Fields with the Density Property

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Communicated by A. Fröhlich

Received November 8, 1973

INTRODUCTION

Let $K$ be a field. Denote by $\mathfrak{G}(K_s|K)$ the Galois group of the separable closure $K_s$ of $K$ over $K$. This group is equipped with a normalized Haar measure $\mu$ with respect to its Krull topology. We are interested in fields of the form $K_s(\sigma)$ which are, by definition, the fixed fields of $e$-tuples $(\sigma) = (\sigma_1, \ldots, \sigma_e) \in \mathfrak{G}(K_s|K)^e$. In [3, p. 76] we have proved the following Theorem:

THEOREM A. If $K$ is a denumerable hilbertian field then almost all $(\sigma) \in \mathfrak{G}(K_s|K)^e$ have the following property: For every nonvoid abstract variety $V$ defined over $K_s(\sigma)$, the set $V(K_s(\sigma))$ of all $K$, $(\sigma)$-rational points of $V$ is Zariski $K$-dense in $\overline{V(K)}$.

In this note we consider a denumerable hilbertian field $K$ equipped with an absolute value $v$ which is either the usual absolute value induced by that of the complex numbers or a non-archimedean valuation with values in a commutative ordered group $T$. The absolute value $v$ is assumed to have been extended in some fixed way to the algebraic closure $\overline{K}$ of $K$. The purpose of this note is to strengthen Theorem A for such $K$ in the following way.

THEOREM B. Let $K$ be a denumerable hilbertian valued field. Then almost all $(\sigma) \in \mathfrak{G}(K_s|K)^e$ have the following property: $V(K_s(\sigma))$ is $v$-dense in $\overline{V(K)}$ for every abstract variety $V$ defined over $K_s(\sigma)$.

* This work was done while the author was at Heidelberg University.
1. Valued Fields

In this note we consider valued fields \((K, v)\) of the following two types:

(i) The archimedean type: \(K\) is a subfield of the field of the complex numbers \(\mathbb{C}\) and \(v\) is the usual absolute value.

(ii) The non-archimedean type: \(K\) is an arbitrary field and \(v\) is a non-trivial valuation of \(K\), i.e., a homomorphism of \(K^*\) into an ordered multiplicative abelian group \(\Gamma\) such that

\[
v(a + b) \leq \max\{v(a), v(b)\},
\]

and \(v(a) \neq 1\) for some \(a \in K^*\) (c.f. Ribenboim [7, p. 27]). As usual we add an element 0 to \(\Gamma\) as a first element with the rule \(0 \cdot \gamma = 0\) for every \(\gamma \in \Gamma\) and put \(v(0) = 0\).

We shall use the notation \(|a|\) instead of \(v(a)\) for elements \(a\) of \(K\) and we keep the notation \(v(A)\) for the value set of a subset \(A\) of \(K\).

In each case \(v\) induces a field topology on \(K\), the basis sets of which are \(\{x \in K \mid |x - a| < \epsilon\}\) where \(a \in K\) and \(\epsilon \in \Gamma\). We shall refer to it as the \(v\)-topology. We denote by \(K_v\), \(K_s\) and \(\tilde{K}\) the \(v\)-completion of \(K\), its separable closure and its algebraic closure respectively. We always assume that \(v\) has been extended first to \(\tilde{K}\) and then to its completion \(\tilde{K}_v\). Every extension of \(K\) will be assumed to lie in \(\tilde{K}_v\) and thus to be a valued field too. \(\Gamma\) will stand for \(v(K_v - \{0\})\). Then for every \(\epsilon \in \Gamma\) there exists an element \(a \in K^*\) such that \(|a| < \epsilon\). This is clear in the archimedean case, since \(\mathbb{Q}\) is dense in \(\mathbb{R}\).

In the non-archimedean case it suffices to consider the case \(0 < \epsilon = |x| < 1\), where \(x \in \tilde{K}\). Now \(x\) lies in a finite extension \(L\) of \(K\). Let \(e = (v(L^*); v(K^*))\) be the ramification index. Then \(e\) is finite (c.f., Ribenboim [7, p. 59]) and hence there exists an \(a \in K^*\) such that \(|a| = |x|^e < |x|\).

**Lemma 1.1.** Let \(K\) be an algebraically closed valued field and let

\[
f(T, X) = f_n(T) X^n + f_{n-1}(T) X^{n-1} + \cdots + f_0(T)
\]

be a polynomial with coefficients in \(K\) in the variables \((T, X) = (T_1, \ldots, T_r, X)\). Let \((t_0, x_0)\) be a \(K\)-rational zero of \(f\) for which \(f_l(t_0) \neq 0\) for some \(0 \leq l \leq n\). Then for every \(\epsilon \in \Gamma\) there exists a \(\delta \in \Gamma\) such that for every \(t_1, \ldots, t_r \in K\) which satisfy

\[
|t_i - t_{0i}| < \delta \quad i = 1, \ldots, r
\]

there exists an \(x \in K\) such that \(f(t, x) = 0\) and \(|x - x_0| < \epsilon\).

**Proof.** Without loss of generality we can assume that \((t_0, x_0) = (0, 0)\). Then \(f_0(0) = 0\) and there exists an \(1 \leq l \leq n\) such that \(f_l(0) \neq 0\). Since


\[ f_0 \text{ and } f_i \text{ are both } \nu \text{-continuous functions we can find a } \delta \in \Gamma \text{ such that } |t_i| < \delta \text{ for } i = 1, ..., r \Rightarrow f_i(t) \neq 0 \text{ and} \]

\[
\left| \frac{f_0(t)}{f_i(t)} \right| < \begin{cases} 
\frac{\epsilon n}{n!} & \text{in the arch. case} \\
\epsilon^n & \text{in the non-arch. case.}
\end{cases}
\]

Suppose now that \( |t_i| < \delta \) for \( i = 1, ..., r \). Let \( m \) be the greatest integer for which \( f_m(t) \neq 0 \). Then \( l \leq m \leq n \) and

\[
f(t, X) = f_m(t)X^m + \cdots + f_i(t)X^i + \cdots + f_0(t) = f_m(t) \prod_{i=1}^{m} (X - x_i)
\]

with \( x_1, ..., x_m \in K \). Then

\[
\frac{f_0(t)}{f_m(t)} = (-1)^m x_1 \cdots x_m, \quad \frac{f_i(t)}{f_m(t)} = (-1)^{m-i} \sum_{\pi} x_{\pi(1)} \cdots x_{\pi(i)},
\]

where \( \pi \) runs over all the injective maps of the set \( \{1, ..., m - l\} \) into the set \( \{1, ..., m\} \). If \( f_0(t) = 0 \) then \( x_i = 0 \) for some \( 1 \leq i \leq m \) and we are done. Suppose therefore that \( f_0(t) \neq 0 \) and extend every \( \pi \) uniquely to a permutation of the set \( \{1, ..., m\} \). Then

\[
\frac{f_i(t)}{f_0(t)} = (-1)^i \sum_{\pi} \frac{1}{x_{\pi(m-l+1)} \cdots x_{\pi(m)}}.
\]

It follows that in both cases there must exist an \( x_i \) such that \( |x_i| < \epsilon \).

**Lemma 1.2.** A separably closed valued field \( K \) is \( \nu \)-dense in \( \tilde{K} \).

**Proof.** We have to prove the Lemma only when \( \text{char}(K) = p \neq 0 \). In this case \( \nu \) is non-archimedean.

Let \( a \in \tilde{K}, a \neq 0 \). Then there exists a power \( q \) of \( p \) such that \( a^q = b \in K \).

Let \( \epsilon \in \Gamma \). Take an element \( c \in K^* \) such that \( |c| < |a|^{-1} \epsilon^q \) and consider the separable polynomial \( X^q - cX - b \). It has \( q \) roots \( x_1, ..., x_q \) in \( K \). Now

\[
ca = a^q - ca - b = \prod_{i=1}^{q} (a - x_i)
\]

\[
\Rightarrow \epsilon^q > \prod_{i=1}^{q} |a - x_i|
\]

\[
\Rightarrow \text{ There exists an } 1 \leq i \leq q \text{ such that } |a - x_i| < \epsilon.
\]
Lemma 1.3. If \( K \) is a complete separably closed valued field then \( K \) is algebraically closed.

Proof. \( K \) is closed in \( \bar{K} \) by completeness and dense in \( \bar{K} \) by Lemma 1.2. It follows that \( K = \bar{K} \).

Lemma 1.4. The completion \( K_v \) of a separably closed valued field \( K \) is algebraically closed.

Proof. By Lemma 1.3 we have only to prove that \( K_v \) is separably closed. Indeed let \( f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \) be a separable polynomial with coefficients in \( K_v \) and let \( x \) be a root of \( f \) in the algebraic closure \( L \) of \( K_v \). Let \( \epsilon \in \mathcal{I} \). Then by Lemma 1.1 if we choose \( b_{n-1}, \ldots, b_0 \) in \( K \) sufficiently \( v \)-close to \( a_{n-1}, \ldots, a_0 \), then the polynomial \( g(X) = X^n + b_{n-1}X^{n-1} + \cdots + b_0 \) is separable and has a root \( y \) such that \( |y - x| < \epsilon \). This \( y \) must belong to \( K \). It follows that \( x \) lies in the \( v \)-closure of \( K \) in \( L \), i.e., in \( K_v \).

Remark. Kürschák proved this lemma for the case where \( K \) is an algebraically closed field and \( v \) is a valuation of rank 1 (c.f. Ribenboim [7, p. 207]).

2. Varieties Over Valued Fields

Let \( V \) be an abstract variety defined over a valued field \( K \). The \( v \)-topology of \( K \) induces in a natural way a \( v \)-topology on the set \( V(K) \) of all \( K \)-rational points of \( V \) (cf. Weil [9, p. 352]). In particular if \( V \) is an affine variety and it is contained in the affine space \( S^n \) then the \( v \)-topology on \( V(K) \) is that which is induced by the \( v \)-topology of \( K^n \). If \( V_0 \) is a Zariski \( K \)-open subset of \( V \) then \( V_0(K) \) is a \( v \)-open subset of \( V(K) \). It follows that if \( L \) is an extension of \( K \) and \( V(K) \) is \( v \)-dense in \( V(L) \) then \( V_0(K) \) is \( v \)-dense in \( V_0(L) \). Again we used the notation \( V_0(K) \) to denote the set of all \( K \)-rational points of \( V_0 \).

Lemma 2.1. Let \( K \) be an infinite field, let \( Z_1, \ldots, Z_m \) be \( m \) sets in the affine space \( S^n \) and let \((a) \in K^n \). Assume that for every \( 1 \leq j \leq m \) there exists a point \((b_j) \in Z_j(\bar{K}) \), \((b_j) \neq (a) \). Then there exists a hyperplane \( L \) which is defined over \( K \), passes through \((a) \) and does not contain any of the \( Z_j \)'s.

Proof. The polynomial \( f(U_1, \ldots, Z_n) = \prod_{j=1}^m \sum_{i=1}^n U_i(b_{ji} - a_i) \) is, by our assumptions, not identically zero. Hence we can find \( u_1, \ldots, u_n \in K \) such that \( f(u_1, \ldots, u_n) \neq 0 \). The hyperplane \( L \) which is defined by the equation

\[
\sum_{i=1}^n u_i(X_i - a_i) = 0
\]

fullfills the requirements.
Lemma 2.2. Let $K$ be an algebraically closed valued field and let $v$ be an abstract variety defined over $K$. If $U$ is a nonempty Zariski $K$-open subset of $V$ then $U(K)$ is $v$-dense in $V(K)$.

Remark. The lemma is well known in the archimedean case (cf. Mumford [6, p. 111]). The following proof holds, however, for every valued field.

Proof. We can assume, without loss of generality that $V$ is an affine irreducible variety. The open set $U$ can be represented in the form $U = V - Z$, where $Z$ is a Zariski $K$-closed subset of $V$ and $\dim Z < \dim V$. We have to prove that if $P \in V(K)$ and $N$ is a $v$-open neighbourhood of $P$ in $V(K)$, then there exists a point $Q \in U(K) \cap N$. We prove this statement in several steps.

(a) $V$ is defined over $K$ by an equation $f(T, X) = 0$, $P = (t, s)$ and $f(T, X) = f_n(T)X^n + \cdots + f_0(T)$ is irreducible. In particular there exists an $0 \leq l \leq n$ such that $f_l(t) \neq 0$, since otherwise $T - t$ would divide $f(T, X)$. In this case $Z$ is reduced to a finite number of points $(t_\mu, x_\mu)$ $\mu = 1, \ldots, m$. We choose a $t' \in K$ $v$-close to $t$ such that $t' \neq t_\mu \mu = 1, \ldots, m$. Then by Lemma 1.1 we can find an $x' \in K$ such that $f(t', x') = 0$ and $(t', x') \in N$.

(b) $V$ is a smooth affine curve. In particular $P$ is a simple point of $V$. Hence there exists a plane curve $W$ and a birational map $\varphi: V \to W$ which are defined over $K$ such that $\varphi$ is biregular in $P$ (cf., Mumford [6, p. 373]). We are therefore reduced to the case (a) which was settled above.

(c) $V$ is an arbitrary affine irreducible curve. Then the normalization $V'$ of $V$ is a smooth affine curve (cf., Weil [9, p. 343]) and there exists a morphism $\varphi$ from $V'$ onto $V$. Since the statement has already been proved for $V'$ it holds also for $V$.

(d) We proceed now by induction on the dimension $r$ of $V$. If $r = 0$ there is nothing to prove. The case $r = 1$ was proved in (c). Assume therefore that $r > 1$ and that the Lemma has already been proved for $r - 1$.

Let $Z_1, \ldots, Z_m$ be the irreducible components of $Z$. By Lemma 2.1 we can find a hyperplane $L$ which passes through $P$ such that $V \subseteq L$ and such that $Z_j \subseteq L$ for every $1 \leq j \leq m$ for which $P \neq Z_j$. Let $V \cap L = V_1 \cup \cdots \cup V_k$ be the decomposition of $V \cap L$ into irreducible components. Assume, for example, that $P \in V_1$. By the Dimension Theorem (cf., Lang [4, p. 36]) $\dim V_1 = r - 1$ and $\dim Z_j \cap L < r - 1$ for every $1 \leq j \leq m$. Hence $\dim Z \cap L < r - 1$. Put $U_1 = V_1 - (Z \cap L \cap V_1)$. Then $U_1$ is a nonempty Zariski $K$-open subset of $V_1$. By the induction hypothesis there exists a point $Q \in U_1(K) \cap N$. This $Q$ lies in $U(K) \cap N$. 


DEFINITION. By a hyper surface we shall mean an absolutely irreducible affine variety \( V \) which is contained in \( S^{r+1} \) and has the dimension \( r \).

For every variety \( V \) we denote by \( V_{\text{sim}} \) the Zariski open subset of \( V \) of all simple points.

**Lemma 2.3.** Let \( K \subseteq L \) be a valued field and let \( M \) be an algebraically closed extension of \( L \) which is contained in \( \bar{K}_v \). If \( W_{\text{sim}}(L) \) is \( v \)-dense in \( W_{\text{sim}}(M) \) for every hyper surface \( W \) defined over \( K \) then \( V(L) \) is \( v \)-dense in \( V(M) \) for every abstract variety \( V \) defined over \( K \).

**Proof.** Let \( V \) be an absolute variety defined over \( K \). Then there exists a hyper surface \( W \) and a birational map \( \varphi : V \to W \) defined over \( K \). (cf. [3, p. 75]). Let \( V_0 \) be a Zariski \( K \)-open subset of \( V_{\text{sim}} \) on which \( \varphi \) is biregular and let \( W_0 \) be the set theoretic image of \( V_0 \) by \( \varphi \). Then \( W_0 \subseteq W_{\text{sim}} \) and \( \varphi \) induces \( v \)-homeomorphisms of \( V_0(L) \), \( V_0(M) \) onto \( W_0(L) \), \( W_0(M) \), respectively. By assumption \( W_{\text{sim}}(L) \) is \( v \)-dense in \( W_{\text{sim}}(M) \), hence \( W_0(L) \) is \( v \)-dense in \( W_0(M) \) and hence \( V_0(L) \) is \( v \)-dense in \( V_0(M) \). By Lemma 2.2 \( V_0(M) \) is \( v \)-dense in \( V(M) \). Hence \( V_0(L) \) is \( v \)-dense in \( V(M) \).

**Lemma 2.4.** Let \( K \) be a separably closed valued field. Then \( V(K) \) is \( v \)-dense in \( V(K_\nu) \) and hence in \( V(\bar{K}) \) for every abstract variety \( V \) defined over \( K \).

**Proof.** By Lemmas 1.4 and 2.3 it suffices to prove that \( W_{\text{sim}}(K) \) is \( v \)-dense in \( W_{\text{sim}}(K_\nu) \) for every hyper surface \( W \) defined over \( K \). Indeed let \( f \in K[T_1, \ldots, T_r, X] \) be an irreducible polynomial and let \( W \) be the hyper surface defined by the equation \( f(T, X) = 0 \). Let \((t, x) \in W_{\text{sim}}(K_\nu)\), then, without loss of generality we can assume that \( (\partial f/\partial X)(t, x) \neq 0 \). This implies that we can use Lemma 1.1 to approximate \((t, x)\) with points \((t', x') \in W_{\text{sim}}(K)\) as in the proof of Lemma 1.4.

3. THE DENSITY PROPERTY

**Definition.** A valued field \( L \) is said to have the density property if \( V(L) \) is \( v \)-dense in \( V(\bar{L}_\nu) \) for every abstract variety \( V \) defined over \( L \).

By Lemma 2.4 every separably closed valued field has the density property. Lemma 2.3 reduces the problem of determining whether a given valued field has the density property to simple points on hyper surfaces. The next Lemma will serve as a further reduction step.

**Lemma 3.1.** Let \( K \) be a valued field and let \( L \) be a separable algebraic extension of \( K \). Then a sufficient (and obviously also necessary) condition for \( L \)
to have the density property is that \( V_{\text{slm}}(L) \) is \( v \)-dense in \( V_{\text{slm}}(L_v) \) for every hyper surface \( v \) defined over \( K \).

**Proof.** Assume that the condition is satisfied. Then by Lemma 2.4, \( V_{\text{slm}}(L) \) is \( v \)-dense in \( V_{\text{slm}}(\bar{K}_v) \) for every hyper surface \( V \) defined over \( K \). Hence, by Lemma 2.3, \( V(L) \) is \( v \)-dense in \( V(\bar{K}_v) \) for every abstract variety \( V \) defined over \( K \).

Now let \( V \) be an abstract variety defined over \( L \). Then by descent theory, there exists an abstract variety \( W \) defined over \( K \) and an epimorphism \( \varphi: W \to V \) which is defined over \( L \) (cf., Weil [8, p. 5]). By what was proved above \( W(L) \) is \( v \)-dense in \( W(\bar{K}_v) \). Hence \( V(L) \) is \( v \)-dense in \( V(\bar{K}_v) \).

**Corollary 3.2.** Every separable algebraic extension of a valued field with the density property has the density property too.

### 4. Hilbertian Valued Fields

Let \( K \) be a field. A **hilbertian** subset \( H \) of \( K^r \) is a set of the form

\[
H = \{(t) \in K^r \mid f_i(t, X) \text{ is defined and irreducible in } K[X], \lambda = 1,\ldots,l,\}
\]

where \( f_1,\ldots,f_l \) are irreducible polynomials in \( K(T_1,\ldots,T_r)[X_1,\ldots,X_n] \).

The field \( K \) is said to be **hilbertian** if all its hilbertian subsets are nonempty. It is known that every number field and every function field is hilbertian (cf., Lang [5, p. 55]). Furthermore, if \( L \) is a finite separable extension of a hilbertian field \( K \), then every hilbertian set of \( L \) contains a hilbertian set of \( K \) (cf., Lang [5, p. 52]).

It follows from the definition that for a hilbertian field \( K \), every hilbertian subset \( H \) of \( K^r \) is dense in \( K^r \) in the Zariski \( K \)-topology. If \( K \) is also valued we can strengthen this statement as follows.

**Lemma 4.1.** Let \( K \) be a hilbertian valued field. Then every hilbertian subset \( H \) of \( K^r \) is \( v \)-dense in \( K^r \).

**Proof.** Let \( H \) be a hilbertian subset of \( K^r \) as above. Let \( (a) \in K^r \) and let \( \gamma \in \Gamma \). Then there exists a \( c \in K^* \) such that \( |c| < \gamma \). Consider the finite set of all polynomials of the form

\[
f_\lambda(a_1 + cT_1^{e_1},\ldots,a_r + cT_r^{e_r}, X),
\]

where \( 1 \leq \lambda \leq l \) and \( e_i = \pm 1 \) for \( i = 1,\ldots,r \). All these polynomials are defined and irreducible in \( K(T)[X] \). Since \( K \) is hilbertian there exist \( s_1,\ldots,s_r \in K \) such that all the polynomials

\[
f_\lambda(a_1 + cs_1^{e_1},\ldots,a_r + cs_r^{e_r}, X)
\]
are defined and irreducible in \( K[X] \). For every \( 1 \leq i \leq r \) we specify \( \epsilon_i \) to be 1 or \(-1\) according to whether \( |s_i| \leq 1 \) or \( |s_i| > 1 \). Then we put \( t_i = a_i + \epsilon_i c_i s_i \), and it is clear that \( |t_i - a_i| < \gamma \), \( i = 1, \ldots, r \) and \((t) \in H\). It follows that \( H \) is \( v \)-dense in \( K^r \).

5. The Haar Measure of \( \mathfrak{G}(K_s/K) \)

It is well known that the absolute Galois group \( \mathfrak{G}(K_s/K) \) of a field \( K \) is compact with respect to its Krull topology. There is therefore a unique way to define a Haar measure \( \mu \) on the Borel field of subsets of \( \mathfrak{G}(K_s/K) \) such that \( \mu(\mathfrak{G}(K_s/K)) = 1 \). If \( L \) is a finite separable extension of \( K \) then \( \mu(\mathfrak{G}(K_s/L)) = 1/[L:K] \). We complete \( \mu \) by adjoining to the Borel field all the subsets having measure 0 and denote the completion also by \( \mu \). More generally, for a positive integer \( e \), we consider the product space \( \mathfrak{G}(K_s/K)^e \) and again denote by \( \mu \) the appropriate completion of the power measure. One can show that it coincides with the completion of the normalized measure of \( \mathfrak{G}(K_s/K)^e \).

A sequence \( \{K_i/K\}_{i=1}^\infty \) of field extensions is said to be linearly disjoint if each \( K_{i+1} \) is linearly disjoint from \( K_1 \cdots K_i \) for every \( i \geq 1 \).

The following lemma is a special case of Lemma 1.10 of [3].

**Lemma 5.1.** Let \( L \) be a finite separable extension of a field \( K \). If \( \{L_i/L\}_{i=1}^\infty \) is a linearly disjoint sequence of finite separable extensions of the same degree then

\[
\mu \left( \bigcup_{i=1}^\infty \mathfrak{G}(K_s/L_i)^e \right) = \frac{1}{[L:K]^e}.
\]

For an \( e \)-tuple \( (\sigma) = (\sigma_1, \ldots, \sigma_e) \) of elements of \( \mathfrak{G}(K_s/K) \) we denote by \( K_s(\sigma) \) its fixed field in \( K_s \).

**Lemma 5.2.** Let \( K \) be a denumerable hilbertian valued field. Then \( K_s(\sigma) \) is \( v \)-dense in \( \hat{K} \) for almost every \((\sigma) \in \mathfrak{G}(K_s/K)^e \).

**Proof.** For \( x \in \hat{K} \) and \( \epsilon \in v(K^*) \) we denote by \( S(x, \epsilon) \) the set of all \((\sigma) \in \mathfrak{G}(K_s/K)^e \) for which there exists an \( y \in K_s(\sigma) \) such that \( |y - x| < \epsilon \). We show that \( \mu(S(x, \epsilon)) = 1 \). This will suffice to prove the lemma, since the set of all \((\sigma) \in \mathfrak{G}(K_s/K)^e \) for which \( K_s(\sigma) \) is \( v \)-dense is the intersection of all the possible \( S(x, \epsilon) \)'s and it is clear that a countable intersection of sets of measure 1 has again the measure 1.

Let \( f(X) = X^n + a_1X^{n-1} + \cdots + a_n \) be a polynomial with coefficients in \( K \) such that \( f(x) = 0 \). We construct by induction a linearly disjoint
sequence, \( \{K_i/K_i \}_{i=1}^{\infty} \), of separable extensions of degree \( n \), such that in every \( K_i \) there exists a \( y \) which satisfies \( |y - x| < \varepsilon \).

Assume that we have already constructed \( K_1, \ldots, K_i \) with the desired properties. Put \( K' = K_1 \cdots K_i \). Then \( K' \) is a finite separable extension of \( K \). Now, the general polynomial of degree \( n \)

\[
f(T, X) = X^n + T_1X^{n-1} + \cdots + T_n
\]

is certainly irreducible over \( K' \). Hence by Lemma 4.1 we can find \( b_1, \ldots, b_n \in K \) arbitrarily \( \nu \)-close to \( a_1, \ldots, a_n \) so that \( f(b, X) \) will be separable and irreducible over \( K' \). If we choose \( b_1, \ldots, b_n \nu \)-close enough to \( a_1, \ldots, a_n \), then, by Lemma 1.1 there exists \( y \in K_s \) such that \( f(b, y) = 0 \) and \( |y - x| < \varepsilon \). Put \( K_{i+1} = K(y) \). Then \( K_{i+1} \) is a separable extension of \( K \) of degree \( n \) and it is linearly disjoint from \( K' \) over \( K \).

It is clear that

\[
\bigcup_{i=1}^{\infty} \mathcal{G}(K_s/K_i)^{\varepsilon} \subseteq S(x, \varepsilon).
\]

By Lemma 5.1 the union has the measure 1, hence \( \mu(S(x, \varepsilon)) = 1 \).

6. The Main Theorem

**Lemma 6.1.** Let \( K \) be a hibertian valued field and let \( f \in K[T_1, \ldots, T_r, X] \) be an absolutely irreducible polynomial. Let \( t_1, \ldots, t_r, x \in K_s \) such that \( f(t, x) = 0 \) and \( (\partial f/\partial X)(t, x) \neq 0 \). Let \( \varepsilon \in \Gamma \) and suppose that \( \delta < \varepsilon \) is an element of \( \Gamma \) such that for every \( t_1', \ldots, t_r' \in K_s \) which satisfy \( |t_i' - t_i| < \delta, \quad i = 1, \ldots, r \), there exists an element \( x' \in K_s \) such that \( f(t', x') = 0 \), \( (\partial f/\partial X)(t', x') \neq 0 \) and \( |x' - x| < \varepsilon \). Let \( L \) be a finite separable extension of \( K \) and suppose that there exist \( t_1', \ldots, t_r' \in L \) which satisfy \( |t_i' - t_i| < \delta/2 \) in the archimedean case and \( |t_i' - t_i| < \delta \) in the non-archimedean case \( i = 1, \ldots, r \). Then for almost all \( (\sigma) \in \mathcal{G}(K_s/L)^{\varepsilon} \) there exist \( a_1, \ldots, a_r, b \in K_s(\sigma) \) such that

\[
\begin{align*}
f(a, b) &= 0, \quad (\partial f/\partial X)(a, b) \neq 0, \quad (1) \\
|a_i - t_i| &< \varepsilon, \quad i = 1, \ldots, r, \quad |b - x| < \varepsilon. \quad (2)
\end{align*}
\]

**Proof.** Let \( d \) be the degree of \( f \) in \( X \). We construct by induction a linearly disjoint sequence \( \{L_j/L_j \}_{j=1}^{\infty} \) of separable extensions of degree \( d \) such that for every \( j \) there exist \( a_1, \ldots, a_r, b \in L_j \) satisfying (1) and (2). Suppose that we have already constructed \( L_1, \ldots, L_{j-1} \) with the desired properties. Put \( L' = L_1 \cdots L_{j-1} \). Then \( L' \) is a finite separable extension of \( L \). By Lemma 4.1 there exist \( a_1, \ldots, a_r \in L \) such that \( |a_i - t_i'| < \delta/2 \) in the archimedean case.
and \(|a_i - t'_i| < \delta\) in the non-archimedean case, \(i = 1, \ldots, r\), and such that

the polynomial \(f(a, X)\) is separable of degree \(d\) and irreducible over \(L'\).

In every case \(|a_i - t_i| < \delta, i = 1, \ldots, r\). Hence by our assumption there exists a \(b \in K_s\) such that (1) and (2) are satisfied. Put \(L_j = L(b)\). Then \(L_j\) is a separable extension of \(L\) of degree \(d\) and it is linearly disjoint from \(L'\) over \(L\).

Now, by Lemma 5.1 \(\bigcup_{j=1}^{\infty} (K_s/L_j) = \mathfrak{G}(K_s/L)^e\) is almost equal to \(\mathfrak{G}(K_s/L)^e\) and every \((\sigma)\) in this union has the desired property.

**Theorem 6.2.** Let \(K\) be hilbertian denumerable valued field \(k\). Then \(K_s(\sigma)\) has the density property for almost all \((\sigma) \in \mathfrak{G}(K_s/k)^e\).

**Proof.** Denote by \(S\) the set of all \((\sigma) \in \mathfrak{G}(K_s/K)^e\) for which \(V_{s, m}(K_s(\sigma))\) is \(v\)-dense in \(V_{s, m}(K_s)\) for every hyper surface \(V\) which is defined over \(K\).

By Lemma 3.1 it suffices to prove that \(\mu(S) = 1\).

Indeed let \(V\) a hyper surface which is defined over \(K\), let \(P \in V_{s, m}(K_s)\) and let \(\epsilon \in \Gamma\). Denote by \(f(T_1, \ldots, T_r, X)\) the absolutely irreducible polynomial in \(K[T_1, \ldots, T_r, X]\), which defines \(V\) and let \(P = (t, x)\). We can assume, without loss of generality, that \((\partial f/\partial X)(t, x) \neq 0\). By Lemma 1.1 there exists a \(\delta \in \Gamma, \delta < \epsilon, \) such that for every \(t'_1, \ldots, t'_r \in \bar{K}\) which satisfy

\[
|t'_i - t_i| < \delta, \quad i = 1, \ldots, r,
\]

there exists an \(x' \in \bar{K}\) such that \(|x' - x| < \epsilon, f(t', x') = 0\) and \(\partial f/\partial X(t', X) \neq 0\). The last condition obviously implies that if \(t'_1, \ldots, t'_r \in K_s\) then \(x' \in K_s\). Let now \(L\) be a finite separable extension of \(K\) and suppose that there exist \(t'_1, \ldots, t'_r \in L\) for which \(|t'_i - t_i| < \delta/2\) in the archimedean case and \(|t'_i - t_i| < \delta\) in the nonarchimedean case, \(i = 1, \ldots, r\). Let \(S(V, P, \epsilon, L)\) be the set of all \((\sigma) \in \mathfrak{G}(K_s/L)^e\) for which there exist \(a_1, \ldots, a_r \in K_s(\sigma)\) such that

\[
f(a, b) = 0, \quad (\partial f/\partial X)(a, b) \neq 0
\]

\[
|a_i - t_i| < \epsilon, \quad i = 1, \ldots, r; \quad |b - x| < \epsilon.
\]

By Lemma 5.1

\[
\mu(\mathfrak{G}(K_s/L)^e - S(V, P, \epsilon, L)) = 0.
\]

Put \(T\) for the set of all \((\sigma) \in \mathfrak{G}(K_s/K)^e\) for which \(K_s(\sigma)\) is \(v\)-dense in \(K_s\).

By Lemma 4.1

\[
\mu(T) = 1.
\]

Clearly \(S \subseteq T\). We claim that

\[
T - S \subseteq \bigcup [\mathfrak{G}(K_s/L)^e - S(V, P, \epsilon, L)],
\]

where the union runs over all possible \(V, P, \epsilon, L\).
Indeed let \((\sigma) \in T - S\). Then there exists a hyper surface \(V\) which is defined over \(K\), a point \(P \in V_{\text{std}}(K_\sigma)\) and an \(\epsilon \in \nu(K^*)\) such that for every \(P' \in V_{\text{std}}(K_\delta(\sigma))\) the maximal value of the differences of the corresponding coordinates of \(P\) and \(P'\) is not smaller than \(\epsilon\). Let \(f(T_1', \ldots, T_r', X)\) be the absolutely irreducible polynomial which defines \(V\) and let \(\delta \in T\) as above. Then there exist \(t_1', \ldots, t_r' \in K_\delta(\sigma)\) which satisfy the condition (3). Put \(L = K(t_1', \ldots, t_r')\). Then \(L\) is a finite separable extension of \(K\) which is contained in \(K_\delta(\sigma)\). Hence \((\sigma) \in G(K_{\delta}/L) = S(V, P, \epsilon, L)\).

Now the number of summands in the right-hand side of (8) is \(\aleph_0\), since \(K\) itself is denumerable. Each summand has by (6) the measure 0. It follows that \(\mu(T - S) = 0\). Hence, by (7) \(\mu(S) = 1\).

7. Remarks

In [2, Section 3] we considered a valued field \(K\) and defined it to be \textit{hilbertian with respect to its valuation} if its hilbertian sets are \(\nu\)-dense in the corresponding powers of \(K\). It appears now that every hilbertian valued field is also hilbertian with respect to its valuation (cf., Lemma 4.1). Theorem 6.1 of [2] can therefore be reformulated as follows:

**Theorem 7.1.** Let \(K\) be a denumerable hilbertian valued field. If \(K_\nu\) is separable over \(K\) then for almost all \((\sigma) \in G(K_{\delta}/K)\) and for every absolute variety \(V\) defined over \(K\), \(V_{\text{std}}(K_\delta(\sigma) \cap K_\nu)\) is \(\nu\)-dense in \(V_{\text{std}}(K_\nu)\). In particular \(G(K_\delta(\sigma) \cap K_\nu)\) is \(\nu\)-dense in \(G(K_\nu)\) for every group variety \(G\) defined over \(K\).

A field \(K\) is said to be \textit{pseudo algebraically closed (P.A.C.)} if every nonvoid absolute variety defined over \(K\) has a \(K\)-rational point. Now, a valued field \(K\) having the density property is certainly P.A.C. Indeed, if \(V\) is a nonvoid absolute variety defined over \(K\) then by Hilbert's Nullstellensatz \(V(\bar{K})\) is not empty. Since \(V(K)\) is \(\nu\)-dense in \(V(\bar{K})\) it is also not empty. In the opposite direction \(G\). Frey proved in [1, Theorem 2] that if \(K\) is a P.A.C. valued field and \(\nu(K) \subseteq \mathbb{R}\), then \(K_\nu\) is algebraically closed and hence \(K\) is \(\nu\)-dense in \(\mathbb{R}_\nu\). This statement can be generalized to finite rank valuations. The following question is therefore very natural:

**Problem 1.** Does every valued P.A.C. field have also the density property?

Till now we considered a valued field \(K\) and a fixed extension of \(\nu\) to \(\bar{K}\) which we have also denoted by \(\nu\). We let now the extension of \(\nu\) to vary and we say that an algebraic extension \(L\) of \(K\) has the density property with respect
to an extension $w$ of $v$ to $\bar{K}$ if $V(L)$ is $w$-dense in $V(\bar{K}_w)$ for every absolute variety $V$ defined over $L$. We propose the following problem:

**Problem 2.** Let $K$ be a denumerable hilbertian $v$-valued field. Is it true that for almost all $(\sigma) \in G(K_s/K)^s K_s(\sigma)$ has the density property with respect to every extension $w$ of $v$ to $\bar{K}$?

Obviously a positive answer to Problem 1 will provide a positive answer to Problem 2. In general there are at least $2^{\aleph_0}$ distinct extensions of $v$ to $\bar{K}$. Hence we can not apply the usual argument of intersecting $\aleph_0$ sets of measure 1 in order to deduce a positive answer to Problem 2 from our main theorem.

**References**