C} \setminus (-\infty, 0]$, the branch of the logarithm is well-defined.) □

In view of Lemma 5.1, this shows

$$\left| \frac{v'''(z)}{\rho_{\overline{\mathbb{H}}}(z)^2} \right| \asymp \left| \frac{(d/dt)|_{t=0} \log \lambda_t}{\log \lambda_0} \right| \asymp \|\mu\|_T, \qquad \frac{5\pi}{4} < \arg z < \frac{7\pi}{4}, \tag{9.4}$$

for radial μ. For an arbitrary Beltrami coefficient $\mu \in M(\mathbb{H})^f$, the *pointwise* lower bound (9.4) need not hold in general. However, we can deduce an averaged version

of (9.4) from the radial case, which suffices for our purposes. Indeed, by replacing $\mu(z)$ with $\mu(rz)$ and averaging over $r \in (r_1, r_2)$, $r_2/r_1 = \lambda_0$ yields

$$\fint_{r_1}^{r_2} \left| \frac{v'''(re^{i\theta})}{\rho_{\mathbb{H}}(re^{i\theta})^2} \right| \cdot \frac{dr}{r} \gtrsim \left| \frac{(d/dt)|_{t=0} \log \lambda_t}{\log \lambda_0} \right| \asymp \|\mu\|_T, \qquad \frac{5\pi}{4} < \theta < \frac{7\pi}{4}. \qquad (9.5)$$

Integrating over θ and applying the Cauchy-Schwarz inequality, we obtain:

Lemma 9.2. *Suppose $\mu \in M(\mathbb{H})$ is invariant under $z \to \lambda_0 z$ and $v = (d/dt)|_{t=0} w_{t\mu}$ as above. For an "annular rectangle" $\mathscr{R} = S_{\theta_1,\theta_2} \cap F_{r_1,r_2}$,*

$$S_{\theta_1,\theta_2} = \{z : \arg z \in (\theta_1, \theta_2)\} \quad and \quad F_{r_1,r_2} = \{z : r_1 < |z| < r_2\},$$

with $(\theta_1, \theta_2) \subseteq (5\pi/4, 7\pi/4)$ and $r_2/r_1 = \lambda_0$, we have

$$\fint_{\mathscr{R}} \left| \frac{v'''(z)}{\rho_{\overline{\mathbb{H}}}(z)^2} \right|^2 \cdot \rho_{\mathbb{H}}^2 |dz|^2 \gtrsim \left| \frac{(d/dt)|_{t=0} \log \lambda_t}{\log \lambda_0} \right|^2 \asymp \|\mu\|_T^2. \qquad (9.6)$$

We can use Lemma 9.2 to study the Weil-Petersson metric on Teichmüller space. Suppose $X \in \mathcal{T}_g$ is a Riemann surface and $\gamma \subset X$ is a simple geodesic whose length is bounded above and below, e.g. $L_1 < L_X(\gamma) < L_2$. Let $p : \mathbb{H} \to X = \mathbb{H}/\Gamma$ be the universal covering map chosen so that the imaginary axis covers γ. By the collar lemma (e.g. see [Hub, Theorem 3.8.3]), there exists an annular rectangle $\mathscr{R}$ with $(r_1, r_2) = (1, e^{L_X(\gamma)})$ and $(\theta_1, \theta_2) = (-\pi/2 - \epsilon_{L_2}, -\pi/2 + \epsilon_{L_2})$ which has definite hyperbolic area, and for which $(p \circ \overline{z})|_{\mathscr{R}}$ is injective. It follows that for a Beltrami coefficient $\mu \in M(\mathbb{H})^\Gamma$, we have $\|p_*\mu\|_{\mathrm{WP}(\mathcal{T}_g)} \gtrsim \|\pi_*\mu\|_{T(\mathcal{T}_1)}$.

For applications to dynamical systems, it is easier to work with round balls instead of annular rectangles. An averaging argument similar to the one above shows:

Lemma 9.3. *Suppose the multiplier $\lambda_0 = f'(0) < M_2$ is bounded from above. Given $0 < R < 1$, one can find a ball $\mathscr{B} = \{w : d_{\mathbb{H}}(-iy_0, w) < R\}$, $1 \leq y_0 \leq \lambda_0$, for which*

$$\fint_{\mathscr{B}} \left| \frac{v'''(z)}{\rho_{\mathbb{H}}(z)^2} \right|^2 \cdot |dz|^2 \gtrsim \left| \frac{(d/dt)|_{t=0} \log \lambda_t}{\log \lambda_0} \right|^2.$$

9.2. Lower bounds in complex dynamics.

For a Blaschke product $f \in \mathcal{B}_2$ and $\mu \in M(\mathbb{D})^f$, we consider the quadratic differential $v''' = v'''_{\mu^+}$ and the two-parameter family $f_{s,t} := w_{\mu_{s,t}} \circ f \circ (w_{\mu_{s,t}})^{-1}$ where $\mu_{s,t} := s\mu + (t\mu)^+$ and $\nu^+ := \overline{(1/\overline{z})^*\nu}$.

Theorem 9.1 (Blowing up)**.** *Suppose $f(z) \in \mathcal{B}_2$ is Blaschke product and $f^{\circ q}(\xi) = \xi$ is a repelling periodic point on the unit circle with $(f^{\circ q})'(\xi) < M_2$. If $\mu(z) \in M(\mathbb{D})^f$ satisfies $\|\mu\|_\infty \leq 1$ and $|\dot{L}_{0,t}(\xi)/L(\xi)| \asymp 1$, then there exist a ball*

$$\mathscr{B} = B\left(\xi \cdot (1 - c_1 \cdot \delta_c), c_2 \cdot \delta_c\right) \quad for which \quad \fint_{\mathscr{B}} \left| \frac{v'''(z)}{\rho(z)^2} \right|^2 \cdot |dz|^2 \asymp 1. \qquad (9.7)$$

Proof. By Lemma 9.3, we can find a small ball $\mathscr{B}_0$ of definite hyperbolic size near ξ for which

$$\fint_{\mathscr{B}_0} \left| \frac{v'''(z)}{\rho(z)^2} \right|^2 \cdot |dz|^2 \asymp |\dot{L}_{0,t}(\xi)/L(\xi)|^2. \tag{9.8}$$

Using the forward iteration of f (and Koebe's distortion theorem), we can blow up this ball so that its Euclidean size is comparable to δ_c. Note that due to the error term in Lemma 2.3, in order for the estimate (9.8) to remain meaningful, we must insist that $|\dot{L}_{0,t}(\xi)/L(\xi)|$ is bounded from below. $\square$

Theorem 9.2 (Blowing down)**.** *In the setting of Theorem 9.1, if the multiplier is bounded from both below and above, $M_1 < (f^{\circ q})'(\xi) < M_2$, then*

$$\limsup_{r \to 1^-} \frac{1}{2\pi} \int_{|z|=r} \left| \frac{v'''(z)}{\rho(z)^2} \right|^2 d\theta \asymp 1. \tag{9.9}$$

Sketch of proof. In view of Lemma 2.3, the estimate (9.7) holds for the inverse images of $\mathscr{B}$. Since the multiplier is bounded from below, the constants c_1 and c_2 in Theorem 9.1 can be chosen small enough so that the repeated inverse images of $\mathscr{B}$ are disjoint from $\mathscr{B}$ (and thus from each other). By Lemmas 7.2 and 7.3, the laminated area $\mathcal{A}(\hat{\mathscr{B}})$ is bounded from below, which proves (9.9). $\square$

In Section 10, we will use the "blowing up" and "blowing down" techniques to give lower bounds for the Weil-Petersson metric when the multiplier of the repelling periodic orbit is small.

Remark. To give lower bounds for the Weil-Petersson metric, we used the gradient of the multiplier of a periodic orbit in the μ direction. In view of the the identities

$$(d/dt)|_{t=0} \log(f_{t,t}^{\circ q})'(\xi_{t,t}) = 2 \operatorname{Re} (d/dt)|_{t=0} \log(f_{0,t}^{\circ q})'(\xi_{0,t}),$$
$$(d/dt)|_{t=0} \log(f_{it,it}^{\circ q})'(\xi_{it,it}) = 2 \operatorname{Im} (d/dt)|_{t=0} \log(f_{0,t}^{\circ q})'(\xi_{0,t}),$$

we can also use the gradient of the multiplier in the Blaschke slice, i.e. in the $\mu + \mu^+$ or $i\mu + (i\mu)^+$ directions.

10. Incompleteness: General Case

In this section, we prove Theorem 1.2 which says that the Weil-Petersson metric is comparable to the model metric $\rho_{1/4}$ in the small horoballs. Note that outside the small horoballs, the upper bound is automatic: see the corollary to Theorem 1.4 or use part (a) of Theorem 2.2.

Unraveling definitions, we need to show that if $f_a \in \mathcal{B}_2$, $a \in \mathcal{H}_{p/q}(\eta)$, $\eta < C_{\text{small}}$ and $\mu = \mu_\lambda = \varphi_a^*(\lambda \cdot z/\bar{z} \cdot d\bar{z}/dz) \in M(\mathbb{D})^{f_a}$ is an optimal Beltrami coefficient with $|\lambda| = 1$, then $\|\mu \cdot \chi_{\mathcal{G}}\|_{\text{WP}}^2 \asymp \eta^{1/2}$.

For $a \in \mathcal{B}_{p/q}(C_{\text{small}})$, the flowers are still well-separated; however, we no longer have uniform control on the quasi-geodesic property. Indeed, when $a^q \in \mathbb{C} \setminus [0, \infty)$, multiplication by a^q traces out a logarithmic spiral $\{a^{qt}, t > 0\}$, and if we take $a^q \to 1$ along a horocycle, this logarithmic spiral tends to $\overline{\mathbb{D}}$ in the Hausdorff topology. Nevertheless, we can still show that $\limsup_{r \to 1^-} |\mathcal{G}(f_a) \cap S_r|$ is small. The following lemma is the key to both the upper and lower bounds for $\|\mu \cdot \chi_{\mathcal{G}}\|_{\text{WP}}^2$:

Lemma 10.1. *Suppose that $\langle \xi_1, \xi_2, \ldots, \xi_q \rangle$ is a repelling periodic orbit of a Blaschke product $f \in \mathcal{B}_2$ whose multiplier is $m < M_{\text{small}} := 1 + \frac{1}{16}$. There exists a constant $K > 0$ sufficiently large such that the branch of $(f^{\circ q})^{-1}$ which takes ξ_i to itself, maps $B(\xi_i, R)$ strictly inside of itself, where $R := \frac{\delta_c}{K\sqrt{m-1}}$.*

Corollary. *For each $i = 1, 2, \ldots, q$, the formula*

$$\varphi_{\xi_i}(z) := \lim_{n \to \infty} m^n \Big((f^{\circ nq})^{-1}(z) - \xi_i \Big) \tag{10.1}$$

defines a univalent holomorphic function on $B(\xi_i, R)$ satisfying

$$\varphi_{\xi_i} \circ (f^{\circ q})^{-1} = m^{-1} \cdot \varphi_{\xi_i}, \quad \varphi_{\xi_i}(\xi_i) = 0, \quad (\varphi_{\xi_i})'(\xi_i) = 1.$$

By Koebe's distortion theorem, Lemma 10.1 implies that the dynamics of $f^{\circ q}$ is nearly linear in the balls $B(\xi_i, R)$, i.e. if $z, f^{\circ q}(z), f^{\circ 2q}(z), \ldots, f^{\circ nq}(z) \in B(\xi_i, t \cdot R)$ with $t \leq 1/2$, then:

$$\left| \frac{|(f^{\circ nq})'(z)|}{m^n} - 1 \right| \lesssim t \qquad \text{and} \qquad \left| \arg(f^{\circ nq}(z) - \xi_i) - \arg(z - \xi_i) \right| \lesssim t. \tag{10.2}$$

Remark. Note that Lemma 10.1 is only significant for repelling periodic orbits with small multipliers. For $m > M_{\text{small}}$, one can apply Koebe's distortion theorem to the inverse branch $(f^{\circ q})^{-1}$ on $B(\xi_i, \delta_c)$ to see that $(f^{\circ q})^{-1}$ maps the ball $B(\xi_i, \delta_c/K)$ inside of itself.

Combining Lemma 10.1 with part (ii) of Lemma 8.1 gives:

Theorem 10.1 (Flower bounds). *There exists a constant $\pi/2 < \theta_1 < \pi$ such that for any $f_a \in \mathcal{B}_2$ with $a \in \mathcal{B}_{p/q}(C_{\text{small}})$,*

$$\mathcal{F} \subset \bigcup_{i=1}^{q} S(\xi_i, \theta_1, R) \cup B(0, 1 - 0.5 \cdot R) =: \bigcup S_i \cup B. \tag{10.3}$$

(The notation $S(\zeta, \theta, R) := \left\{ z : \arg(z/\zeta - 1) \in (\pi - \frac{\theta}{2}, \pi + \frac{\theta}{2}) \right\} \cap B(\zeta, R)$ denotes the central sector at $\zeta \in S^1$ of opening θ.)

Remark. We do not need to know *any* information about the behavior of the flower within the ball $B(0, 1 - 0.5 \cdot R)$.

With the help of Theorem 10.1, we extend the flower separation and structure lemmas to the wider class of parameters. Since the statements are interrelated, we state them as a single theorem:

Theorem 10.2. *For $a \in \mathcal{H}_{p/q}(\eta)$ with $\eta < C_{\mathrm{small}}$,*

   (a) The hyperbolic distance $d_{\mathbb{D}}(\mathcal{F}, c) \geq \frac{1}{2} \log(1/\eta) - O(1)$.

   (b) The hyperbolic distance $d_{\mathbb{D}}(\mathcal{F}, \mathcal{F}_*) \geq \log(1/\eta) - O(1)$.

   (c) The hyperbolic distance between any two pre-flowers exceeds $\log \eta - O(1)$.

   (a′) The critically-centered flower $\tilde{\mathcal{F}} \subset B(-\hat{c}, \mathrm{const} \cdot \eta^{1/2})$.

   (b′) The immediate pre-flower $\mathcal{F}_* \subset B(\hat{c}, \mathrm{const} \cdot \delta_c \cdot \eta^{1/2})$.

Using Theorems 10.1 and 10.2, it is easy to deduce Theorem 1.2. We give the details in Section 10.3.

10.1. **Linearization at repelling periodic orbits.** To show Lemma 10.1, we recall a formula for the derivative of a Blaschke product on the unit circle:

Lemma 10.2 (Equation (3.1) of [McM4]). *Given a Blaschke product $f_{\mathbf{a}} \in \mathcal{B}_d$, for $\zeta \in S^1$, we have*

$$|f_{\mathbf{a}}'(\zeta)| = 1 + \sum_{i=1}^{d-1} \frac{1 - |a_i|^2}{|\zeta + a_i|^2}. \tag{10.4}$$

In particular, the absolute value of the derivative of a Blaschke product is always greater than 1 on the unit circle. Specifying Lemma 10.2 to degree 2 and rearranging, we obtain:

Lemma 10.3. *Suppose $f \in \mathcal{B}_2$ is degree 2 Blaschke product and $\zeta \in S^1$. Then,*

$$|\zeta + a| = \sqrt{\frac{1 - |a|^2}{|f'(\zeta)| - 1}}. \tag{10.5}$$

Lemma 10.3 says that if $|f'(\zeta)|$ is close to 1, then ζ is far away from the point $-a$. For example, the condition $|f'(\zeta)| < 1 + \frac{1}{16}$ guarantees that $|\zeta + a| \geq 4\,\delta_c$ and $|\zeta - \hat{c}| \geq 3\,\delta_c$. (Since the critical point c is the hyperbolic midpoint of $[0, -a]$, we have $(\frac{1}{2} + \frac{1}{\sqrt{2}})\delta_c \geq \sqrt{1 - |a|^2} \geq \delta_c$.)

Lemma 10.4. *There exists a constant $K > 0$ such that for any degree 2 Blaschke product $f_a \in \mathcal{B}_2$ and $\zeta \in S^1 \setminus B(\hat{c}, 3\delta_c)$,*

$$|f'(z) - f'(\zeta)| \leq \frac{|f'(\zeta)| - 1}{2}, \qquad z \in B\left(\zeta, \frac{|\zeta + a|}{K}\right). \tag{10.6}$$

In particular, f is injective on $B\left(\zeta, \frac{|\zeta+a|}{K}\right)$ with $f\left(B(\zeta, \frac{|\zeta+a|}{K})\right) \supset B\left(f(\zeta), \frac{|\zeta+a|}{K}\right)$, and the branch of f^{-1} defined on $B\left(f(\zeta), \frac{|\zeta+a|}{K}\right)$ which takes $f(\zeta) \to \zeta$ is a contraction.

Proof. Differentiating twice gives $f''(z) = \frac{2(1-|a|^2)}{(1+\bar{a}z)^3}$ which implies that

$$|f''(z)| \asymp \frac{1 - |a|^2}{|z + a|^3}, \qquad z \in \mathbb{D} \setminus B(\hat{c}, 2\delta_c). \tag{10.7}$$

In view of (10.5), this gives

$$|f''(z)| \leq \frac{C}{|\zeta + a|} \cdot (|f'(\zeta)| - 1), \qquad z \in B\left(\zeta, \frac{|\zeta + a|}{3}\right).$$

Therefore, (10.6) holds with $K = \min(1/3, 1/(2C))$. □

Proof of Lemma 10.1. Let $m_i = |f'(\xi_i)|$ so that $m = |(f^{\circ q})'(\xi_1)| = m_1 m_2 \cdots m_q$. Since each $1 \leq m_i \leq m \leq 1 + \frac{1}{16}$, by Lemma 10.4, f^{-1} is a contraction on each ball $B(\xi_i, R)$. Therefore, the composition $(f^{\circ q})^{-1}$ is a contraction as well. □

10.2. **Separation and structure revisited.** The following lemma provides a convenient way for estimating hyperbolic distances between points in the unit disk:

Lemma 10.5. *Suppose $z_1, z_2 \in \mathbb{D}$ and z_0 is the point on the hyperbolic geodesic $[z_1, z_2]$ closest to origin. If z_0 does not coincide with either endpoint, then*

$$d_{\mathbb{D}}(z_1, z_2) = d_{\mathbb{D}}(|z_1|, |z_0|) + d_{\mathbb{D}}(|z_0|, |z_2|) + O(1). \tag{10.8}$$

Corollary. *If $\zeta_1, \zeta_2 \in S^1$ and the balls $B(\zeta_1, 2r_1)$ are $B(\zeta_2, 2r_2)$ are disjoint, then*

$$d_{\mathbb{D}}\left(B(\zeta_1, r_1), \ B(\zeta_2, r_2)\right) = d_{\mathbb{D}}\left(\zeta_1(1 - r_1), \ \zeta_2(1 - r_2)\right) + O(1). \tag{10.9}$$

To deduce the corollary from lemma, it suffices to observe that for two points $z_1 \in \partial B(\zeta_1, r_1) \cap \mathbb{D}$ and $z_2 \in \partial B(\zeta_2, r_2) \cap \mathbb{D}$, the highest point z_0 on $[z_1, z_2]$ satisfies $1 - |z_0| \asymp |\zeta_1 - \zeta_2|$.

Recall that the condition $|f'(\zeta)| < 1 + \frac{1}{16}$ guarantees that $|\zeta - \hat{c}| \geq 3\delta_c$. Applying the corollary with $B(\hat{c}, 2 \cdot \delta_c)$ and $B\left(\zeta, 2 \cdot \frac{|\zeta+a|}{K}\right)$ and $K \geq 2$ shows:

Lemma 10.6. *Suppose that $f_a \in \mathcal{B}_2$ is a degree 2 Blaschke product and $\zeta \in S^1$ is such that $|f'(\zeta)| < 1 + \frac{1}{16}$. For $K \geq 2$, we have*

$$d_{\mathbb{D}}\Big(c,\, B(\zeta, R_\zeta)\Big) = \frac{1}{2} \log \frac{1}{|f'(\zeta)| - 1} + \log K + O(1), \quad R_\zeta := \frac{|\zeta + a|}{K}.$$

We now deduce Theorem 10.2 from Theorem 10.1:

Proof of Theorem 10.2. Recall from Theorem 8.1 that $\eta \asymp (m - 1)$. By Theorem 10.1, the flower $\mathcal{F}$ is contained in $\mathbb{D} \setminus B(\hat{c}, R/2)$. This implies that

$$d_{\mathbb{D}}(\mathcal{F}, c) \geq d_{\mathbb{D}}\big(\hat{c}(1 - R/2),\, c\big) = \log\Big(\frac{1}{K\sqrt{m-1}}\Big) + O(1) = \log \frac{1}{\eta^{1/2}} + O(1)$$

and $\tilde{\mathcal{F}} \subset m_{c \to 0}\big(\mathbb{D} \setminus B(\hat{c}, R/2)\big) \subset B(-\hat{c}, \text{const} \cdot \eta^{1/2})$. This proves (a) and (a').

Applying Koebe's distortion theorem to the appropriate branch of $\tilde{f}^{-1}$ on $B(-\hat{c}, 1)$, we see that $\tilde{\mathcal{F}}_*$ is a nearly-affine copy of $\tilde{\mathcal{F}}$. Furthermore, since c is the midpoint of the hyperbolic geodesic $[0, -a]$, we must have $\tilde{\mathcal{F}}_* \approx -\tilde{\mathcal{F}}$. Therefore,

$$d_{\mathbb{D}}(\mathcal{F}, \mathcal{F}_*) = d_{\mathbb{D}}(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}_*) \geq \log \frac{1}{\eta} + O(1)$$

and $\mathcal{F}_* \subset m_{0 \to c}\big(B(\hat{c}, \text{const} \cdot \eta^{1/2})\big) \subset B(\hat{c}, \text{const} \cdot \delta_c \cdot \eta^{1/2})$. This proves (b) and (b').

Finally, (c) follows from the Schwarz lemma and the trick used in the proof of part (b) of Theorem 6.2. $\qquad\square$

10.3. **Proof of the main theorem.** We are now ready to show that

$$\|\mu \cdot \chi_{\mathcal{G}}\|_{\mathrm{WP}}^2 \lesssim \eta^{1/2} \qquad \text{for } a \in \mathcal{H}_{p/q}(\eta) \text{ with } \eta < C_{\mathrm{small}}.$$

We first prove the upper bound. Reflecting (10.3) about the critical point, we see that the immediate pre-flower $\mathcal{F}_*$ is contained in the union of the reflections $\bigcup S_i^* \cup B^*$. We claim that

$$\int_{\mathcal{F}_*} \frac{|dz|^2}{1 - |z|} \lesssim \delta_c \sqrt{m - 1}. \tag{10.10}$$

Assuming the claim, Lemmas 7.2 and 7.3 tell us that the laminated area

$$\mathcal{A}(\mathcal{G}(f_a)) \lesssim \frac{\delta_c \sqrt{m - 1}}{h(f_a, m)} \asymp \sqrt{m - 1} \asymp \eta^{1/2},$$

which by Theorem 2.1 implies $\|\mu \cdot \chi_{\mathcal{G}}\|_{\mathrm{WP}}^2 \lesssim \eta^{1/2}$ as desired. To prove (10.10), we need to carefully reflect the flower about the critical point.

The reflection B^* of the ball $B(0, 1 - 0.5 \cdot R)$ is contained in a horoball of diameter $\asymp \delta_c \sqrt{m - 1}$, resting on $\hat{c}$. Therefore, $\int_{B^*} \frac{|dz|^2}{1 - |z|} \lesssim \delta_c \sqrt{m - 1}$. Similar reasoning shows that the reflection S_i^* of S_i is contained in a sector $S(\xi_i^*, \theta_2, R_i^*)$, with $\theta_1 < \theta_2 < \pi$

and

$$R_i^* \asymp \delta_c \cdot \sqrt{m_i - 1} \cdot \frac{\sqrt{m_i - 1}}{\sqrt{m - 1}} = \delta_c \cdot \frac{m_i - 1}{\sqrt{m - 1}}. \tag{10.11}$$

The total contribution of these sectors to the integral (10.10) is roughly

$$\int_{\bigcup S_i^*} \frac{|dz|^2}{1 - |z|} \asymp \delta_c \sum \frac{m_i - 1}{\sqrt{m - 1}} \asymp \delta_c \sqrt{m - 1}. \tag{10.12}$$

This proves the upper bound.

For the lower bound, observe that by the blowing up technique of Theorem 9.1 together with Theorem 8.1, there exist balls

$$\mathscr{B}_i = B\left(\xi_i \cdot \left(1 - c_1 \cdot \frac{\delta_c}{\sqrt{m - 1}}\right), \ c_2 \cdot \frac{\delta_c}{\sqrt{m - 1}}\right) \tag{10.13}$$

lying in the sectors S_i for which $\int_{\mathscr{B}_i} |v'''/\rho^2(z)|^2 \cdot |dz|^2 \asymp 1$. The reflection $\mathscr{B}_i^*$ of $\mathscr{B}_i$ is a ball of definite hyperbolic size whose Euclidean center is located roughly at height

$$\asymp \delta_c \cdot \sqrt{m_i - 1} \cdot \frac{\sqrt{m_i - 1}}{\sqrt{m - 1}} = \delta_c \cdot \frac{m_i - 1}{\sqrt{m - 1}}.$$

Since the (repeated) pre-images of the $\mathscr{B}_i^*$ are disjoint, and each repeated pre-image is a near-affine copy of $\mathscr{B}_i^*$, by Lemmas 7.2 and 7.3,

$$\mathcal{A}\left(\bigcup_i \hat{\mathscr{B}}_i^*\right) \asymp \sum \frac{m_i - 1}{\sqrt{m - 1}} \asymp \sqrt{m - 1} \asymp \eta^{1/2}.$$

Thus, the lower bounds match the upper bounds up to a multiplicative constant. This concludes the proof of Theorem 1.2.

11. LIMITING VECTOR FIELDS

In this section, we study the convergence of Blaschke products to vector fields. For a Blaschke product $f_{\mathbf{a}}(z) = z \prod_{i=1}^{d-1} \frac{z + a_i}{1 + \overline{a_i} z}$, set $z_i := -a_i$. By a *radial degeneration*, we mean a sequence of Blaschke products $f_{\mathbf{a}} \in \mathcal{B}_d$ such that:

(1) The multiplier of the attracting fixed point tends (asymptotically) *radially* to $e(p/q)$, i.e. $\arg(e(p/q) - a) \to \arg(e(p/q))$.
(2) Each z_i converges to some point $e(\theta_i) \in S^1$.
(3) The limiting ratios of speeds at which the zeros escape are well-defined, i.e.

$$1 - |z_i| \sim \rho_i \cdot (1 - |a|)$$

with $\rho_i > 0$ and $\sum \rho_i = 1$.

To a radial degeneration, one can associate a natural probability measure μ on the unit circle which takes the escape rates into account: μ gives mass ρ_i/q to $e(\theta_i + j/q)$. Here, we use the convention that if some of the points coincide, we sum the masses.

Theorem 11.1. *One can compute:*

$$\kappa(z) = \lim_{a \to 1} \frac{f_{\mathbf{a}}^{\circ q}(z) - z}{1 - |a|^q} \to -z \int \frac{\zeta - z}{\zeta + z} d\mu_\zeta. \tag{11.1}$$

Furthermore,

$$f_{\mathbf{a}}^{\circ q}(z) - z - (1 - |a|^q)\kappa(z) = O\left((1 - |a|^q)^2\right) \tag{11.2}$$

uniformly in the closed unit disk away from $\operatorname{supp}\mu$.

Examples:

(1) As $a \to 1$ radially in $\mathcal{B}_2$, $f_a \to \kappa_1 := z \cdot \frac{z-1}{z+1} \cdot \frac{\partial}{\partial z}$.
(2) As $a \to e(p/q)$ radially in $\mathcal{B}_2$, $f_a^{\circ q} \to \kappa_q := q \cdot ((-1)^{q+1} \cdot z^q)^* \kappa_1$.

Let $\{g^\eta\}_{0 < \eta < 1}$ be the semigroup generated by κ written in multiplicative notation, i.e. $g^{\eta_1} \circ g^{\eta_2} = g^{\eta_1\eta_2}$, normalized so that $(g^\eta)'(0) = \eta$. Using (11.2), we promote the algebraic convergence in (11.1) to the dynamical convergence of the high iterates of $f_{\mathbf{a}}$ to the flow generated by $\kappa(z)$:

Theorem 11.2. *For* $0 < \eta < 1$, *if we choose* $T_{a,\eta}$ *so that*

$$(f_{\mathbf{a}}^{\circ q T_{a,\eta}})'(0) \to \eta,$$

then $f_{\mathbf{a}}^{\circ q T_{a,\eta}} \to g^\eta$ *uniformly in the closed unit disk away from* $\operatorname{supp}\mu$.

For applications, it is convenient to use the convergence of linearizing coordinates:

Corollary. *As* $a \to e(p/q)$ *radially, the linearizing coordinates* $\varphi_{\mathbf{a}} : \mathbb{D} \to \mathbb{C}$ *converge to the linearizing coordinate* $\varphi_\kappa := \lim_{\eta \to 0^+}(1/\eta) \cdot g^\eta(z)$ *of the semigroup generated by the limiting vector field* κ.

Remark. More generally, one can consider *linear degenerations* where $a \to e(p/q)$ asymptotically along a linear ray, i.e. with $a \approx e(p/q)(1 - \delta + \delta \cdot Ti)$ and δ small. In this case, the limiting vector field takes the more general form

$$\kappa(z) = \lim_{a \to 1} \frac{f_{\mathbf{a}}^{\circ q}(z) - z}{1 - |a|^q} \to -z \int \frac{\zeta - z}{\zeta + z} d\mu_\zeta + Ti \cdot z. \tag{11.3}$$

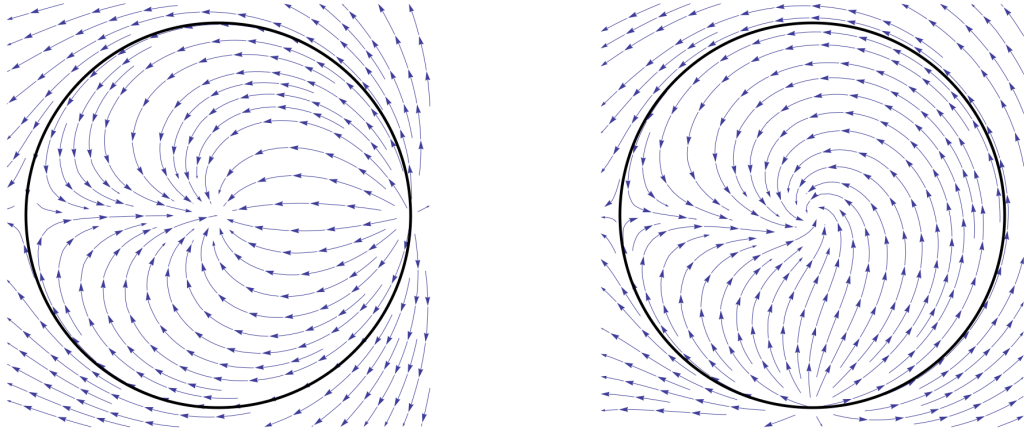We call μ the *driving measure* and T the *rotational factor*.

FIGURE 7. The vector fields $z \cdot \frac{z-1}{z+1} \cdot \frac{\partial}{\partial z}$ and $\left(z \cdot \frac{z-1}{z+1} + iz\right)\frac{\partial}{\partial z}$.

11.1. **Blaschke vector fields.** Before proving Theorems 11.1 and 11.2, let us examine the vector fields that may be obtained by this process. Recall that for a holomorphic vector field κ, the poles of κ are the *saddle points*, while the zeros are *sources* if $\text{Re } \kappa'(z) > 0$ and *sinks* if $\text{Re } \kappa'(z) < 0$ (if $\text{Re } \kappa'(z) = 0$, then z is a *center* but in our setting, this possibility does not occur).

Observe that for $\zeta \in S^1$, the map $z \to -\frac{\zeta+z}{\zeta-z}$ takes the unit disk onto the left half-plane. Therefore, as a function of z on the unit circle, $-\int \frac{\zeta+z}{\zeta-z}d\mu_\zeta$ takes purely imaginary values and (its imaginary part) is monotone increasing in $\arg z$ (except at the poles of κ). It follows that $\kappa = -z \int \frac{\zeta+z}{\zeta-z}d\mu_\zeta$ is tangent to the unit circle, has simple poles and in between any two adjacent poles has a unique zero. Since $\kappa'(0) = -1$, the point 0 is a sink. Conversely, it can be shown that any vector field with the above properties comes from a radial degeneration of some sequence of Blaschke products. Since we will not need this fact, we omit the proof.

Lemma 11.1. *Let* $M_a(z) = \frac{z+a}{1+\bar{a}z}$. *Suppose* $a \approx A \in S^1$ *with* $a = A(1 - \delta + \delta \cdot Ti)$ *and* $\delta > 0$. *Then,*

$$\frac{M_a(z)/A - 1}{1 - |a|} = \left(-\frac{A-z}{A+z} + Ti\right) + O\left((1-|a|)^2\right) \tag{11.4}$$

where the estimate is uniform over a *in any non-tangential sector at* A.

Proof. This is an exercise in differentiation. One simply needs to compute

$$\frac{\partial}{\partial\delta}\bigg|_{\delta=0} \frac{1}{A} \cdot \frac{z + A(1 - \delta + \delta \cdot Ti)}{1 + (1/A)(1 - \delta - \delta \cdot Ti)z} = \frac{z-A}{z+A} + Ti$$

and use the fact that $1 - |a| \approx \delta$. $\square$

We first prove Theorem 11.1 in the case when $a \to 1$. For a Blaschke product $f_{\mathbf{a}}(z) = z \prod_{i=1}^{d-1} \frac{z+a_i}{1+\overline{a_i}z}$, let $A_i = \hat{a}_i$, $A = \hat{a}$ and $T = T(f_{\mathbf{a}}) = -i \cdot \frac{A-1}{1-|a|}$. The idea is to compare $f_{\mathbf{a}}(z)$ to the vector field $\kappa(f_{\mathbf{a}})$ given by (11.3) with driving measure $\mu(f_{\mathbf{a}}) = \sum \frac{1-|a_i|}{1-|a|} \cdot \delta_{-A_i}$ and rotational factor $T(f_{\mathbf{a}})$:

Lemma 11.2. *The estimate*

$$f_{\mathbf{a}}(z) - z - (1-|a|)\kappa(z) = O\left((1-|a|)^2\right) \tag{11.5}$$

holds uniformly for z in the closed unit disk away from $\operatorname{supp} \mu$.

Proof. Using that $\prod(1+\delta_i) = 1 + \sum \delta_i + O(\max |\delta_i|^2)$ gives

$$
\begin{aligned}
f_{\mathbf{a}}(z) - z &= z\left(\prod \frac{z+a_i}{1+\overline{a_i}z} - \prod A_i\right) + z(A-1) \\
&\approx Az\sum\left(\frac{1}{A_i}\frac{z+a_i}{1+\overline{a_i}z} - 1\right) + z(A-1).
\end{aligned}
$$

Therefore,

$$\frac{f_{\mathbf{a}}(z) - z}{1 - |a|} \approx -Az\sum \rho_i \cdot \left(\frac{A_i - z}{A_i + z}\right) + Ti \cdot z = -Az \int \frac{-\zeta + z}{-\zeta - z} d\mu_\zeta + Ti \cdot z$$

as desired. $\qquad \square$

Theorem 11.1 now follows in the case when $a \to 1$ since for radial degenerations, the rotational factor $T(f_{\mathbf{a}}) \to 0$.

11.2. **Radial degenerations with** $a \to e(p/q)$. As noted above, for a radial degeneration with $a \to e(p/q)$, we consider the limiting vector field of $f_{\mathbf{a}}^{\circ q}$. In view of Lemma 11.2, to show that $f_{\mathbf{a}}^{\circ q}$ converges to a vector field κ whose driving measure gives mass ρ_i/q to each point $e(\theta_i + j/q)$, it suffices to analyze the zero set of $f_{\mathbf{a}}^{\circ q}$.

Let us first consider the case of a generic radial degeneration, i.e. when the points $e(\theta_i + j/q)$ with $1 \le i \le d-1$ and $0 \le j \le q-1$ are all different. The zero set of $f_{\mathbf{a}}^{\circ q}$ consists of the zeros of $f_{\mathbf{a}}$ and their $1, 2, \ldots, (q-1)$-fold pre-images. We omit the trivial zero at the origin and split the remaining zeros of $f_{\mathbf{a}}^{\circ q}$ into two groups: the *dominant zeros* and *subordinate zeros*. The dominant zeros are the zeros $z_i = z_{i,0}$ of $f_{\mathbf{a}}(z)$ and their shadows $z_{i,j}$ near $e(-jp/q)z_i$. We will refer to all other zeros as the subordinate zeros. From formula (7.3), it follows that the heights of the subordinate zeros are insignificant compared to the heights of the dominant zeros. Thus, only the dominant zeros contribute to the limiting vector field.

Let us now consider the general case. For a point $z \in \mathbb{D}$, call w a *dominant pre-image* of z under $f_{\mathbf{a}}$ if it is located near $e(-p/q)\hat{z}$, i.e. if $|\hat{w} - e(-p/q)\hat{z}| \le \epsilon$.

Otherwise, we say that w is a *subordinate pre-image*. We define a *dominant zero* of $f_{\mathbf{a}}^{\circ q}$ to be a point $z \in \mathbb{D}$ which is the j-fold dominant pre-image of some z_i, with $0 \leq j \leq q - 1$. To show that the driving measure μ has the desired expression, it suffices to show that the subordinate zeros have negligible height. We prove this in two lemmas:

Lemma 11.3. *Suppose $f_{\mathbf{a}}(z) = z \prod \frac{z + a_i}{1 + \overline{a_i} z} \in \mathcal{B}_d$ with $|a| = |f_{\mathbf{a}}'(0)| \approx 1$. For a point $z \in B\big(0, 1 - K\sqrt{1 - |a|}\big)$ with $K \geq 1$, the hyperbolic distance $d_{\mathbb{D}}(f_{\mathbf{a}}(z), az) < C/K^2$.*

Proof. The map $z \rightarrow \frac{z + a_i}{1 + \overline{a_i} z}$ takes the ball $B\big(0, 1 - K\sqrt{1 - |a|}\big)$ inside the ball

$$B\left(a_i, \ (C_1/K) \cdot \sqrt{1 - |a|} \cdot \frac{1 - |a_i|}{1 - |a|}\right).$$

Multiplying over $i = 1, 2, \ldots, d - 1$, we see that $\prod \frac{z + a_i}{1 + \overline{a_i} z} \in B\big(a, (C_2/K)\sqrt{1 - |a|}\big)$ which shows that $|f_{\mathbf{a}}(z) - az| \leq (C_2/K)\sqrt{1 - |a|}$ as desired. $\qquad \square$

Lemma 11.4. *Suppose w satisfies $f(w) = z$ yet $|\hat{w} - e(-p/q)\hat{z}| \geq \epsilon$. Then,*

$$\frac{G(w)}{G(z)} = O_\epsilon(1 - |a|). \tag{11.6}$$

Proof. Consider the hyperbolic geodesic $[0, w]$. Set $w_0 := (1 - K\sqrt{1 - |a|}) \cdot w$ and write $[0, w] = [0, w_0] \cup [w_0, w]$. Since f restricted to the first segment $[0, w_0]$ is nearly a rotation by $e(p/q)$, we see that during the first part of the journey from $f(0) = 0$ to $f(w) = z$ along $f([0, w])$ we have moved in the wrong direction, i.e.

$$
\begin{aligned}
d_{\mathbb{D}}(f(w_0), f(w)) &= d_{\mathbb{D}}(f(w_0), 0) + d_{\mathbb{D}}(0, f(w)) - O_\epsilon(1), \\
&= d_{\mathbb{D}}(w_0, 0) + d_{\mathbb{D}}(0, z) - O_\epsilon(1),
\end{aligned}
$$

as Lemma 10.5 shows. By the Schwarz lemma, $d_{\mathbb{D}}(w_0, w) \geq d_{\mathbb{D}}(f(w_0), f(w))$. Therefore, we must have $d_{\mathbb{D}}(0, w) = d_{\mathbb{D}}(0, w_0) + d_{\mathbb{D}}(w_0, w) \geq 2d_{\mathbb{D}}(0, w_0) + d_{\mathbb{D}}(0, z) - O_\epsilon(1)$ to make up for this detour. $\qquad \square$

11.3. **Asymptotic semigroups.** By an *asymptotic semigroup* on a domain Ω, written in additive notation, we mean a family of holomorphic maps $\{f_t\}_{t \geq 0} : \Omega_t \rightarrow \mathbb{C}$, with $\Omega_t \rightarrow \Omega$ in the Carathéodory topology, satisfying

$$f_t(z) = z + t \cdot \kappa(z) + O_K(t^2), \qquad 0 < t < t_K, \tag{11.7}$$

for some holomorphic vector field κ. (To convert to multiplicative notation, write $f_t = g^{\eta(t)}$ with $\eta(t) = e^{-t}$.) The notation O_K denotes that the implicit constant is

uniform on compact subsets of Ω. The condition (11.7) implies that

$$\left| f_t(z) - f_{t_1}(f_{t_2}(z)) \right| \leq O_K(t^2), \qquad t = t_1 + t_2. \tag{11.8}$$

We now show that the short term iteration of f_t approximates the flow of κ:

Theorem 11.3. *Given a ball $B(z_0, R)$ compactly contained in Ω, one can find a $t_0 > 0$, so that for $z \in B(z_0, R)$ and $t < t_0$, the limit*

$$g_t(z) := \lim_{\|\mathscr{P}\| \to 0} f_{\mathscr{P}}(z) := \lim_{\max t_i \to 0} f_{t_n}(f_{t_{n-1}}(\cdots (f_{t_1}(z)) \cdots)) \tag{11.9}$$

over all possible partitions $t_1 + t_2 + \cdots + t_n = t$ exists, and defines a holomorphic function on $B(z_0, R)$.

Above, the notation $f_{\mathscr{P}}(z)$ denotes the expression $f_{t_n}(f_{t_{n-1}}(\cdots (f_{t_1}(z)) \cdots))$ where $\mathscr{P}$ is a partition of the interval $[0, t]$ by the points $\tau_k = \sum_{j \leq k} t_j$. The existence of the limit in (11.9) implies that $\{g_t\}$ satisfies $g_s \circ g_t = g_{s+t}$ as long as g_{s+t} is well-defined. Clearly, the vector field κ is the generator of the semigroup $\{g_t\}$.

Proof. Choose two balls $B(z_0, R_2) \supset B(z_0, R_1) \supset B(z_0, R)$ compactly contained in Ω. We then choose $t_0^* > 0$ so that $|f_t(z) - z| \leq C_{R_1} t$ for $z \in B(z_0, R_1)$ and $t \leq t_0^*$. By taking $t_0 := \min\left(\frac{R_1 - R}{C_{R_1}}, t_0^*\right)$, we guarantee that all computations $f_{t_k} \circ \cdots \circ f_{t_1}(z)$ with $z \in B(z_0, R)$ and $t_1 + t_2 + \cdots + t_k < t_0$ stay within $B(z_0, R_1)$. Therefore, for $0 < t < t_0$, a finite partition $\mathscr{P}$ of the interval $[0, t]$ defines a holomorphic function $f_{\mathscr{P}}(z)$ on $B(z_0, R)$.

To prove the convergence of (11.9), it suffices to show that if $\mathscr{Q}$ is a refinement of $\mathscr{P}$, then $|f_{\mathscr{P}}(z) - f_{\mathscr{Q}}(z)| \leq Ct\|\mathscr{P}\|$. Actually, it suffices to show that for an arbitrary partition $\mathscr{P}$ of $[0, t]$, one has $|f_{\mathscr{P}}(z) - f_t(z)| \leq Ct^2$. For this purpose, we introduce some bookkeeping: in view of (11.8), we say that the cost of splitting an interval of length T into two intervals is $C \cdot T^2$. Using the greedy algorithm, its not hard to show that the minimal cost of any partition of $[0, t]$ is at most $O(t^2)$.

To combine the "costs," we use the fact that on $B(z_0, R_1)$, the hyperbolic metric $\rho_{B(z_0, R_2)}$ is comparable to the Euclidean metric. Therefore, by the Schwarz lemma,

$$d_{B(z_0, R_2)}\left(z, f_{t_n} \circ \cdots \circ f_{t_1}(z)\right) \leq \sum_{k=1}^{n} d_{B(z_0, R_2)}\left(f_{t_k} \circ \cdots \circ f_{t_1}(z), \ f_{t_{k-1}} \circ \cdots \circ f_{t_1}(z)\right)$$

$$\lesssim \sum_{k=1}^{n} \left| f_{t_k} \circ \cdots \circ f_{t_1}(z) - f_{t_{k-1}} \circ \cdots \circ f_{t_1}(z) \right|$$

which gives the claim. $\qquad \square$

Clearly, Theorem 11.2 is a special case of Theorem 11.3, where $\Omega = \mathbb{C} \setminus P(\kappa)$ is the complement of the set of poles of κ. By the Schwarz lemma, inside the unit disk, $g^t(z)$ can be defined for *all* time, whereas on the unit circle, one can only define $g^t(z)$ until one hits a pole of κ.

12. Asymptotics of the Weil-Petersson metric

In this section, we prove Theorem 1.3 which says that as $a \to e(p/q)$ radially in $\mathcal{B}_2$, the ratio $\omega_B/\rho_{1/4}$ tends to a constant, depending on the denominator q. In the language of half-optimal Beltrami coefficients, we need to show that for a fixed λ with $|\lambda| = 1$, $\|\mu_\lambda \cdot \chi_{\mathcal{G}}\|_{\mathrm{WP}} \sim C'_q(1 - |a|)^{1/4}$. As noted in the introduction, the key observation is the convergence of Blaschke products to vector fields. The convergence of the linearizing coordinates (the corollary to Theorem 11.2) gives:

Theorem 12.1. *As $a \to e(p/q)$ radially,*

(i) *The flowers $\mathcal{F}_{p/q}(f_a) \to \mathcal{F}_{p/q}(\kappa_q)$ in the Hausdorff topology.*

(ii) *The optimal Beltrami coefficients $\mu_\lambda(f_a) = \varphi_a^*(\lambda \cdot z/\overline{z} \cdot d\overline{z}/dz)$ converge uniformly to $\varphi_{\kappa_q}^*(\lambda \cdot z/\overline{z} \cdot d\overline{z}/dz)$ on compact subsets of $\mathbb{D} \setminus \{0\}$.*

Together with Lemma 10.1, which controls the shapes of flowers near the unit circle, Theorem 12.1 implies the quasi-geodesic property:

Lemma 12.1 (Quasi-geodesic property)**.** *As $a \to e(p/q)$ radially, each petal $\mathcal{P}_{\xi_i(f_a)}(f_a)$ lies within a bounded distance of the geodesic ray $[0, \xi_i(f_a)]$. Alternatively, the flower $\mathcal{F}(f_a)$ lies within a bounded neighbourhood of the hyperbolic convex hull of the origin and the ends $\xi_i(f_a)$.*

For convenience of the reader, we give an alternative proof of the strong linearization property of Lemma 10.1 using the existence of the limiting vector field $\kappa = \kappa_q$. Observe that as $a \to e(p/q)$ radially, the p/q-cycle $\langle \xi_1(f_a), \xi_2(f_a), \ldots, \xi_q(f_a) \rangle$ converges to the set of sources $\langle \xi_1, \xi_2, \ldots, \xi_q \rangle$ of κ. Note that $\kappa'(\xi_i) > 0$ is a positive real number since κ is tangent to the unit circle.

Choose a ball $B(\xi_i, R')$ on which $\left| \frac{\kappa'(z)}{\kappa'(\xi_i)} - 1 \right| < 1/10$. From the uniform convergence $\frac{f_a^{\circ q}(z) - z}{1 - |a|^q} \to \kappa(z)$ on $B(\xi_i, R')$, it follows that on $B(\xi_i, R'/2)$, we have

$$(f_a^{\circ q})'(z) \approx 1 + \kappa'(z)(1 - |a|^q), \qquad \left| (f_a^{\circ q})''(z) \right| \leq C_1(1 - |a|^q),$$

where $C_1 = \max_{z \in B(\xi_i, R')} |\kappa''(z)| + \epsilon$. Therefore, for a sufficiently close to $e(p/q)$, we have $|\xi_i(f_a) - \xi_i| < R'/4$ and $|(f_a^{\circ q})''(z)| \leq C_2 |(f_a^{\circ q})'(\xi_1(f_a)) - 1|$ for $z \in B(\xi, R'/2)$, which implies that $(f_a^{\circ q})^{-1}$ maps $B(\xi_i(f_a), R)$ into itself, with $R = \min(R'/4, 1/(2C_2))$.

Flower counting hypothesis. From Theorem 10.2, we know that the immediate pre-flower is approximately the image of the flower under the Möbius involution about the critical point, while the pre-flowers are nearly-affine copies of the immediate pre-flower. Therefore, the pre-flowers of all maps

$$\{f_a, \ a \in e(p/q) \cdot [1 - \epsilon, 1)\}$$

also have nearly the same shape up to affine scaling. Let $n(r, f_a)$ denote the number of repeated pre-images of $-a$ that lie in $B(0, r)$. By Theorem 7.2,

$$\mathfrak{c}(f_a) \ := \ \lim_{r \to 1} \frac{n(r, f_a)}{1 - r} \ = \ \frac{\log(1/|a|)}{h(f_a, m)} \ \sim \ \sqrt{1 - |a|}, \qquad \text{as } |a| \to 1. \qquad (12.1)$$

The quantity $n(r, f_a)$ roughly counts the number of pre-flowers that intersect S_r. By renewal theory, for r close to 1, the circle S_r intersects pre-flowers at "hyperbolically random" locations. Therefore, is reasonable to hypothesize that as $a \to e(p/q)$ radially, $\frac{1}{\mathfrak{c}(f_a)} \cdot \|\mu_\lambda \cdot \chi_{\mathcal{G}}\|_{\mathrm{WP}}^2$ converges to a constant. To justify this, we must show three things:

(1) The contributions of the pre-flowers are more or less independent.
(2) All pre-flowers of the same size contribute roughly the same amount.
(3) Most of the integral $\mathcal{I}_r[\mu] = \frac{1}{2\pi} \int_{|z|=r} |v_{\mu^+}'''/\rho^2|^2 d\theta$ comes from pre-flowers whose size is comparable to $1 - r$.

12.1. **Decay of correlations.** In this section, we use "flower" to mean either a flower or a pre-flower. Write the half-optimal coefficient as $\mu_{\mathrm{half}} = \sum_{\mathcal{F}} \mu_{\mathcal{F}}$ with $\mu_{\mathcal{F}}$ supported on $\mathcal{F}$. Set

$$v_{\mathcal{F}}'''(z) = -\frac{6}{\pi} \int_{\mathcal{F}^+} \frac{\mu^+(\zeta)}{(\zeta - z)^4} \cdot |d\zeta|^2.$$

Then $v'''(z) = \sum_{\mathcal{F}} v_{\mathcal{F}}'''(z)$. We wish to show that the integral (1.4) is proportional to the flower count. The main difficulty is that (1.4) features the L^2 norm so we have "correlations" $\sum_{\mathcal{F}_1 \neq \mathcal{F}_2} \int_{|z|=r} \frac{v_{\mathcal{F}_1}'''}{\rho^2} \cdot \frac{\overline{v_{\mathcal{F}_2}'''}}{\rho^2} d\theta$. We now show that these correlations are insignificant compared to the main term $\sum_{\mathcal{F}} \int_{|z|=r} \left|\frac{v_{\mathcal{F}}'''}{\rho^2}\right|^2 d\theta$.

For a point $z \in \mathbb{D}$, let $\mathcal{F}_z$ be the flower which is closest to z in the hyperbolic metric (in case of a tie, we pick $\mathcal{F}_z$ arbitrarily) and $\mathcal{R}_z$ be the union of all the other flowers. The integral (1.4) splits into four parts:

$$\int_{|z|=r} \left\{ \left|\frac{v_{\mathcal{F}_z}'''(z)}{\rho(z)^2}\right|^2 + \frac{v_{\mathcal{F}_z}'''(z)}{\rho(z)^2} \cdot \frac{\overline{v_{\mathcal{R}_z}'''(z)}}{\rho(z)^2} + \frac{v_{\mathcal{R}_z}'''(z)}{\rho(z)^2} \cdot \frac{\overline{v_{\mathcal{F}_z}'''(z)}}{\rho(z)^2} + \left|\frac{v_{\mathcal{R}_z}'''(z)}{\rho(z)^2}\right|^2 \right\} d\theta$$

By the lower bound established in Section 10, the first term is bounded below by the flower count which decays roughly like $\asymp \sqrt{1 - |a|}$, while each of the other three

terms contribute on the order of $O(1 - |a|)$, and so are negligible. Take for instance the second term. By the triangle inequality, for any $z \in \mathbb{D}$,

$$
\frac{v'''_{\mathcal{F}_z}(z)}{\rho(z)^2} \cdot \frac{\overline{v'''_{\mathcal{R}_z}(z)}}{\rho(z)^2} \lesssim e^{-d_{\mathbb{D}}(z, \mathcal{F}_z)} \cdot e^{-d_{\mathbb{D}}(z, \mathcal{R}_z)} \leq e^{-d_{\mathbb{D}}(\mathcal{F}_z, \mathcal{R}_z)}
$$

which is bounded by $e^{-d_{\mathbb{D}}(0, a)} \asymp (1 - |a|)$. The estimate for the other two error terms is similar.

12.2. **Convergence of Beltrami coefficients.** For a Blaschke product $f_a \in \mathcal{B}_2$ with $a \approx e(p/q)$, we define an *idealized garden* $\mathcal{G}^{\mathrm{id}}(f_a)$ where the pre-flowers have the model shape. First, we define the idealized flower $\mathcal{F}^{\mathrm{id}}(f_a) := \mathcal{F}(g^\eta)$ to be the flower of the limiting vector field. We then define the idealized immediate pre-flower $\mathcal{F}^{\mathrm{id}}_*(f_a)$ as the image of $\mathcal{F}(g^\eta)$ under the Möbius involution about $c(f_a)$. Finally, for a pre-flower $\mathcal{F}_z(f_a)$, we define its idealized version $\mathcal{F}^{\mathrm{id}}_z(f_a)$ to be the affine copy of $\mathcal{F}^{\mathrm{id}}_*(f_a)$, which has the same A-point z.

The *idealized half-optimal Beltrami coefficient* μ_{id} is defined similarly: on $\mathcal{F}^{\mathrm{id}}(f_a)$, we let $\mu_{\mathrm{id}} \cdot \chi_{\mathcal{F}^{\mathrm{id}}}$ be the half-optimal Beltrami coefficient for the limiting vector field; while on the pre-flowers, we define $\mu_{\mathrm{id}} \cdot \chi_{\mathcal{F}^{\mathrm{id}}_z}$ by scaling $\mu_{\mathrm{id}} \cdot \chi_{\mathcal{F}^{\mathrm{id}}}$ appropriately. Our current objective is to show that the integral averages for the model coefficient and the half-optimal coefficient μ_{half} are approximately the same:

Lemma 12.2. *As $a \to e(p/q)$ radially,*

$$
\mathcal{I}[\mu_{\mathrm{id}}] - \mathcal{I}[\mu_{\mathrm{half}}] = o\big(\sqrt{1 - |a|}\big).
$$

There are two sources of error. First, the pre-flowers don't quite match up with their idealized counterparts. Secondly, since the linearizing maps φ_a and φ_κ are slightly different, the Beltrami coefficients μ_{half} and μ_{id} themselves are slightly different. To prove Lemma 12.2, we we split $\mathcal{F}^\alpha$ into three parts:

$$
\begin{aligned}
A^{\alpha, \delta} &= \mathcal{F}^\alpha \cap \{z : |z| < \delta\}, \\
B^{\alpha, \delta} &= \mathcal{F}^\alpha \cap \{z : \delta < |z| < 1 - \delta\}, \\
C^{\alpha, \delta} &= \mathcal{F}^\alpha \cap \{z : 1 - \delta < |z|\}.
\end{aligned}
$$

Taking pre-images, we obtain ABC decompositions of pre-flowers. As usual, we take $\alpha = 1/2$ unless specified otherwise. We typically omit the parameter $\delta > 0$ from the notation.

Estimating the symmetric difference. For any $\epsilon > 0$, if a is sufficiently close to $e(p/q)$, the symmetric difference of the flower $\mathcal{F}$ and its idealized version $\mathcal{F}^{\mathrm{id}}$ is

contained in the set

$$\Delta(\mathcal{F}) := A(f_a) \cup A(g^\eta) \; \cup \; \left(B^{1/2+\epsilon}(g^\eta) \setminus B^{1/2-\epsilon}(g^\eta)\right) \; \cup \; C(f_a) \cup C(g^\eta). \quad (12.2)$$

Taking pre-images, we obtain sets $\Delta(\mathcal{F}_z)$ which contain symmetric differences of pre-flowers and their idealized versions. Let $\Delta = \bigcup \Delta(\mathcal{F}_z)$. Observe that the proportion

$$\frac{\mathcal{A}(\Delta)}{\mathcal{A}(\mathcal{G})} = \frac{\limsup_{r\to 1} |\Delta \cap S_r|}{\lim_{r\to 1} |\mathcal{G} \cap S_r|} \quad (12.3)$$

can be made arbitrarily small by choosing $\delta, \epsilon > 0$ small. By Theorem 2.1, it follows that $\mathcal{I}[\mu_{\text{half}} \cdot \chi_\Delta] = o\big(\sqrt{1-|a|}\big)$ and $\mathcal{I}[\mu_{\text{id}} \cdot \chi_\Delta] = o\big(\sqrt{1-|a|}\big)$.

Estimating the difference between Beltrami Coefficients. From the convergence of the linearizing maps $\varphi_a \to \varphi_\kappa$, when $a \approx e(p/q)$, $|\mu_{\text{half}} - \mu_{\text{id}}|$ is arbitrarily small on $B^{1/2+\epsilon}(g^\eta)$. By Koebe's distortion theorem, the same estimate holds on pre-flowers. Therefore, the difference $|\mu_{\text{half}} \cdot \chi_{\Delta^c} - \mu_{\text{id}} \cdot \chi_{\Delta^c}|$ is small in L^∞ sense. Theorem 2.1 implies $\mathcal{I}[\mu_{\text{half}} \cdot \chi_{\Delta^c} - \mu_{\text{id}} \cdot \chi_{\Delta^c}] = o\big(\sqrt{1-|a|}\big)$.

Combining the errors. In [Hed], Hedenmalm observed that Minkowski's inequality implies that $\mathcal{I}[\mu]$ behaves like a semi-norm:

$$\left|\sqrt{\mathcal{I}[\mu]} - \sqrt{\mathcal{I}[\nu]}\right| \le \sqrt{\mathcal{I}[\mu - \nu]}, \quad (12.4)$$

where in the definition of $\mathcal{I}[\mu]$, we use $\limsup$ if necessary (if μ is not invariant). Returning to the task at hand, since $\mathcal{I}[\mu_{\text{id}}]$ and $\mathcal{I}[\mu_{\text{half}}]$ are $\gtrsim \sqrt{1-|a|}$ and the errors are $o\big(\sqrt{1-|a|}\big)$, Minkowski's inequality (12.4) completes the proof of Lemma 12.2.

12.3. **Flowers: large and small.** Finally, we must show that most of the integral average $\int_{S_r} |v'''/\rho^2|^2 d\theta$ comes from flowers whose size is $\asymp (1-r)$. In view of (12.1), given $\epsilon > 0$, there exists $0 < r_{\text{mix}} = r_{\text{mix}}(f_a) < 1$ such that $\frac{n(r,f_a)}{1-r} \approx_\epsilon \mathfrak{c}(f_a)$ for $r \in (r_{\text{mix}}, 1)$. For $r \in (r_{\text{mix}}, 1)$, we decompose

$$\mu_{\text{half}} = \mu_{\text{small}} + \mu_{\text{med}} + \mu_{\text{large}} + \mu_{\text{huge}} \quad (12.5)$$

where

$$\begin{cases} \text{small flowers} & \text{have size } s \le (1-r)/k, \\ \text{medium flowers} & \text{have size } (1-r)/k \le s \le k(1-r), \\ \text{large flowers} & \text{have size } k(1-r) \le s \le 1 - r_{\text{mix}}, \\ \text{huge flowers} & \text{have size } s \ge 1 - r_{\text{mix}}. \end{cases}$$

(The *size* of a flower $\mathcal{F}_z$ may be defined as either its diameter or as $1 - |z|$. The two quantities are comparable along radial degenerations.)

From the lower bound, we know that

$$\mathcal{I}[\mu_{\text{med}}] \asymp \mathfrak{c}(f_a).$$

We claim that if the "tolerance" $k > 1$ is large, then

$$\left| \mathcal{I}[\mu_{\text{half}}] - \mathcal{I}[\mu_{\text{med}}] \right| \lesssim \mathfrak{c}(f_a)/\sqrt{k}. \tag{12.6}$$

Since there are only finitely many huge flowers and they satisfy the quasi-geodesic property, $|\mathcal{G}_{\text{huge}} \cap S_r| \to 0$ as $r \to 1$. By counting the number of large flowers, we can conclude $|\mathcal{G}_{\text{large}} \cap S_r| \lesssim \mathfrak{c}(f_a)/k$ as well. Therefore by Theorem 2.1,

$$\mathcal{I}[\mu_{\text{huge}}] + \mathcal{I}[\mu_{\text{large}}] \lesssim \mathfrak{c}(f_a)/k$$

for r close to 1. It remains to estimate the contribution of the small flowers. This can be done by combining the Fubini argument from the proof of Theorem 2.1 with part (b) of Theorem 2.2. This leads to the estimate

$$\int_{|z|=r} |v_{\text{small}}'''/\rho^2|^2 d\theta \lesssim \frac{\|\mu\|_\infty}{k} \cdot \limsup_{R \to 1^+} \frac{1}{2\pi} \left| \operatorname{supp} \mu^+ \cap S_R \right| \lesssim \mathfrak{c}(f_a)/k.$$

Using Minkowski's inequality (12.4) as before proves (12.6). This completes the proof of Theorem 1.3.

References

[AIM] K. Astala, T. Iwaniec, G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton University Press, 2009.

[B] C. J. Bishop, *Big deformations near infinity*, Illinois J. Math. **47**(2003), 977–996.

[E] A. Epstein, *Bounded hyperbolic components of quadratic rational maps*, Ergod. Th. & Dynam. Sys. **20**(2000), 727–748.

[GM] J. B. Garnett, D. E. Marshall, *Harmonic Measure*, New Mathematical Monographs 2, Cambridge University Press, 2005.

[Ha] D. H. Hamilton, *Absolutely Continuous Conjugacies of Blaschke Products*, Advances in Math. **121**(1996), 1–20.

[Hed] H. Hedenmalm, *Bloch functions and asymptotic tail variance*, preprint, arXiv:1509.06630, 2015.

[Hub] J. H. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*, Matrix Editions, 2006.

[IT] Y. Imayoshi, M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, 1992.

[Ivr] O. Ivrii, *Rescaling limits of Blaschke products*, preprint, 2015.

[La] S. P. Lalley, *Renewal Theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits.* Acta Math. **163**(1989), 1–55.

[Mak] N. G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. **51**(1985), 369–384.

[McM1] C. T. McMullen, *Cusps are dense.* Annals of Math. **133**(1991) 217–247.

[McM2] C. T. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric.* Inv. math. **173**(2008), 365–425.

[McM3] C. T. McMullen, *A compactification of the space of expanding maps on the circle*, Geom. Funct. Anal. **18**(2008), 2101–2119.

[McM4] C. T. McMullen, *Dynamics on the unit disk: Short geodesics and simple cycles*, Comm. Math. Helv. **85**(2010), 723–749.

[McM5] C. T. McMullen, *Ribbon $\mathbb{R}$-trees and holomorphic dynamics on the unit disk*, J. Topol. **2**(2009), 23–76.

[Mil] J. Milnor, *Hyperbolic Components*, arXiv:1205.2668, 2012.

[MS] C. T. McMullen, D. P. Sullivan, *Quasiconformal Homeomorphisms and Dynamics III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math. **135**(1998), 351–395.

[PrSm] I. Prause, and S. Smirnov, *Quasisymmetric distortion spectrum*, Bull. London Math. Soc. **43**(2011) 267–277.

[PP] W. Parry, M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, Astérisque, vol. 187–188, 1990.

[PUZ1] F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps I*, Ann. of Math., **130**(1989), 1–40.

[PUZ2] F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps II*, Studia Math., **97**(1991), 189–225.

[S] S. Smirnov, *Dimension of Quasicircles*, Acta Math., **205**(2010), 189–197.

[SS] M. Shub, D. P. Sullivan, *Expanding endomorphisms of the circle revisited*, Ergod. Th. & Dynam. Sys. **5**(1985), 285–289.

[Wol] S. Wolpert, *Families of Riemann surfaces and Weil-Petersson geometry*, AMS, 2010.

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