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Strong Approximation Theorem for Absolutely Integral Varieties over PSC Galois Extensions of Global Fields

by

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Abstract: Let K be a global field, $\mathcal{V}$ a proper subset of the set of all primes of K, $\mathcal{S}$ a finite subset of $\mathcal{V}$, and $\tilde{K}$ (resp. K_{sep}) a fixed algebraic (resp. separable algebraic) closure of K. Let $\mathrm{Gal}(K) = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ be the absolute Galois group of K. For each $\mathfrak{p} \in \mathcal{V}$ we choose a Henselian (respectively, a real or algebraic) closure $K_\mathfrak{p}$ of K at $\mathfrak{p}$ in $\tilde{K}$ if $\mathfrak{p}$ is non-archimedean (respectively, archimedean). Then, $K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p}\in\mathcal{S}} \bigcap_{\tau\in\mathrm{Gal}(K)} K_\mathfrak{p}^\tau$ is the maximal Galois extension of K in K_{sep} in which each $\mathfrak{p} \in \mathcal{S}$ totally splits. For each $\mathfrak{p} \in \mathcal{V}$ we choose a $\mathfrak{p}$-adic absolute value $|\ |_\mathfrak{p}$ of $K_\mathfrak{p}$ and extend it in the unique possible way to $\tilde{K}$.

For $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_e) \in \mathrm{Gal}(K)^e$ let $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ be the maximal Galois extension of K in $K_{\mathrm{tot},\mathcal{S}}$ fixed by $\sigma_1,\ldots,\sigma_e$. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ (with respect to the Haar measure), the field $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ satisfies the following local-global principle:

Let V be an absolutely integral affine variety in $\mathbb{A}_K^n$. Suppose that for each $\mathfrak{p} \in \mathcal{S}$ there exists $\mathbf{z}_\mathfrak{p} \in V_{\mathrm{simp}}(K_\mathfrak{p})$ and for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{S}$ there exists $\mathbf{z}_\mathfrak{p} \in V(\tilde{K})$ such that in both cases $|\mathbf{z}_\mathfrak{p}|_\mathfrak{p} \le 1$ if $\mathfrak{p}$ is non-archimedean and $|\mathbf{z}_\mathfrak{p}|_\mathfrak{p} < 1$ if $\mathfrak{p}$ is archimedean. Then, there exists $\mathbf{z} \in V(K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}])$ such that for all $\mathfrak{p} \in \mathcal{V}$ and for all $\tau \in \mathrm{Gal}(K)$ we have: $|\mathbf{z}^\tau|_\mathfrak{p} \le 1$ if $\mathfrak{p}$ is archimedean and $|\mathbf{z}^\tau|_\mathfrak{p} < 1$ if $\mathfrak{p}$ is non-archimedean.

Introduction

The strong approximation theorem for a global field K gives an $x \in K$ that lies in given $\mathfrak{p}$-adically open discs for finitely many given primes $\mathfrak{p}$ of K such that the absolute $\mathfrak{p}$-adic value of x is at most 1 for all other primes $\mathfrak{p}$ except possibly one [CaF67, p. 67]. A possible generalization of that theorem to an arbitrary absolutely integral affine variety V over K fails, because in general, $V(K)$ is a small set. For example, if V is a curve of genus at least 2, then $V(K)$ is finite (by Faltings). This obstruction disappears as soon as we switch to "large Galois extensions" of K. We prove in this work a strong approximation theorem for absolutely integral affine varieties over each "large Galois extension" of K.

To be more precise, let $\tilde{K}$ be an algebraic closure of K, K_{sep} the separable closure of K in $\tilde{K}$, $\mathrm{Gal}(K) = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ the absolute Galois group of K, and e a non-negative integer. We equip $\mathrm{Gal}(K)^e$ with the normalized Haar measure [FrJ08, Section 18.5] and use the expression "for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$" to mean "for all $\boldsymbol{\sigma}$ in $\mathrm{Gal}(K)^e$ outside a set of measure zero". For each $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_e) \in \mathrm{Gal}(K)^e$ let $K_{\mathrm{sep}}(\boldsymbol{\sigma}) = \{x \in K_{\mathrm{sep}} \mid x^{\sigma_i} = x, \text{ for } i = 1,\ldots,e\}$ and let $K_{\mathrm{sep}}[\boldsymbol{\sigma}]$ be the maximal Galois extension of K in $K_{\mathrm{sep}}(\boldsymbol{\sigma})$.

Let $\mathbf{P}_K$ be the set of all primes of K, $\mathbf{P}_{K,\mathrm{fin}}$ the set of all finite (i.e. non-archimedean) primes and $\mathbf{P}_{K,\mathrm{inf}}$ the set of all infinite (i.e. archimedean) primes. We fix a proper subset $\mathcal{V}$ of $\mathbf{P}_K$, a finite subset $\mathcal{T}$ of $\mathcal{V}$, and a subset $\mathcal{S}$ of $\mathcal{T}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. For each $\mathfrak{p} \in \mathcal{V}$ we fix a completion $\hat{K}_\mathfrak{p}$ of K at $\mathfrak{p}$ and embed $\tilde{K}$ in an algebraic closure $\widetilde{\hat{K}_\mathfrak{p}}$ of $\hat{K}_\mathfrak{p}$. Then, we extend the normalized absolute value $|\ |_\mathfrak{p}$ of $\hat{K}_\mathfrak{p}$ to $\widetilde{\hat{K}_\mathfrak{p}}$ in the unique possible way. In particular, this defines $|x|_\mathfrak{p}$ for each $x \in \tilde{K}$. As usual, if $\mathbf{x} = (x_1,\ldots,x_n) \in \tilde{K}^n$, we write $|\mathbf{x}|_\mathfrak{p} = \max(|x_1|_\mathfrak{p},\ldots,|x_n|_\mathfrak{p})$.

We set $K_\mathfrak{p} = \tilde{K} \cap \hat{K}_\mathfrak{p}$ and note that $K_\mathfrak{p}$ is a Henselian closure of K at $\mathfrak{p}$ if $\mathfrak{p} \in \mathbf{P}_{K,\mathrm{fin}}$ and a real or the algebraic closure of K at $\mathfrak{p}$ if $\mathfrak{p} \in \mathbf{P}_{K,\mathrm{inf}}$. Thus,

$$K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p}\in\mathcal{S}} \bigcap_{\tau\in\mathrm{Gal}(K)} K_\mathfrak{p}^\tau$$

is the maximal Galois extension of K in which each $\mathfrak{p}\in\mathcal{S}$ totally splits. For each $\boldsymbol{\sigma}\in\mathrm{Gal}(K)^e$ we set $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})=K_{\mathrm{sep}}(\boldsymbol{\sigma})\cap K_{\mathrm{tot},\mathcal{S}}$ and $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]=K_{\mathrm{sep}}[\boldsymbol{\sigma}]\cap K_{\mathrm{tot},\mathcal{S}}$.

For each extension M of K in $\tilde{K}$ and every $\mathfrak{p}\in\mathbf{P}_{\mathrm{fin}}\cap\mathcal{V}$ we consider the valuation ring $\mathcal{O}_{M,\mathfrak{p}}=\{x\in M \mid |x|_{\mathfrak{p}}\le 1\}$ of M at $\mathfrak{p}$. For each $\mathcal{U}\subseteq\mathcal{V}$ we define $\mathcal{O}_{M,\mathcal{U}}$ to be the set of all $x\in M$ such that $|x^\tau|_{\mathfrak{p}}\le 1$ for all $\mathfrak{p}\in\mathcal{U}$ and $\tau\in\mathrm{Gal}(K)$. Note that if $\mathcal{U}\subseteq\mathbf{P}_{K,\mathrm{fin}}$, then $\mathcal{O}_{M,\mathcal{U}}$ is an intersection of valuation rings, hence it is an integrally closed domain. Note however that $\mathcal{O}_{M,\{\mathfrak{p}\}}$ is different from $\mathcal{O}_{M,\mathfrak{p}}$.

In this notation the following result is a reformulation of [JaR08, Thm. 2.2]. Throughout this paper, for each positive integer n, by an **affine variety in** $\mathbb{A}^n_K$ we mean a closed subscheme of $\mathbb{A}^n_K$ (Subsection 4.2).

Proposition A: *For almost all $\boldsymbol{\sigma}\in\mathrm{Gal}(K)^e$ the field $M=K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ satisfies the following strong approximation theorem: Let V be an affine absolutely integral variety in $\mathbb{A}^n_K$ for some positive integer n. For each $\mathfrak{p}\in\mathcal{S}$ let $\mathbf{z}_{\mathfrak{p}}\in V_{\mathrm{simp}}(K_{\mathfrak{p}})$, for each $\mathfrak{p}\in\mathcal{T}\smallsetminus\mathcal{S}$ let $\mathbf{z}_{\mathfrak{p}}\in V(\tilde{K})$, and for each $\mathfrak{p}\in\mathcal{V}\smallsetminus\mathcal{T}$ let $\mathbf{z}_{\mathfrak{p}}\in V(\mathcal{O}_{\tilde{K},\mathfrak{p}})$. Then, for every $\varepsilon>0$ there exists $\mathbf{z}\in V(M)$ such that $|\mathbf{z}-\mathbf{z}_{\mathfrak{p}}^\tau|_{\mathfrak{p}}<\varepsilon$ for all $\mathfrak{p}\in\mathcal{T}$ and $\tau\in\mathrm{Gal}(K)$ and $|\mathbf{z}^\tau|_{\mathfrak{p}}\le 1$ for all $\mathfrak{p}\in\mathcal{V}\smallsetminus\mathcal{T}$ and $\tau\in\mathrm{Gal}(K)$.*

When $e=0$, we have $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})=K_{\mathrm{tot},\mathcal{S}}$ and we retrieve [MoB89, Thm. 1.3]. For arbitrary $e\ge 0$, Proposition A implies the following analog of Rumely's local-global principle for almost all fields $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$:

Proposition B: *For almost all $\boldsymbol{\sigma}\in\mathrm{Gal}(K)^e$ the field $M=K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ satisfies the following local-global principle: Let V be an affine absolutely integral variety in $\mathbb{A}^n_K$ for some positive integer n. Suppose for each $\mathfrak{p}\in\mathcal{S}$ there exists $\mathbf{z}_{\mathfrak{p}}\in V_{\mathrm{simp}}(K_{\mathfrak{p}})$ and for each $\mathfrak{p}\in\mathcal{V}\smallsetminus\mathcal{S}$ there exists $\mathbf{z}_{\mathfrak{p}}\in V(\tilde{K})$ such that in each case the following holds: $|\mathbf{z}_{\mathfrak{p}}|_{\mathfrak{p}}\le 1$ if $\mathfrak{p}\in\mathbf{P}_{K,\mathrm{fin}}$ and $|\mathbf{z}_{\mathfrak{p}}|_{\mathfrak{p}}<1$ if $\mathfrak{p}\in\mathbf{P}_{K,\mathrm{inf}}$.*

Then, there exists $\mathbf{z}\in V(M)$ such that for all $\tau\in\mathrm{Gal}(K)$ we have: $|\mathbf{z}^\tau|_{\mathfrak{p}}\le 1$ for each $\mathfrak{p}\in\mathcal{V}\cap\mathbf{P}_{K,\mathrm{fin}}$ and $|\mathbf{z}^\tau|_{\mathfrak{p}}<1$ for each $\mathfrak{p}\in\mathcal{V}\cap\mathbf{P}_{K,\mathrm{inf}}$.

For $K=\mathbb{Q}$, $e=0$, and $\mathcal{V}=\mathbf{P}_{\mathrm{fin}}$, Proposition B specializes to Rumely's local-global principle for the ring $\tilde{\mathbb{Z}}$ of all algebraic integers [Rum86]. That principle yields an affirmative answer to Hilbert's 10th problem for $\tilde{\mathbb{Z}}$ [Rum86, p. 130, Thm. 2], answering a question of Julia Robinson from the 1970'ties. L. v. d. Dries applies the local-global principle to prove that the elementary theory of $\tilde{\mathbb{Z}}$ is decidable [Dri88, p. 190, Cor.].

The proof of Proposition A is carried out along the lines of the proof of the local-global principle for $K_{\mathrm{tot},\mathcal{S}}$ of [GPR95]. In addition it uses that for almost all $\boldsymbol{\sigma}\in\mathrm{Gal}(K)^e$ the field $K_{\mathrm{sep}}(\boldsymbol{\sigma})$ is **PAC over** $\mathcal{O}_{K,\mathcal{V}}$. This means that for every absolutely irreducible polynomial $f\in K[X,Y]$ which is separable in Y there exist infinitely many points $(a,b)\in\mathcal{O}_{K,\mathcal{V}}\times K_{\mathrm{sep}}(\boldsymbol{\sigma})$ such that $f(a,b)=0$. This implies that $K_{\mathrm{sep}}(\boldsymbol{\sigma})$ is also PAC over $\mathcal{O}_{L,\mathcal{V}}$ for every extension L of K in $K_{\mathrm{sep}}(\boldsymbol{\sigma})$.

Unfortunately, as [BaJ08, Thm. B] proves, no Galois extension of K except K_{sep} is PAC over K, let alone over $\mathcal{O}_{K,\mathcal{V}}$. In particular, if $\boldsymbol{\sigma}\ne 1$, then $K_{\mathrm{sep}}[\boldsymbol{\sigma}]$ is not PAC over $\mathcal{O}_{K,\mathcal{V}}$. Thus, the proof of Proposition A breaks down for the fields $K_{\mathrm{sep}}[\boldsymbol{\sigma}]$. However, almost all of the fields $M=K_{\mathrm{sep}}[\boldsymbol{\sigma}]$ have a weaker property than being PAC over $\mathcal{O}_{K,\mathcal{V}}$, namely they are "weakly K-stably PAC over $\mathcal{O}_{K,\mathcal{V}}$" (Definition12.1 for $\mathcal{S}=\emptyset$). This would almost help to adjust the proof of Proposition A given in [JaR08] to a proof of the analogous theorem for almost all of the fields $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$. However, as in [JaR08], we would need to replace K somewhere along the proof by a finite extension L that lies in $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ and then proceed with $L_{\mathrm{tot},\mathcal{S}_L}[\boldsymbol{\sigma}]$, where $\mathcal{S}_L$ is the set of all primes of L lying over $\mathcal{S}$. Although it is still true that $L_{\mathrm{sep}}(\boldsymbol{\sigma})=K_{\mathrm{sep}}(\boldsymbol{\sigma})$ and $K_{\mathrm{sep}}(\boldsymbol{\sigma})$ is weakly L-stably PAC over $\mathcal{O}_{L,\mathcal{V}}$ (for almost all $\boldsymbol{\sigma}\in\mathrm{Gal}(L)^e$), the field $L_{\mathrm{sep}}[\boldsymbol{\sigma}]$ may properly contain $K_{\mathrm{sep}}[\boldsymbol{\sigma}]$ even if we choose L to be Galois over K, so nothing that we prove on $L_{\mathrm{tot},\mathcal{S}_L}[\boldsymbol{\sigma}]$ would apply to $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$.

Fortunately, the proof of [MoB89, Thm. 1.3] does not enlarge K as [JaR08] does. We combine the method of that proof with the method of the proof of the main result of [GeJ02]. In our case the latter result says that $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ is P$\mathcal{S}$C for almost all $\boldsymbol{\sigma}\in\mathrm{Gal}(K)^e$. This means that if V is an absolutely integral affine variety in $\mathbb{A}^n_{K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]}$ for some positive integer n and $V_{\mathrm{simp}}(K_{\mathfrak{p}}^\tau)\ne\emptyset$ for every $\mathfrak{p}\in\mathcal{S}$ and $\tau\in\mathrm{Gal}(K)$, then $V(K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}])\ne\emptyset$. One of the main ingredients of the proof of that theorem is the main result of [GJR17] which produces a "symmetrically stabilizing" element t for a given function field F of one variable over K with zeros and poles in given $\mathcal{S}$-adically open neighborhoods in $V(K_{\mathrm{tot},\mathcal{S}})$.

The construction of t in the present work has to be done with extra care. We prove the following analog of Proposition A (see Theorem 13.7):

THEOREM C (Strong approximation theorem): *Let $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, e, \mathbf{P}_{K,\mathrm{fin}}$ be as above. In particular, K is a global field and $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $M = K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ satisfies the* **strong approximation theorem**, *that is M has the following property:*

Let V be an absolutely integral affine variety in $\mathbb{A}_K^n$ for some positive integer n. For each $\mathfrak{p} \in \mathcal{S}$ let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $V_{\mathrm{simp}}(K_\mathfrak{p})$. For each $\mathfrak{p} \in \mathcal{T} \smallsetminus \mathcal{S}$ let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $V(\tilde{K})$, invariant under the action of $\mathrm{Gal}(K_\mathfrak{p})$. Finally, for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{T}$ we assume that $V(\mathcal{O}_{\tilde{K},\mathfrak{p}}) \neq \emptyset$. Then,

$$V(\mathcal{O}_{M,\mathcal{V} \smallsetminus \mathcal{T}}) \cap \bigcap_{\mathfrak{p}\in\mathcal{T}} \bigcap_{\tau\in\mathrm{Gal}(K)} \Omega_\mathfrak{p}^\tau \neq \emptyset. \tag{1}$$

The first three sections of this work introduce necessary prerequisites. Section 4 reduces the proof of the strong approximation theorem for an intermediate field M of $K_{\mathrm{tot},\mathcal{S}}/K$ from absolutely integral affine varieties over the given global field K to absolutely integral affine curves over K. In particular it allows us to increase $\mathcal{T}$ within $\mathcal{V}$ and replace V by a non-empty Zariski-open subset, if necessary. Given an absolutely integral affine curve C over K, we use this flexibility in Section 5 to construct a principal ideal domain $R = \mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}}$ with quotient field K and a smooth affine curve X over R such that $X_K = C$. Then, following [MoB89], we embed X as a Zariski-open subset of a projective regular curve $\bar{X} = \mathrm{Proj}(R[t_0,\dots,t_r])$, where $R[t_0,\dots,t_r] = \sum_{k=0}^\infty R[t_0\dots,t_r]_k$ is a graded integral domain over R such that $R[t_0,\dots,t_r]_0 = R$ and $R[t_0,\dots,t_r]_1 = \sum_{i=0}^r Rt_i$ (Lemma 5.6).

The main result of [MoB89] produces for every large positive integer k a section $s_0 \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$ such that each of the irreducible components of the effective divisor $\mathrm{div}(s_0)$ yields distinct points of $X(K_{\mathrm{tot},\mathcal{S}})$ that belong to the left hand side of (1) with C replacing V and $K_{\mathrm{tot},\mathcal{S}}$ replacing M. In particular, s_0 does not vanish on $Z = \bar{X} \smallsetminus X$ (essentially Proposition 7.6 and Lemma 7.8).

In order to find such points in $C(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$, we construct a surjective morphism φ from $\bar{X}_K$ onto a projective curve $Y = \mathrm{Proj}(K[s_0,\dots,s_l])$, where $s_0, s_1, \dots, s_l$ are elements of $R[t_0,\dots,t_r]_k$ for an appropriately chosen large k and s_0 is as in the preceding paragraph. Moreover, $s_1,\dots,s_l$ vanish on Z. Changing the base from R to $\tilde{K}$, the curve $Y_{\tilde{K}}$ has some special properties. It is a non-strange curve with only finitely many inflection points and finitely many double tangents, and it has cusps with a given large multiplicity q such that the multiplicities of all other points of $Y_{\tilde{K}}$ are at most q (Proposition 10.5).

Choosing q as a large prime number, the main result of [GJR17] and Proposition 11.2 give an element

$$t = \frac{s_0 + a_1 s_1 + \ldots + a_l s_l}{s_0 + b_1 s_1 + \cdots + b_l s_l}$$

of the function field F of $\bar{X}_K$ such that $F/K(t)$ is a finite separable extension and the Galois closure $\hat{F}$ of $F/K(t)$ is a regular extension of K (we call t a "stabilizing element" of F/K). Moreover, $a_1,\dots,a_l,b_1,\dots,b_l \in R$, $b_1 = 1 + a_1$, and $(a_1,\dots,a_l,b_2,\dots,b_l)$ can be chosen in a $\mathcal{T}$-open subset of R^{2l-1}.

By a result of [GJR00] (quoted here as Lemma 13.6), for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, every extension M of $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ in $K_{\mathrm{tot},\mathcal{S}}$ is "weakly K-stably P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$" (Definition 12.1). If we take $a_1,\dots,a_l$ in R sufficiently close to 0 in the $\mathcal{T}$-adic topology and $b_2,\dots,b_l \in R$, then that property yields an M-rational place of FM with residue field M such that, with $s' = s_0 + a_1 s_1 + \cdots + a_l s_l$, the zero of $\mathrm{div}(s')$ that corresponds to this place belongs to $C(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}}) \cap \bigcap_{\mathfrak{p}\in\mathcal{T}} \bigcap_{\tau\in\mathrm{Gal}(K)} \Omega_\mathfrak{p}^\tau \neq \emptyset$ (Proposition 12.3). Thus, M satisfies the strong approximation theorem.

Finally, we denote the compositum of all finite Galois extensions of K with symmetric Galois groups by K_{symm}. In a forthcoming work we prove the following result:

THEOREM D: *In the notation of Theorem C, the field $K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}$ satisfies the strong approximation theorem.*

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1. Twisted Sheaves

Recall that a ring A (commutative with 1) is **graded** if $A = \bigoplus_{k=0}^{\infty} A_k$, where each summand A_k is a commutative group under the addition of A and $A_k A_l \subseteq A_{k+l}$ for all $k, l \geq 0$. In particular, A_0 is a subring of A and each A_k is an A_0-module. We then say that A is a **graded ring over** A_0. Each non-zero $s \in A$ has a unique presentation $s = \sum_{k=0}^{\infty} s_k$, where $s_k \in A_k$ for each $k \geq 0$ and $s_k = 0$ for all large k. The elements of $\bigcup_{k=0}^{\infty} A_k$ are said to be **homogeneous** and the elements s_k above are the **homogeneous components** of s.

If a homogeneous element s of A belongs to A_k, we say that the A**-degree** of s is k and write $\deg_A(s) = k$. If s' is an additional homogeneous element of A, then $\deg_A(ss') = \deg_A(s) + \deg_A(s')$.

If $s_0, \ldots, s_l$ are elements of A_k for some $k \geq 0$, then $T = A_0[s_0, \ldots, s_l]$ is a graded ring over A_0 with T_m being the A_0-module generated by all of the monomials in $s_0, \ldots, s_l$ whose A-degree is km. In particular, $T_0 = A_0$ and $T_1 = \sum_{i=0}^{l} A_0 s_i$.

An A-module M is **graded** if $M = \bigoplus_{k=0}^{\infty} M_k$, where each M_k is an additive subgroup of M and $A_k M_l \subseteq M_{k+l}$ for all k, l.

An ideal $\mathfrak{a}$ of A is **homogeneous** if $\mathfrak{a}$ is homogeneous as a graded A-module; alternatively, if $\mathfrak{a} = \bigoplus_{k=0}^{\infty} (\mathfrak{a} \cap A_k)$; alternatively, if each of the homogeneous components of every $a \in \mathfrak{a}$ belongs to $\mathfrak{a}$; alternatively, if $\mathfrak{a}$ is generated by homogeneous elements. An example of a homogeneous ideal is $A_+ = \bigoplus_{k=1}^{\infty} A_k$. The homogeneous prime ideals of A not containing A_+ form a set $\mathrm{Proj}(A)$ that has a natural sheaf structure [Liu06, p. 52, Prop. 2.3.38].

If $(\mathfrak{a}_i)_{i \in I}$ is a family of homogeneous ideals of A, then each of the following ideals is homogeneous: $\sum_{i \in I} \mathfrak{a}_i$, $\prod_{i \in I} \mathfrak{a}_i$ (= the set of all finite sums of finite products $a_{i_1} \cdots a_{i_n}$ with $a_{i_1} \in \mathfrak{a}_{i_1}, \ldots, a_{i_n} \in \mathfrak{a}_{i_n}$ and $i_1, \ldots, i_n$ distinct elements of I), and $\bigcap_{i \in I} \mathfrak{a}_i$.

Setup 1.1: Let $A = \bigoplus_{k=0}^{\infty} A_k$ be a Noetherian graded ring. Then, the ideal A_+ of A is finitely generated, so $A_1 = \sum_{i=0}^{r} A_0 t_i$ is a finitely generated A_0-module. We assume that $A = A_0[t_0, \ldots, t_r]$. Then, we set $V = \mathrm{Proj}(A)$ and consider for each k the twisted sheaf $O_V(k)$ [Har77, pp. 116–117] and the abelian group $\Gamma(V, \mathcal{O}_V(k))$ of its global sections. Each $t \in \Gamma(V, \mathcal{O}_V(k))$ can be viewed as an element of the direct product $\prod_{P \in V} A_P$ which is locally a fraction of degree k. This means that each $P_0 \in V$ has a Zariski-open neighborhood V_0 and there exist homogeneous elements f and g of A such that $\deg_A(f) - \deg_A(g) = k$, $g \notin P$, and $t_P = \frac{f}{g}$ in A_P for each $P \in V_0$. If $a \in A_j$, then at is an element of $\Gamma(V, \mathcal{O}_V(j+k))$, which is defined in the latter notation by $(at)_P = \frac{af}{g}$ for each $P \in V_0$. This definition makes $\bigoplus_{k=0}^{\infty} \Gamma(V, \mathcal{O}_V(k))$ into a graded A-module. It also gives a natural homomorphism $\beta = \beta_V \colon A \to \bigoplus_{k=0}^{\infty} \Gamma(V, \mathcal{O}_V(k))$ of graded A-modules mapping each $s \in A_k$ onto the element of $\prod_{P \in V} A_P$ whose Pth coordinate is $\frac{s}{1}$. Let $\beta_k = \beta_{V,k} \colon A_k \to \Gamma(V, \mathcal{O}_V(k))$ be the kth homogeneous component of β. ∎

For the convenience of the reader we supply a proof to a special case of [Gro61III, p. 446, Thm. 2.3.1]. It says that β_k is an isomorphism for all large k.

LEMMA 1.2: *The following statements hold under Setup 1.1:*
(a) *Let I be an ideal of A such that $A_1 \subseteq \sqrt{I}$. Then, $A_m \subseteq I$ for all large m.*
(b) *Let s be a homogeneous element of A whose annihilator $I = \{a \in A \mid as = 0\}$ is contained in no $P \in \mathrm{Proj}(A)$. Then, $A_m \subseteq I$ for all large m.*

Proof of (a): For each $0 \leq i \leq r$ there exists e_i such that $t_i^{e_i} \in I$. Let $e = \sum_{i=0}^{r} (e_i - 1)$ and let $m > e$. If $\prod_{i=0}^{r} t_i^{m_i} \in A_m$, then $\sum_{i=0}^{r} m_i = m > \sum_{i=0}^{r} (e_i - 1)$, so there exists $0 \leq i \leq r$ with $m_i \geq e_i$, hence $\prod_{i=0}^{r} t_i^{m_i} \in I$. Since A_m is generated as an A-module by the monomials of degree m in $t_0, \ldots, t_r$, we conclude that $A_m \subseteq I$.

Proof of (b): First note that $I = 0{:}As = \{a \in A \mid as = 0\}$ is a homogeneous ideal of A [ZaS75II, p. 152, Thm. 8]. Therefore, by the same theorem, $\sqrt{I}$ is also homogeneous. By [Bou89, p. 283, Prop. 1], $\sqrt{I}$ is an intersection of homogeneous prime ideals P of A. By assumption, none of those P is in $\mathrm{Proj}(A)$, so all of them contain A_+, hence also A_1. It follows that $A_1 \subseteq \sqrt{I}$. By Part (a), $A_m \subseteq I$ for all large m. ∎

Lemma 1.3: *Under Setup 1.1, the natural homomorphism $\beta_k\colon A_k \to \Gamma(V, \mathcal{O}_V(k))$ is an isomorphism for all large k.*

Proof: We break up the proof into two parts.

Part A: *For all large k, the map β_k is injective.* Since $\beta\colon A \to \bigoplus_{k=0}^{\infty} \Gamma(V, \mathcal{O}_V(k))$ is a homomorphism of graded A-modules, $I = \mathrm{Ker}(\beta)$ is a homogeneous ideal of A. Since A is Noetherian, $I = \sum_{i=1}^{n} Ab_i$ with $b_i \in A_{k_i}$ for some distinct non-negative integers k_i, $i = 1, \ldots, n$. By the convention in Setup 1.1, $\left(\frac{b_i}{1}\right)_{P \in V} = \beta_{k_i}(b_i) = 0$, where for each $P \in V$, the quotient $\frac{b_i}{1}$ is taken in the local ring A_P. Thus, there exists $b \in A \smallsetminus P$ with $bb_i = 0$. It follows that $N_i = \{a \in A \mid ab_i = 0\} \not\subseteq P$. Lemma 1.2(b) gives an l_i such that $A_k \subseteq N_i$ for all $k > l_i$. Let $l_0 = \max(k_1 + l_1, \ldots, k_n + l_n)$. For each $l > l_0$ and for each $1 \le i \le n$ we have $l - k_i > l_i$, so $A_{l-k_i} \subseteq N_i$, hence $A_{l-k_i} b_i = 0$. Using the presentation $I = \sum_{i=1}^{n} Ab_i$ and the homogeneity of I, we get $I_l = \sum_{i=1}^{n} A_{l-k_i} b_i$. Therefore, $I_l = 0$ for each $l > l_0$. This means that β_l is injective for all $l > l_0$.

Part B: *For all large k, the map β_k is surjective.* Let $X = \mathbb{P}^r_{A_0} = \mathrm{Proj}(R)$, with $R = A_0[T_0, \ldots, T_r]$, be the projective space of dimension r over $\mathrm{Spec}(A_0)$. Let J be the kernel of the A_0-epimorphism $R \to A$ that maps each T_i onto t_i, $i = 0, \ldots, r$. Let $\mathcal{J}$ be the sheaf of ideals associated with J, that is the sheaf appearing in the following exact sequence of sheafs:

(1)
$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_X \xrightarrow{j^\#} j_*\mathcal{O}_V \longrightarrow 0,$$

where $j\colon V \to X$ is the inclusion map [Har77, p. 115, Definition]. Since $\mathcal{O}_X(k)$ is an invertible sheaf on X [Har77, p. 117, Prop. II.5.12(a)], the tensor product of (1) with $\mathcal{O}_X(k)$ remains exact. In other words, the sequence $0 \to \mathcal{J}(k) \to \mathcal{O}_X(k) \to j_*\mathcal{O}_V(k) \to 0$ is exact. Indeed, one may check the exactness locally at each $P \in X$ [GoW10, p. 172] using that $O_X(k)_P$ is a free $O_{X,P}$-module. This yields an exact sequence of cohomology groups:

(2)
$$0 \to \Gamma(X, \mathcal{J}(k)) \to \Gamma(X, \mathcal{O}_X(k)) \to \Gamma(X, (j_*\mathcal{O}_V)(k)) \to H^1(X, \mathcal{J}(k))$$

[Har77, p. 208, Prop. III.2.6 or Liu06, p. 184, Prop. 5.2.15]. Since $\mathcal{J}(k)$ is a coherent sheaf on X [Har77, p. 116, Prop. II.5.9], a theorem of Serre [Har77, p. 228, Thm. III.5.2(b) or Liu06, p. 195, Thm. 5.3.2(b)] asserts that $H^1(X, \mathcal{J}(k)) = 0$ for all large k. By [Har77, p. 117, Prop. II.5.12(c)] applied to the A_0-epimorphism $R \to A$ that maps T_i onto t_i, $i = 0, \ldots, r$, we have $j_*(\mathcal{O}_V(k)) \cong (j_*\mathcal{O}_V)(k)$. It follows from the definition of the direct image [Har77, p. 65, Def.] that $\Gamma(X, (j_*\mathcal{O}_V)(k)) \cong \Gamma(V, \mathcal{O}_V(k))$. Thus, (2) becomes:

(3)
$$0 \to \Gamma(X, \mathcal{J}(k)) \to \Gamma(X, \mathcal{O}_X(k)) \to \Gamma(V, \mathcal{O}_V(k)) \to 0.$$

Adding the maps $\beta_{X,k}$ and $\beta_{V,k}$ of Setup 1.1 to (3), we get the following commutative diagram:

(4)
$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathcal{J}(k)) \arrow[r] & \Gamma(X, \mathcal{O}_X(k)) \arrow[r] & \Gamma(V, \mathcal{O}_V(k)) \arrow[r] & 0 \\
 & & R_k \arrow[u, "\beta_{X,k}"] \arrow[r] & A_k \arrow[u, "\beta_{V,k}"'] &
\end{tikzcd}$$

By [Har77, p. 118, Prop. II.5.13], $\beta_{X,k}$ is an isomorphism for all k. Since the two horizontal maps of the commutative square of (4) are surjective, $\beta_{V,k}$ is surjective for all large k. ∎

Remark 1.4: Under Setup 1.1, let V' be a closed subscheme of V and let I be a homogeneous ideal of A such that $V' = \mathrm{Proj}(A/I)$ [Liu06, p. 168, Prop. 5.1.30]. Then, $A' = A/I = \bigoplus_{k=0}^{\infty}(A_k/A_k \cap I)$ is a graded ring over $A'_0 = A_0/A_0 \cap I$. Moreover, $A'_1 = \sum_{i=0}^{r} A'_0 t'_i$ with $t'_i = t_i + I$, and $A' = A'_0[t'_0, \ldots, t'_r]$.

For each integer $k \geq 0$ let $\pi_{V,V'}^{(k)} \colon A_k \to A'_k$ be the epimorphism of abelian groups induced by the epimorphism $A \to A/I$ of rings and let $\rho_{V,V'}^{(k)} \colon \Gamma(V, \mathcal{O}_V(k)) \to \Gamma(V', \mathcal{O}_{V'}(k))$ be the restriction homomorphism induced by the closed immersion $V' \subseteq V$. We set $\beta_k = \beta_{V,k}$ and $\beta'_k = \beta_{V',k}$ (Setup 1.1). By Lemma 1.3, we have for each large k that β_k and β'_k are isomorphisms. Since $\beta_{V,k}$ is natural in V, we have $\rho_{V,V'}^{(k)} \circ \beta_k = \beta'_k \circ \pi_{V,V'}^{(k)}$. It follows that β_k maps the kernel $A_k \cap I$ of $\pi_{V,V'}^{(k)}$ onto $\mathrm{Ker}(\rho_{V,V'}^{(k)})$. Also, since $\pi_{V,V'}^{(k)}$ is surjective, so is $\rho_{V,V'}^{(k)}$. This gives the following commutative diagram with two short exact sequences:

(5)
$$\begin{tikzcd}
0 \arrow[r] & A_k \cap I \arrow[r] \arrow[d, "\beta_k"'] & A_k \arrow[r, "\pi_{V,V'}^{(k)}"] \arrow[d, "\beta_k"'] & A'_k \arrow[r] \arrow[d, "\beta'_k"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\rho_{V,V'}^{(k)}) \arrow[r] & \Gamma(V, \mathcal{O}_V(k)) \arrow[r, "\rho_{V,V'}^{(k)}"] & \Gamma(V', \mathcal{O}_{V'}(k)) \arrow[r] & 0
\end{tikzcd}$$

The maps $\pi_{V,V'}^{(k)}$ and $\rho_{V,V'}^{(k)}$ combine to epimorphisms of A-modules $\pi_{V,V'} \colon A \to A'$ and $\rho_{V,V'} \colon \bigoplus_{k=0}^{\infty} \Gamma(V, \mathcal{O}_V(k)) \to \bigoplus_{k=0}^{\infty} \Gamma(V', \mathcal{O}_{V'}(k))$ that satisfy $\rho_{V,V'} \circ \beta_V = \beta_{V'} \circ \pi_{V,V'}$.

Following this observation, we categorically identify A_k with $\Gamma(V, \mathcal{O}_V(k))$ via $\beta_{V,k}$ and identify $A_k \cap I$ with $\mathrm{Ker}(\rho_{V,V'}^{(k)})$ for all large k. ■

LEMMA 1.5: *In the notation of Setup 1.1, let $V_1, \ldots, V_m$ be closed pairwise disjoint subschemes of the projective scheme V and let k be a sufficiently large positive integer. For each $1 \leq i \leq m$ let $s_i \in \Gamma(V_i, \mathcal{O}_{V_i}(k))$. Then, there exists an $s \in \Gamma(V, \mathcal{O}_V(k))$ such that $s|_{V_i} = s_i$ for $i = 1, \ldots, m$.*

Proof: We consider the closed subscheme $V' = \bigcup_{i=1}^{m} V_i$ of V. The sets $V_1, \ldots, V_m$ are closed and disjoint in V'. Hence, they are also open in V'. If $i \neq j$, then the restrictions of both s_i and s_j to $\Gamma(\emptyset, \mathcal{O}_{V'}(k))$ is the unique element 0 of the latter module. By the basic property of sheaves, there exists $s' \in \Gamma(V', \mathcal{O}_{V'}(k))$ such that $s'|_{V_i} = s_i$ for $i = 1, \ldots, m$. Since V' is a closed subscheme of V, the surjectivity of $\rho_{V,V'}^{(k)}$ in (5) gives an $s \in \Gamma(V, \mathcal{O}_V(k))$ such that $s|_{V'} = s'$, hence $s|_{V_i} = s_i$ for $i = 1, \ldots, m$. ■

Example 1.6: Let K be a field and $t_0, \ldots, t_r$ non-zero elements of a field extension of K. We set $\mathbf{t} = (t_0, \ldots, t_r)$ and assume that $K[\mathbf{t}]$ is a graded ring over K such that $K[\mathbf{t}]_1 = \sum_{i=0}^{r} K t_i$. Then, for all distinct integers i, j between 0 and r the element t_i is transcendental over $K\big(\frac{t_0}{t_j}, \ldots, \frac{t_r}{t_j}\big)$ [ZaS75II, p. 168, Lemma]. Also, for each $k \geq 0$, $K[\mathbf{t}]_k$ is the vector space over K generated by all monomials in $t_0, \ldots, t_r$ of degree k with coefficients in K.

A **homogeneous element** of the quotient field $K(\mathbf{t})$ of $K[\mathbf{t}]$ is a quotient $\frac{f}{g}$ of homogeneous elements of $K[\mathbf{t}]$ with $g \neq 0$. We set $\deg_{K[\mathbf{t}]}\big(\frac{f}{g}\big) = \deg_{K[\mathbf{t}]}(f) - \deg_{K[\mathbf{t}]}(g)$ and observe that $\deg_{K[\mathbf{t}]}$ is a well defined homomorphism from the multiplicative group of homogeneous elements of $K(\mathbf{t})^\times$ onto $\mathbb{Z}$.

We consider the integral projective variety $V = \mathrm{Proj}(K[\mathbf{t}])$ over K. Then, for each $0 \leq i \leq r$, $F = K\big(\frac{t_0}{t_i}, \ldots, \frac{t_r}{t_i}\big)$ is the function field of V. It can also be described as the set of all homogeneous elements of $K(\mathbf{t})$ of $K[\mathbf{t}]$-degree 0. Indeed, if $f(\mathbf{t}), g(\mathbf{t})$ are homogeneous elements of $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree k with $g \neq 0$, then $\frac{f(\mathbf{t})}{g(\mathbf{t})} = \frac{f(t_0/t_i, \ldots, t_r/t_i)}{g(t_0/t_i, \ldots, t_r/t_i)} \in F$.

Recall that the local ring of V at a point P is the ring $\mathcal{O}_{V,P}$ of all quotients $\frac{f}{g}$, where f and g are homogeneous elements of $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree and $g \notin P$. Likewise for each $k \geq 0$ the stalk $\mathcal{O}_V(k)_P$ is the K-vector-space that consists of all quotients $\frac{f}{g}$, where f and g are homogeneous elements of $K[\mathbf{t}]$ such that $\deg_{K[\mathbf{t}]}(f) - \deg_{K[\mathbf{t}]}(g) = k$ and $g \notin P$. By Lemma 1.3,

(a) for every large positive integer k an element x of $K(\mathbf{t})$ belongs to $K[\mathbf{t}]_k$ if and only if $x \in \mathcal{O}_V(k)_P$ for all $P \in V$.

Next we assume that V is an integral normal projective curve over K. Then,

(b) for each closed point P of V, the local ring $\mathcal{O}_{V,P}$ is a valuation ring of F [Lan58, p. 151, Thm. 1]. We denote the corresponding normalized discrete valuation of F by ord_P. By definition, $\mathcal{O}_{V,P}$ is the subring of F that consists of all quotients $\frac{s}{u}$, where s, u are homogeneous elements of $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree with $u \notin P$. Thus, each of them satisfies $\mathrm{ord}_P\left(\frac{s}{u}\right) \geq 0$. Since $\mathcal{O}_{V,P}$ is the valuation ring of ord_P, each $x \in F$ with $\mathrm{ord}_P(x) \geq 0$ can be written as $\frac{s}{u}$ with s, u as above. In particular, if both s and u as above do not belong to P, then $\mathrm{ord}_P\left(\frac{s}{u}\right) = 0$.

(c) If $\pi \in F$ satisfies $\mathrm{ord}_P(\pi) \geq 1$ and we write $\pi = \frac{p}{v}$ with p and v homogeneous elements of $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree with $v \notin P$, then $p \in P$ (otherwise $\pi^{-1} = \frac{v}{p} \in \mathcal{O}_{V,P}$, so $\mathrm{ord}_P(\pi) = 0$, in contrast to our assumption).

Conversely, if f and u are homogeneous elements of $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree, $f \in P$, and $u \notin P$, then $\frac{f}{u} \in \mathcal{O}_{V,P}$, hence $\mathrm{ord}_P\left(\frac{f}{u}\right) \geq 0$. If $\mathrm{ord}_P\left(\frac{f}{u}\right) = 0$, then $\frac{u}{f} \in \mathcal{O}_{V,P}$. This gives homogeneous elements g, v in $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree such that $v \notin P$ and $\frac{u}{f} = \frac{g}{v}$, hence $uv = fg \in P$ in contrast to the assumption that P is a prime ideal. It follows that $\mathrm{ord}_P\left(\frac{f}{u}\right) \geq 1$.

(d) If x is a homogeneous element of $K(\mathbf{t})$ of $K[\mathbf{t}]$-degree k, $h \in K[\mathbf{t}]_k \smallsetminus P$, and $\mathrm{ord}_P\left(\frac{x}{h}\right) \geq 0$, then by (b), $\frac{x}{h} = \frac{f}{g}$, where f and g are homogeneous elements of $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree with $g \notin P$. Thus, $x = \frac{fh}{g} \in \mathcal{O}_V(k)_P$.

(e) Let x and u be homogeneous elements of $K[\mathbf{t}]$ of the same $K[\mathbf{t}]$-degree such that $u \notin P$ and $x \in P^q$ for some positive integer q. Since P is a homogeneous ideal of $K[\mathbf{t}]$, there exist a positive integer l and homogeneous elements $t_{i1}, \ldots, t_{iq} \in K[\mathbf{t}]$ that belong to P, $i = 1, \ldots, l$, such that $x = \sum_{i=1}^{l} \prod_{j=1}^{q} t_{ij}$, and under the setting $d = \deg_{K[\mathbf{t}]}(x)$ and $d_{ij} = \deg_{K[\mathbf{t}]}(t_{ij})$ we have $\sum_{j=1}^{q} d_{ij} = d$ for all i. We choose a homogeneous element $v \in K[\mathbf{t}]_1$ with $v \notin P$ (e.g. one of the t_i's), divide x by v^d and obtain

$$\frac{x}{v^d} = \sum_{i=1}^{l} \prod_{j=1}^{q} \frac{t_{ij}}{v^{d_{ij}}} \ .$$

By (c), $\mathrm{ord}_P\left(\frac{t_{ij}}{v^{d_{ij}}}\right) \geq 1$ for all i, j. Hence, $\mathrm{ord}_P\left(\frac{x}{v^d}\right) \geq q$. It follows that $\mathrm{ord}_P\left(\frac{x}{u}\right) = \mathrm{ord}_P\left(\frac{x}{v^d}\right) + \mathrm{ord}_P\left(\frac{v^d}{u}\right) \geq q$. ∎

2. Global Sections of Invertible Sheaves and Cartier Divisors

Following [Liu06, p. 266, Exer. 7.1.13], we associate effective Cartier divisors to global sections of invertible sheaves on integral schemes and introduce their degrees.

2.1 Divisors on curves over a field. We consider a **curve** C over a field L. Thus, C is a separated scheme of finite type over L, each of its irreducible components is of dimension 1. We assume that C is integral and projective and let F be the function field of C. For each closed point $\mathbf{p}$ of C and each non-zero $f \in \mathcal{O}_{C,\mathbf{p}}$ we write $\mathrm{ord}_{\mathbf{p}}(f)$ for the length of the $\mathcal{O}_{C,\mathbf{p}}$-module $\mathcal{O}_{C,\mathbf{p}}/\mathcal{O}_{C,\mathbf{p}}f$ [AtM69, p. 77]. This function satisfies

$$\mathrm{ord}_{\mathbf{p}}(fg) = \mathrm{ord}_{\mathbf{p}}(f) + \mathrm{ord}_{\mathbf{p}}(g), \tag{1}$$

hence it extends to a function $\mathrm{ord}_{\mathbf{p}}$ on $F^\times$ satisfying (1) for all $f, g \in F^\times$ [BLR90, p. 237]. If $\mathbf{p}$ is a closed normal point of C, then $\mathrm{ord}_{\mathbf{p}}$ coincides with the normalized valuation attached to the discrete valuation ring $\mathcal{O}_{C,\mathbf{p}}$ as introduced in Example 1.6(b).

If $(U_i, f_i)_{i\in I}$ is data that represent a Cartier divisor D on C, we define $\mathrm{ord}_{\mathbf{p}}(D)$ as $\mathrm{ord}_{\mathbf{p}}(f_i)$ for each $i \in I$ such that $\mathbf{p} \in U_i$. Then, the Weil divisor that corresponds to D is $D_{\mathrm{Weil}} = \sum \mathrm{ord}_{\mathbf{p}}(D)\mathbf{p}$, where $\mathbf{p}$ ranges over all closed points of C. The **degree** of D (and of D_{Weil}) is then

$$\deg(D) = \sum_{\mathbf{p}} \mathrm{ord}_{\mathbf{p}}(D)[L(\mathbf{p}) : L]. \tag{2}$$

Here, $L(\mathbf{p})$ is the residue field $\mathcal{O}_{C,\mathbf{p}}/\mathfrak{m}_{C,\mathbf{p}}$ of C at $\mathbf{p}$. If an affine neighborhood of $\mathbf{p}$ in C is embedded in $\mathbb{A}^n_L$ and one views $\mathbf{p}$ as an n-tuple of elements of $\tilde{L}$, then the field obtained from L by adjoining those elements is L-isomorphic to $L(\mathbf{p})$.

By (1), $\deg(D_1 + D_2) = \deg(D_1) + \deg(D_2)$ for any two Cartier (or Weil) divisors D_1 and D_2 on C. A Cartier divisor on C that can be represented by a pair (C, f) with $f \in F^\times$ is said to be **principal** and is denoted by $\mathrm{div}(f)$. By [GoW10, p. 498, Thm. 15.32], $\deg(\mathrm{div}(f)) = 0$.

Recall that a Cartier divisor D on C which is represented by data $(U_i, f_i)_{i\in I}$ naturally corresponds to an invertible sheaf $\mathcal{L}$ on C such that $\Gamma(U_i, \mathcal{L}) = \Gamma(U_i, \mathcal{O}_C)f_i^{-1}$ for each $i \in I$. Two Cartier divisors that correspond to isomorphic invertible sheaves on C differ by a principal divisor [GoW10, p. 303, Prop. 11.26]. By the preceding paragraph, they have the same degree. Hence, one defines $\deg(\mathcal{L}) = \deg(D)$ for each Cartier divisor D on C that corresponds to $\mathcal{L}$. Since addition of divisors corresponds to tensor products of the corresponding invertible sheaves, we have $\deg(\mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{L}') = \deg(\mathcal{L}) + \deg(\mathcal{L}')$.

By [GoW10, p. 498, Remark 15.30(2)], the degree of divisors (hence of invertible sheaves) on C is invariant under a change of the base field.

2.2 Curves over schemes. Let $f\colon C \to S$ be an S-curve, i.e. f is a morphism of schemes of finite presentation with one dimensional fibers. Under the assumption that f is flat and proper and that both S and C are integral, [BLR90, p. 238, Prop. 2] generalizes the definition of the degree to invertible sheaves on C (hence the definition of the degree of divisors on C). We restrict ourselves to the only case we use in this work, where for each $s \in S$, the fiber $C_s = \mathrm{Spec}(\Bbbk(s)) \times_S C$ is an integral curve over the residue field $\Bbbk(s) = \mathcal{O}_{S,s}/\mathfrak{m}_{S,s}$ of S at s. Let $i_s\colon C_s \to C$ be the canonical morphism. We consider an invertible sheaf $\mathcal{L}$ on C and for each $s \in S$ let $\mathcal{L}_s$ be the pull-back $i_s^*\mathcal{L}$. It is an invertible sheaf on the fiber C_s [BLR90, p. 238, last paragraph before Prop. 2]. Since S is integral, [BLR90, p. 238, Prop. 2] implies that $\deg(\mathcal{L}_s)$ (defined in Subsection 2.1) has a unique value on S, which we define as $\deg(\mathcal{L})$. It follows from Subsection 2.1 that the degree is additive and invariant under base change. In particular, if $S = \mathrm{Spec}(R)$ for some integral domain R with quotient field K, and we take s to be the generic point of S, we get that $\deg(D) = \deg(D_K)$ for each Cartier divisor D on C.

Finally we note that the assumptions on $f\colon C \to S$ to be flat and proper are satisfied if $S = \mathrm{Spec}(R)$ (resp. $S = \mathrm{Spec}(L)$), where R is a Dedekind domain (resp. L is a field), and f is projective and surjective (or at least dominating). See for example [Liu06, p. 137, Prop. 3.9] and [Liu06, p. 108, Thm. 3.30]. These are the cases we consider in this work.

2.3 Subschemes attached to divisors. As in Subsection 2.2, let $f\colon C \to S$ be an S-curve. Recall that a Cartier divisor D on C represented by data $(U_i, f_i)_{i\in I}$ is said to be **effective** if $f_i \in \Gamma(U_i, \mathcal{O}_C)$ for each $i \in I$. In this case, D gives rise to a closed subscheme $C(D)$ of C such that $\Gamma(U_i, \mathcal{O}_{C(D)}) = \Gamma(U_i, \mathcal{O}_C)/f_i\Gamma(U_i, \mathcal{O}_C)$ for each $i \in I$. We say that D is **flat** (resp. finite) over S if $C(D)$ is flat (resp. finite) over S. We say that a subset C_0 of C is **disjoint from** D, if $C_0 \cap C(D) = \emptyset$. Finally note that if $S = \mathrm{Spec}(L)$ for some field L, then $\deg(D) = \dim_L \Gamma(C(D), \mathcal{O}_{C(D)})$ [GoW10, p. 497, (15.9.1)].

2.4 Divisors of global sections. Let C be an integral scheme with function field F. We consider an invertible sheaf $\mathcal{L}$ on C and a non-zero global section $s \in \Gamma(C, \mathcal{L})$, and elaborate on [Liu06, p. 266, Exer. 7.1.13] to associate an effective Cartier divisor $\mathrm{div}(s)$ to s.

By definition, C can be covered by open subsets U_i, $i \in I$, such that $\mathcal{L}|_{U_i}$ is a free $\mathcal{O}_C|_{U_i}$-module of rank 1. Thus, for each $i \in I$ there exists $e_i \in \Gamma(U_i, \mathcal{L})$ such that for each Zariski-open subset U of U_i, the element $e_i|_U$ is a free generator of the $\Gamma(U, \mathcal{O}_C)$-module $\Gamma(U, \mathcal{L})$. In particular, there exists a unique $f_i \in \Gamma(U_i, \mathcal{O}_C)$ such that $s|_{U_i} = f_i e_i$. Moreover, for each additional $j \in I$ there exists $u_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_C)^\times$ such that $e_i|_{U_i\cap U_j} = u_{ij} \cdot e_j|_{U_i \cap U_j}$, hence $u_{ij} \cdot f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Thus, the data $(U_i, f_i)_{i\in I}$ define an effective Cartier divisor $\mathrm{div}(s)$ on C.

For later use we say that e_i is a **free $\mathcal{O}_C|_{U_i}$-generator** of $\mathcal{L}|_{U_i}$.

By [Har77, p. 144, Def.], the invertible sheaf $\mathcal{L}(\mathrm{div}(s))$ associated with $\mathrm{div}(s)$ satisfies $\mathcal{L}(\mathrm{div}(s))|_{U_i} = (\mathcal{O}_C|_{U_i})f_i^{-1}$ for each $i \in I$. It follows from the construction made in the preceding paragraph that the $\Gamma(U_i, \mathcal{O}_C)$-isomorphisms $\varphi_i\colon \Gamma(U_i, \mathcal{L}(\mathrm{div}(s))) \to \Gamma(U_i, \mathcal{L})$ defined by $\varphi_i(s' f_i^{-1}) = s' e_i$ for each $s' \in \Gamma(U_i, \mathcal{O}_C)$ combine to an isomorphism $\varphi\colon \mathcal{L}(\mathrm{div}(s)) \to \mathcal{L}$ of invertible sheaves on C.

Now we assume that C is an integral locally factorial proper curve over a Noetherian domain R (possibly a field). As in [Har77, p. 141, first part of the proof of Prop. 6.11] or [GoW10, p. 307, (11.13.4) and Thm. 11.38(2)], the Weil divisor that corresponds to $\mathrm{div}(s)$ is

$$\mathrm{div}_{\mathrm{Weil}}(s) = \sum_P \mathrm{ord}_P(\mathrm{div}(s))P, \tag{3}$$

where P ranges over all prime divisors of C such that $P \cap U_i \neq \emptyset$ and $\mathrm{ord}_P(\mathrm{div}(s)) = \mathrm{ord}_P(f_i)$ for each $i \in I$. Here, in analogy to the notation introduced in Example 1.6(b), ord_P is the normalized discrete valuation of F that corresponds to the valuation ring $\mathcal{O}_{C,P}$. Thus, $\mathrm{ord}_P(f_i)$ is non-negative and independent of the i that satisfies $P \cap U_i \neq \emptyset$, so $\mathrm{div}_{\mathrm{Weil}}(s)$ is an effective Weil divisor. The finitely many prime divisors P of C with $\mathrm{ord}_P(\mathrm{div}(s)) > 0$ are called the **zeros** of s. In the notation of Subsection 2.3, the set of zeros of s is the underlying topological set of $C(\mathrm{div}(s))$. Hence, $\mathrm{div}(s)$ is disjoint to a subset C_0 of C if each of the zeros of s is disjoint to C_0. We say that $\mathrm{div}(s)$ is flat and finite over an integral domain R if $C(\mathrm{div}(s))$ is flat and finite over R.

In addition to the assumptions made on C above we now assume that $C_\mathfrak{p}$ is integral over $\mathbb{k}(\mathfrak{p}) = \mathrm{Quot}(R/\mathfrak{p})$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$ (this is the only case we use in this work). The degree of $\mathrm{div}(s)$ is defined as in Subsection 2.1 if C is a curve over a field. If C is a curve over R, then by Subsection 2.2, $\deg(\mathrm{div}(s)) = \deg(\mathrm{div}(s)_K)$, where $K = \mathrm{Quot}(R)$. Since $\mathcal{L}(\mathrm{div}(s)) \cong \mathcal{L}$, we deduce that $\deg(\mathrm{div}(s)) = \deg(\mathcal{L})$. It follows that $\deg(\mathrm{div}(s')) = \deg(\mathrm{div}(s))$ for each non-zero $s' \in \Gamma(C, \mathcal{L})$.

If the zeros of s belong to a Zariski-open subscheme C_0 of C, we may consider $\mathrm{div}(s)$ also as a divisor on C_0.

2.5 The section 1_D. Let C be an integral scheme with function field F. Let D be a Cartier divisor on C with representing data $(U_i, f_i)_{i\in I}$. One attaches an invertible sheaf $\mathcal{L}(D)$ on C such that $\mathcal{L}(D)|_{U_i} = \mathcal{O}_C|_{U_i} f_i^{-1}$, hence $\Gamma(U_i, \mathcal{L}(D)) = \Gamma(U_i, \mathcal{O}_C) f_i^{-1}$ for every $i \in I$ [Har77, p. 144, Def.]. If D is an effective divisor, then $f_i \in \Gamma(U_i, \mathcal{O}_C)$, so the unit of F, $1 = f_i f_i^{-1}$ belongs to $\Gamma(U_i, \mathcal{L}(D))$ for each $i \in I$. Hence, there exists a global section $1_D \in \Gamma(C, \mathcal{L}(D))$ such that $1_D|_{U_i} = 1$ for each $i \in I$.

In the notation of Subsection 2.4, the Cartier divisor on C that corresponds to 1_D has $(U_i, f_i)_{i\in I}$ as representing data. Hence, $\mathrm{div}(1_D) = D$.

2.6 The ample sheaves $\mathcal{O}_C(k)$. Let A_0 be a Noetherian integral domain and let $A = \bigoplus_{k=0}^\infty A_k$ be a graded integral domain over A_0 such that $A_1 = \sum_{i=0}^r A_0 t_i$ and $A = A_0[\mathbf{t}]$ with $\mathbf{t} = (t_0, \ldots, t_r)$. Then, $C = \mathrm{Proj}(A)$ is isomorphic to a closed subscheme of $\mathbb{P}^r_{A_0}$ [Liu06, p. 53, Lemma 2.3.41], so C is projective over A_0. Hence, C is proper over A_0 [Liu06, p. 108, Thm. 3.3.30]. We assume that C is a regular curve over A_0, in particular C is locally factorial [Liu06, p. 130, Thm. 4.2.16(b)]. As above, we also assume that $C_\mathfrak{p}$ is integral for each $\mathfrak{p} \in \mathrm{Spec}(A_0)$. Let F be the function field of C. Following Subsection 2.4, we attach to each non-zero $s \in A_k$ with k large an effective Weil divisor $\mathrm{div}_{\mathrm{Weil}}(s)$ as follows:

We set $U = A_1 \smallsetminus \{0\}$ and consider $u \in U$. Recall that $D_+(u) = \{\mathbf{p} \in C \mid u \notin \mathbf{p}\}$ and the ring $\Gamma(D_+(u), \mathcal{O}_C)$ consists of all the quotients $\frac{s}{u^l}$, where s is a homogeneous element of A and $\deg_A(s) = l$. The $\Gamma(D_+(u), \mathcal{O}_C)$-module $\Gamma(D_+(u), \mathcal{O}_C(k))$ consists of all quotients $\frac{s}{u^j}$, where s is a homogeneous element of A and $\deg_A(s) - j = k$ (see the proof of [Har77, p. 117, Prop. II.5.12(a)]). Writing $\frac{s}{u^j} = \frac{s}{u^{j+k}} u^k$, we see that u^k is a free $\mathcal{O}_C|_{D_+(u)}$-generator of $\mathcal{O}_C(k)|_{D_+(u)}$. In particular, $\mathcal{O}_C(k)$ is an invertible sheaf on C [Har77, p. 117, Prop. II.5.12(a)].

For large k, Lemma 1.3 identifies $\Gamma(C, \mathcal{O}_C(k))$ with A_k. Following Subsection 2.4, the Cartier divisor that corresponds to an element $s \in A_k$ (which we write as $\frac{s}{u^k} u^k$) is $(D_+(u), \frac{s}{u^k})_{u\in U}$. By our assumptions on A, for each prime divisor P of C and, with $\mathbf{p}$ the homogeneous prime ideal of A underlying P, there exists $u \in U \smallsetminus \mathbf{p}$, so $\mathrm{ord}_P\left(\frac{s}{u^k}\right)$ is a non-negative integer that does not depend on u. Hence, $\mathrm{div}_{\mathrm{Weil}}(s) = \sum \mathrm{ord}_P\left(\frac{s}{u^k}\right) P$, where P ranges over all prime divisors of C.

It follows from this definition that if s' is another homogeneous element of A of large A-degree, then $\mathrm{div}_{\mathrm{Weil}}(ss') = \mathrm{div}_{\mathrm{Weil}}(s) + \mathrm{div}_{\mathrm{Weil}}(s')$.

2.7 Divisors of function fields. We assume in this subsection that the ring A_0 introduced in Subsection 2.6 is a field L. Then, the scheme C introduced in that section is a projective normal curve over L. We identify the prime divisors P of F/L with the closed points of C such that the valuation ring of P, considered as a prime divisor, coincides with the local ring of C at P, considered as a point of C. In particular, the degree of P over L as a prime divisor coincides with its degree over L as a point of C. Then, a **divisor** of F/L is a formal sum $D = \sum k_P P$, where P ranges over all prime divisors of F/L and all but finitely many of the integral coefficients k_P are zero [FrJ08, Section 3.1]. As in (2), $\deg(D) = \sum_P k_P[L(P) : L]$.

If $f \in F^\times$, we write $\mathrm{div}(f) = \sum_P \mathrm{ord}_P(f)P$ (in accordance with Subsection 2.1). We also write

$$\mathrm{div}_0(f) = \sum_{\mathrm{ord}_P(f)>0} \mathrm{ord}_P(f)P \text{ and } \mathrm{div}_\infty(f) = -\sum_{\mathrm{ord}_P(f)<0} \mathrm{ord}_P(f)P$$

for the **zero divisor** and the **pole divisor**, respectively, of f. Since $\mathrm{div}(f) = \mathrm{div}_0(f) - \deg_\infty(f)$ and $\deg(\mathrm{div}(f)) = 0$ [Che51, p. 18, Thm. 5], we have $\deg(\mathrm{div}_0(f)) = \deg(\mathrm{div}_\infty(f))$. Note that if s and s' are non-zero homogeneous elements of A of the same A-degree, then $f = \frac{s'}{s} \in F^\times$, so $s' = fs$. For each divisor P of C we choose $u \in U \smallsetminus P$. Then, by Section 2.6, $\mathrm{ord}_P(\mathrm{div}_{\mathrm{Weil}}(s')) = \mathrm{ord}_P\big(\frac{s'}{u^k}\big) = \mathrm{ord}_P(f) + \mathrm{ord}_P\big(\frac{s}{u^k}\big) = \mathrm{ord}_P(\mathrm{div}_{\mathrm{Weil}}(f)) + \mathrm{ord}_P(\mathrm{div}_{\mathrm{Weil}}(s))$. Hence,

$$\mathrm{div}_{\mathrm{Weil}}(s') = \mathrm{div}_{\mathrm{Weil}}(f) + \mathrm{div}_{\mathrm{Weil}}(s). \tag{4}$$

Therefore, $\deg(\mathrm{div}_{\mathrm{Weil}}(s')) = \deg(\mathrm{div}_{\mathrm{Weil}}(s))$.

In the sequel we omit the subscript "Weil" from Weil divisors. However, occasionally we add a subscript L for the divisors of elements of $F^\times$ to indicate the field of constants of F.

3. Continuity of Divisors

We apply the identification of global sections of high degrees of twisted sheaves on a projective scheme with homogeneous polynomials to the case of a curve over a local field and prove a theorem about continuity of divisors of functions.

Throughout this section we consider a field L and a graded ring $A = \bigoplus_{k=0}^\infty A_k$ over $L = A_0$ such that $A_1 = \sum_{i=0}^r Lt_i$ and $A = L[t_0, \ldots, t_r]$, with $t_0, \ldots, t_r \neq 0$. We assume that $C = \mathrm{Proj}(A)$ is an absolutely integral normal projective curve over L with function field F. In particular, F is a regular extension of L [FrJ08, p. 175, Cor. 10.2.2(b)].

3.1 Continuity. We assume in this section that L is a field equipped with an absolute value $|\ |$ which is either non-archimedean and Henselian or $|\ |$ is archimedean and L is either real closed or algebraically closed with $\mathbb{C}$ as the $|\ |$-completion. Note that, if L is separably closed, then L is Henselian with respect to every non-archimedean absolute value [Jar91, Cor. 11.3].

We consider a normal absolutely integral projective curve C over L with function field F. We extend $|\ |$ to the algebraic closure $\tilde{L}$ of L in the unique possible way and prove that for each large k the map $s \mapsto \mathrm{div}(s)$ from $\Gamma(C, \mathcal{O}_C(k))$ to the set of divisors on C is $|\ |$-continuous in a sense that will become clear in Lemma 3.4.

Following Subsection 2.7, we identify the set of L-rational points $C(L)$ of C with the set of prime divisors of F/L of degree 1. The absolute value $|\ |$ of L induces a topology on $C(L)$ (see [Mum88, p. 57, Sec. I.10] or [GPR95, p. 68, Sec. 7]), so we may speak of an $|\ |$**-open neighborhood** U of a point $\mathbf{p}$ in $C(L)$. The set U is defined by inequalities involving $|\ |$ and elements of L. If L' is an algebraic extension of L, then the same inequalities define a neighborhood $U(L')$ of the unique point $\mathbf{p}_{L'}$ of $C(L')$ that lies over $\mathbf{p}$. To simplify notation, we also write $\mathbf{p}$ rather than $\mathbf{p}_{L'}$.

Here are some useful remarks about the interaction of the $|\ |$-topology with the Zariski-topology.

(a) Let V be an absolutely integral affine variety in $\mathbb{A}_L^n$ for some positive integer n. If U is a Zariski-open subset of V, then $U(L)$ is $|\ |$-open in $V(L)$ [Mum88, p. 57, (i)]. On the other hand, if $U(L)$ is a $|\ |$-open subset of $V(L)$ that contains a **simple point** (= non-singular point) of V, then $U(L)$ is Zariski-dense in V [GeJ02, Prop. 8.2(b)].

(b) If L is algebraically closed, and U is a non-empty Zariski-open subset of V, then $U(L)$ is $|\ |$-dense in $V(L)$ [GeJ75, Lemma 2.2].

(c) If L is separably closed and U is a non-empty Zariski-open subset of V, then $U(L)$ contains a simple point of V [Lan58, p. 76, Prop. 9]. Hence, by (a), $U(L)$ is $|\ |$-dense in $V(L)$.

3.2 Total splitting. Let D be an effective divisor of F/L and N a finite separable extension of L. We say that D **totally splits in** FN if the extension D_N of D to N is the sum $\sum_{i=1}^m P_i$ of distinct prime divisors of degree 1 of FN/N. In this case we also say that $D_N = \sum_{i=1}^m P_i$ is a **total splitting** of D in FN. Note that P_i has in this case a unique extension to a prime divisor $P_{i,N'}$ of N' for every separable algebraic extension N' of N [Deu73, p. 128, Thm.]. Hence, if L' is a separable algebraic extension of L and we set $N' = NL'$, then $D_{N'} = \sum_{i=1}^m P_{i,N'}$ is a total splitting of D in FN'.

Given a divisor D of F/L, we consider the vector space

$$\mathfrak{L}(D) = \{f \in F^\times \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\}$$

over L.

Lemma 3.3: *In the above notation, let f be an element of $F^\times$ with a total splitting $\operatorname{div}_0(f)_N = \sum_{i=1}^m P_i$ of $\operatorname{div}_0(f)$ in FN. For each i let U_i be an $|\ |$-open neighborhood of P_i in $C(N)$. Let $u_1, \ldots, u_l$ be elements of $\mathfrak{L}(\operatorname{div}_\infty(f))$ and let $b_1, \ldots, b_l$ be elements of L satisfying $f = \sum_{\lambda=1}^l b_\lambda u_\lambda$.*

Then, there exists a real $\gamma > 0$ such that every separable algebraic extension L' of L has the following property: if $b'_1, \ldots, b'_l \in L'$ satisfy $|b'_\lambda - b_\lambda| < \gamma$ for $1 \leq \lambda \leq l$ and we set $f' = \sum_{\lambda=1}^l b'_\lambda u_\lambda$ and $N' = NL'$, then $\operatorname{div}_\infty(f')_{N'} = \operatorname{div}_\infty(f)_{N'}$ and $\operatorname{div}_0(f')_{N'} = \sum_{i=1}^m P'_i$ is a total splitting of $\operatorname{div}_0(f')_{L'}$ in FN' with $P'_i \in U_i(N')$ for all i.

Proof: We may assume that $L' = L$ and $N' = N$. Then, we choose an L-basis $v_1, \ldots, v_d$ for $\mathfrak{L}(\operatorname{div}_\infty(f))$ and set $u_\lambda = \sum_{\delta=1}^d a_{\lambda\delta} v_\delta$ for some $a_{\lambda\delta} \in L$ and $\lambda = 1 \ldots, l$. This gives $f = \sum_{\delta=1}^d (\sum_{\lambda=1}^l b_\lambda a_{\lambda\delta}) v_\delta$ and $f' = \sum_{\delta=1}^d (\sum_{\lambda=1}^l b'_\lambda a_{\lambda\delta}) v_\delta$. Since the map

$$(b'_1, \ldots, b'_l) \mapsto \Big(\sum_{\lambda=1}^l b'_\lambda a_{\lambda 1}, \ldots, \sum_{\lambda=1}^l b'_\lambda a_{\lambda d}\Big)$$

is $|\ |$-continuous, we may replace $u_1, \ldots, u_l$ by $v_1, \ldots, v_d$, if necessary, to assume that $u_1, \ldots, u_l$ form a basis of $\mathfrak{L}(\operatorname{div}_\infty(f))$. Now we may apply [JaR08, Prop. 4.3] to conclude the existence of $\gamma > 0$ that has the properties of the conclusion of the lemma. ∎

Lemma 3.4: *As above we consider an absolute valued field $(L, |\ |)$ which is Henselian, real closed, or algebraically closed. We also consider the normal absolutely integral projective curve $C = \operatorname{Proj}(L[t_0, \ldots, t_r])$ over L with function field F introduced at the beginning of this section.*

Next we consider a finite Galois extension N of L, sections $s, s_1, \ldots, s_e \in \Gamma(C, \mathcal{O}_C(k))$ with k large as in Remark 1.4, and elements $a_1, \ldots, a_e \in L$ such that $s = \sum_{\varepsilon=1}^e a_\varepsilon s_\varepsilon$ and $\operatorname{div}(s)_N = \sum_{i=1}^m P_i$ is a total splitting of $\operatorname{div}(s)$ in FN. For each i let U_i be an $|\ |$-open neighborhood of P_i in $C(N)$.

Then, there exists a real $\gamma > 0$ such that if L' is a separable algebraic extension of L and $a'_1, \ldots, a'_e \in L'$ satisfy $|a'_\varepsilon - a_\varepsilon| < \gamma$ for $\varepsilon = 1, \ldots, e$ and we set $s' = \sum_{\varepsilon=1}^e a'_\varepsilon s_\varepsilon$ and $N' = NL'$, then $\operatorname{div}(s')_{N'} = \sum_{i=1}^m P'_i$ is a total splitting of $\operatorname{div}(s')_{L'}$ in FN' with $P'_i \in U_i(N')$ for all i. Moreover, $\deg(\operatorname{div}(s')_{L'}) = \deg(\operatorname{div}(s)_L)$.

Proof: Again, we may assume that $L' = L$ and hence that $N' = N$. Since t_0 is non-zero, it vanishes at only finitely many points of C. Applying an invertible linear transformation over L on the coordinates $t_0, \ldots, t_r$, we may assume that

(1) $$t_0(P_i) \neq 0 \text{ for all } i.$$

Under this assumption we set $t = \frac{s}{t_0^k} = \sum_{\varepsilon=1}^e a_\varepsilon \frac{s_\varepsilon}{t_0^k}$.

Claim: $\operatorname{div}_0(t) = \operatorname{div}(s)$. By (4) in Subsection 2.7,

(2) $$k \cdot \operatorname{div}(t_0) + \operatorname{div}(t) = \operatorname{div}(s).$$

Consider a point $\mathbf{p} \in C(N)$. Since C is normal and N is a separable extension, C_N is also normal [Lan58, p. 146, Thm. 7], so the notation $\operatorname{ord}_\mathbf{p}$ makes sense. By (2),

(3) $$k \cdot \operatorname{ord}_\mathbf{p}(\operatorname{div}(t_0)_N) + \operatorname{ord}_\mathbf{p}(t) = \operatorname{ord}_\mathbf{p}(\operatorname{div}(s)_N).$$

By Subsection 2.4, $\mathrm{div}(t_0) \ge 0$. If $\mathrm{ord}_{\mathbf{p}}(\mathrm{div}(t_0)_N) > 0$, then $t_0(\mathbf{p}) = 0$, so by (1), $\mathbf{p} \ne P_1, \dots, P_m$. Hence, $s(\mathbf{p}) \ne 0$, that is $\mathrm{ord}_{\mathbf{p}}(\mathrm{div}(s)_N) = 0$. Hence, by (3), $\mathrm{ord}_{\mathbf{p}}(t) < 0$. Therefore, $\mathrm{ord}_{\mathbf{p}}(\mathrm{div}_0(t)_N) = 0$. If $\mathrm{ord}_{\mathbf{p}}(\mathrm{div}(t_0)_N) = 0$, then by (3), $\mathrm{ord}_{\mathbf{p}}(t) = \mathrm{ord}_{\mathbf{p}}(\mathrm{div}(s)_N) \ge 0$, so $\mathrm{ord}_{\mathbf{p}}(\mathrm{div}_0(t)_N) = \mathrm{ord}_{\mathbf{p}}(\mathrm{div}(s)_N)$. Thus, the latter equality holds for all $\mathbf{p} \in C(N)$. This implies that $\mathrm{div}_0(t)_N = \mathrm{div}(s)_N$. Since the map of the group of divisors of C into the group of divisors of C_N given by $D \mapsto D_N$ is injective, we conclude that $\mathrm{div}_0(t) = \mathrm{div}(s)$, as claimed.

Lemma 3.3 gives a real $\gamma > 0$ such that if $a'_1, \dots, a'_e \in L$ satisfy $|a'_\varepsilon - a_\varepsilon| < \gamma$ for $\varepsilon = 1, \dots, e$, and we set $t' = \sum_{\varepsilon=1}^{e} a'_\varepsilon \frac{s_\varepsilon}{t_0^k}$, then

(4a) $\mathrm{div}_0(t')_N = \sum_{i=1}^{m} P'_i$ is a total splitting of $\mathrm{div}_0(t')$ in FN and $P'_i \in U_i(N)$ for $i = 1, \dots, m$, and
(4b) $\mathrm{div}_\infty(t')_N = \mathrm{div}_\infty(t)_N$.

Finally we observe that $s' = \sum_{\varepsilon=1}^{e} a'_\varepsilon s_\varepsilon$ satisfies $t' = \frac{s'}{t_0^k}$. As in (2), $k \cdot \mathrm{div}(t_0)_N + \mathrm{div}(t')_N = \mathrm{div}(s')_N$. Hence, by (2), $\mathrm{div}(s')_N - \mathrm{div}(t')_N = \mathrm{div}(s)_N - \mathrm{div}(t)_N$, so $\mathrm{div}(s')_N - \mathrm{div}_0(t')_N + \mathrm{div}_\infty(t')_N = \mathrm{div}(s)_N - \mathrm{div}_0(t)_N + \mathrm{div}_\infty(t)_N$. It follows from the claim and from (4b) that $\mathrm{div}(s')_N = \mathrm{div}_0(t')_N$. We conclude from (4a) that $\mathrm{div}(s')_N = \sum_{i=1}^{m} P'_i$ is a total splitting of $\mathrm{div}(s')$ in FN. Moreover, since F/L is regular, the degree of divisors is preserved under the extension of the base field from L to N [Deu73, p. 126, Thm.]. Hence, $\deg(\mathrm{div}(s')) = \deg(\mathrm{div}(s')_N) = m = \deg(\mathrm{div}(s)_N) = \deg(\mathrm{div}(s))$, as claimed. ∎

4. Reduction Steps

We set up the arithmetical objects that appear in the proof of Theorem C and prove two reduction lemmas. They allow us to replace V by an open subvariety and $\mathcal{T}$ by a larger finite subset of $\mathcal{V}$. Finally we reduce Theorem C to the case where V is a curve.

4.1 A global field. Let K be a global field, that is K is either a number field or an algebraic function field of one variable over a finite field. Following Weil's Foundation [Wei62], we choose an algebraically closed field $\mho$ that contains K and has a sufficiently large transcendence degree to contain all of the field extensions of K that appear in this work. If F is a subfield of $\mho$, then F_{sep} and $\tilde{F}$ denote the unique separable closure and the unique algebraic closure of F, respectively, in $\mho$. In particular, if F' is an extension of F in $\mho$, then $\tilde{F} \subseteq \widetilde{F'}$. We denote the absolute Galois group $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ of F by $\mathrm{Gal}(F)$.

4.2 Convention for affine varieties. We follow [Liu06, p. 55, Def. 3.47] to define an affine variety over K as an affine scheme associated to a finitely generated algebra over K.

Let V be an absolutely integral affine variety over K which we assume to be a closed K-subscheme of $\mathbb{A}_K^n$ for some n (in which case we also say that V is an **absolutely integral affine variety in $\mathbb{A}_K^n$**). Thus, $V = \mathrm{Spec}(K[\mathbf{x}])$, where $K[\mathbf{x}] = K[\mathbf{X}]/I$ with $\mathbf{X} = (X_1, \dots, X_n)$, I is a prime ideal of $K[\mathbf{X}]$ such that $\tilde{K}[\mathbf{X}]/\tilde{K}I$ is an integral domain, and $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i = X_i + I$ for $i = 1, \dots, n$. In the classical algebraic geometry V is said to be (or more accurately, closely related to) the **absolutely irreducible affine variety defined over** K by I. Thus, in the classical language, V is just the set of all $\mathbf{a} \in \mho^n$ such that $f(\mathbf{a}) = 0$ for all $f \in I$. This is the language used in our previous papers [FrJ08], [GeJ75], [GeJ89], [GeJ02], [GJR00], [JaR94], [JaR95], [JaR98], and [JaR08] that we use in this work. Following that convention, for each subset A of $\mho$ we set $V(A) = \{\mathbf{a} \in A^n \mid f(\mathbf{a}) = 0 \text{ for all } f \in I\}$. Each $\mathbf{a}$ in $V(A)$ is an **A-rational point** of V. Embedding $F = K(\mathbf{x})$ in $\mho$, the n-tuple $\mathbf{x}$ is then a **generic point** of V and $F = K(\mathbf{x})$ is a function field of V. It is a regular extension of K [FrJ08, p. 175, Cor. 10.2.2(a)]. As usual, if $\dim(V) = 1$, we speak about a "curve" rather than a "variety".

We also write V_{simp} for the Zariski-open subset of V that consists of all **simple** (= non-singular) points of V.

4.3 Convention for projective varieties. By an **absolutely integral projective variety in $\mathbb{P}_K^r$** we mean a closed absolutely integral subscheme W of $\mathbb{P}_K^r$. Thus, $W = \mathrm{Proj}(K[\mathbf{T}]/I)$, where $\mathbf{T} = (T_0, \dots, T_r)$, I is a homogeneous prime ideal of the graded ring $K[\mathbf{T}]$ that does not contain every T_i, and $\tilde{K}[\mathbf{T}]/\tilde{K}I$ is an integral domain. For each extension L of K, we use the classical notation and identify $W(L) = \mathrm{Mor}_K(\mathrm{Spec}(L), W)$ with the set of all equivalence classes $\mathbf{a} = (a_0{:}\cdots{:}a_r)$ of $(r+1)$-tuples of elements of L with respect to multiplication by an element of $L^\times$ such that there exists $0 \le j \le r$ with $a_j \ne 0$ and $(a_0, \dots, a_r)$ is a zero of I. In this case $K(\mathbf{a}) = K\left(\frac{a_0}{a_j}, \dots, \frac{a_r}{a_j}\right)$ is the **residue field** of $\mathbf{a}$.

In particular, a point $\mathbf{t} = (t_0{:}\cdots{:}t_r)$ of $W(\mho)$ is **generic** if the map $(T_0,\dots,T_r) \mapsto (t_0,\dots,t_r)$ induces a K-isomorphism $K[T_0,\dots,T_r]/I \to K[t_0,\dots,t_r]$. Equivalently, for each $\mathbf{a} \in W(\mho)$ the map $(t_0,\dots,t_r) \mapsto (a_0,\dots,a_r)$ uniquely extends to a K-homomorphism $K[t_0,\dots,t_r] \to K[a_0,\dots,a_r]$. In this case $F = K(\mathbf{t})$ is the **function field** of W. This notation is independent of the representative $(t_0,\dots,t_r)$ of $\mathbf{t}$. However, $K[t_0,\dots,t_r]$ does depend on that representative of $\mathbf{t}$. Nevertheless, we abuse our notation and abbreviate $K[t_0,\dots,t_r]$ by $K[\mathbf{t}]$ whenever $t_0,\dots,t_r$ are given.

The points of W are the homogeneous prime ideals of $K[\mathbf{t}]$ that do not contain $K[\mathbf{t}]_+$, i.e. do not contain the set $\{t_0,\dots,t_r\}$. If $P \in W$, then $K(P) = \mathcal{O}_{W,P}/\mathfrak{m}_{W,P}$ is the **residue field** of P. In particular, if $K(P) = K$, then P is a K**-rational** point of K that corresponds to a point $\mathbf{a} \in W(K)$ such that the map $\mathbf{t} \to \mathbf{a}$ defines a K-isomorphism $K[\mathbf{t}]/P \cong K$.

For a field extension L of K, a point Q of $W_L = W \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ **lies over** P (equivalently, over $\mathbf{a}$) if $Q \cap K[\mathbf{t}] = P$.

4.4 Local Fields. We denote the set of all primes of K by $\mathbf{P}_K$. For each $\mathfrak{p} \in \mathbf{P}_K$ we fix a completion $\hat{K}_\mathfrak{p}$ of K at $\mathfrak{p}$ in $\mho$ and an absolute $\mathfrak{p}$-adic value $|\ |_\mathfrak{p}$ of $\hat{K}_\mathfrak{p}$. Then, we extend $|\ |_\mathfrak{p}$ to $\widetilde{\hat{K}_\mathfrak{p}}$ in the unique possible way. In particular, $|\ |_\mathfrak{p}$ is now also defined on $\tilde{K}$.

Let V be an absolutely integral affine variety in $\mathbb{A}_K^n$ (Subsection 4.2). The $\mathfrak{p}$-adic topology on $\widetilde{\hat{K}_\mathfrak{p}}$ defines a $\mathfrak{p}$-adic topology on $V(\widetilde{\hat{K}_\mathfrak{p}})$ (Subsection 3.1). For each extension L of K in $\widetilde{\hat{K}_\mathfrak{p}}$ we refer to a $\mathfrak{p}$-adically open (resp. closed) subsets of $V(L)$ as $\mathfrak{p}$**-open** (resp. $\mathfrak{p}$**-closed**). Each $\mathfrak{p}$-open subset Ω of $V(L)$ is a union of open $\mathfrak{p}$-balls defined by parameters from L. If L' is an extension of L in $\widetilde{\hat{K}_\mathfrak{p}}$, then the same parameters define open $\mathfrak{p}$-balls in $V(L')$. Their union is a $\mathfrak{p}$-open subset of $V(L')$ that we denote by $\Omega(L')$. Note that a change in the parameters that define Ω does not effect the set $\Omega(L')$. In particular, $\Omega(L') \cap V(L) = \Omega(L)$.

Next we consider the field $K_\mathfrak{p} = K_{\mathrm{sep}} \cap \hat{K}_\mathfrak{p}$ and call it a $\mathfrak{p}$**-closure** of K at $\mathfrak{p}$. It is a Henselian closure of K at $\mathfrak{p}$ if $\mathfrak{p} \in \mathbf{P}_K$ is non-archimedean, a real closure of K if $\mathfrak{p}$ is archimedean and real, and $\tilde{K}$ if $\mathfrak{p}$ is archimedean and complex.

If K is a number field, then $\mathrm{char}(K) = 0$, so $K_{\mathrm{sep}} = \tilde{K}$, hence $K_\mathfrak{p} = \tilde{K} \cap \hat{K}_\mathfrak{p}$. If K is a function field of one variable over a finite field, then $\hat{K}_\mathfrak{p}$ is a regular extension of $K_\mathfrak{p}$ [Jar94, Lemma 2.2], in particular $K_\mathfrak{p} = \tilde{K} \cap \hat{K}_\mathfrak{p}$. Thus, the latter relation holds in both cases.

4.5 Holomorphy domains. For each $\mathfrak{p} \in \mathbf{P}_K$ and a subfield M of $\widetilde{\hat{K}_\mathfrak{p}}$ we consider the closed disc

$$\mathcal{O}_{M,\mathfrak{p}} = \{x \in M \mid\ |x|_\mathfrak{p} \le 1\}$$

of M at $\mathfrak{p}$. We omit $\mathfrak{p}$ from $\mathcal{O}_{M,\mathfrak{p}}$ if $\hat{K}_\mathfrak{p} \subseteq M \subseteq \widetilde{\hat{K}_\mathfrak{p}}$. If $\mathfrak{p}$ is non-archimedean, then $\mathcal{O}_{M,\mathfrak{p}}$ is a valuation ring of rank 1 of M.

Next we consider a subset $\mathcal{U}$ of $\mathbf{P}_K$ and a field $K \subseteq M \subseteq \tilde{K}$. Let $\mathcal{U}_M$ be the set of all primes of M that lie over $\mathcal{U}$. If $\mathfrak{q} \in \mathcal{U}_M$ lies over $\mathfrak{p} \in \mathcal{U}$, then we denote the unique absolute value of M that extends $|\ |_\mathfrak{p}$ to M and represents $\mathfrak{q}$ by $|\ |_\mathfrak{q}$. In this case there exists $\tau \in \mathrm{Gal}(K)$ such that $|x|_\mathfrak{q} = |x^\tau|_\mathfrak{p}$ for each $x \in M$. Conversely, the latter condition defines $\mathfrak{q}$. We set

$$\mathcal{O}_{M,\mathcal{U}} = \bigcap_{\mathfrak{q} \in \mathcal{U}_M} \{x \in M \mid\ |x|_\mathfrak{q} \le 1\}$$

for the $\mathcal{U}$**-holomorphy domain** of M *. If $\mathcal{U}$ consists of non-archimedean primes, then $\mathcal{O}_{M,\mathcal{U}}$ is the integral closure of $\mathcal{O}_{K,\mathcal{U}}$ in M [Lan58, p. 12, Prop. 4]. If $\mathcal{U}$ is arbitrary but M is Galois over K, then

$$\mathcal{O}_{M,\mathcal{U}} = \bigcap_{\mathfrak{p} \in \mathcal{U}} \bigcap_{\tau \in \mathrm{Gal}(K)} \mathcal{O}_{M,\mathfrak{p}}^\tau .$$

Note that
(1) if $\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathbf{P}_K$, then $\mathcal{O}_{M,\mathcal{U}'} \subseteq \mathcal{O}_{M,\mathcal{U}}$.

* Note that in general $\mathcal{O}_{M,\{\mathfrak{p}\}} \ne \mathcal{O}_{M,\mathfrak{p}}$.

4.6 Basic Objects. In the number field case (i.e. $\mathrm{char}(K) = 0$), we denote the set of all non-archimedean primes of K by $\mathbf{P}_{K,\mathrm{fin}}$. In the function field case, where $p = \mathrm{char}(K) > 0$, we fix a separating transcendence element t_K for $K/\mathbb{F}_p$ and let $\mathbf{P}_{K,\mathrm{fin}} = \{\mathfrak{p} \in \mathbf{P}_K \mid |t_K|_\mathfrak{p} \le 1\}$. In both cases $\mathbf{P}_{K,\mathrm{fin}}$ is cofinite in $\mathbf{P}_K$ and we set

$$\mathcal{O}_K = \mathcal{O}_{K,\mathbf{P}_{K,\mathrm{fin}}} = \{x \in K \mid |x|_\mathfrak{p} \le 1 \text{ for all } \mathfrak{p} \in \mathbf{P}_{K,\mathrm{fin}}\}.$$

If K is a number field, then $\mathcal{O}_K$ is the integral closure of $\mathbb{Z}$ in K. In the function field case $\mathcal{O}_K$ is the integral closure of $\mathbb{F}_p[t_K]$ in K. In both cases $\mathcal{O}_K$ is a Dedekind domain [CaF67, p. 13, Prop. 1]. Following the convention in algebraic number theory, we call $\mathcal{O}_K$ the **ring of integers** of K.

Next we choose a finite (possibly empty) subset $\mathcal{S}$ of $\mathbf{P}_K$, set

$$K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p}\in\mathcal{S}} \bigcap_{\tau\in\mathrm{Gal}(K)} K_\mathfrak{p}^\tau,$$

and observe that $K_{\mathrm{tot},\mathcal{S}}$ is the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally splits.

We also choose a non-empty proper subset $\mathcal{V}$ of $\mathbf{P}_K$ that contains $\mathcal{S}$.

4.7 Strong Approximation. Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ that contains $\mathcal{S}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. Thus, by (1), $\mathcal{O}_K \subseteq \mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}}$.

Given an absolutely integral affine variety V in $\mathbb{A}_K^n$ for some positive integer n, we consider for each $\mathfrak{p} \in \mathcal{T}$

(3a) a finite Galois extension $L_\mathfrak{p}$ of $K_\mathfrak{p}$, such that $L_\mathfrak{p} = K_\mathfrak{p}$ if $\mathfrak{p} \in \mathcal{S}$, and

(3b) a non-empty $\mathfrak{p}$-open subset $\Omega_\mathfrak{p}$ of $V_{\mathrm{simp}}(L_\mathfrak{p})$, invariant under the action of $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$.

Assuming that

(3c) $V(\mathcal{O}_{\tilde{K},\mathfrak{p}}) \ne \emptyset$, equivalently that $V(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}}) \ne \emptyset$ [GeJ75, Lemma 2.4], for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{T}$,

we say that $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ is **approximation data** for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, V$.

Given a field $K \subseteq M \subseteq K_{\mathrm{tot},\mathcal{S}}$, we write $(M, K, \mathcal{S}, \mathcal{V}, V, \mathcal{T}, (L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}) \models \mathrm{SAT}$ if

(4) there exists $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau \in \Omega_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}$ and all $\tau \in \mathrm{Gal}(K)$.

We write $(M, K, \mathcal{S}, \mathcal{V}, V) \models \mathrm{SAT}$ if

(5) $(M, K, \mathcal{S}, \mathcal{V}, V, \mathcal{T}, (L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}) \models \mathrm{SAT}$ for all finite subsets $\mathcal{T}$ of $\mathcal{V}$ that contain $\mathcal{S}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$ and for all approximation data $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, V$.

Finally, we write $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$ and say that M satisfies the **strong approximation theorem** for $K, \mathcal{S}, \mathcal{V}$ if

(6) $(M, K, \mathcal{S}, \mathcal{V}, V) \models \mathrm{SAT}$ for every absolutely integral affine variety V in $\mathbb{A}_K^n$ for some positive integer n.

Note that all $\mathfrak{p}$-closures of K at a given $\mathfrak{p} \in \mathbf{P}_K$ are K-isomorphic. Hence, Conditions (3), (4), (5), and (6) are independent of the choices of the closures.

4.8 Fixing K, $\mathcal{S}$, and $\mathcal{V}$. For the rest of the work we fix the global field K, the proper subset $\mathcal{V}$ of $\mathbf{P}_K$, and the finite subset $\mathcal{S}$ of $\mathcal{V}$, as in Subsection 4.6. Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ that contains $\mathcal{S}$ and satisfies $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. Let V be an absolutely integral affine variety over K in $\mathbb{A}_K^n$ for some positive integer n and let $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ be approximation data for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, V$.

Remark 4.9: Condition (3) can be reformulated in terms of completions instead of closures at primes of K. Indeed, suppose that for each $\mathfrak{p} \in \mathcal{T}$ we are given a finite Galois extension $\hat{L}_\mathfrak{p}$ of $\hat{K}_\mathfrak{p}$, such that $\hat{L}_\mathfrak{p} = \hat{K}_\mathfrak{p}$ if $\mathfrak{p} \in \mathcal{S}$, and a non-empty $\mathfrak{p}$-open subset $\hat{\Omega}_\mathfrak{p}$ of $V_{\mathrm{simp}}(\hat{L}_\mathfrak{p})$, invariant under the action of $\mathrm{Gal}(\hat{L}_\mathfrak{p}/\hat{K}_\mathfrak{p})$. Then, with $L_\mathfrak{p} = \hat{L}_\mathfrak{p} \cap K_{\mathrm{sep}}$, the $\mathfrak{p}$-open subset $\Omega_\mathfrak{p} = \hat{\Omega}_\mathfrak{p} \cap V(L_\mathfrak{p})$ of $V(L_\mathfrak{p})$ is non-empty.

Indeed, if $\mathfrak{p} \in \mathbf{P}_{K,\mathrm{fin}}$, then by [JaR98, Remark 1.6], $V(L_\mathfrak{p})$ is $\mathfrak{p}$-dense in $V(\hat{L}_\mathfrak{p})$. If $\mathfrak{p} \in \mathbf{P}_{K,\mathrm{inf}}$ is real, then $L_\mathfrak{p}$ and $\hat{L}_\mathfrak{p}$ are real closed, so $\hat{L}_\mathfrak{p}$ is an elementary extension of $L_\mathfrak{p}$ as ordered fields [Pre84, p. 51, Cor. 5.2]. In particular, $V(L_\mathfrak{p})$ is $\mathfrak{p}$-dense in $V(\hat{L}_\mathfrak{p})$. Finally, if $\mathfrak{p} \in \mathbf{P}_{K,\mathrm{inf}}$ is complex, then $L_\mathfrak{p} = \tilde{\mathbb{Q}}$, $\hat{L}_\mathfrak{p}$ is isomorphic to $\mathbb{C}$ and there exists a real closed field $L_{\mathfrak{p},0}$ such that $L_\mathfrak{p} = L_{\mathfrak{p},0}(\sqrt{-1})$, and the pair $(\hat{L}_\mathfrak{p}, \hat{L}_{\mathfrak{p},0})$, with $\hat{L}_{\mathfrak{p},0}$ being the $\mathfrak{p}$-closure of $L_{\mathfrak{p},0}$ in $\hat{L}_\mathfrak{p}$, is isomorphic to $(\mathbb{C}, \mathbb{R})$. The $\mathfrak{p}$-density of $V(L_\mathfrak{p})$ in $V(\hat{L}_\mathfrak{p})$ follows in this case from the fact that $\hat{L}_{\mathfrak{p},0}$ is an elementary extension of $L_{\mathfrak{p},0}$ as ordered fields.

Now we choose $\hat{\mathbf{z}} \in \hat{\Omega}_\mathfrak{p}$ and $\varepsilon > 0$ such that $\{\mathbf{z} \in V(\hat{L}_\mathfrak{p}) \mid |\mathbf{z} - \hat{\mathbf{z}}|_\mathfrak{p} < \varepsilon\} \subseteq \hat{\Omega}_\mathfrak{p}$. Since $L_\mathfrak{p}$ is $\mathfrak{p}$-dense in $\hat{L}_\mathfrak{p}$, there exists $\mathbf{a} \in L_\mathfrak{p}^n$ that satisfies $|\mathbf{a} - \hat{\mathbf{z}}|_\mathfrak{p} < \frac{\varepsilon}{2}$. Since $\hat{L}_\mathfrak{p}$ is an elementary extension of $L_\mathfrak{p}$ as ordered fields, there exists $\mathbf{z} \in V(L_\mathfrak{p})$ such that $|\mathbf{z} - \mathbf{a}|_\mathfrak{p} < \frac{\varepsilon}{2}$. Then, $|\mathbf{z} - \hat{\mathbf{z}}|_\mathfrak{p} < \varepsilon$, so $\mathbf{z} \in \hat{\Omega}_\mathfrak{p} \cap V(L_\mathfrak{p})$, as desired.

Conversely, given $L_\mathfrak{p}$ and $\Omega_\mathfrak{p}$ as in (3b), we may consider $\hat{L}_\mathfrak{p} = \hat{K}_\mathfrak{p} L_\mathfrak{p}$ and let $\hat{\Omega}_\mathfrak{p} = \Omega_\mathfrak{p}(\hat{L}_\mathfrak{p})$. Then, $\hat{\Omega}_\mathfrak{p}$ is a non-empty $\mathfrak{p}$-open subset of $V_{\mathrm{simp}}(\hat{L}_\mathfrak{p})$.

By Abraham Robinson, the theory of algebraically closed valued fields (with nontrivial valuation) is model complete [Pre86, p. 240, Kor. 4.18]. Hence, we could have replaced Condition (3c) by the condition: $V(\mathcal{O}_{\widetilde{\hat{K}_\mathfrak{p}}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{T}$. ∎

In proving the strong approximation theorem for $K, \mathcal{S}, \mathcal{V}$, we may choose $\mathcal{T}, V, (L_\mathfrak{p}, \Omega_\mathfrak{p})$ with some extra properties. This is proved in the following lemma.

LEMMA 4.10: *Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ that contains $\mathcal{S}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$, V an absolutely integral affine variety in $\mathbb{A}_K^n$ for some positive integer n, and $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ approximation data for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, V$. We consider a field extension M of K in $K_{\mathrm{tot},\mathcal{S}}$. Then, in order to prove that*

$$(M, K, \mathcal{S}, \mathcal{V}, V, \mathcal{T}, (L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}) \models \mathrm{SAT},$$

we may

(a) *replace $\Omega_\mathfrak{p}$, for each $\mathfrak{p} \in \mathcal{T}$, by $\Omega_\mathfrak{p} \cap U(L_\mathfrak{p})$, where U is a given non-empty Zariski-open affine subset of V defined by polynomial inequalities with coefficients in K,*
(b) *replace $\mathcal{T}$ by any larger finite subset $\mathcal{T}'$ of $\mathcal{V}$ and extend $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ to any approximation data $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}'}$ for $K, \mathcal{S}, \mathcal{T}', \mathcal{V}, V$,*
(c) *replace V by any absolutely integral affine variety V' in $\mathbb{A}_K^{n'}$, for some positive integer n', which is birationally equivalent to V, and*
(d) *replace V by any non-empty Zariski-open affine subvariety V_0 of V defined by polynomial inequalities with coefficients in K, considered as an affine variety in $\mathbb{A}_K^{n+1}$; in other words, if $V = \mathrm{Spec}(B)$ is an affine variety over K, replace V by the Zariski-open subset $D(f) = \{\mathfrak{p} \in B \mid f \notin \mathfrak{p}\}$, for some non-zero $f \in B$, and identify $D(f)$ with $\mathrm{Spec}(B[f^{-1}])$.*

Proof of (a): Since $U(L_\mathfrak{p})$ is $\mathfrak{p}$-open in $V(L_\mathfrak{p})$ (Statement (a) of Subsection 3.1), $\Omega'_\mathfrak{p} = \Omega_\mathfrak{p} \cap U(L_\mathfrak{p})$ is also $\mathfrak{p}$-open in $V(L_\mathfrak{p})$. Since $\Omega_\mathfrak{p}$ contains a simple point of V (by (3b)), $\Omega_\mathfrak{p}$ is Zariski-dense in V (Statement (a) of Subsection 3.1), hence $\Omega'_\mathfrak{p} \neq \emptyset$. Moreover, since $\Omega_\mathfrak{p}$ is invariant under $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$, so is $\Omega'_\mathfrak{p}$. Finally, if $\mathbf{z} \in U(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ and $\mathbf{z}^\tau \in \Omega'_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}$ and $\tau \in \mathrm{Gal}(K)$, then $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ and $\mathbf{z}^\tau \in \Omega_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}$ and $\tau \in \mathrm{Gal}(K)$, as desired.

Proof of (b): Consider $\mathfrak{p} \in \mathcal{T}' \smallsetminus \mathcal{T}$. By assumption, $\mathfrak{p}$ is finite. By (3c), $V(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}}) \neq \emptyset$. Since V_{simp} is non-empty and Zariski-open in V and $V(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}})$ is $\mathfrak{p}$-open in $V(K_{\mathrm{sep}})$, we have by Subsection 3.1(c), that $V_{\mathrm{simp}}(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}}) \neq \emptyset$. Hence, we may choose a finite Galois extension $L_\mathfrak{p}$ of $K_\mathfrak{p}$ such that $\Omega_\mathfrak{p} = V_{\mathrm{simp}}(\mathcal{O}_{L_\mathfrak{p},\mathfrak{p}}) \neq \emptyset$. Since V_{simp} is Zariski-open in V and $V(\mathcal{O}_{L_\mathfrak{p},\mathfrak{p}})$ is $\mathfrak{p}$-open in $V(L_\mathfrak{p})$, the set $\Omega_\mathfrak{p}$ is $\mathfrak{p}$-open in $V(L_\mathfrak{p})$ (Subsection 3.1(a)). Since V_{simp} is defined over K, the set $\Omega_\mathfrak{p}$ is invariant under the action of $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$.

Thus, $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}'}$ is approximation data for $K, \mathcal{S}, \mathcal{T}', \mathcal{V}, V$. If $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}'})$ and $\mathbf{z}^\tau \in \Omega_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}'$ and $\tau \in \mathrm{Gal}(K)$, then $\mathbf{z}^\tau \in \Omega_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}$ and $\tau \in \mathrm{Gal}(K)$, and $\mathbf{z}^\tau \in \Omega_\mathfrak{p} \subseteq V(\mathcal{O}_{L_\mathfrak{p},\mathfrak{p}})$ for all $\mathfrak{p} \in \mathcal{T}' \smallsetminus \mathcal{T}$ and $\tau \in \mathrm{Gal}(K)$. It follows that $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$, as desired.

Proof of (c): Since V and V' are birationally equivalent over K, there exists a K-isomorphism φ of a non-empty Zariski-open affine subset V_0 of V onto a non-empty Zariski-open affine subset V'_0 of V'. Both V_0 and V'_0 are absolutely integral affine varieties over K. Hence, φ corresponds to an isomorphism from the coordinate ring of V'_0 onto the coordinate ring of V_0 [Liu06, p. 48, Lemma 2.3.23]. Thus, both φ and φ^{-1} are defined by polynomials with coefficients in K. We choose a finite subset $\mathcal{T}'$ of $\mathcal{V}$ that contains $\mathcal{T}$ such that all of those coefficients belong to $\mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}'}$.

Next we choose $\mathbf{z}_0 \in V'(\tilde{K})$ and extend $\mathcal{T}'$ within $\mathcal{V}$ to assume that $\mathbf{z}_0 \in V'(\mathcal{O}_{\tilde{K},\mathcal{V}\smallsetminus\mathcal{T}'})$. By (3b), for each $\mathfrak{p} \in \mathcal{T}$, $\Omega_\mathfrak{p}$ is a non-empty $\mathfrak{p}$-open subset of $V_{\mathrm{simp}}(L_\mathfrak{p})$ which is invariant under $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$. Hence, by Subsection 3.1(a), $\Omega_\mathfrak{p} \cap V_{0,\mathrm{simp}}(L_\mathfrak{p})$ is a non-empty $\mathfrak{p}$-open subset of $V_{0,\mathrm{simp}}(L_\mathfrak{p})$ which is invariant under $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$. Moreover, φ maps $V_{0,\mathrm{simp}}(L_\mathfrak{p})$ $\mathfrak{p}$-homeomorphically onto $V'_{0,\mathrm{simp}}(L_\mathfrak{p})$, so $\Omega'_\mathfrak{p} = \varphi(\Omega_\mathfrak{p} \cap V_{0,\mathrm{simp}}(L_\mathfrak{p}))$ is a non-empty $\mathfrak{p}$-open subset of $V'_{0,\mathrm{simp}}(L_\mathfrak{p})$, hence also of $V'_{\mathrm{simp}}(L_\mathfrak{p})$, which is invariant under $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$.

By Condition (3c), for each $\mathfrak{p} \in \mathcal{T}' \smallsetminus \mathcal{T}$, $V(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}}) \neq \emptyset$. By Subsection 3.1(c), there exists $\mathbf{z}_\mathfrak{p} \in V_{0,\mathrm{simp}}(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}})$. Let $L_\mathfrak{p}$ be a finite Galois extension of $K_\mathfrak{p}$ with $\mathbf{z}_\mathfrak{p} \in V_{0,\mathrm{simp}}(\mathcal{O}_{L_\mathfrak{p},\mathfrak{p}})$. Then, $\Omega'_\mathfrak{p} = \varphi(V_{0,\mathrm{simp}}(\mathcal{O}_{L_\mathfrak{p},\mathfrak{p}}))$ is a non-empty $\mathfrak{p}$-open subset of $V'_{0,\mathrm{simp}}(L_\mathfrak{p})$, hence also of $V'_{\mathrm{simp}}(L_\mathfrak{p})$, which is invariant under the action of $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$. Thus, $(L_\mathfrak{p}, \Omega'_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}'}$ is approximation data for $K, \mathcal{S}, \mathcal{T}', \mathcal{V}, V'$.

We assume that there exists $\mathbf{z}' \in V'(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}'})$ such that $(\mathbf{z}')^\tau \in \Omega'_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}'$ and $\tau \in \mathrm{Gal}(K)$. Since $\mathcal{T}'$ is non-empty and $\Omega'_\mathfrak{p} \subseteq V'_{0,\mathrm{simp}}(L_\mathfrak{p})$ for $\mathfrak{p} \in \mathcal{T}'$, we have $\mathbf{z}' \in V'_0(\tilde{K})$. Moreover, since the coordinates of $\mathbf{z}'$ belong to $\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}'}$, we have $\mathbf{z}' \in V'_0(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}'})$. By the choice of $\mathcal{T}'$,

$$\mathbf{z} = \varphi^{-1}(\mathbf{z}') \in V_0(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}'}) \cap \bigcap_{\mathfrak{p}\in\mathcal{T}'\smallsetminus\mathcal{T}} \bigcap_{\tau\in\mathrm{Gal}(K)} V_{0,\mathrm{simp}}(\mathcal{O}^\tau_{L_\mathfrak{p},\mathfrak{p}}) \cap \bigcap_{\mathfrak{p}\in\mathcal{T}} \bigcap_{\tau\in\mathrm{Gal}(K)} \Omega^\tau_\mathfrak{p}.$$

Hence, $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ and $\mathbf{z}^\tau \in \Omega_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}$ and $\tau \in \mathrm{Gal}(K)$, as desired.

Proof of (d): V_0 is birationally equivalent over K to V, so we may use rule (c). ∎

Example 4.11: Units. Let c be a non-zero element of K_{sep}, let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ that contains $\mathcal{S}$ such that $\mathcal{V}\smallsetminus\mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$, and let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$. Consider the finite subset $\mathcal{T}' = \mathcal{T} \cup \{\mathfrak{p} \in \mathcal{V} \mid |c^\tau|_\mathfrak{p} \neq 1$ for at least one $\tau \in \mathrm{Gal}(K)\}$ of $\mathcal{V}$. Thus, $|c^\tau|_\mathfrak{p} = 1$ for all $\mathfrak{p} \in \mathcal{V}\smallsetminus\mathcal{T}'$ and all $\tau \in \mathrm{Gal}(K)$. Hence, c is a unit of $\mathcal{O}_{K(c),\mathcal{V}\smallsetminus\mathcal{T}'}$. It follows from Lemma 4.10 that in order to prove that $(M, K, \mathcal{S}, \mathcal{V}, V, \mathcal{T}, (L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}) \models \mathrm{SAT}$ for a given absolutely integral affine variety V in $\mathbb{A}^n_K$ for some positive integer n and approximation data $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, V$, we may assume that c is a unit of $\mathcal{O}_{K(c),\mathcal{V}\smallsetminus\mathcal{T}}$. ∎

We apply Lemma 4.10 to reduce the strong approximation theorem to the case of curves.

LEMMA 4.12: *Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$. Suppose $(M, K, \mathcal{S}, \mathcal{V}, C) \models \mathrm{SAT}$ for every positive integer m and every absolutely integral affine curve C in $\mathbb{A}^m_K$. Then, $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$.*

Proof: Let V be an absolutely integral affine variety in $\mathbb{A}^n_K$ for some positive integer n. Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ that contains $\mathcal{S}$ such that $\mathcal{V}\smallsetminus\mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. Let $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ be approximation data for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, V$. We choose a finite separable extension K' of K and a point $\mathbf{z}_0 \in V(K')$. Then, we choose a finite subset $\mathcal{T}'$ of $\mathcal{V}$ that contains $\mathcal{T}$ such that $\mathbf{z}_0 \in V(\mathcal{O}_{K',\mathcal{V}\smallsetminus\mathcal{T}'})$, hence also $\mathbf{z}_0 \in V(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}})$, for each $\mathfrak{p} \in \mathcal{V}\smallsetminus\mathcal{T}'$. By Lemma 4.10, we may replace $\mathcal{T}$ by $\mathcal{T}'$ to assume that $\mathbf{z}_0 \in V(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{V}\smallsetminus\mathcal{T}$.

Now we choose for each $\mathfrak{p} \in \mathcal{T}$ a point $\mathbf{z}_\mathfrak{p} \in \Omega_\mathfrak{p} \subseteq V(L_\mathfrak{p})$. Then we apply [JaR98, Lemma 10.1] to find an absolutely integral affine curve C on V over K that goes through $\mathbf{z}_0$ and $\mathbf{z}_\mathfrak{p}$ for every $\mathfrak{p} \in \mathcal{T}$. Moreover, since by (3b) each of the points $\mathbf{z}_\mathfrak{p}$ with $\mathfrak{p} \in \mathcal{T}$ is simple on V, that lemma allows us to choose C such that each of those $\mathbf{z}_\mathfrak{p}$ is also simple on C. Thus, $\mathbf{z}_0 \in C(\mathcal{O}_{K_{\mathrm{sep}},\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{V}\smallsetminus\mathcal{T}$ and $\mathbf{z}_\mathfrak{p} \in \Omega_\mathfrak{p} \cap C_{\mathrm{simp}}(L_\mathfrak{p}) \subseteq C_{\mathrm{simp}}(L_\mathfrak{p})$ for each $\mathfrak{p} \in \mathcal{T}$.

It follows that $(L_\mathfrak{p}, \Omega_\mathfrak{p} \cap C_{\mathrm{simp}}(L_\mathfrak{p}))_{\mathfrak{p}\in\mathcal{T}}$ is approximation data for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, C$. By assumption, there exists $\mathbf{z} \in C(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau \in \Omega_\mathfrak{p} \cap C_{\mathrm{simp}}(L_\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{T}$ and $\tau \in \mathrm{Gal}(K)$. Therefore, $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ and $\mathbf{z}^\tau \in \Omega_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}$ and $\tau \in \mathrm{Gal}(K)$. We conclude that $(M, K, \mathcal{S}, \mathcal{V}, V) \models \mathrm{SAT}$. It follows that $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$, as claimed. ∎

5. Curves

Following Lemma 4.12, we now concentrate on curves. We extend a given affine curve C over K to an affine curve X over a subring R of K and complete X to an integral projective curve $\bar{X}$ over R. We apply Lemma 4.10 several times to make convenient assumptions on the associated data. These assumptions are used in the sequel to prove the strong approximation theorem.

5.1 An Affine Curve. Let $K, \mathbf{P}_K, K_\mathfrak{p}, \hat{K}_\mathfrak{p}, \mho, \mathcal{S}, \mathcal{T}, \mathcal{V}, M, \mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}}$ be as in Section 4. In particular, $\mathcal{V}\smallsetminus\mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. Let C be an absolutely integral affine curve in $\mathbb{A}^n_K$. We choose a generic point $\mathbf{x} = (x_1,\ldots,x_n)$ for C over K with $x_1,\ldots,x_n \in \mho$ (Subsection 4.2). Moreover, enlarging $\mho$ if necessary, we choose $x_1,\ldots,x_n$ such that $\mathrm{trans.deg}(K(\mathbf{x})/K) = \mathrm{trans.deg}(\hat{K}_\mathfrak{p}(\mathbf{x})/\hat{K}_\mathfrak{p})$ for each $\mathfrak{p} \in \mathbf{P}_K$. Then, $K(\mathbf{x})/K$ is a regular extension of transcendence degree 1, $F = K(\mathbf{x})$ is the function field of C over K. Moreover, for each $\mathfrak{p} \in \mathbf{P}_K$, the field F is linearly disjoint from $\hat{K}_\mathfrak{p}$ over K, so $\hat{K}_\mathfrak{p}(\mathbf{x})/\hat{K}_\mathfrak{p}$ is also a regular extension [FrJ08, Lemma 2.6.7].

We apply Lemma 4.10 to replace C by a Zariski-open subset of simple points and assume that
(1) C is smooth.

For each $\mathfrak{p} \in \mathcal{T}$ let $L_\mathfrak{p}$ be a finite Galois extension of $K_\mathfrak{p}$ such that $L_\mathfrak{p} = K_\mathfrak{p}$ if $\mathfrak{p} \in \mathcal{S}$. Then, let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $C(L_\mathfrak{p})$, invariant under the action of $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$. We also assume that
(2) $C(\mathcal{O}_{\tilde{K},\mathfrak{p}}) \neq \emptyset$ for each $\mathfrak{p} \in \mathcal{V}\smallsetminus\mathcal{T}$.
Thus, $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ is approximation data for $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, C$.

5.2 Principal Ideal Domain. Recall that the class group of the ring of integers $\mathcal{O}_K = \mathcal{O}_{K,\mathbf{P}_{\mathrm{fin}}}$ of K is finite (see [CaF67, p. 71] for the number field case and [Ros02, p. 243, Prop. 14.2] for the function field case). Let $\mathfrak{a}_1,\ldots,\mathfrak{a}_h$ be ideals of $\mathcal{O}_K$ that represent the group of fractional ideals of $\mathcal{O}_K$ modulo principal fractional ideals. Denote the union of $\mathcal{T}$ with the set of all prime divisors of $\mathfrak{a}_1,\ldots,\mathfrak{a}_h$ that belong to $\mathcal{V}$ by $\mathcal{T}'$. Then, $\mathfrak{a}_i\mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}'} = \mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}'}$ for $i = 1,\ldots,h$. Each ideal $\mathfrak{a}$ of $\mathcal{O}_K$ can be represented as $\mathfrak{a} = b\cdot\mathfrak{a}_i$ for some i between 1 and h and $b \in K^\times$, so $\mathfrak{a}\mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}'} = b\cdot\mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}'}$. Thus, $\mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}'}$ is a principal ideal domain (see also [IsR05, p. 211, Prop. 8.9.7]).

Using Lemma 4.10, we replace $\mathcal{T}$ by $\mathcal{T}'$, if necessary, to assume
(3) $R = \mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}}$ is a principal ideal domain. In particular, R is integrally closed, hence a Dedekind domain. Therefore, $R_\mathfrak{p}$ is a regular local ring for each $\mathfrak{p} \in \mathrm{Spec}(R)$.
Note that whenever we replace $\mathcal{T}$ by a larger finite subset $\mathcal{T}'$ of $\mathcal{V}$, we also replace R by its quotient ring $R' = \mathcal{O}_{K,\mathcal{V}\smallsetminus\mathcal{T}'}$, which is still a principal ideal domain.

In the case where $\mathcal{V} = \mathcal{T}$, the ring R is an intersection of an empty set of local subrings of K, so $R = K$. In this case our results overlap with those of [GeJ02].

5.3 Nagata Rings. A Noetherian ring A (commutative with 1) is called a **Nagata ring** if for every prime ideal P of A and every finite extension L of $\mathrm{Quot}(A/P)$ the integral closure of A/P in L is a finitely generated A/P-module [Mat80, p. 231]. In particular, every field is a Nagata ring. The main theorem in this area, due to Nagata, says that each finitely generated ring extension of a Nagata ring is again a Nagata ring [Mat80, p. 240, Thm. 72].

Lemma 5.4:
(a) *Every Dedekind ring A of characteristic 0 is a Nagata ring.*
(b) *Suppose that A is a Dedekind ring and a Nagata ring. Then, every subring B of $\mathrm{Quot}(A)$ that contains A is also a Dedekind ring and a Nagata ring.*
(c) *R is a Nagata ring.*

Proof of (a): See [Liu06, p. 340, Example 8.2.28(b)].

Proof of (b) (Moret-Bailly): That B is a Dedekind ring is a classical theorem of Noether-Grell [FrJ08, p. 32, Prop. 2.4.7]. We prove that B is also a Nagata ring.

Consider a prime ideal $\mathfrak{q}$ of B. If $\mathfrak{q}$ is maximal, then $B/\mathfrak{q}$ is a field. Hence, if F is a finite extension of $B/\mathfrak{q}$, then F is the integral closure of $B/\mathfrak{q}$ in F and F is a finitely generated $B/\mathfrak{q}$-module.

Otherwise, $\mathfrak{q} = 0$ (because B is a Dedekind ring). Let L be a finite extension of $\mathrm{Quot}(A)$ and consider the integral closures A_L and B_L of A and B, respectively, in L.

We consider a maximal ideal Q of B and set $P = A\cap Q$. Since $\mathrm{Quot}(A)$ is the quotient field of both A and B, we have $P \neq 0$. Hence, A_P is a proper subring of $\mathrm{Quot}(A)$. Moreover, $A_P \subseteq B_Q \subset \mathrm{Quot}(A)$. Since A is Dedekind, A_P is a discrete valuation ring. Hence, $A_P = B_Q$ [FrJ08, p. 23, Lemma 2.2.5].

Next let $A_{L,P}$ be the localization of the A-module A_L at P and let $B_{L,Q}$ be the localization of the B-module B_L at Q. Since as a ring, $B_{L,Q}$ is integral over $B_Q = A_P$ and $A_{L,P}$ is the integral closure of A_P in L, we have $B_{L,Q} \subseteq A_{L,P}$. Hence, $B_{L,Q} \subseteq A_{L,P} = A_LA_P \subseteq (A_LB)_Q \subseteq B_{L,Q}$, Thus, $B_{L,Q} = (A_L B)_Q$ for all maximal ideals Q of B. It follows from [AtM69, p. 40, Prop. 3.9] that $B_L = A_L B$. Since A is a Nagata ring, A_L is a finitely generated A-module. Hence, B_L is a finitely generated B-module. We conclude that B is a Nagata ring, as claimed.

Proof of (c): By Subsection 4.6, $\mathcal{O}_K$ is a Dedekind ring. If K is a number field, then $\mathcal{O}_K$ is also a Nagata ring, by (a). If K is a function field of one variable over a finite field of characteristic p, then by Subsection 4.6, $\mathcal{O}_K$ is an integral closure of $\mathbb{F}_p[t_K]$ in K. Since $\mathbb{F}_p$ is a Nagata ring, Nagata's theorem implies that $\mathcal{O}_K$ is a Nagata ring.

Thus, (c) is a special case of (b) for $A = \mathcal{O}_K$ and $B = R$. ∎

5.5 Affine Scheme. Using the above notation, we consider the affine integral schemes $\mathrm{Spec}(R)$ and $X = \mathrm{Spec}(R[\mathbf{x}])$, and let $f\colon X \to \mathrm{Spec}(R)$ be the structure morphism given by $f(P) = P \cap R$. Then, $\mathrm{Spec}(R)$ is a regular scheme of dimension 1 if $R \neq K$ (resp. 0, if $R = K$) and $\dim(X) = 2$ if $R \neq K$ (resp. 1 if $R = K$), because $\mathrm{trans.deg}(K(\mathbf{x})/K) = 1$. By (3), $R[\mathbf{x}]$ is a Noetherian ring, hence X is a Noetherian scheme.

By (2), for each non-zero $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists a point $\mathbf{a} \in C(\mathcal{O}_{\tilde{K},\mathfrak{p}})$, where $\mathfrak{p}$ is considered here as an element of $\mathcal{V} \smallsetminus \mathcal{T}$. That point is an R-specialization of $\mathbf{x}$. It follows that $1 \notin \mathfrak{p}R[\mathbf{x}]$. Otherwise there exist $b_i \in \mathfrak{p}$ and $h_i \in R[\mathbf{X}]$, $i = 1, \ldots, l$, such that $1 = \sum_{i=1}^{l} b_i h_i(\mathbf{x})$. Then, $1 = \sum_{i=1}^{l} b_i h_i(\mathbf{a}) \in \mathfrak{p}\mathcal{O}_{\tilde{K},\mathfrak{p}}$, a contradiction. Hence, the prime ideal $\mathfrak{p}$ of R (which is actually a maximal ideal) extends to a prime ideal of $R[\mathbf{x}]$. Since the generic point of X is mapped onto the generic point of $\mathrm{Spec}(R)$, this implies that

(4) the morphism $f\colon X \to \mathrm{Spec}(R)$ is surjective.

In fact, (4) also implies (2). But, as we don't use this implication, we do not prove it here.

By Subsection 5.1, F/K is a regular extension of transcendence degree 1. We choose a separating transcendence element $t_F \in R[\mathbf{x}]$ for F/K. Then, $R[t_F]$ is an integrally closed domain [ZaS75II, p. 85, Thm. 29(a)] and $F/K(t_F)$ is a finite separable extension. Let $z \in R[\mathbf{x}]$ be a primitive element for $F/K(t_F)$, integral over $R[t_F]$. The discriminant g of $\mathrm{irr}(z, K(t_F))$ is a non-zero element of $R[t_F]$, hence g is invertible in the ring $R[t_F, g^{-1}]$. Multiply g, if necessary, by a non-zero element of $R[t_F]$ to assume that each x_i is integral over $R[t_F, g^{-1}]$. By [FrJ08, p. 109, Lemma 6.1.2], $R[t_F, g^{-1}, z]$ is the integral closure of $R[t_F, g^{-1}]$ in F. Hence, $R[\mathbf{x}, g^{-1}] = R[t_F, g^{-1}, z]$ and the ring extension $R[\mathbf{x}, g^{-1}]/R[t_F, g^{-1}]$ is étale [Ray70, p. 18, Remarques].

By Lemma 4.10(c), we may replace C by the affine curve with the generic point $(\mathbf{x}, g^{-1})$ over K. Thus, we may assume without loss that g^{-1} is one of the coordinates of $\mathbf{x}$, hence

(5) the ring $R[\mathbf{x}] = R[t_F, g^{-1}, z]$ is integrally closed. Thus, X is normal.

Moreover, $R[\mathbf{x}]$ is étale over $R[t_F, g^{-1}]$. Since $\mathrm{Spec}(R[t_F, g^{-1}])$ is étale over $\mathrm{Spec}(R[t_F])$ [Liu06, p. 140, Prop. 4.3.22(b)] and $\mathrm{Spec}(R[t_F])$ is smooth over $\mathrm{Spec}(R)$, we conclude from [Liu06, p. 143, Prop. 4.3.38] that

(6) the morphism $f\colon X \to \mathrm{Spec}(R)$ is smooth.

Note that (5) and (6) remain true if we replace $\mathcal{T}$ by a larger finite subset of $\mathcal{V}$, because integral closedness and smoothness are preserved under a change of the base ring by a quotient ring.

For each $\mathfrak{p} \in \mathrm{Spec}(R)$ we consider the fiber $X_\mathfrak{p} = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\bar{K}_\mathfrak{p})$ of f at $\mathfrak{p}$, where $\bar{K}_\mathfrak{p} = R/\mathfrak{p}$. Then, $X_\mathfrak{p} = \mathrm{Spec}(R[\mathbf{x}]/R[\mathbf{x}]\mathfrak{p}) = \mathrm{Spec}(R[t_F, g^{-1}, z]/R[t_F, g^{-1}, z]\mathfrak{p})$. Now we consider a polynomial $h \in R[X_0, X_{n+1}]$ such that $h(t_F, X_{n+1}) = \mathrm{irr}(z, K(t_F))$. Since F/K is regular, h is absolutely irreducible [FrJ08, p. 175, Cor. 10.2.2]. Since h is absolutely irreducible, it remains absolutely irreducible modulo $\mathfrak{p}$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$ [FrJ08, p. 170, Prop. 9.4.3]. Moreover, $g \neq 0$ modulo $\mathfrak{p}$ for almost all $\mathfrak{p} \in \mathrm{Spec}(R)$. Adding the finitely many prime divisors of K that belong to $\mathcal{V}$ and correspond to the exceptional $\mathfrak{p}$'s to $\mathcal{T}$, we may assume by Lemma 4.10 that

(7) each of the fibers $X_\mathfrak{p}$ of X over $\mathrm{Spec}(R)$ is absolutely integral.

Lemma 5.6: *Starting from the Zariski-closed affine subscheme X of $\mathbb{A}^n_R$, we consider the Zariski-closure X' of X in $\mathbb{P}^n_R$ and let $\bar{X}$ be the normalization of X' in F. Then:*

(a) *$\bar{X}$ may be identified with $\mathrm{Proj}(R[\mathbf{t}])$, with $\mathbf{t} = (t_0, \ldots, t_r)$, where $R[\mathbf{t}]$ is a graded ring over R with $R[\mathbf{t}]_1 = \sum_{i=0}^{r} Rt_i$. In particular, $\bar{X}$ is a Noetherian scheme.*

(b) *Each of the elements $t_0, \ldots, t_r$ is transcendental over F. Thus, $K(\mathbf{t})/K$ is a regular extension of transcendence degree 2 and $t_0, \ldots, t_r \neq 0$.*

(c) *$R\left[\frac{\mathbf{t}}{t_0}\right]$ is integrally closed with quotient field F.*

(d) *The scheme X may be identified with a Zariski-open subset of $\bar{X}$ and $f\colon X \to \mathrm{Spec}(R)$ lifts to a surjective morphism $\bar{f}\colon \bar{X} \to \mathrm{Spec}(R)$.*

Proof: We write $X' = \mathrm{Proj}(R[\mathbf{s}'])$, where $\mathbf{s}' = (s'_0, \ldots, s'_n)$, $R[\mathbf{s}']$ is a graded ring over R with $R[\mathbf{s}']_1 = \sum_{i=0}^n Rs'_i$ such that $s'_0 \neq 0$ and $x_i = \frac{s'_i}{s'_0}$ for $i = 1, \ldots, n$. Then, the inclusion map $\xi\colon X' \to \mathbb{P}^n_R$ is a closed immersion. Let $\pi_n\colon \mathbb{P}^n_R \to \mathrm{Spec}(R)$ be the canonical morphism and let $f'\colon X' \to \mathrm{Spec}(R)$ be the restriction of π_n to X'. By definition, f' is a projective morphism that extends f. Let $\pi\colon \bar{X} \to X'$ be the normalization of X' [Liu06, p. 120, Prop. 4.1.22]. In particular, $\bar{X}$ is an absolutely integral normal scheme over R whose function field coincides with that of X', namely F. Moroever, π is an integral morphism.

Claim A: *π is a finite morphism.* The scheme X' is covered by the affine Noetherian Zariski-open sets $\mathrm{Spec}(R\big[\frac{\mathbf{s}'}{s'_i}\big])$, where i ranges over all integers between 0 and n with $s'_i \neq 0$. Each of the integral domain $R\big[\frac{\mathbf{s}'}{s'_i}\big]$ is a finitely generated R-algebra. Hence, for each Zariski-open affine subset U of X' the ring $\Gamma(U, \mathcal{O}_{X'})$ is a finitely generated R-algebra whose quotient field is F [Mum88, p. 122, Def. 3 and Prop. 1]. Moreover, the open set $\pi^{-1}(U)$ of $\bar{X}$ is also affine [Liu06, p. 120, Def. 4.1.20] and $\Gamma(\pi^{-1}(U), \mathcal{O}_{\bar{X}})$ is the integral closure of $\Gamma(U, \mathcal{O}_{X'})$ in F [Liu06, p. 121, comment following Definition 4.1.24]. By Lemma 5.4(c), R is a Nagata ring, so $\Gamma(\pi^{-1}(U), \mathcal{O}_{\bar{X}})$ is finitely generated as a $\Gamma(U, \mathcal{O}_{X'})$-module. We conclude that π is finite, as claimed.

Claim B: *The map π is a projective morphism in the sense of [Har77, p. 103, Def.].* Indeed X' is a closed subscheme of $\mathbb{P}^n_R$, so the above mentioned definition of [Har77] coincides with that of [Gro61II, p. 104, Def. 5.5.2]. Thus, by Claim A and [Gro61II, p. 113, Cor. 6.1.11], π is projective. See also, [GoW10, p. 401, Cor. 13.77].

It follows from [Liu06, p. 108, Cor. 3.3.32(b)] that $f' \circ \pi\colon \bar{X} \to \mathrm{Spec}(R)$ is a projective morphism. Thus, there exist a positive integer r and a closed immersion $\varphi\colon \bar{X} \to \mathbb{P}^r_R$ such that $\bar{f} = f' \circ \pi = \pi_r \circ \varphi$, where π_r is the canonical morphism $\mathbb{P}^r_R \to \mathrm{Spec}(R)$. This gives the following commutative diagram:

(8)
$$\begin{tikzcd}
\pi^{-1}(X) \arrow[r] \arrow[d] & \bar{X} \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathbb{P}^r_R \arrow[ddl, bend left, "\pi_r"] \\
X \arrow[r, "\iota"] \arrow[dr, "f"'] & X' \arrow[r, "\xi"] \arrow[d, "f'"] & \mathbb{P}^n_R \arrow[dl, "\pi_n"] \\
& \mathrm{Spec}(R) &
\end{tikzcd}$$

where $\iota\colon X \to X'$ is the inclusion map. Since X is normal (by (5)), the restriction of π to $\pi^{-1}(X)$ is an isomorphism onto X [GoW10, p. 340, Rem. 12.46]. We use that isomorphism to identify X with $\pi^{-1}(X)$. Then, we identify $\bar{X}$ with the closed subscheme $\varphi(\bar{X})$ of $\mathbb{P}^r_R$. By [Liu06, p. 168, Prop. 5.1.30], $R[T_0, \ldots, T_r]$ has a homogeneous ideal J such that $\bar{X} = \mathrm{Proj}(R[T_0, \ldots, T_r]/J)$. For each $0 \leq i \leq r$ let $t_i = T_i + J$ and set $\mathbf{t} = (t_0, \ldots, t_r)$. Then, $R[\mathbf{t}]$ is a graded ring over R with $R[\mathbf{t}]_1 = \sum_{i=0}^r Rt_i$. By (3), R is Noetherian, hence so is $R[\mathbf{t}]$. Therefore, $\bar{X}$ is a Noetherian scheme, as (a) asserts.

We omit all of the i's between 0 and r with $t_i = 0$, change r, and reenumerate the indices, if necessary, to assume that $t_i \neq 0$ for each $0 \leq i \leq r$. Then, by Example 1.6, each t_i is transcendental over F. Since $t_j = \frac{t_i t_j}{t_i}$ for all $0 \leq i, j \leq r$, we have $K(\mathbf{t}) = F(t_i)$ for all $0 \leq i \leq r$. Hence, $\mathrm{trans.deg}(K(\mathbf{t})/K) = \mathrm{trans.deg}(F/K) + 1 = 2$. Since by Subsection 5.1, F/K is a regular extension, so is $K(\mathbf{t})/K$ [FrJ08, p. 41, Cor. 2.6.8(b)], as claimed by (b).

Since $\bar{X}$ is normal and $R\big[\frac{\mathbf{t}}{t_0}\big]$ is the coordinate ring of the open affine subscheme of $\bar{X}$ defined by $T_0 \neq 0$, we have that $R\big[\frac{\mathbf{t}}{t_0}\big]$ is an integrally closed ring with quotient field F, as claimed in (c).

Finally, we deduce from Diagram (8) that the morphism $\bar{f} = f' \circ \pi$ from $\bar{X}$ to $\mathrm{Spec}(R)$ extends $f\colon X \to \mathrm{Spec}(R)$. Since, by (4), f is surjective, so is $\bar{f}$, as asserted by (d). ∎

5.7 Boundary. We consider the closed subset $\bar{X} \smallsetminus X$ of $\bar{X}$. Since $\bar{X}$ is irreducible of dimension 2 if $R \neq K$ (resp. 1, if $R = K$) and X is open in $\bar{X}$ and non-empty, $\dim(\bar{X} \smallsetminus X) \leq 1$. Let Z be the unique reduced subscheme of $\bar{X}$ with support $\bar{X} \smallsetminus X$. Thus, $\dim(Z) \leq 1$. Since $K[\mathbf{x}]$ is not finite over K, the affine scheme $C = X_K$ is not proper [Liu06, p. 104, Lemma 3.3.17]. In particular, Z_K (hence also Z) is non-empty. Since $\dim(\mathrm{Spec}(R)) = 1$ if $R \neq K$ (resp. $\dim(\mathrm{Spec}(R)) = 0$ if $R = K$), we conclude that

(9) $\dim(Z) = 1$ if $R \neq K$ (resp. $\dim(Z) = 0$ if $R = K$).

Let $Z = \bigcup_{i=1}^{d(Z)} Z_i$, with $d(Z) \geq 1$, be the decomposition of Z into its irreducible components over R. We prove that, after a possible enlargement of $\mathcal{T}$ inside $\mathcal{V}$,

(10) for each $1 \leq i \leq d(Z)$, Z_i is a regular scheme over R with $\dim(Z_i) = 1$ if $R \neq K$ (resp. $\dim(Z_i) = 0$ if $R = K$) and the restriction of $\bar{f}$ to Z_i is a finite, flat, and surjective morphism.

Indeed, for each $1 \leq i \leq d(Z)$ let $f_i\colon Z_i \to \mathrm{Spec}(R)$ be the restriction of $\bar{f}$ to Z_i. Thus, f_i is the restriction of the natural morphism $\mathbb{P}^r_R \to \mathrm{Spec}(R)$ to the closed subset Z_i of $\mathbb{P}^r_R$. It follows that f_i is a projective morphism. By [Liu06, p. 108, Thm. 3.3.30], f_i is proper. In particular, f_i is a closed map, so $f_i(Z_i)$ is a closed subset of $\mathrm{Spec}(R)$. Since $\mathrm{Spec}(R)$ is an irreducible scheme of dimension ≤ 1, $f_i(Z_i)$ is either a closed point of $\mathrm{Spec}(R)$ or all of $\mathrm{Spec}(R)$. If in the first case the prime of K that corresponds to $f_i(Z_i)$ is in $\mathcal{V}$, we adjoin it to $\mathcal{T}$. Since $R = \mathcal{O}_{K,\mathcal{V} \smallsetminus \mathcal{T}}$ (by (3)), Z_i won't be an irreducible component of Z any more. Having done so for all of those i's, we may assume that $f_i(Z_i) = \mathrm{Spec}(R)$ for all i. Since Z is non-empty, the above procedure does not eliminate all of the Z_i's. In other words, we may still assume that $d(Z) \geq 1$.

The fiber of the generic point of $\mathrm{Spec}(R)$ (i.e. of the zero ideal) is the generic point of Z_i. For each closed point $\mathfrak{p} \in \mathrm{Spec}(R)$ the set $f_i^{-1}(\mathfrak{p})$ of Z_i is closed. Since Z_i is irreducible of dimension ≤ 1, $f_i^{-1}(\mathfrak{p})$ is either a finite set or $f_i^{-1}(\mathfrak{p}) = Z_i$. In the latter case we have $f_i(Z_i) = \{\mathfrak{p}\}$, in contrast to the preceding paragraph. It follows that the fibers of f_i are finite.

We have therefore proved that the morphism f_i is projective with finite fibers. By [Liu06, p. 152, Cor. 4.4.7], f_i is a finite morphism. Since Z_i is reduced, we get by the definition of a finite morphism and by the fact that $f_i\colon Z_i \to \mathrm{Spec}(R)$ is surjective that $Z_i = \mathrm{Spec}(R_i)$ is an affine scheme, where R_i is an integral domain, finitely generated and integral over R. Since R is a Dedekind domain (by (3)), [Liu06, p. 11, Cor. 1.2.14] implies that R_i is flat over R. Hence, f_i is flat. Since the integral closure of R in $\mathrm{Quot}(R_i)$ is also a finitely generated R-module (because R is a Nagata ring), we may enlarge $\mathcal{T}$ in $\mathcal{V}$ to assume that R_i is integrally closed, hence a Dedekind domain. Thus, Z_i is a Dedekind scheme [Liu06, p. 116, Example 4.1.7] and therefore regular [Liu06, p. 117, Prop. 4.1.12 and p. 128, Example 4.2.9]. Moreover, since R_i is a finitely generated R-module, $\dim(Z_i) = \dim(R) = 1$ if $R \neq K$ (resp. $\dim(Z_i) = 0$ if $R = K$). This complete the proof of Statement (10).

Next we prove that, after another possible enlargement of $\mathcal{T}$ in $\mathcal{V}$ (Lemma 4.10),

(11) Z is a regular scheme over R of dimension 1 if $R \neq K$ (resp. 0, if $R = K$) and the restriction f_Z of $\bar{f}$ to Z is a finite, flat, and surjective morphism.

Indeed, if $1 \leq i < j \leq d(Z)$, then $Z_i \cap Z_j$, as an intersection of distinct irreducible subschemes of Z of dimension ≤ 1, is a scheme of dimension 0, hence finite. Therefore, $f_Z(Z_i \cap Z_j)$ is a finite subset of $\mathrm{Spec}(R)$. Adding the primes in $\mathcal{V}$ that correspond to this subset to $\mathcal{T}$, we may assume that $Z_i \cap Z_j = \emptyset$. In other words, we may assume that $Z = \dot{\bigcup}_{i=1}^{d(Z)} Z_i$. Since each of the sets Z_i is closed in Z, it is also open.

As a disjoint union of open regular subschemes Z_i (by (10)), the scheme Z is itself regular. Moreover, the natural map $f_Z\colon Z \to \mathrm{Spec}(R)$, inducing for each i the map f_i on Z_i, is finite, flat, and surjective, because by (10), f_i has these properties for each i. This concludes the proof of (11).

5.8 The ideals I and I_i. Since Z is a closed subscheme of $\bar{X} = \mathrm{Proj}(R[\mathbf{t}])$, we may identify Z with $\mathrm{Proj}(R[\mathbf{t}]/I)$, where I is a homogeneous ideal of $R[\mathbf{t}]$ [Liu06, p. 168, Prop. 5.1.30]. Similarly, for each $1 \leq i \leq d(Z)$, there exists a homogeneous prime ideal I_i of $R[\mathbf{t}]$ that contains I and $R[\mathbf{t}]_+ \not\subseteq I_i$ such that $Z_i = V_+(I_i)$. Since Z is reduced, I is equal to its radical and the latter is equal to the intersection of all homogeneous prime ideals that contain I and are minimal with this property [ZaS75II, p. 152, Thm. 8 and Corollary]. The set $\mathcal{P}$ of all these prime ideals is finite (because $R[\mathbf{t}]/I$ is Noetherian). The ideals $I_1, \ldots, I_{d(Z)}$ belong to $\mathcal{P}$. Let $P_1, \ldots, P_m$ be all the other ideals in $\mathcal{P}$ and note that each of them contains $R[\mathbf{t}]_+$. For each $P \in \mathcal{P}$ with $P \cap R \neq 0$, we add the elements of $\mathcal{V}$ that correspond to prime ideals of

R that divide a generator of $P \cap R$ (use (3)) to $\mathcal{T}$. After this enlargement, $P \cap R = 0$, so $P \subseteq R[\mathbf{t}]_+$ for each $P \in \mathcal{P}$. In particular, $P_j = R[\mathbf{t}]_+$ for $j = 1, \ldots, m$. Note that for $1 \le i \le d(Z)$, the property $\bar{f}(Z_i) = \mathrm{Spec}(R)$, which (10) guarantees, implies that $I_i \cap R = 0$, so $I_i \subseteq R[\mathbf{t}]_+$. Hence, $I_i \subseteq P_j$ for each $i = 1, \ldots, d(Z)$ and $j = 1, \ldots, m$. It follows from the minimality of the elements in $\mathcal{P}$ that $m = 0$. Therefore, $\bigcap_{i=1}^{d(Z)} I_i = I$.

5.9 The boundary over K. The quotient ring of $R[\mathbf{t}]$ with respect to the multiplicative set $R \smallsetminus \{0\}$ is $K[\mathbf{t}]$. By Subsection 5.8, $I_i \cap R = 0$ for $i = 1, \ldots, d(Z)$. Hence, $KI_1, \ldots, KI_{d(Z)}$ are distinct points of $\bar{X}_K$. It follows that the generic fiber $Z_K = \mathrm{Proj}(K[\mathbf{t}]/KI)$ of Z consists of $d(Z)$ distinct points $Z_{1,K}, \ldots, Z_{d(Z),K}$, corresponding to the points $KI_1, \ldots, KI_{d(Z)}$ of $\bar{X}_K$. Each of these points is closed, so $KI_j \not\subseteq KI_i$ if $j \neq i$. It follows that $\bigcap_{j \neq i} KI_j \not\subseteq KI_i$ for every $1 \le i \le d(Z)$. By Subsection 5.8, $\bigcap_{i=1}^{d(Z)} KI_i = KI$.

We denote the degree of the divisor $\sum_{i=1}^{d(Z)} Z_{i,K}$ attached to Z_K by $\deg_K(Z_K)$.

5.10 Special fibers. We let $\bar{X}_{\mathrm{sing}}$ be the closed subset of all singular points of $\bar{X}$. Since $\bar{X}$ is normal, each of its points of codimension 1 is nonsingular [Liu06, p. 268, Example 7.2.6]. Hence, $\bar{X}_{\mathrm{sing}}$ has dimension 0, so $\bar{X}_{\mathrm{sing}}$ is finite. Following [MoB89, p. 187, (3.1.2)], we add the finitely many primes in $\mathcal{V}$ corresponding to the finite subset $\bar{f}(\bar{X}_{\mathrm{sing}})$ of $\mathrm{Spec}(R)$ to $\mathcal{T}$ and assume that

(12) $\bar{X}$ is regular.

Finally, we may apply the arguments that prove (7) to each of the finitely many affine Zariski-open parts of $\bar{X}$ and conclude, possibly after an additional enlargement of $\mathcal{T}$ in $\mathcal{V}$, that

(13) each of the fibers $\bar{X}_{\mathfrak{p}}$ of $\bar{X}$ over $\mathrm{Spec}(R)$ is an absolutely integral projective curve.

5.11 Generic fibers. We consider the generic fibers $X_K = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K) = \mathrm{Spec}(K[\mathbf{x}])$ and $\bar{X}_K = \bar{X} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ of X and $\bar{X}$, respectively. Then, X_K is an affine K-scheme which is actually isomorphic to our original curve C. Since C is smooth (by (1)),

(14) X_K is smooth.

Moroever, $\bar{X}_K$ is the normalization of the projective closure of X_K in $\mathbb{P}^r_K$ [Eis95, p. 126, Prop. 4.4.13, and p. 127, last paragraph]. In particular, $\bar{X}_K$ is normal.

By (7) and (13),

(15) X_K and $\bar{X}_K$ are absolutely integral.

Moreover, for each $\mathfrak{p} \in \mathcal{T}$ we may view the subset $\Omega_{\mathfrak{p}}$ of $C(L_{\mathfrak{p}})$ introduced in Subsection 5.1 also as a $\mathfrak{p}$-open subset of $X_K(L_{\mathfrak{p}})$.

6. Closed Separable Point

We choose a closed separable point $\mathbf{b}$ of X over K, let $E = K(\mathbf{b})$, denote the integral closure of R in E by R_E, choose a point B' of $\bar{X}_{R_E}$ that lies over $\mathbf{b}$, use the conjugates of B' over K to construct a homogeneous ideal B'' of $R_E[\mathbf{t}]$, and prove that $V_+(B') \cap V_+(B'') = \emptyset$. We use the homogeneous ideals B' and B'' of $R_E[\mathbf{t}]$ in Section 9 to produce homogeneous coordinates $s_0, s_1, \ldots, s_l \in R[\mathbf{t}]$ of large degree of a projective curve $Y = \mathrm{Proj}(K[s_0, \ldots, s_l])$ (Lemma 9.5), and to construct in Section 10 a birational morhism $\varphi\colon \bar{X}_K \to Y$ which maps the smooth affine curve X_K minus the point corresponding to $B = R[\mathbf{t}] \cap B'$ isomorphically onto a Zariski-open smooth affine subset Y_0 of Y, maps Z_K onto a point $\mathbf{y}_0 \in Y(K)$, and maps the point of $\bar{X}_K$ corresponding to B onto cusps $\mathbf{y}_1, \ldots, \mathbf{y}_e \in Y(\tilde{K})$ of multiplicity q, where q is a large prime number, such that $Y(\tilde{K}) = Y_0(\tilde{K}) \cup \{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_e\}$ (Lemmas 10.3 and 10.4). We use that curve to construct a symmetrically stabilizing element for F/K using the main result of [GRJ17] quoted here as Proposition 8.6.

6.1 Separable integral point. We assume without loss that t_0, t_1 form a separable transcendence base for $K(\mathbf{t})/K$ (Lemma 5.6(b) and [FrJ08, p. 38, Lemma 2.6.1]). Let $h_0, h_1, \ldots, h_n \in R[T_0, \ldots, T_r]$ be homogeneous polynomials of the same degree such that $h_0(\mathbf{t}) \neq 0$ and $x_j = \frac{h_j(\mathbf{t})}{h_0(\mathbf{t})}$ for $j = 1, \ldots, n$. Then, we choose $b_1 \in K_{\mathrm{sep}} \smallsetminus K$ and extend the map $(t_0, t_1) \mapsto (1, b_1)$ to a K-homomorphism $\varphi\colon K[\mathbf{t}] \to K_{\mathrm{sep}}$ such that with $b_i = \varphi(t_i)$ for $i = 2, \ldots, r$ and $\mathbf{b} = (1{:}b_1{:}\cdots{:}b_r)$ we have $h_0(\mathbf{b}) \neq 0$. It follows that $\mathbf{b} \in X(K_{\mathrm{sep}}) \smallsetminus X(K)$. From a geometric point of view we can choose a separating transcendence base of F/K that leads to a nonconstant morphism $f\colon C \to \mathbb{A}^1$, so there is a dense open U in $\mathbb{A}^1$ such that $f^{-1}(U) \to U$ is finite étale, and choose $b_1 \in U(K_{\rm sep}) \smallsetminus U(K)$ and $\mathbf{b} \in f^{-1}(b_1)(K_{\rm sep})$. Since X is smooth (Subsection 5.1, (1)), $\mathbf{b} \in \bar{X}_{\rm simp}(K_{\rm sep})$. Let $E = K(b_1, \ldots, b_r)$, set $e = [E : K]$, and note that $e \geq 2$, by the choice of b_1. We choose a non-zero element b' of R such that $b'b_i$ is integral over R for $i = 1, \ldots, r$. Adjoining the prime divisors of b' that are in $\mathcal{V}$ to $\mathcal{T}$ and using Lemma 4.10, we may assume that $b_1, \ldots, b_r$ are integral over R. Geometrically, we can consider the point $\mathbf{b}$ as a section $\mathrm{Spec}(E) \to C$. Then, after enlarging $\mathcal{T}$ if necessary, it extends to a section $\mathrm{Spec}(R_E) \to X$.

For each ideal $\mathfrak{a}$ of a graded ring A we let $\mathfrak{a}^h$ be the ideal generated by all of the homogeneous elements of $\mathfrak{a}$. Then, $\mathfrak{a}^h$ is the maximal homogeneous ideal of A contained in $\mathfrak{a}$. By [Liu06, p. 51, Lemma 2.3.35(a)], $\mathfrak{a}^h$ is a prime ideal, if $\mathfrak{a}$ is.

Having made this definition, we consider the homogeneous prime ideal $B = \mathrm{Ker}(\varphi)^h \cap R[\mathbf{t}]$ of $R[\mathbf{t}]$. Geometrically, B is the generic point of the image of the section $\mathrm{Spec}(R_E) \to X$. Note that $t_0 \notin \mathrm{Ker}(\varphi)$ (because $\varphi(t_0) = 1$), hence $t_0 \notin B$. Thus, B can be also considered as a point of $\bar{X}$ that belongs to X. Moreover, B lies under $\mathbf{b}$.

Since $\mathrm{Ker}(\varphi)^h$ is a prime ideal of $K[\mathbf{t}]$, its intersection with $K^\times$ is empty, hence

(1) $B \cap R = 0$.

Since $K[\mathbf{t}]$ is the quotient ring of $R[\mathbf{t}]$ with respect to the multiplicative set $R \smallsetminus \{0\}$ and B is disjoint to that set (by (1)), we have

(2) $KB \cap R[\mathbf{t}] = B$ and $KB = \mathrm{Ker}(\varphi)^h$.

6.2 The ring R_E. Following Subsection 6.1, we consider the separable field extension $E = K(\mathbf{b})$ of K and let R_E be the integral closure of R in E. By Subsection 6.1, $b_1, \ldots, b_r$ are integral over R, so $b_1, \ldots, b_r \in R_E$.

Since R is a principal ideal domain (Subsection 5.2) and E/K is a finite separable extension, R_E is a finitely generated free R-module [Wae91, p. 175, Sec. 17.3]. Then R_E has an R-basis $w_1, \ldots, w_e$ which is also a basis for E/K.

We choose $\sigma_1, \ldots, \sigma_e \in \mathrm{Aut}(\tilde{K}/K)$ whose restrictions to E are the distinct K-embeddings of E into $\tilde{K}$ and σ_1 is the identity map of E. Since $K(\mathbf{t})/K$ is a regular extension (Lemma 5.6(b)), we may extend $\sigma_1, \ldots, \sigma_e$ to elements of $\mathrm{Aut}(\tilde{K}(\mathbf{t})/K(\mathbf{t}))$ having the same names.

Since E/K is a separable extension $\det(w_i^{\sigma_j}) \neq 0$ [Lan93, p. 286, Cor. 5.4]. Moreover, $\det(w_i^{\sigma_j})_{i,j=1,\ldots,e}$ belongs to the integral closure $\tilde{R}$ of R in $\tilde{K}$. We use Lemma 4.10 to enlarge $\mathcal{T}$ such that

(3) $\det(w_i^{\sigma_j})$ is invertible in $\tilde{R}$.

Having made this assumption, we prove that

(4) $R_E\big[\frac{\mathbf{t}}{t_0}\big]$ is integrally closed.

Indeed, let $f \in E\big(\frac{\mathbf{t}}{t_0}\big)$ be integral over $R_E\big[\frac{\mathbf{t}}{t_0}\big]$. Since $E\big(\frac{\mathbf{t}}{t_0}\big) = E \cdot K\big(\frac{\mathbf{t}}{t_0}\big) = \sum_{i=1}^e K w_i \cdot K\big(\frac{\mathbf{t}}{t_0}\big) = \sum_{i=1}^e w_i K\big(\frac{\mathbf{t}}{t_0}\big)$, we may write $f = \sum_{i=1}^e w_i f_i$ with $f_1, \ldots, f_e \in K\big(\frac{\mathbf{t}}{t_0}\big)$. Applying σ_j on the latter equality, we get $f^{\sigma_j} = \sum_{i=1}^e w_i^{\sigma_j} f_i$, $j = 1, \ldots, e$. Applying Kramer's rule to the latter system of equations, we find for each $1 \leq k \leq e$ that $f_k = f'_k / \det(w_i^{\sigma_j})$ with f'_k in the integral closure of $R\big[\frac{\mathbf{t}}{t_0}\big]$ in $K\big(\frac{\mathbf{t}}{t_0}\big)$. It follows from (3) that f_k belongs to the integral closure of $R\big[\frac{\mathbf{t}}{t_0}\big]$ in $K\big(\frac{\mathbf{t}}{t_0}\big)$. Since $R\big[\frac{\mathbf{t}}{t_0}\big]$ is integrally closed (Lemma 5.6(c)), $f_k \in R\big[\frac{\mathbf{t}}{t_0}\big]$. It follows that $f \in R_E\big[\frac{\mathbf{t}}{t_0}\big]$.

Notation 6.3: We consider the homogeneous ideals

$$\tilde{B}_j = \sum_{i=1}^r \tilde{K}[\mathbf{t}](t_i - b_i^{\sigma_j} t_0), \ j = 1, \ldots, e,$$

$$B' = \sum_{i=1}^r R_E[\mathbf{t}](t_i - b_i t_0), \quad B'' = \bigcap_{j=2}^e (R_E[\mathbf{t}] \cap \tilde{B}_j)$$

of $\tilde{K}[\mathbf{t}]$ and $R_E[\mathbf{t}]$, respectively, and note that $\tilde{B}_j = \tilde{B}_1^{\sigma_j}$ for $j = 1, \ldots, e$. Note also that $\tilde{K}[\mathbf{t}]/\tilde{B}_j$ is isomorphic to the integral domain $\tilde{K}[t_0]$, so $\tilde{B}_j$ is a prime ideal of $\tilde{K}[\mathbf{t}]$ for $j = 1, \ldots, e$. Similarly $R_E[\mathbf{t}]/B' \cong R_E[t_0]$, so B' is a prime ideal of $R_E[\mathbf{t}]$. $\blacksquare$

Lemma 6.4: $B' = R_E[\mathbf{t}] \cap \tilde{B}_1$.

Proof: It suffices to prove that each $f \in R_E[\mathbf{t}] \cap \tilde{B}_1$ belongs to B'. To that end we choose a basis $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \dots)$ for $\tilde{K}/E$ with $\tilde{w}_1 = 1$. Then, we note that since $K(\mathbf{t})/K$ is a regular extension (Lemma 5.6(b)), also $E(\mathbf{t})/E$ is a regular extension. Hence, $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \dots$ also form a basis for $\tilde{K}[\mathbf{t}]$ over $E[\mathbf{t}]$. By definition, $f = \sum_{i=1}^r f_i(t_i - b_i t_0)$ with $f_i \in \tilde{K}[\mathbf{t}]$ for $i = 1, \dots, r$. For each $1 \le i \le r$ we write $f_i = \sum_{k=1}^\infty f_{ik}\tilde{w}_k$ with $f_{ik} \in E[\mathbf{t}]$ for all k and all but finitely many of the f_{ik}'s are 0. Then, $f = \sum_{k=1}^\infty \big(\sum_{i=1}^r f_{ik}(t_i - b_i t_0)\big)\tilde{w}_k$. Comparing the coefficients of $\tilde{w}_1$ on both sides, we have $f = \sum_{i=1}^r f_{i1}(t_i - b_i t_0) \in E[\mathbf{t}]B' \cap R_E[\mathbf{t}]$.

Since the $R_E[\mathbf{t}]$-degree of each non-zero element of B' is at least 1, we have $B' \cap R_E = 0$. In addition observe that $E[\mathbf{t}]$ is the quotient ring of $R_E[\mathbf{t}]$ with respect to the multiplicative subset $R_E \smallsetminus \{0\}$. Since B' is a prime ideal of $R_E[\mathbf{t}]$ (Notation 6.3), it follows that $E[\mathbf{t}]B' \cap R_E[\mathbf{t}] = B'$, so $f \in B'$, as claimed. ∎

Lemma 6.5: *$B = R[\mathbf{t}] \cap \tilde{B}_j$ for $j = 1, \dots, e$, $B \subseteq B' \cap B''$, and $\tilde{K}B = \bigcap_{j=1}^e \tilde{B}_j$. Thus, $\tilde{B}_1, \dots, \tilde{B}_e$ are exactly the points of $X_{\tilde{K}}$ that lie over B. Each of them is simple. Moreover, $\tilde{B}_j \not\subseteq \tilde{B}_{j'}$ if $j \ne j'$.*

Proof: Since $K(\mathbf{t})/K$ is a regular extension, we may uniquely extend the K-homomorphism φ introduced in Subsection 6.1 to a $\tilde{K}$-homomorphism $\tilde{\varphi}\colon \tilde{K}[\mathbf{t}] \to \tilde{K}$. Then, $\mathrm{Ker}(\tilde{\varphi})^h$ is a homogeneous prime ideal of $\tilde{K}[\mathbf{t}]$ that belongs to $X_{\tilde{K}}$ and $\tilde{\varphi}(\mathbf{t}) = \mathbf{b}$. For each $f \in \mathrm{Ker}(\tilde{\varphi})^h$ we apply the Taylor expansion around $\frac{\mathbf{b}}{b_0}$ to $f\big(\frac{\mathbf{t}}{t_0}\big)$ (with $b_0 = 1$) and then multiply the resulting expression by $t_0^{\deg(f)}$. We find that $\mathrm{Ker}(\tilde{\varphi})^h = \sum_{i=1}^r \tilde{K}[\mathbf{t}](t_i - b_i t_0)$. It follows that $B = R[\mathbf{t}] \cap \mathrm{Ker}(\varphi)^h = R[\mathbf{t}] \cap \mathrm{Ker}(\tilde{\varphi})^h = R[\mathbf{t}] \cap \sum_{i=1}^r \tilde{K}[\mathbf{t}](t_i - b_i t_0)$. Applying σ_j on both sides, we get

(5) $B = R[\mathbf{t}] \cap \sum_{i=1}^r \tilde{K}[\mathbf{t}](t_i - b_i^{\sigma_j} t_0) = R[\mathbf{t}] \cap \tilde{B}_j$ for $j = 1, \dots, e$.

The point $\tilde{B}_1$ of $X_{\tilde{K}}$ corresponds to $\mathbf{b}$, so $\tilde{B}_1$ is simple. Hence, $\tilde{B}_j = \tilde{B}_1^{\sigma_j}$ is also simple for $j = 1, \dots, e$.

By (5) and by Lemma 6.4, $B = R[\mathbf{t}] \cap \tilde{B}_1 \subseteq R_E[\mathbf{t}] \cap \tilde{B}_1 = B'$. Also, $B \subseteq \bigcap_{j=2}^e (R_E[\mathbf{t}] \cap \tilde{B}_j) = B''$.

Let P' be a point of $X_{\tilde{K}}$ that contains $\tilde{K}B$ and let $\mathbf{b}' = (1{:}b_1'{:}\cdots{:}b_r')$ be the corresponding point in $X(\tilde{K})$ (note that $t_0 \notin P'$, otherwise $t_0 \in K[\mathbf{t}] \cap P' = KB$). Let $\varphi'\colon K[\mathbf{t}] \to \tilde{K}$ be the K-homomorphism mapping $\mathbf{t}$ onto $\mathbf{b}'$. Then, $\mathrm{Ker}(\varphi')^h = KB = \mathrm{Ker}(\varphi)^h$ (because KB is a closed point of X_K). It follows that there exists $\sigma \in \mathrm{Aut}(\tilde{K}/K)$ such that $\mathbf{b}' = \mathbf{b}^\sigma$. Hence, P' is one of the $\tilde{B}_j$'s, as claimed.

Finally, we prove that $\tilde{B}_j \not\subseteq \tilde{B}_{j'}$ if $j \ne j'$. Indeed, it suffices to prove that $\tilde{B}_j \subseteq \tilde{B}_1$ implies that $j = 1$. Indeed, the latter assumption implies that for all $1 \le i, i' \le r$ there exist $f_{i,i'} \in \tilde{K}[T_0, \dots, T_r]$ such that $t_i - b_i^{\sigma_j} t_0 = \sum_{i'=1}^r f_{i,i'}(\mathbf{t})(t_{i'} - b_{i'} t_0)$. Applying $\tilde{\varphi}$ on both sides, we get $b_i - b_i^{\sigma_j} = 0$ for $i = 1, \dots, r$. Since $E = K(b_1, \dots, b_r)$, we conclude from the choice of $\sigma_1, \dots, \sigma_e$ in Subsection 6.2 that $j = 1$, as claimed. ∎

Since $t_0 \in R[\mathbf{t}]$, $\tilde{B}_j \cap R[\mathbf{t}] = B$ (Lemma 6.5), and $t_0 \notin B$ (Subsection 6.1), we have:

Corollary 6.6: *For each $1 \le j \le e$ we have $t_0 \notin \tilde{B}_j$.*

Notation 6.7: By the choice of $\sigma_1, \dots, \sigma_e$, the r-tuples $(b_1^{\sigma_j}, \dots, b_r^{\sigma_j})$, $j = 1, \dots, e$ are distinct. Since the ring R is infinite, it contains $c_1, \dots, c_r$ such that

$$\sum_{i=1}^r c_i(b_i - b_i^{\sigma_j}) \ne 0, \qquad j = 2, \dots, e.$$

We consider the non-zero element $c = \prod_{j=2}^e \sum_{i=1}^r c_i(b_i - b_i^{\sigma_j})$ of $\tilde{R}$. By Example 4.11, we may add finitely many primes in $\mathcal{V}$ to $\mathcal{T}$, if necessary, to assume that c is invertible in $\tilde{R}$. ∎

Lemma 6.8: $V_+(B') \cap V_+(B'') = \emptyset$.

Proof: We break up the proof into several parts.

Part A: *The elements* $\tilde t_1,\dots,\tilde t_e$. For each $1\le j\le e$ let $\tilde t_j=\sum_{i=1}^r c_i(t_i-b_i^{\sigma_j}t_0)$. Since $b_1,\dots,b_r$ are separable over K, integral over R, and $\tilde t_j\in\tilde R[\mathbf{t}]_1$,

(6) $\frac{\tilde t_j}{t_0}$ is separable over $K\big(\frac{\mathbf{t}}{t_0}\big)$ and integral over $R\big[\frac{\mathbf{t}}{t_0}\big]$, $j=1,\dots,e$.

By definition,

(7) $\tilde t_j\in\tilde B_j$ for $j=1,\dots,e$.

We claim that

(8) there exists a positive integer k_0 such that $t_0^{k_0}\prod_{j=2}^e\tilde t_j\in B''$.

Indeed, each $\sigma\in\mathrm{Gal}(K(\mathbf{t}))$ permutes $\frac{\tilde t_1}{t_0},\dots,\frac{\tilde t_e}{t_0}$, so by (6), $\prod_{j=1}^e\frac{\tilde t_j}{t_0}\in K\big(\frac{\mathbf{t}}{t_0}\big)$. In addition, $\frac{\tilde t_1}{t_0}\in E\big(\frac{\mathbf{t}}{t_0}\big)$. Therefore,

$$\prod_{j=2}^e\frac{\tilde t_j}{t_0}=\Big(\prod_{j=1}^e\frac{\tilde t_j}{t_0}\Big)\Big/\frac{\tilde t_1}{t_0}\in E\Big(\frac{\mathbf{t}}{t_0}\Big).$$

By (6), $\prod_{j=2}^e\frac{\tilde t_j}{t_0}$ is integral over $R\big[\frac{\mathbf{t}}{t_0}\big]$, hence also over $R_E\big[\frac{\mathbf{t}}{t_0}\big]$. Since by (4), $R_E\big[\frac{\mathbf{t}}{t_0}\big]$ is integrally closed in $E\big(\frac{\mathbf{t}}{\mathbf{t}_0}\big)$, we have $\prod_{j=2}^e\frac{\tilde t_j}{t_0}\in R_E\big[\frac{\mathbf{t}}{t_0}\big]$. Hence, there exists a positive integer k_0 such that $t_0^{k_0}\prod_{j=2}^e\tilde t_j\in R_E[\mathbf{t}]$. It follows from (7) that $t_0^{k_0}\prod_{j=2}^e\tilde t_j\in\bigcap_{j=2}^e\tilde B_j\cap R_E[\mathbf{t}]=B''$, as claimed.

Part B: *A power of* t_0. We note that

$$\begin{aligned}t_0^{k_0}\prod_{j=2}^e\tilde t_j&=t_0^{k_0}\prod_{j=2}^e\sum_{i=1}^r c_i(t_i-b_i^{\sigma_j}t_0)\\&=t_0^{k_0}\prod_{j=2}^e\sum_{i=1}^r c_i(t_i-b_it_0+b_it_0-b_i^{\sigma_j}t_0)\\&=t_0^{k_0}u+t_0^{k_0}\prod_{j=2}^e\sum_{i=1}^r c_i(b_i-b_i^{\sigma_j})t_0\\&=t_0^{k_0}u+t_0^{k_0+e-1}c,\end{aligned}$$

where c is the invertible element of $\tilde R$ introduced in Notation 6.7, and u is a sum of products of $e-1$ elements of $\tilde R[\mathbf{t}]$, one of which is $c_i(t_i-b_it_0)$ for some $1\le i\le r$, so belongs to B', and the others have the form $c_i(b_i-b_i^{\sigma_j})t_0$, so they belong to $\tilde R[\mathbf{t}]$. Thus, $u\in\tilde R[\mathbf{t}]B'$. Since c is invertible in $\tilde R$, we have, by (8), that

(9) $t_0^{k_0+e-1}=-c^{-1}t_0^{k_0}u+c^{-1}t_0^{k_0}\prod_{j=2}^e\tilde t_j\in\tilde R[\mathbf{t}]B'+\tilde R[\mathbf{t}]B''$.

End of proof: We recall that $V_+(B')$ (resp. $V_+(B'')$) is the set of all homogeneous prime ideals of $R_E[\mathbf{t}]$ that contain B' (resp. B'') but do not contain the set $\{t_0,\dots,t_r\}$. If $P\in V_+(B')\cap V_+(B'')$, then $B'+B''\subseteq P$. Since $\tilde R[\mathbf{t}]$ is an integral extension of $R_E[\mathbf{t}]$, there exists a prime ideal $\tilde P$ of $\tilde R[\mathbf{t}]$ whose intersection with $R_E[\mathbf{t}]$ is P. In particular, $\tilde R[\mathbf{t}]B'+\tilde R[\mathbf{t}]B''\subseteq\tilde P$. By (9), $t_0^{k_0+e-1}\in\tilde P$. Hence, $t_0^{k_0+e-1}\in\tilde P\cap R_E[\mathbf{t}]=P$, so $t_0\in P$. Since for each $1\le i\le e$, we have $t_i-b_it_0\in B'\subseteq P$, we have $t_i=(t_i-b_it_0)+b_it_0\in P$. Thus, $\{t_0,\dots,t_r\}\subseteq P$. This contradiction implies that P as above does not exist. ∎

Remark 6.9: We could save the introduction of this section if X_K had a K-rational point. But in view of Falting's theorem, many of the absolutely integral curves over K have no K-rational points, if K is a number field. Still, we could simplify the proof of the properties of B if we could choose $\mathbf{b}$ as **Galois over** K, that is such that $E=K(\mathbf{b})$ is a Galois extension of K. But unfortunately, it seems to be unknown if each absolutely integral curve over a global field has a Galois point [JaP16]. So, we have chosen $\mathbf{b}$ as a separable algebraic point over K which is not K-rational. The latter condition makes the proofs of the properties of $\mathbf{b}$ somewhat simpler in that we need not distinguish between the cases where $\mathbf{b}$ is K-rational or not. ∎

6.10 THE CLOSED SUBSCHEMES Z_{qB}. Along with the closed subscheme Z of $\bar{X}$ we consider also the closed subscheme $Z_B = \mathrm{Proj}(R[\mathbf{t}]/B)$ and for each positive integer q the closed subscheme $Z_{qB} = \mathrm{Proj}(R[\mathbf{t}]/B^q)$ of $\bar{X}$. All of the subschemes Z_{qB} are actually contained in X and have the same underlying topological space. As for Z, we have $\dim(Z_{qB}) = 1$ if $R \neq K$ (resp. $\dim(Z_{qB}) = 0$ if $R = K$) and the extensions $Z_{qB,K} = \mathrm{Proj}(K[\mathbf{t}]/KB^q)$ and $Z_{qB,\tilde{K}} = \mathrm{Proj}(\tilde{K}[\mathbf{t}]/\tilde{K}B^q)$ have dimension 0. Moreover, since $X \cap Z = \emptyset$, we have $Z_{qB} \cap Z = \emptyset$. In particular, $Z_{B,K} \cap Z_K = \emptyset$ and $I \not\subseteq B$.

7. From Picard Group to Free Modules

We present a result of [MoB89, Section 3] that gives a big set of effective Cartier divisors on X whose irreducible components are finite and surjective over $\mathrm{Spec}(R)$ and satisfy certain approximation conditions at each $\mathfrak{p} \in \mathcal{T}$. Lemma 7.10 then says that the above mentioned big set is in a sense $\mathcal{T}$-open.

7.1 DIVISORS. For each positive integer d we consider the fiber product $X^d = X \times_{\mathrm{Spec}(R)} \cdots \times_{\mathrm{Spec}(R)} X = \mathrm{Spec}(R[\mathbf{x}] \otimes_R \cdots \otimes_R R[\mathbf{x}])$ of d copies of X (resp. d copies of $R[\mathbf{x}]$). Let the symmetric group $\mathfrak{S}_d$ act on X^d by permutation. Then, the quotient

$$X^{(d)} = X^d/\mathfrak{S}_d$$

is an affine scheme over $\mathrm{Spec}(R)$ and $\mathfrak{S}_d$ acts transitively on each fiber of $X^d \to X^{(d)}$. Moreover, since $\mathrm{Spec}(R)$ is a Noetherian scheme, the natural projection $X^d \to X^{(d)}$ is finite [GoW10, p. 331, Prop. 12.27(4)].

The **fat diagonal** Δ of X^d is the closed subscheme such that

$$\Delta(L) = \bigcup_{i \neq j} \{(\mathbf{p}_1, \ldots, \mathbf{p}_d) \in X^d(L) \mid \mathbf{p}_i = \mathbf{p}_j\}$$

for every ring extension L of R. Note that $\mathfrak{S}_d$ leaves Δ invariant. Hence, it makes sense to set

$$U_d = (X^d \smallsetminus \Delta)/\mathfrak{S}_d.$$

Also, note that the inertia group in $\mathfrak{S}_d$ of each $(\mathbf{p}_1, \ldots, \mathbf{p}_d) \in X^d \smallsetminus \Delta$ is trivial. Hence, by [Liu06, p. 147, Exer. 4.3.19], the map $X^d \to X^{(d)}$ is étale along $X^d \smallsetminus \Delta$.

Now let S be an R-scheme. Since X is smooth over $\mathrm{Spec}(R)$ (Statement (6) of Section 5), [MoB89, (3.2.3)] says that there is a functorial bijection between $X^{(d)}(S)$ and

(1) the set of all effective Cartier divisors D on $X_S = X \times_{\mathrm{Spec}(R)} S$ that are finite and flat of degree d over S,

with $\deg(D)$ as defined in Subsection 2.2.

7.2 GLOBAL SECTIONS. We consider again the graded ring $R[\mathbf{t}] = R[t_0, \ldots, t_r]$ introduced in Lemma 5.6 such that $\bar{X} = \mathrm{Proj}(R[\mathbf{t}])$. We also consider the closed reduced subscheme $Z = \bar{X} \smallsetminus X$ introduced in Subsection 5.7 and recall that $Z = \mathrm{Proj}(R[\mathbf{t}]/I)$, where I is a homogeneous ideal of $R[\mathbf{t}]$ (Subsection 5.8). For each large positive integer k, Remark 1.4 gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R[\mathbf{t}]_k \cap I \arrow[r] \arrow[d, equal] & R[\mathbf{t}]_k \arrow[r, "\pi^{(k)}_{\bar{X},Z}"] \arrow[d, equal] & (R[\mathbf{t}]/I)_k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\rho^{(k)}_{\bar{X},Z}) \arrow[r] & \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k)) \arrow[r, "\rho^{(k)}_{\bar{X},Z}"] & \Gamma(Z, \mathcal{O}_Z(k)) \arrow[r] & 0
\end{tikzcd} \tag{2}$$

where the upper and lower rows are short exact sequences which have been identified via canonical maps. Also, $\pi^{(k)}_{\bar{X},Z}$ is the quotient map and $\rho^{(k)}_{\bar{X},Z}$ is the restriction map from $\bar{X}$ to Z. Changing the base from R

to a field L that contains K, Diagram (2) becomes

$$
\begin{tikzcd}
0 \arrow[r] & L[\mathbf{t}]_k \cap LI \arrow[r] \arrow[d, equal] & L[\mathbf{t}]_k \arrow[r, "\pi^{(k)}_{\bar{X}_L, Z_L}"] \arrow[d, equal] & (L[\mathbf{t}]/LI)_k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\rho^{(k)}_{\bar{X}_L, Z_L}) \arrow[r] & \Gamma(\bar{X}_L, \mathcal{O}_{\bar{X}_L}(k)) \arrow[r, "\rho^{(k)}_{\bar{X}_L, Z_L}"] & \Gamma(Z_L, \mathcal{O}_{Z_L}(k)) \arrow[r] & 0\,.
\end{tikzcd}
\tag{3}
$$

7.3 Generalized Picard functor. In this subsection we let L be a ring extension of R and consider the category $\mathcal{C}(L)$ whose objects are the couples $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible sheaf on $\bar{X}_L$ and $\alpha\colon \mathcal{O}_{Z_L} \to \mathcal{L}|_{Z_L}$ is an isomorphism. A morphism $(\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ between two objects of $\mathcal{C}(L)$ is an isomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ such that $\varphi|_{Z_L} \circ \alpha = \alpha'$.

In particular, if D is a Cartier divisor on $\bar{X}_L$ which is disjoint from Z_L (Subsection 2.3) and $(U_m, f_m)_{m\in M}$ is data that represent D, then for all $m \in M$ and $\mathbf{p} \in U_m \cap Z_L$, the image $f_{m,\mathbf{p}}$ of f_m in $\mathcal{O}_{\bar{X}_L,\mathbf{p}}$ is invertible, so $f_{m,\mathbf{p}}^{-1}\mathcal{O}_{\bar{X}_L,\mathbf{p}} = \mathcal{O}_{\bar{X}_L,\mathbf{p}}$. On the other hand, $\mathcal{L}(D)_\mathbf{p} = f_{m,\mathbf{p}}^{-1}\mathcal{O}_{\bar{X}_L,\mathbf{p}}$ (with $\mathcal{L}(D)$ as in Subsection 2.5), so $\mathcal{L}(D)_\mathbf{p} = \mathcal{O}_{\bar{X}_L,\mathbf{p}}$. It follows that $\mathcal{L}(D)|_{Z_L} \cong \mathcal{O}_{Z_L}$. Finally, since for each $m \in M$, $1_D|_{U_m}$ is the unit element of $\Gamma(U_m, \mathcal{L}(D))$ (Subsection 2.5), we may consider $1_D|_{Z_L}$ as the identity map $\mathcal{O}_{Z_L} \to \mathcal{L}(D)|_{Z_L}$. Thus, $(\mathcal{L}_{\bar{X}_L}(D), 1_D)$ is one of the objects of $\mathcal{C}(L)$ mentioned in the preceding paragraph.

If $(\mathcal{L}, \alpha), (\mathcal{L}', \alpha') \in \mathcal{C}(L)$, then $(\mathcal{L} \otimes_{\mathcal{O}_{\bar{X}_L}} \mathcal{L}', \alpha \otimes \alpha') \in \mathcal{C}(L)$ and morphisms of objects of $\mathcal{C}(L)$ commute with tensor products.

If L is a field extension of K, we denote for each non-negative integer d the subcategory of $\mathcal{C}(L)$ of all objects $(\mathcal{L}, \alpha)$ with $\deg(\mathcal{L}) = d$ by $\mathcal{C}_d(L)$.

We note in passing that [MoB89, Subsection 3.4] denotes the group of isomorphism classes of objects of $\mathcal{C}_d(L)$ by $\mathrm{PG}_d(\bar{X}, Z)(L)$ and call it the **generalized Picard functor** relative to Z.

7.4 Generalized Picard functors over $\hat{K}_\mathfrak{p}$. We use the convention of Subsection 5.1. For each $\mathfrak{p} \in \mathcal{T}$ let $\hat{L}_\mathfrak{p} = L_\mathfrak{p}\hat{K}_\mathfrak{p}$ and let $\hat{\Omega}_\mathfrak{p}^{[d]}$ be the set of effective Weil divisors D on $X_{\hat{K}_\mathfrak{p}}$ of degree d with $D_{\hat{L}_\mathfrak{p}} = \sum_{i=1}^d \mathbf{p}_i$, where $\mathbf{p}_1, \ldots, \mathbf{p}_d$ are distinct points in $\Omega_\mathfrak{p}(\hat{L}_\mathfrak{p})$ (notation of Subsection 4.4). Thus, D totally splits in $F\hat{L}_\mathfrak{p}$ in the sense of Subsection 3.2, where F is the function field of X_K introduced in Subsection 5.1. Moreover, each $D \in \hat{\Omega}_\mathfrak{p}^{[d]}$ can be considered as a point of $U_d(\hat{L}_\mathfrak{p})$ which is fixed under the action of $\mathrm{Gal}(\hat{L}_\mathfrak{p}/\hat{K}_\mathfrak{p})$. Therefore, $\hat{\Omega}_\mathfrak{p}^{[d]}$ may be viewed as a subset of $U_d(\hat{K}_\mathfrak{p})$ (notation of Subsection 7.1), hence of $X^{(d)}(\hat{K}_\mathfrak{p})$.

Next we let $W_\mathfrak{p}^{[d]}$ be the set of all pairs $(\mathcal{L}, \alpha) \in \mathcal{C}_d(\hat{K}_\mathfrak{p})$ that are equivalent to $(\mathcal{L}_{\bar{X}_{\hat{K}_\mathfrak{p}}}(D), 1_D)$ for some $D \in \hat{\Omega}_\mathfrak{p}^{[d]}$. We quote two results from [MoB89] that rely on the assumptions we made on X, $\bar{X}$, Z, and f in Section 5.

Lemma 7.5: *The following statements hold for each $\mathfrak{p} \in \mathcal{T}$.*

(a) *$\hat{\Omega}_\mathfrak{p}^{[d]}$ is $\mathfrak{p}$-open in $U_d(\hat{K}_\mathfrak{p})$ [MoB89, Lemma 3.3(a)].*

(b) *Let d and d' be non-negative integers such that $d \geq 2 \cdot \mathrm{genus}(\bar{X}_K) + \deg_K(Z_K)$ (see Subsection 5.9 for the definition of $\deg_K(Z_K)$). Then, $W_\mathfrak{p}^{[d]} W_\mathfrak{p}^{[d']} \subseteq W_\mathfrak{p}^{[d+d']}$, where the product on the left hand side is defined by the tensor product introduced in Subsection 7.3 [MoB89, Lemma 3.7.2(ii)].*

Next we draw a consequence of [MoB89, Lemma 3.8] and [MoB89, Lemma 3.9]. To that end we use [Har77, p. 117, Prop. II.5.12(c)] to identify $\mathcal{O}_{\bar{X}}(k)|_Z$ (which implicitly appears in the above mentioned lemmas of [MoB89]) with $\mathcal{O}_Z(k)$.

Proposition 7.6: *There exist a positive integer k_0 and an isomorphism $\alpha^{(k_0)}\colon \mathcal{O}_Z \to \mathcal{O}_Z(k_0)$ such that for each integral positive multiple k of k_0 the positive integer $d_k = \deg(\mathcal{O}_{\bar{X}_K}(k))$ (in the notation of Subsection 2.1) and the isomorphism $\alpha^{(k)} = (\alpha^{(k_0)})^{\otimes(k/k_0)}\colon \mathcal{O}_Z \to \mathcal{O}_Z(k)$ satisfy the following condition:*

There is a section $s_0^{(k)} \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$ such that, in the notation of Diagram (2),

(a) $\alpha^{(k)}(Z)(1) = \rho^{(k)}_{\bar{X},Z}(s^{(k)}_0)$, *where* $\alpha^{(k)}(Z)\colon \Gamma(Z,\mathcal{O}_Z) \to \Gamma(Z,\mathcal{O}_Z(k))$ *is the corresponding isomorphism of* $\Gamma(Z,\mathcal{O}_Z)$*-modules,*

(b) $(\mathcal{O}_{\bar{X}_{\hat{K}_\mathfrak{p}}}(k), \alpha^{(k)}_\mathfrak{p}) \in W^{[d_k]}_\mathfrak{p}$, *and*

(c) $\mathrm{div}(s^{(k)}_{0,\mathfrak{p}}) \in \hat{\Omega}^{[d_k]}_\mathfrak{p}$, *for each* $\mathfrak{p} \in \mathcal{T}$,

where 1 *is the unit element of the ring* $\Gamma(Z,\mathcal{O}_Z)$, *and* $\alpha^{(k)}_\mathfrak{p}$ *and* $s^{(k)}_{0,\mathfrak{p}}$ *are the isomorphism and the section obtained from* $\alpha^{(k)}$ *and* s_0 *by base change from* R *to* $\hat{K}_\mathfrak{p}$.

In addition, the identifications made in Diagrams (2) and (3) and their commutativity are valid for R *and for every field extension* L *of* K.

Proof: By [MoB89, Lemma 3.9], applied to the ample invertible sheaf $\mathcal{O}_{\bar{X}}(1)$ on $\bar{X}$ [GoW10, p. 386, Example 13.45] rather than to $\mathcal{M}_0$, there exist a positive integer k_0 and an isomorphism $\alpha^{(k_0)}\colon \mathcal{O}_Z \to \mathcal{O}_Z(k_0)$ such that

(4a) $d_0 = d_{k_0} = \deg(\mathcal{O}_{\bar{X}_K}(k_0)) \geq 2 \cdot \mathrm{genus}(\bar{X}_K) + \deg_K(Z_K)$ and

(4b) $(\mathcal{O}_{\bar{X}_{\hat{K}_\mathfrak{p}}}(k_0), \alpha^{(k_0)}_\mathfrak{p}) \in W^{[d_0]}_\mathfrak{p}$ for each $\mathfrak{p} \in \mathcal{T}$.

Now consider an integral positive multiple k of k_0 and let $k_1 = k/k_0$. Recall that $\mathcal{O}_{\bar{X}}(k)$ is naturally isomorphic to $\mathcal{O}_{\bar{X}}(k_0)^{\otimes k_1}$ [Har77, p. 117, Prop. II.5.12(b)] and $\mathcal{O}_{\bar{X}}(k_0)$ is a free $\mathcal{O}_{\bar{X}}$-module of rank 1, so $\alpha^{(k)} = (\alpha^{(k_0)})^{\otimes k_1}$ is an isomorphism of $\mathcal{O}_Z$ onto $\mathcal{O}_Z(k)$ and $d_k = k_1 d_0 = \deg(\mathcal{O}_{\bar{X}_K}(k))$ (Subsection 2.1). By (4a) and Lemma 7.5(b), $(W^{[d_0]}_\mathfrak{p})^{k_1} \subseteq W^{[d_k]}_\mathfrak{p}$ for each $\mathfrak{p} \in \mathcal{T}$. Hence, by (4b), Condition (b) holds for each $\mathfrak{p} \in \mathcal{T}$.

By [MoB89, Lemma 3.8], there exists $s^{(k)}_0 \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$ such that (a) and (c) are satisfied, as claimed.

The last assertion of the proposition holds if we eventually replace k_0 by a sufficiently large integral positive multiple of itself. ∎

7.7 Generators of global sections. In the notation of Proposition 7.6 let k be an integral positive multiple of k_0 and let

$$\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k), \alpha^{(k)}) = \{s \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k)) \mid \rho^{(k)}_{\bar{X},Z}(s) = \alpha^{(k)}(Z)(1)\}.$$

The bijection given in (1) for the scheme $\mathrm{Spec}(R)$ and the bijection given in [MoB89, p. 189, (3.5.4)] prove part (a) of the following result:

Lemma 7.8: *If* $s \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k), \alpha^{(k)})$, *then*

(a) $\mathrm{div}(s)$ *is an effective Cartier divisor on* X, *finite and flat over* $\mathrm{Spec}(R)$ *of degree* $d_k = \deg(\mathcal{O}_{\bar{X}_K}(k))$, *and*

(b) *each irreducible component of* $\mathrm{div}(s)$ *is finite and surjective over* $\mathrm{Spec}(R)$.

Proof of (b): Let $f\colon X \to \mathrm{Spec}(R)$ be the morphism introduced in Subsection 5.5, let Y be the closed subscheme of X attached to $\mathrm{div}(s)$ (Subsection 2.3). By Subsection 5.5, X is Noetherian, hence so is Y [Liu06, p. 55, Prop. 2.3.46(a)]. Consider an irreducible component Y' of Y. Since $f|_Y$ is finite, it is proper [GoW10, p. 344, Example 12.56(3)], hence closed. By (a), f is flat on Y. Hence, by [Liu06, p. 136, Lemma 4.3.7], $f(Y')$ is dense in $\mathrm{Spec}(R)$, so $f(Y') = \mathrm{Spec}(R)$. By [GoW10, p. 325, Prop. 12.11(1)], the closed immersion $Y' \to Y$ is finite. Composing it with $f|_Y$, we conclude that $f|_{Y'}$ is a finite morphism [GoW10, p. 325, Prop. 12.11(2)]. ∎

7.9 Divisors of sections in open sets. Let k be an integral positive multiple of k_0 and consider elements $s_1, \ldots, s_l$ in $\mathrm{Ker}(\rho^{(k)}_{\bar{X},Z})$, that is elements of $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$ that vanish on Z. Let $s^{(k)}_0$ be the section introduced in Proposition 7.6. We set $\mathbf{s} = (s^{(k)}_0, s_1, \ldots, s_l)$ and

$$\Gamma^{(k)}_\mathbf{s} = \{s^{(k)}_0 + \sum_{i=1}^{l} a_i s_i \mid a_1, \ldots, a_l \in R\}. \tag{5}$$

Then, $\Gamma_{\mathbf{s}}^{(k)} \subseteq \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k), \alpha^{(k)})$, hence Lemma 7.8 holds for every $s \in \Gamma_{\mathbf{s}}^{(k)}$.

For each $\mathfrak{p} \in \mathcal{T}$ and every algebraic extension K' of K let $\hat{\Omega}_{\mathfrak{p},K'}^{[d_k]}$ be the set of Cartier divisors D on $X_{\hat{K}_\mathfrak{p}K'}$ that are effective of degree $d_k = \deg(\mathcal{O}_{\bar{X}_K}(k))$, étale, totally split in $F\hat{L}_\mathfrak{p}K'$ in the sense of Subsection 3.2 (where F is the function field of X_K introduced in Subsection 5.1), whose irreducible $\hat{L}_\mathfrak{p}K'$-components are in $\Omega_\mathfrak{p}(\hat{L}_\mathfrak{p}K')$. We also set

(6) $\Gamma_{\mathbf{s},\mathfrak{p},K'}^{(k)}$ to be the set of all $s \in \Gamma(\bar{X}_{\hat{K}_\mathfrak{p}K'}, \mathcal{O}_{\bar{X}_{\hat{K}_\mathfrak{p}K'}}(k))$ of the form $s = s_0^{(k)} + \sum_{i=1}^{l} a_i s_i$ with $a_1, \ldots, a_l \in \hat{K}_\mathfrak{p}K'$ such that $\mathrm{div}(s) \in \hat{\Omega}_{\mathfrak{p},K'}^{[d_k]}$.

By Lemma 7.5(a), $\hat{\Omega}_{\mathfrak{p},K'}^{[d_k]}$ is $\mathfrak{p}$-open in $U_d(\hat{K}_\mathfrak{p}K')$. Hence, an application of Lemma 3.4 to the Galois extension $\hat{L}_\mathfrak{p}K'/\hat{K}_\mathfrak{p}K'$, with $\mathfrak{p} \in \mathcal{T}$, yields the following result:

Lemma 7.10: *Let k_0 be the integer introduced in Proposition 7.6, let k be an integral positive multiple of k_0, and let $s_0^{(k)}$ be an element of $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$ that Proposition 7.6 gives. In addition let $s_1, \ldots, s_l$ be elements of $\mathrm{Ker}(\rho_{\bar{X},Z}^{(k)})$ and set $\mathbf{s} = (s_0^{(k)}, s_1, \ldots, s_l)$. Then, there exists a positive real number γ such that if K' is a separable algebraic extension of K, then the following holds: If $a_1, \ldots, a_l \in K'$ satisfy $|a_i^\sigma|_\mathfrak{p} < \gamma$ for all $1 \le i \le l$, $\sigma \in \mathrm{Gal}(K)$, and $\mathfrak{p} \in \mathcal{T}$, then $s_0^{(k)} + \sum_{i=1}^{l} a_i s_i \in \Gamma_{\mathbf{s},\mathfrak{p},K'}^{(k)}$ for each $\mathfrak{p} \in \mathcal{T}$.*

8. A Stabilizing Element

Let K, F, R, X, $\bar{X}$, and Z be as in Subsections 4.1, 5.1, 5.2, 5.5, Lemma 5.6, and Subsection 5.7, respectively. In particular F is a finitely generated regular extension of K of transcendence degree 1. Thus, F has a transcendental element t over K such that $F/K(t)$ is a finite separable extension. Let $\hat{F}$ be the Galois closure of $F/K(t)$. We say that t **symmetrically stabilizes** F/K if $\mathrm{Gal}(\hat{F}\tilde{K}/\tilde{K}(t))$ is isomorphic to the symmetric group of rank $[F : K(t)]$. In this case $\mathrm{Gal}(\hat{F}\tilde{K}/\tilde{K}(t)) \cong \mathrm{Gal}(\hat{F}/K(t))$ [FrJ08, p. 391, Lemma 18.9.2], hence $\hat{F}/K$ is a regular extension. The existence of symmetrically stabilizing elements is proved in [GeJ89] in the case where F/K is conservative (in particular, if K is perfect), and in [Neu98] in the general case. In [GeJ02, Thm. 16.2] we prove that t can be chosen as a quotient of linear combinations of a basis of the linear space $\mathfrak{L}(D)$ (introduced just before Lemma 3.3) attached to a certain very ample divisor D of F/K. In this section and in the three following ones we refine that construction and choose the coefficient of the first element of the basis to be 1, keeping the other coefficients in given non-empty $\mathcal{T}$-open subsets of R, where $\mathcal{T}$ is a finite subset of $\mathcal{V}$ that contains $\mathcal{S}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. Here we call a subset U of R **$\mathcal{T}$-open** if U is the union of **basic $\mathcal{T}$-open sets**. The latter are intersections of $\mathfrak{p}$-open discs of K, where $\mathfrak{p}$ ranges over all elements of $\mathcal{T}$.

Our construction depends on the main result of [GJR17] that we now start to explain.

8.1 Matrices. Let $\mho$ be the universal field extension of K chosen in Subsection 4.1. For each pair (i, j) of positive integers we consider the affine variety $\mathbb{M}_{ij}$ over K such that the set $\mathbb{M}_{ij}(\mho)$ consists of all $i \times j$ matrices with entries in $\mho$. Thus, $\mathbb{M}_{ij}$ is naturally isomorphic to the affine space $\mathbb{A}_K^{ij}$. If $i \le j$, we write $\mathbb{M}_{ij}^*$ for the non-empty Zariski-open subset of $\mathbb{M}_{ij}$ consisting of all matrices in $\mathbb{M}_{ij}$ of rank i, i.e. with linearly independent rows. We fix a positive integer l for this section, let

$$\mathbb{M}^{(l)} = \mathbb{M}_{2,3}^* \times \mathbb{M}_{3,4}^* \times \cdots \times \mathbb{M}_{l,l+1}^*,$$

and define a morphism $\mu^{(l)} \colon \mathbb{M}^{(l)} \to \mathbb{M}_{2,l+1}$ by multiplication:

$$\mu^{(l)}(A_2, A_3, \ldots, A_l) = A_2 A_3 \cdots A_l\,,$$

and observe that actually $\mu^{(l)}$ maps $\mathbb{M}^{(l)}(K)$ onto $\mathbb{M}_{2,l+1}^*(K)$ [GeJ02, §3]. For each $i \ge 2$, we define a map $\psi_i \colon \mathbb{M}_{i,i+1}^* \to \mathbb{P}^i$ mapping each $A \in \mathbb{M}_{i,i+1}^*(\mho)$ onto the unique point $(y_0 : \cdots : y_i)$ of $\mathbb{P}^i(\mho)$ that satisfies $A \begin{pmatrix} y_0 \\ \vdots \\ y_i \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Kramer's rule implies that ψ_i is a morphism. Let $\mathbb{P}^{(l)} = \mathbb{P}^2 \times \cdots \times \mathbb{P}^l$ and

$\psi^{(l)} = \psi_2 \times \cdots \times \psi_l \colon \mathbb{M}^{(l)} \to \mathbb{P}^{(l)}$. Then $\psi^{(l)}$ is a morphism that maps $\mathbb{M}^{(l)}(K)$ onto $\mathbb{P}^{(l)}(K)$ [GeJ02, §3]. Both maps from $\mathbb{M}^{(l)}$ appear in the following row:

$$\mathbb{P}^{(l)} \xleftarrow{\psi^{(l)}} \mathbb{M}^{(l)} \xrightarrow{\mu^{(l)}} M^*_{2,l+1}\,. \tag{1}$$

8.2 F. K. Schmidt's derivatives. Let $\Delta = \mathrm{Proj}(K[s_0, s_1, \dots, s_l])$ be an absolutely integral projective curve in $\mathbb{P}^l_K$ with function field F, where $K[s_0, s_1, \dots, s_l]$ is a graded domain over K with $K[s_0, s_1, \dots, s_l]_1 = \sum_{i=0}^{l} Ks_i$. We set $\tilde{\Delta} = \Delta_{\tilde{K}} = \mathrm{Proj}(\tilde{K}[s_0, s_1, \dots, s_l])$.

Over each point $\mathbf{p} \in \tilde{\Delta}(\tilde{K})$ there lie only finitely many prime divisors $P_1, \dots, P_e$ of $F\tilde{K}/\tilde{K}$ (alternatively, finitely many points of the normalization of $\tilde{\Delta}$), with $e \geq 1$. For each $1 \leq i \leq e$ let $\mathfrak{m}_i$ be the maximal ideal of the discrete valuation ring $\mathcal{O}_i$ of $F\tilde{K}$ that corresponds to P_i and let π_i be a generator of $\mathfrak{m}_i$. Then, $\mathcal{O}_{\tilde{\Delta},\mathbf{p}} \subseteq \mathcal{O}_i$ and $\mathfrak{m}_i \cap \mathcal{O}_{\tilde{\Delta},\mathbf{p}} = \mathfrak{m}_{\tilde{\Delta},\mathbf{p}}$. We identify $\mathcal{O}_i/\mathfrak{m}_i$ with $\tilde{K}$. If an element f of $F\tilde{K}$ belongs to $\mathcal{O}_i$, we denote its residue modulo $\mathfrak{m}_i$ in $\tilde{K}$ by $f(P_i)$, otherwise we set $f(P_i) = \infty$. In the former case, one may express f as a formal power series $f = \sum_{k=0}^{\infty} \frac{D^k f}{D\pi_i^k}(P_i)\pi_i^k$, with coefficients in $\tilde{K}$, where $\frac{D^k f}{D\pi_i^k}$ is an element of $\mathcal{O}_i$ called the **F. K. Schmidt derivative** of degree k of f with respect to P_i [GJR17, Section 4].

8.3 Characteristic-0 like curves. For each $1 \leq i \leq e$ there exists $u_i \in \tilde{K}(s_0, \dots, s_l)$ such that for each $0 \leq j \leq l$ we have $u_i s_j \in \mathcal{O}_i$ and there is $0 \leq j' \leq l$ such that $u_i s_{j'} \notin \mathfrak{m}_i$. Then, we write $\mathbf{s}(P_i)$ for the point

$$\mathbf{p} = (u_i\mathbf{s})(P_i) = ((u_i s_0)(P_i){:}(u_i s_1)(P_i){:}\cdots{:}(u_i s_l)(P_i))$$

of $\tilde{\Delta}(\tilde{K})$ and note that $\mathbf{p}$ does not depend on u_i. However, for each $k \geq 1$, the expression $\frac{D^k(u_i\mathbf{s})}{D\pi_i^k}(P_i)$ may depend on u_i and on π_i. Nevertheless, we denote it by $\mathbf{s}^{[k]}(P_i)$ and make sure that each of the objects that depend on this symbol does not depend on u_i nor on π_i.

For example, by [GJR17, Lemma 4.2], the condition

$$\mathrm{rank}(\mathbf{p}\ \mathbf{s}^{[1]}(P_i)) = 2 \tag{2}$$

(where both $\mathbf{p}$ and $\mathbf{s}^{[1]}(P_i)$ are considered here as columns of height $l+1$ and $(\mathbf{p}\ \mathbf{s}^{[1]}(P_i))$ is the corresponding $(l+1) \times 2$ matrix) is independent of u_i and π_i. By [GJR17, Lemma 5.1], Condition (2) is equivalent to the condition that $\mathbf{s}^{[1]}(P_i)$ is not a column of zeros. Thus, the latter condition is also independent of u_i and π_i. By [GJR17, Lemma 5.2], $\mathbf{p}$ is a simple point of $\tilde{\Delta}$ if and only if $F\tilde{K}/\tilde{K}$ has a unique prime divisor P over $\mathbf{p}$ and $\mathbf{s}^{[1]}(P)$ is not a column of zeros. In this case we write $\mathbf{s}^{[1]}(\mathbf{p})$ for $\mathbf{s}^{[1]}(P)$. Then, the linear form $\mathbf{p}Y_0 + \mathbf{s}^{[1]}(\mathbf{p})Y_1$ is a parametric presentation of the tangent $T_{\tilde{\Delta},\mathbf{p}}$ to $\tilde{\Delta}$ at $\mathbf{p}$.

We say that $\mathbf{p}$ is an **inflection point** of $\tilde{\Delta}$ if $\mathbf{p}$ is simple and

$$\mathrm{rank}(\mathbf{p}\ \mathbf{s}^{[1]}(\mathbf{p})\ \mathbf{s}^{[2]}(\mathbf{p})) = 2.$$

Again, by [GJR17, Lemma 4.2], this condition is independent of the parameters. By [GeJ89, Lemma 3.1 and the paragraph before Lemma 1.1], our definition of an inflection point coincides with the traditional one if $\tilde{\Delta}$ is a plane curve [Har77, p. 148].

If $\mathrm{char}(K) = 0$, then $\tilde{\Delta}$ has only finitely many **double tangents** (i.e. tangents at two simple points or more) and only finitely many inflection points. Moreover, if $\tilde{\Delta}$ is not a line, it is **non-strange**. This means that there exists no point in $\mathbb{P}^l(\tilde{K})$ through which infinitely many tangents to $\tilde{\Delta}$ at simple points go [GJR17, first paragraph of Section 11]. In positive characteristic one or more of these properties may fail for some curves. So, we say for arbitrary characteristic that Δ is a **characteristic-0-like curve** if $\tilde{\Delta}$ has only finitely many double tangents, finitely many inflection points, and it is non-strange.

The point $\mathbf{p}$ is a **cusp** of $\tilde{\Delta}$ if $\mathbf{p}$ is singular and $F\tilde{K}/\tilde{K}$ has a unique prime divisor that lies over $\mathbf{p}$.

8.4 MULTIPLICITIES. Consider a point $\mathbf{p} \in \tilde\Delta(\tilde K)$ and let $\mathfrak{m} = \mathfrak{m}_{\tilde\Delta,\mathbf{p}}$ be the maximal ideal of the local ring $\mathcal{O}_{\tilde\Delta,\mathbf{p}}$. Let $P_1, \ldots, P_e$ be the distinct prime divisors of $F\tilde K/\tilde K$ that lie over $\mathbf{p}$. For each $1 \le i \le e$ we define the **multiplicity** of $\tilde\Delta$ at P_i as

$$\mathrm{mult}(\tilde\Delta, P_i) = \min_{a\in\mathfrak{m}} \mathrm{ord}_{P_i}(a),$$

where ord_{P_i} is the normalized discrete valuation of $F\tilde K/\tilde K$ attached to P_i. We also note that $\dim_{\tilde K} \mathfrak{m}^k/\mathfrak{m}^{k+1}$ becomes a constant positive integer for all large positive integers k [GJR17, Remark 6.2]. We call that integer the **multiplicity of $\tilde\Delta$ at $\mathbf{p}$** and denote it by $\mathrm{mult}(\tilde\Delta, \mathbf{p})$. Thus,

$$\mathrm{mult}(\tilde\Delta, \mathbf{p}) = \dim_{\tilde K} \mathfrak{m}^k/\mathfrak{m}^{k+1}$$

for each large k. By [GJR17, Lemma 6.4],

$$\mathrm{mult}(\tilde\Delta, \mathbf{p}) = \sum_{i=1}^{e} \mathrm{mult}(\tilde\Delta, P_i).$$

In particular, if $\mathbf{p}$ is normal (i.e., in this case, simple), $F\tilde K/\tilde K$ has a unique prime divisor P over $\mathbf{p}$ and

$$\mathrm{mult}(\tilde\Delta, \mathbf{p}) = \mathrm{mult}(\tilde\Delta, P) = \min_{a\in\mathfrak{m}} \mathrm{ord}_P(a) = 1.$$

If $F\tilde K/\tilde K$ has a unique prime divisor P that lies over $\mathbf{p}$ and $\mathrm{mult}(\tilde\Delta, P) > 1$, then $\mathcal{O}_{\tilde\Delta,\mathbf{p}}$ is a proper subring of the valuation ring of $F\tilde K/\tilde K$ at P, so $\mathcal{O}_{\tilde\Delta,\mathbf{p}}$ is not a discrete valuation ring of $F\tilde K/\tilde K$. Hence, $\mathbf{p}$ is a singular point of $\tilde\Delta$, so $\mathbf{p}$ is a cusp of $\tilde\Delta$.

Definition 8.5: Let q be a positive integer. A **q-curve** over $\tilde K$ is an integral projective curve $\tilde\Delta$ over $\tilde K$ which
(3a) is characteristic-0-like,
(3b) has a cusp of multiplicity q, and
(3c) $\max_{\mathbf{q}\in\tilde\Delta(\tilde K)} \mathrm{mult}(\tilde\Delta, \mathbf{q}) = q$. ∎

We may now quote [GJR17, Thm. 16.1] for our global field K:

PROPOSITION 8.6: *Let $\Delta = \mathrm{Proj}(K[s_0, \ldots, s_l])$ be an absolutely integral projective curve in $\mathbb{P}^l_K$, where $K[s_0, \ldots, s_l]$ is a graded ring over K with $K[s_0, \ldots, s_l]_1 = \sum_{i=0}^l K s_i$. Let F be the function field of Δ and suppose that $\tilde\Delta = \Delta_{\tilde K}$ is a q-curve for some prime number q.*

Then, there exists a non-empty Zariski-open subset U_i of $\mathbb{P}^i_{\tilde K}$, $i = 2, 3, \ldots, l$, such that with $U = U_2 \times U_3 \times \cdots \times U_l \subseteq \mathbb{P}^{(l)}$, for each $A \in (\psi^{(l)})^{-1}(U(K))$ and with $\mu^{(l)}(A) = \begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$, the element $t = \sum_{i=0}^l a_i s_i / \sum_{i=0}^l b_i s_i$ $[F : K(t)]$-symmetrically stabilizes F/K.

Remark 8.7: Theorem 16.1 of [GJR17] assumes that $s_0, s_1, \ldots, s_l$ are elements of F. We may achieve this condition by choosing a non-zero element s' of $\sum_{i=0}^l K s_i$. Then, $(\frac{s_0}{s'} : \cdots : \frac{s_l}{s'})$ is a generic point of Δ with coordinates in F and $\sum_{i=0}^l a_i \frac{s_i}{s'} / \sum_{i=0}^l b_i \frac{s_i}{s'} = t$. ∎

9. Homogeneous Generic Point

In the next section we construct a birational morphism of $\bar X_K$ onto a q-curve Y over K, with a large prime number q, on which Proposition 8.6 will be applied. The aim of this section is to construct the homogeneous coordinates of the generic point of Y.

Recall from Lemma 5.6 that $\bar X = \mathrm{Proj}(R[\mathbf{t}])$, where $\mathbf{t} = (t_0, \ldots, t_r)$, $R[\mathbf{t}]$ is a graded ring with $R[\mathbf{t}]_0 = R$, $R[\mathbf{t}]_1 = \sum_{i=0}^r R t_i$, and $t_0, \ldots, t_r \neq 0$.

LEMMA 9.1: *Let k be a positive integer and $s_0, \dots, s_{r'}$ non-zero generators of the K-vector-space $K[\mathbf{t}]_k$. Then, for every $0 \le i \le r$ and $0 \le i' \le r'$ we have*

$$K\Big(\frac{t_0}{t_i}, \dots, \frac{t_r}{t_i}\Big) = K\Big(\frac{s_0}{s_{i'}}, \dots, \frac{s_{r'}}{s_{i'}}\Big). \tag{1}$$

Proof: The left hand side of (1) is the function field F of X and of $\bar{X}$. For each $0 \le i' \le r'$ there exists a homogeneous polynomial $f_{i'} \in K[T_0, \dots, T_r]$ of degree k with $s_{i'} = f_{i'}(\mathbf{t})$. Hence, for each $0 \le j' \le r'$ we have

$$\frac{s_{j'}}{s_{i'}} = \frac{f_{j'}(t_0, \dots, t_r)}{t_0^k} \Big/ \frac{f_{i'}(t_0, \dots, t_r)}{t_0^k} = f_{j'}\big(1, \frac{t_1}{t_0}, \dots, \frac{t_r}{t_0}\big) \Big/ f_{i'}\big(1, \frac{t_1}{t_0}, \dots, \frac{t_r}{t_0}\big) \in F. \tag{2}$$

Conversely, we denote the right hand side of (1) by F'. For each $0 \le i \le r$ there exist $a_0, \dots, a_{r'}$ and $b_0, \dots, b_{r'}$ in K such that $t_i t_0^{k-1} = a_0 s_0 + \cdots + a_{r'} s_{r'}$ and $t_0^k = b_0 s_0 + \cdots + b_{r'} s_{r'}$. Then,

$$\frac{t_i}{t_0} = \frac{t_i t_0^{k-1}}{t_0^k} = \frac{a_0 s_0 + \cdots + a_{r'} s_{r'}}{s_{i'}} \Big/ \frac{b_0 s_0 + \cdots + b_{r'} s_{r'}}{s_{i'}} \in F'. \tag{3}$$

It follows from (2) and (3) that $F = F'$, as claimed. ∎

The following result is [GJR17, Prop. 19.1]:

PROPOSITION 9.2: *Let F be an algebraic function field of one variable over $\tilde{K}$ and consider an element $t \in F^\times$. Let $\mathbf{s} = (s_0{:}s_1{:}\cdots{:}s_m)$ be a generic point of an integral projective curve Δ in $\mathbb{P}^m_{\tilde{K}}$ with $s_0, s_1, \dots, s_m \in F$. Let $\mathbf{x}' = (x'_0{:}x'_1{:}\cdots{:}x'_{n'})$ be a generic point of an integral projective curve Λ in $\mathbb{P}^{n'}_{\tilde{K}}$ with $x'_0, x'_1, \dots, x'_{n'} \in F$. Suppose Δ is characteristic-0-like. In addition suppose that for each $(j,k) \in \{0, \dots, m\} \times \{0, \dots, n'\}$ there exists $a_{jk} \in \tilde{K}$ such that $ts_j = \sum_{k=0}^{n'} a_{jk} x'_k$. Then, Λ is also characteristic-0-like curve.*

Setup 9.3: Let R be the principal ideal domain with quotient field K introduced in Subsection 5.2, X the affine scheme over R introduced in Subsection 5.5, and $\bar{X}$ the projective scheme over R introduced in Lemma 5.6. Subsection 6.1 introduces a separable point B of X that we consider as a homogeneous prime ideal of $R[\mathbf{t}]$ and a point $\mathbf{b} = (1{:}b_1{:}\cdots{:}b_r)$ of $X(K_{\mathrm{sep}})$ that lies over B with $b_1, \dots, b_r$ integral over R. As in Subsection 6.2, we set $E = K(\mathbf{b}) = K(B)$ and let $R_E = \mathcal{O}_{E,\mathcal{V} \smallsetminus \mathcal{T}}$ be the integral closure of R in E.

As in Subsection 6.2, let $w_1, \dots, w_e$ be an R-basis of R_E (hence, also a K-basis of E) and let $\sigma_1, \dots, \sigma_e$ be elements of $\mathrm{Aut}(\tilde{K}(\mathbf{t})/K(\mathbf{t}))$ whose restrictions to E are the distinct K-embeddings of E into $\tilde{K}$ and σ_1 is the identity map of E. The choices made in that subsection imply that $\det(w_i^{\sigma_j})_{i,j=1,\dots,e}$ is invertible in the integral closure $\tilde{R}$ of R in $\tilde{K}$ and the ring $R_E\big[\frac{\mathbf{t}}{t_0}\big]$ is integrally closed. Finally,

(4a) we consider the simple points $\tilde{B}_j = \sum_{i=1}^r \tilde{K}[\mathbf{t}](t_i - b_i^{\sigma_j} t_0)$, $j = 1, \dots, e$, of $X_{\tilde{K}}$ that lie over B and the corresponding points $\mathbf{b}_j = (1{:}b_1^{\sigma_j}{:}\cdots{:}b_r^{\sigma_j}) = \mathbf{b}^{\sigma_j}$ of $X(\tilde{K})$ (so, in the notation of Subsection 6.10, $Z_B(\tilde{K}) = \{\mathbf{b}_1, \dots, \mathbf{b}_e\}$). Note that since $E = K(\mathbf{b})$ is a separable extension of K of degree e, the points $\mathbf{b}_1, \dots, \mathbf{b}_e$ form a complete system of conjugate separable points of $\bar{X}(\tilde{K})$ that lie over B and none of the ideals $\tilde{B}_1, \dots, \tilde{B}_e$ of $\tilde{K}[\mathbf{t}]$ contains another one,

(4b) we consider the homogeneous ideals $B' = \sum_{i=1}^r R_E[\mathbf{t}](t_i - b_i t_0) = R_E[\mathbf{t}] \cap \tilde{B}_1$ (Lemma 6.4) and $B'' = \bigcap_{j=2}^e R_E[\mathbf{t}] \cap \tilde{B}_j$ of $R_E[\mathbf{t}]$ introduced in Notation 6.3 that satisfy $V_+(B') \cap V_+(B'') = \emptyset$ (Lemma 6.8),

(4c) we consider the positive integer k_0 mentioned in Proposition 7.6,

(4d) we recall that $Z = \mathrm{Proj}(R[\mathbf{t}]/I)$, where I is a non-zero homogeneous ideal of $R[\mathbf{t}]$ (Subsection 5.8) such that $I \not\subseteq B$ (Subsection 6.10), choose a non-zero homogeneous element s_I of $I \smallsetminus B$, and set $k_I = \deg_{K[\mathbf{t}]}(s_I)$, and

(4e) for each large multiple k of k_0, we consider the isomorphism $\alpha^{(k)}(Z){:}\ \Gamma(Z, \mathcal{O}_Z) \to \Gamma(Z, \mathcal{O}_Z(k))$ that appears in Proposition 7.6 and the homomorphism

$$\rho^{(k)}_{\bar{X},Z}{:}\ \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k)) \to \Gamma(Z, \mathcal{O}_Z(k))$$

that appears in the commutative diagram (2) in Subsection 7.2. ∎

LEMMA 9.4: *Under Setup 9.3, let $a_1,\dots,a_r$ be elements of R and set*

$$\tilde{s}=\prod_{j=1}^{e}(a_1(t_1-b_1^{\sigma_j}t_0)+\cdots+a_r(t_r-b_r^{\sigma_j}t_0)).$$

Then, $\tilde{s}\in R[\mathbf{t}]$.

Proof: We consider the independent variables $T_0,\dots,T_r$ and the element

$$\tilde{S}=\prod_{j=1}^{e}(a_1(T_1-b_1^{\sigma_j}T_0)+\cdots+a_r(T_r-b_r^{\sigma_j}T_0)) \tag{5}$$

of $\tilde{K}(\mathbf{T})$, where $\mathbf{T}=(T_0,\dots,T_r)$. Using the distributive law we may rewrite (5) as

$$\tilde{S}=\sum_{i=1}^{m}h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})\mu_i(\mathbf{T}), \tag{6}$$

where $h_1,\dots,h_m$ are polynomials with coefficients in R, $\underline{b}=(b_1,\dots,b_r)$, and $\mu_1(\mathbf{T}),\dots,\mu_m(\mathbf{T})$ are distinct monomials in $T_0,\dots,T_r$ of degree e.

We extend $\sigma_1,\dots,\sigma_e$ to elements of $G=\mathrm{Aut}(\tilde{K}(\mathbf{t},\mathbf{T})/K(\mathbf{t},\mathbf{T}))$ with the same names. Since $b_1,\dots,b_r\in E$ (Setup 9.3), the choice of $\sigma_1,\dots,\sigma_e$, implies for each $\tau\in G$ that the e-tuple $(\underline{b}^{\sigma_1\tau},\dots,\underline{b}^{\sigma_e\tau})$ is a permutation of $(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})$. Therefore, applying τ on (5) gives $\tilde{S}^\tau=\tilde{S}$. On the other hand, applying τ on (6) gives $\tilde{S}^\tau=\sum_{i=1}^m h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})^\tau\mu_i(\mathbf{T})$. Hence, $\sum_{i=1}^m h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})\mu_i(\mathbf{T})=\tilde{S}=\tilde{S}^\tau=\sum_{i=1}^m h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})^\tau\mu_i(\mathbf{T})$. Since $\mu_1(\mathbf{T}),\dots,\mu_m(\mathbf{T})$ are linearly independent over $\tilde{K}$, we get $h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})^\tau=h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})$ for $i=1,\dots,m$. Since $b_1,\dots,b_r\in K_{\mathrm{sep}}$ (Setup 9.3), we get that $h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})\in K$ for $i=1,\dots,m$. Since $h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})$ are integral over R (because $b_1,\dots,b_r$ are integral over R, as mentioned in Setup 9.3) and R is integrally closed (Subsection 5.2), we have $h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})\in R$ for $i=1,\dots,m$.

Finally, we observe that the specialization $\mathbf{T}\to\mathbf{t}$, extends to a $\tilde{K}$-homomorphism $\varphi\colon\tilde{K}[\mathbf{T}]\to\tilde{K}[\mathbf{t}]$. It follows from (5) and (6) that $\tilde{s}=\varphi(\tilde{S})=\sum_{i=1}^m h_i(\underline{b}^{\sigma_1},\dots,\underline{b}^{\sigma_e})\mu_i(\mathbf{t})\in R[\mathbf{t}]$, as claimed. ∎

LEMMA 9.5: *Under Setup 9.3, let q be a positive integer and let k be a large multiple of k_0. Then, $R[\mathbf{t}]_k=\Gamma(\bar{X},\mathcal{O}_{\bar{X}}(k))$ has elements $s_0^{(k)},s_1^{(k)},\dots,s_{l(k)}^{(k)}$ with $l(k)\ge e$ such that the following holds:*

(a) $s_0^{(k)}|_Z=\rho_{\bar{X},Z}^{(k)}(s_0^{(k)})=\alpha^{(k)}(Z)(1)\neq 0$. *Moreover, $s_0^{(k)}$ vanishes at no point of $Z(\tilde{K})$ and $\mathrm{div}(s_{0,\mathfrak{p}}^{(k)})\in\hat{\Omega}_{\mathfrak{p}}^{[d_k]}$ (notation of Proposition 7.6) for each $\mathfrak{p}\in\mathcal{T}$.*

(b) $s_0^{(k)}\notin\tilde{B}_j$ *for $j=1,\dots,e$.*

(c) $s_i^{(k)}|_Z=0$, *so $s_i^{(k)}\in I$, hence $s_i^{(k)}\in I_j$ for $i=1,\dots,l(k)$ and $j=1,\dots,d(Z)$ (in the notation of Subsection 5.8).*

(d) $s_i^{(k)}\equiv w_i^{\sigma_j}s_0^{(k)}\mod\tilde{B}_j^q$, *in particular $s_i^{(k)}(\tilde{B}_j)=w_i^{\sigma_j}s_0^{(k)}(\tilde{B}_j)$, for $i,j=1,\dots,e$.*

(e) $s_i^{(k)}\in\tilde{B}_j^q$ *for $i=e+1,\dots,l(k)$ and $j=1,\dots,e$.*

(f) $s_{e+1}^{(k)},\dots,s_{l(k)}^{(k)}$ *form an R-basis for the free R-module $L^{(k)}=\mathrm{Ker}(\rho^{(k)}_{\bar{X},Z\cup Z_{qB}})=R[\mathbf{t}]_k\cap I\cap B^q$, hence also a K-basis for the vector space $L_K^{(k)}=\mathrm{Ker}(\rho^{(k)}_{\bar{X}_K,Z_K\cup Z_{qB,K}})=K[\mathbf{t}]_k\cap KI\cap KB^q$ over K.*

(g) *The function field of $\mathrm{Proj}(K[s_0^{(k)},\dots,s_{l(k)}^{(k)}])$ is F.*

(h) $\mathrm{Proj}(\tilde{K}[s_0^{(k)},\dots,s_{l(k)}^{(k)}])$ *is a characteristic-0-like integral projective curve in $\mathbb{P}_{\tilde{K}}^{l(k)}$.*

Proof: We break up the proof into several parts.

Part A: *Choosing* $s_0^{(k)} \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$. Let k be a large multiple of k_0. Let Z_B be the closed reduced subscheme $\mathrm{Proj}(R[\mathbf{t}]/B)$ of $\bar{X}$ (introduced in Subsection 6.10). Since X and Z are disjoint (Subsection 5.7) and Z_B is a closed subscheme of $\bar{X}$ which is contained in X (Subsection 6.10), restriction of sections gives rise (by Lemma 1.5) to an epimorphism

$$\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k)) \longrightarrow \Gamma(Z, \mathcal{O}_Z(k)) \times \Gamma(Z_B, \mathcal{O}_{Z_B}(k)). \tag{7}$$

Recall that we are identifying $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$, $\Gamma(Z, \mathcal{O}_Z(k))$, and $\Gamma(Z_B, \mathcal{O}_{Z_B}(k))$ with $R[\mathbf{t}]_k$, $R[\mathbf{t}]_k/(R[\mathbf{t}]_k \cap I)$, and $R[\mathbf{t}]_k/(R[\mathbf{t}]_k \cap B)$, respectively (Remark 1.4). The restriction maps of (7) are replaced under these identifications by the quotient maps. Thus, in these terms, the epimorphism (7) is given by

$$s \mapsto (s + (R[\mathbf{t}]_k \cap I), s + (R[\mathbf{t}]_k \cap B)).$$

By Proposition 7.6, there exists an isomorphism of sheaves $\alpha^{(k)}\colon \mathcal{O}_Z \to \mathcal{O}_Z(k)$ of $\mathcal{O}_Z$-modules such that $\Gamma(Z, \mathcal{O}_Z(k)) = \alpha^{(k)}(Z)(1) \cdot \Gamma(Z, \mathcal{O}_Z)$, where 1 is the unit element of the ring $\Gamma(Z, \mathcal{O}_Z)$. Moreover, there exists $s_0^{(k)} \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k)) = R[\mathbf{t}]_k$ with

(8) $s_0^{(k)}|_Z = \rho_{\bar{X},Z}^{(k)}(s_0^{(k)}) = \alpha^{(k)}(Z)(1) \neq 0$. Also, the germ $\alpha^{(k)}(Z)(1)_P$ of $\alpha^{(k)}(Z)(1)$ at each point $P \in Z$ is non-zero, so $s_0^{(k)}$ vanishes at no point of $Z(\tilde{K})$,

as stated in (a). Moreover,

(9) $\mathrm{div}(s_{0,\mathfrak{p}}^{(k)}) \in \hat{\Omega}_{\mathfrak{p}}^{[d_k]}$ for each $\mathfrak{p} \in \mathcal{T}$, where $d_k = \deg(\mathcal{O}_{\bar{X}_K}(k))$.

We choose by (7) a section $s_{IB} \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k)) = R[\mathbf{t}]_k$ that belongs to I but not to B. By Lemma 7.10, we may replace $s_0^{(k)}$, if necessary, by $s_0^{(k)} + as_{IB}$ with $a \in R$ which is sufficiently $\mathcal{T}$-close to 0 to assume that, in addition to (8) and (9),

(10) $s_0^{(k)} \notin B$. Hence, by Lemma 6.5, $s_0^{(k)} \notin \tilde{B}_j$ for $j = 1, \ldots, e$,

so (b) holds.

Part B: *Choosing* $s'_1, \ldots, s'_e \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$. We use Setup 9.3(4b) to set

$$Z_{qB'} = \mathrm{Proj}(R_E[\mathbf{t}]/(B')^q) \text{ and } Z_{qB''} = \mathrm{Proj}(R_E[\mathbf{t}]/(B'')^q).$$

Both are disjoint closed subschemes of $\bar{X}_{R_E}$ which are contained in X_{R_E}, so are disjoint from the closed subscheme Z_{R_E} of $\bar{X}_{R_E}$. Hence, by Lemma 1.5, restriction of sections gives rise to an epimorphism

$$\Gamma(\bar{X}_{R_E}, \mathcal{O}_{\bar{X}_{R_E}}(k)) \to \Gamma(Z_{R_E}, \mathcal{O}_{Z_{R_E}}(k)) \times \Gamma(Z_{qB'}, \mathcal{O}_{Z_{qB'}}(k)) \times \Gamma(Z_{qB''}, \mathcal{O}_{Z_{qB''}}(k)). \tag{11}$$

Thus, there exists $s'_1 \in \Gamma(\bar{X}_{R_E}, \mathcal{O}_{\bar{X}_{R_E}}(k)) = R_E[\mathbf{t}]_k$ such that $s'_1|_{Z_{R_E}} = 0$, $s'_1|_{Z_{qB'}} = s_0^{(k)}|_{Z_{qB'}}$, and $s'_1|_{Z_{qB''}} = 0$. Then, for each large multiple k of k_0, we have by Remark 1.4 that

$$s'_1 \in R_E I, \quad s'_1 - s_0^{(k)} \in (B')^q, \quad \text{and} \quad s'_1 \in (B'')^q. \tag{12}$$

This implies that $s'_1 \notin \tilde{B}_1$ (otherwise it would follow from $s'_1 - s_0^{(k)} \in (B')^q \subseteq B' \subseteq \tilde{B}_1$ that $s_0^{(k)} \in R[\mathbf{t}] \cap \tilde{B}_1 = B$ (Lemma 6.5), which contradicts (10)) and $s'_1 \in \tilde{B}_j^q$ for $j = 2, \ldots, e$.

Next we write $s'_1 = f'_1(\mathbf{t})$, where $f'_1 \in R_E[T_0, \ldots, T_r]_k$ and recall that $\tilde{R} = \mathcal{O}_{\tilde{K},\mathcal{V} \smallsetminus \mathcal{T}}$ is the integral closure of R in $\tilde{K}$ (Subsection 4.5). We set $s'_j = (s'_1)^{\sigma_j} = (f'_1)^{\sigma_j}(\mathbf{t}) \in \tilde{R}[\mathbf{t}]_k$ for $j = 2, \ldots, e$. Then, by the preceding paragraph,

(13) $s'_j|_{Z_{\tilde{R}}} = 0$, $s'_j - s_0^{(k)} \in \tilde{B}_j^q$, and $s'_{j'} \in \tilde{B}_j^q$ for $j' \neq j$. In particular, by (10), $s'_j(\tilde{B}_j) = s_0^{(k)}(\tilde{B}_j) \neq 0$ for $j = 1, \ldots, e$ and $s'_j(\tilde{B}_{j'}) = 0$ for $j' \neq j$.

Part C: *Choosing* $s_1^{(k)},\dots,s_e^{(k)}$. For each $1\le i\le e$ let

(14) $s_i^{(k)}=\sum_{j=1}^e w_i^{\sigma_j}s'_j=\sum_{j=1}^e (w_if'_1)^{\sigma_j}(\mathbf{t})$.

Then, each of the coefficients of the monomials in $t_0,\dots,t_r$ on the right hand side of (14) is an element of K which is integral over R. Since the latter ring is integrally closed (Subsection 5.2), each of those coefficients belong to R. Hence, $s_i^{(k)}\in R[\mathbf{t}]$. Moreover, since $f'_1\in R_E[T_0,\dots,T_r]_k$, we have $s_i^{(k)}\in R[\mathbf{t}]_k$. By (13),

(15) $s_i^{(k)}|_Z=0$ for $i=1,\dots,e$,

as stated in (c). Again, by (13),

(16) $s_i^{(k)}=\sum_{j'=1}^e w_i^{\sigma_{j'}}s'_{j'}\equiv w_i^{\sigma_j}s'_j\equiv w_i^{\sigma_j}s_0^{(k)} \mod \tilde{B}_j^q$ for $i,j=1,\dots,e$,

as stated in (d).

Part D: *The free modules* $L^{(k)}$ *and the linear spaces* $L_K^{(k)}$. We choose a non-zero homogeneous element s_B of B and let $k_B=\deg_{K[\mathbf{t}]}(s_B)$ (Section 1).

(17) We choose a large multiple k_1 of k_0 such that $k_1\ge k_I+qk_B+1$, where k_I is as in (4d).

For each large integer k let

(18) $L^{(k)}=\mathrm{Ker}(\rho^{(k)}_{\bar{X},Z\cup Z_{qB}})=R[\mathbf{t}]_k\cap I\cap B^q$ (Remark 1.4).

Since $\Gamma(\bar{X},\mathcal{O}_{\bar{X}}(k))=R[\mathbf{t}]_k$ is a finitely generated R-module and R is Noetherian, $L^{(k)}$ is a finitely generated R-module. Moreover, since both R and $L^{(k)}$ are submodules of the field $K(\mathbf{t})$, $L^{(k)}$ is torsion-free as an R-module. In addition, R is a principal ideal domain (Setup 9.3). So, $L^{(k)}$ is a finitely generated free R-module [Lan93, p. 148, Thm. 7.1]. It satisfies the following rule:

(19) If $s\in L^{(k)}$ and $s'\in R[\mathbf{t}]_{k'}$, then $ss'\in L^{(k+k')}$.

Similarly we consider the vector space

(20) $L_K^{(k)}=\mathrm{Ker}(\rho^{(k)}_{\bar{X}_K,Z_K\cup Z_{qB,K}})=K[\mathbf{t}]_k\cap KI\cap(KB)^q$

over K and observe that Rule (19) holds also for these vector spaces.

Let $s_0^{[k_1]},\dots,s_m^{[k_1]}$ be an R-basis of $L^{(k_1)}$ and consider the scheme $\Lambda=\mathrm{Proj}(R[s_0^{[k_1]},\dots,s_m^{[k_1]}])$. By (19) and (18),

(21) $s_Is_B^qK[\mathbf{t}]_{k_1-k_I-qk_B}\subseteq K[\mathbf{t}]_{k_1}\cap KI\cap KB^q=L_K^{(k_1)}$, where s_I is introduced in (4d).

Since $k_1-k_I-qk_B\ge 1$ (by (17)), Lemma 9.1 implies that the quotients of the elements of $K[\mathbf{t}]_{k_1-k_I-qk_B}$ by a chosen non-zero element of this K-vector-space generate the field F over K. Since $s_Is_B^q\ne 0$, Relation (21) implies that the function field of Λ_K is F.

Part E: *Characteristic-0-like curve.* We follow [GJR17, Remark 18.1] and let $s_0^{[3k_1]},\dots,s_{m^*}^{[3k_1]}$ be all of the elements of the form $s_h^{[k_1]}s_i^{[k_1]}s_j^{[k_1]}$ with $0\le h,i,j\le m$. By (19), $s_0^{[3k_1]},\dots,s_{m^*}^{[3k_1]}\in L^{(3k_1)}\subseteq R[\mathbf{t}]_{3k_1}$. Thus, $R[s_0^{[3k_1]},\dots,s_{m^*}^{[3k_1]}]$ is a graded ring over R with $R[s_0^{[3k_1]},\dots,s_{m^*}^{[3k_1]}]_1=\sum_{i=0}^{m^*}Rs_i^{[3k_1]}$ and $\Lambda^*=\mathrm{Proj}(R[s_0^{[3k_1]},\dots,s_{m^*}^{[3k_1]}])$ is the image of Λ under the 3-fold Veronese embedding. In particular, the function field of Λ_K^* is F. Also, $\Lambda^*_{\tilde{K}}=\mathrm{Proj}(\tilde{K}[s_0^{[3k_1]},\dots,s_{m^*}^{[3k_1]}])$ is the image of $\Lambda_{\tilde{K}}=\mathrm{Proj}(\tilde{K}[s_0^{[k_1]},\dots,s_m^{[k_1]}])$ under the 3-fold Veronese embedding. Therefore, by [GJR17, Prop. 18.6],

(22) the curve $\Lambda^*_{\tilde{K}}$ is characteristic-0-like.

Let $k\ge 3k_1$ be a large multiple of k_0. For each $0\le i\le m^*$ we set $s_i^*=t_0^{k-3k_1}s_i^{[3k_1]}\in L^{(k)}$ (by (19)). In addition, we choose $s_{e+1}^{(k)},\dots,s_{l(k)}^{(k)}$ in $R[\mathbf{t}]_k$ that form an R-basis of $L^{(k)}$ (as stated in (f)). In particular,

(23) $s_{e+1}^{(k)},\dots,s_{l(k)}^{(k)}$ vanish on Z.

Together with (15), Statement (23) verifies (c). Also, $R[s_0^{(k)},\dots,s_{l(k)}^{(k)}]$ is a graded ring over R with $R[s_0^{(k)},\dots,s_{l(k)}^{(k)]}]_1=\sum_{i=0}^{l(k)}Rs_i^{(k)}$. Since $s_{e+1}^{(k)},\dots,s_{l(k)}^{(k)}$ generate $L_K^{(k)}$ over K, we have $s_0^*,\dots,s_{m^*}^*\in\sum_{i=0}^{l(k)}Ks_i^{(k)}$. Hence,

(24) $\frac{s_i^{[3k_1]}}{t_0^{3k_1}}=\frac{s_i^*}{t_0^k}\in\sum_{j=0}^{l(k)}K\frac{s_j^{(k)}}{t_0^k}$ for $i=0,\dots,m^*$.

Since the function field of Λ^*_K is F, it follows from (24) that F is contained in the function field of $\mathrm{Proj}(K[s_0^{(k)},\ldots,s_{l(k)}^{(k)}])$. The latter is contained in F. Hence, the function field of $\mathrm{Proj}(K[s_0^{(k)},\ldots,s_{l(k)}^{(k)}])$ is F, as stated in (g).

Now observe that $\left(\frac{s_0^{[3k_1]}}{t_0^{3k_1}}:\cdots:\frac{s_{m^*}^{[3k_1]}}{t_0^{3k_1}}\right)$ is a generic point of $\Lambda^*_{\tilde K}$ with coordinates in F, hence in $F\tilde K$. Also, $\left(\frac{s_0^{(k)}}{t_0^k}:\cdots:\frac{s_{l(k)}^{(k)}}{t_0^k}\right)$ is a generic point of $\mathrm{Proj}(\tilde K[s_0^{(k)},\ldots,s_{l(k)}^{(k)}])$ with coordinates in F, hence in $F\tilde K$. Therefore, by (22), (24), and Proposition 9.2, $\mathrm{Proj}(\tilde K[s_0^{(k)},\ldots,s_{l(k)}^{(k)}])$ is a characteristic-0-like curve, as (h) claims.

By the definition of $L^{(k)}$ in Part D, $s_{e+1}^{(k)},\ldots,s_{l(k)}^{(k)}$ vanish on Z_{qB}, hence they all belong to B^q and therefore to $\tilde B_j^q$, $j=1,\ldots,e$, as claimd by (e). ∎

LEMMA 9.6: *In the notation of Lemma 9.5, the following holds for each large multiple k of k_0:*
(a) *the sections $s_0^{(k)},s_1^{(k)},\ldots,s_{l(k)}^{(k)}$ have no common zero in $\bar X(\tilde K)$ and*
(b) *the sections $s_{e+1}^{(k)},\ldots,s_{l(k)}^{(k)}$ have no common zero in $\bar X(\tilde K)\smallsetminus(Z(\tilde K)\cup Z_B(\tilde K))$.*

Proof: By Lemma 9.5(a),(b), $s_0^{(k)}$ vanishes at no point of $Z(\tilde K)\cup Z_B(\tilde K)$. Hence, in order to complete the proof of the claim, it suffices to prove (b).

Since $Z(\tilde K)\cup Z_B(\tilde K)$ is a finite subset of $\bar X(\tilde K)$, there exists a homogeneous polynomial $h_0\in K[T_0,\ldots,T_r]$ that vanish on $Z(\tilde K)\cup Z_B(\tilde K)$ but not on $\bar X(\tilde K)$. Replacing h_0 by its qth power (with q as in Lemma 9.5), we may assume that $h_0(\mathbf t)\in KB^q$. Then, we choose $\mathbf r_0\in\bar X(\tilde K)\smallsetminus(Z(\tilde K)\cup Z_B(\tilde K))$ such that $h_0(\mathbf r_0)\neq 0$.

Since $\dim(\bar X_K)=1$, the polynomial h_0 vanishes only at finitely many points of $\bar X(\tilde K)$. Let $\mathbf r_1,\ldots,\mathbf r_m$ be the finitely many points in $X(\tilde K)\smallsetminus Z_B(\tilde K)$ at which h_0 vanishes. For each i between 1 and m we choose a homogeneous polynomial $h_i\in K[T_0,\ldots,T_r]$ that vanishes on $Z(\tilde K)$ but not at $\mathbf r_i$ such that $h_i(\mathbf t)\in KB^q$. Then we set $k_2=\max(\deg(h_0),\ldots,\deg(h_m))$.

We consider a positive multiple k of k_0 with $k\ge k_2$. Given a point $\mathbf p\in X(\tilde K)\smallsetminus Z_B(\tilde K)$, we choose an index $0\le j\le r$ such that $t_j(\mathbf p)\neq 0$. If $\mathbf p=\mathbf r_i$ for some i between 1 and m, then $h_i(\mathbf p)\neq 0$ (by the choice of h_i). If $\mathbf p\neq \mathbf r_1,\ldots,\mathbf r_m$, then, $h_0(\mathbf p)\neq 0$ (by the defining property of $\mathbf r_1,\ldots,\mathbf r_m$). Thus, in any case, there exists $0\le i\le m$ with $h_i(\mathbf p)\neq 0$. It follows that $h(T_0,\ldots,T_r)=T_j^{k-\deg(h_i)}h_i(T_0,\ldots,T_r)$ is a homogeneous polynomial of degree k with coefficients in K that vanishes on $Z(\tilde K)$, hence on Z_K, but not at $\mathbf p$. Moreover, $h(\mathbf t)\in KB^q$. In particular, $h(\mathbf t)\in\mathrm{Ker}(\rho^{(k)}_{\bar X_K,Z_K\cup Z_{qB,K}})$. By Lemma 9.5(f), the set $\{s_{e+1}^{(k)},\ldots,s_{l(k)}^{(k)}\}$ is a K-basis of $\mathrm{Ker}(\rho^{(k)}_{\bar X_K,Z_K\cup Z_{qB,K}})$. Hence, $h(\mathbf t)$ is a linear combination of $s_{e+1}^{(k)},\cdots,s_{l(k)}^{(k)}$ with coefficients in K, so $h(\mathbf p)$ is a linear combination of $s_{e+1}^{(k)}(\mathbf p),\ldots,s_{l(k)}^{(k)}(\mathbf p)$ with coefficients in K. Therefore, at least one of the elements $s_{e+1}^{(k)}(\mathbf p),\ldots,s_{l(k)}^{(k)}(\mathbf p)$ of $\tilde K$ is non-zero. This proves (b) and completes the proof of the lemma. ∎

10. The Curve Y

We construct a birational morphism of $\bar X_K$ onto a projective q-curve Y over K for each given positive integer $q\ge 2$. Choosing q to be a large prime number, we then apply Proposition 8.6 to construct a symmetrically stabilizing element for F/K with a special form.

Setup 10.1: We replace k_0 (Proposition 7.6) by a large multiple of itself to assume that Lemmas 9.5 and 9.6 hold for each positive multiple k of k_0. Under Setup 9.3, we consider a large multiple k of k_0, a positive integer q, and the elements $s_0^{(k)},\ldots,s_{l(k)}^{(k)}$ of $R[\mathbf t]_k$ that Lemma 9.5 produces. In particular, $K[s_0^{(k)},\ldots,s_{l(k)}^{(k)}]$ is a graded ring over K such that $K[s_0^{(k)},\ldots,s_{l(k)}^{(k)}]_1=\sum_{i=0}^{l(k)}Ks_i^{(k)}$. Let $Y=\mathrm{Proj}(K[s_0^{(k)},\ldots,s_{l(k)}^{(k)}])$ and let $\varphi=\varphi^{(k)}\colon\bar X_K\to Y$ be the rational map defined by $\varphi(\mathbf t)=\mathbf s^{(k)}$. Since $s_0^{(k)},\ldots,s_{l(k)}^{(k)}$ have no common zero in $\bar X(\tilde K)$ (Lemma 9.6), φ is a morphism.

Let $\tilde{\varphi} = \varphi^{(k)}_{\tilde{K}} \colon \bar{X}_{\tilde{K}} \to Y_{\tilde{K}}$ be the extension of φ created by changing the base field from K to $\tilde{K}$. We consider the points

$$\begin{aligned} \mathbf{y}_0 &= (1{:}0{:}\cdots{:}0) \\ \mathbf{y}_1 &= (1{:}w_1^{\sigma_1}{:}\cdots{:}w_e^{\sigma_1}{:}0{:}\cdots{:}0) \\ &\cdots \\ \mathbf{y}_e &= (1{:}w_1^{\sigma_e}{:}\cdots{:}w_e^{\sigma_e}{:}0{:}\cdots{:}0) \end{aligned}$$

of $\mathbb{P}^{l(k)}(\tilde{K})$. ∎

LEMMA 10.2: *Let Γ be an absolutely integral projective curve over a field L and let Γ_0 be a non-empty Zariski-open subset of Γ with $\Gamma_0 \neq \Gamma$. Then, Γ_0 is an absolutely integral affine curve over L.*

Proof: By a result of Goodman, $\Gamma_{0,\tilde{L}}$ is affine [Goo69, p. 167, Prop. 5]. It follows from [GoW10, p. 442, Prop. 14.51(6)] that Γ_0 is also affine. (We are indebted to Ulrich Görtz for this argument.)

Alternatively, we may point out that Γ_0 is not a proper scheme and use [Liu06, Exer. 7.5.5, p. 315].

Another possibility communicated to us by David Harbater is to construct an effective Cartier divisor D on Γ whose support is $\Gamma \smallsetminus \Gamma_0$ and then conclude from [Liu06, Prop. 7.5.5, p. 305] that D is ample. Thus, for some positive integer n_0, the divisor $n_0 D$ is very ample. Hence, $\mathcal{L}(n_0 D)$ admits a set of global sections that provide an embedding of Γ into some projective space $\mathbb{P}^m_L$ such that D is the (set-theoretic) inverse image of the hyperplane at infinity. Therefore, Γ_0 is the inverse image of $\mathbb{A}^m_L$, hence is affine, because closed immersions are finite [GoW10, p. 325, Prop. 12.11(1)]. ∎

LEMMA 10.3: *The morphism φ of Setup 10.1 maps the affine curve $X_K \smallsetminus Z_{B,K}$ isomorphically onto a Zariski-open smooth affine subset Y_0 of Y. Moreover,*

(a) *the morphism $\varphi\colon \bar{X}_K \to Y$ is birational,*

(b) *$\mathbf{y}_0 \in Y(K)$ and $\varphi^{-1}(\mathbf{y}_0) = Z_K$,*

(c) *$\mathbf{y}_j \in Y(\tilde{K})$ and $\tilde{\varphi}^{-1}(\mathbf{y}_j) = \mathbf{b}_j$ for $j = 1, \ldots, e$,*

(d) *$Y_0 = Y \smallsetminus \big(\varphi(Z_K) \cup \varphi(Z_{B,K})\big)$, and*

(e) *$Y_0(\tilde{K}) = Y(\tilde{K}) \smallsetminus \{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_e\}$.*

Proof: Recall that $X_K = \mathrm{Spec}(K[x_1, \ldots, x_n])$ (Subsection 5.11). By Lemma 10.2, the Zariski-open subset

$$X_K \smallsetminus Z_{B,K} = \bar{X}_K \smallsetminus (Z_K \cup Z_{B,K})$$

of X_K (with $Z_{B,K}$ as introduced in Subsection 6.10) is an absolutely integral affine curve over K, hence may be written as $\mathrm{Spec}(K[x_1, \ldots, x_{n'}])$, for some $n' \geq n$ and elements $x_{n+1}, \ldots, x_{n'}$ of F that do not vanish on $Z_{B,K}$. The rest of the proof breaks up into several parts.

PART A: *The affine subset Y_0 of Y.* By Subsection 5.11, $\bar{X}_K$ is a normal curve. Hence, for each $Q \in \bar{X}_K$, the local ring $\mathcal{O}_{\bar{X}_K,Q}$ is a discrete valuation ring of F [Lan58, p. 151, Thm. 1]. In particular, this statement holds for each of the points $KI_1, \ldots, KI_{d(Z)}$ of $\bar{X}_K$ that correspond to the points $Z_{1,K}, \ldots, Z_{d(Z),K}$ of Z_K and which are introduced in Subsection 5.9. The statement also applies to the point KB of X_K introduced in Subsection 6.1. We choose a positive integer e' that satisfies the following condition:

(1) $\mathrm{ord}_{KI_i}(x_{j'}) + e' \geq 0$ and $\mathrm{ord}_{KB}(x_{j'}) + e'q \geq 0$ for $i = 1, \ldots, d(Z)$ and $j' = 1, \ldots, n'$.

Now we set $k' = e'k_0$ and suppose that $k \geq k'$. For each $0 \leq i \leq l(k)$ we choose (by Setup 10.1) a homogeneous polynomial $f_i \in K[T_0, \ldots, T_r]$ of degree k such that $s_i^{(k)} = f_i(\mathbf{t})$. By Setup 10.1, the morphism $\tilde{\varphi} = \varphi^{(k)}_{\tilde{K}} \colon \bar{X}_{\tilde{K}} \to Y_{\tilde{K}}$ is defined by

(2) $\tilde{\varphi}(\mathbf{t}) = (f_0(\mathbf{t}){:}f_1(\mathbf{t}){:}\cdots{:}f_{l(k)}(\mathbf{t}))$.

By Lemma 9.5(a), $s_0^{(k)}$ does not vanish on Z, by Lemma 9.5(c), $s_j^{(k)}|_Z = 0$ for $j = 1, \ldots, l(k)$. Hence,

(3) $\tilde{\varphi}(Z_{\tilde{K}}) = \{\mathbf{y}_0\}$, so $\varphi(Z_K) = \{\mathbf{y}_0\}$ and $\tilde{\varphi}(Z(\tilde{K})) = \{\mathbf{y}_0\}$, in particular $\mathbf{y}_0 \in Y(K)$.

Next note in the notation of Setup 9.3 that

$$Z_B(\tilde{K}) = \{\mathbf{b}_1, \ldots, \mathbf{b}_e\}. \tag{4}$$

We consider j between 1 and e. By Lemma 9.5(b), $s_0^{(k)}(\tilde{B}_j) \neq 0$. Also, for $i = 1, \ldots, e$ we have by Lemma 9.5(d) that $s_i^{(k)}(\tilde{B}_j) = w_i^{\sigma_j} s_0^{(k)}(\tilde{B}_j)$. By Lemma 9.5(e), $s_i^{(k)}(\tilde{B}_j) = 0$ for $i = e+1, \ldots, l(k)$. Hence,

$$
\begin{aligned}
\text{(5)} \qquad \mathbf{y}_j &= (1 : w_1^{\sigma_j} : \cdots : w_e^{\sigma_j} : 0 : \cdots : 0) \\
&= (s_0^{(k)}(\tilde{B}_j) : w_1^{\sigma_j} s_0^{(k)}(\tilde{B}_j) : \cdots : w_e^{\sigma_j} s_0^{(k)}(\tilde{B}_j) : 0 : \cdots : 0) \\
&= (s_0^{(k)}(\tilde{B}_j) : s_1^{(k)}(\tilde{B}_j) : \cdots : s_e^{(k)}(\tilde{B}_j) : s_{e+1}^{(k)}(\tilde{B}_j) : \cdots : s_{l(k)}^{(k)}(\tilde{B}_j)) \\
&= (f_0(\mathbf{b}_j) : f_1(\mathbf{b}_j) : \cdots : f_e(\mathbf{b}_j) : f_{e+1}(\mathbf{b}_j) : \cdots : f_{l(k)}(\mathbf{b}_j)) = \tilde{\varphi}(\mathbf{b}_j) \in Y(\tilde{K}).
\end{aligned}
$$

It follows from (3), (4), and (5) that

(6) $\tilde{\varphi}(Z_B(\tilde{K})) = \{\mathbf{y}_1, \ldots, \mathbf{y}_e\}$ and $\tilde{\varphi}(Z(\tilde{K}) \cup Z_B(\tilde{K})) = \{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_e\}$.

By Setup 9.3, $\mathbf{b}_1, \ldots, \mathbf{b}_e$ form a complete system of K-conjugate separable points of $\bar{X}(\tilde{K})$ that lie over KB, so they are all of the points of $\bar{X}(\tilde{K})$ that lie over KB. Similarly, $\mathbf{y}_1, \ldots, \mathbf{y}_e$ form a complete system of K-conjugate separable points of $Y(\tilde{K})$ that lie over $\varphi(KB)$. By [Lan58, p. 74, the equivalence of C6 and C7], Y has a Zariski-closed subset $Y_{1,1}$ with $Y_{1,1}(\tilde{K}) = \{\mathbf{y}_1, \ldots, \mathbf{y}_e\}$. Then, $Y_1 = \{\mathbf{y}_0\} \cup Y_{1,1}$ is a Zariski-closed subset of Y, $Y_0 = Y \smallsetminus Y_1$ is a non-empty Zariski-open subset of Y and $Y_0(\tilde{K}) = Y(\tilde{K}) \smallsetminus \{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_e\}$.

Part B: *Inclusion of coordinate rings.* We consider a point $\mathbf{p}$ of $X(\tilde{K}) \smallsetminus Z_B(\tilde{K})$. For each positive multiple k of k_0, Lemma 9.6(b) gives $e+1 \leq i \leq l(k)$ such that $f_i(\mathbf{p}) \neq 0$. Hence, by (2) and the definition of the $\mathbf{y}_j$'s,

$$
\tilde{\varphi}(\mathbf{p}) = (f_0(\mathbf{p}){:}f_1(\mathbf{p}){:}\cdots{:}f_{l(k)}(\mathbf{p})) \neq \mathbf{y}_j, \quad j = 0, 1, \ldots, e,
$$

so $\tilde{\varphi}(\mathbf{p}) \in Y_0(\tilde{K})$. Thus,

(7) $\tilde{\varphi}(X(\tilde{K}) \smallsetminus Z_B(\tilde{K})) \subseteq Y_0(\tilde{K})$.

By (3), $\tilde{\varphi}(Z(\tilde{K})) = \{\mathbf{y}_0\} \not\subseteq Y_0(\tilde{K})$, hence the morphism $\tilde{\varphi}\colon \bar{X}_{\tilde{K}} \to Y_{\tilde{K}}$ of integral projective curves over $\tilde{K}$ is non-constant. Since morphisms of projective curves are closed [Mum88, p. 77, Thm. I.9.1], $\tilde{\varphi}(\bar{X}_{\tilde{K}}) = Y_{\tilde{K}}$. It follows from (7) and (6) that $\tilde{\varphi}(X(\tilde{K}) \smallsetminus Z_B(\tilde{K})) = Y_0(\tilde{K})$, hence also

(8) $\varphi(X_K \smallsetminus Z_{B,K}) = Y_0$. It follows from (6) that $\varphi^{-1}(Y_0) = X_K \smallsetminus Z_{B,K}$.

By Lemma 10.2, Y_0 is an affine curve over K. Hence, there is an inclusion

$$
K[Y_0] \subseteq K[X_K \smallsetminus Z_{B,K}] \tag{9}
$$

of the coordinate rings of the affine schemes Y_0 and $X_K \smallsetminus Z_{B,K}$ [Liu06, p. 48, Prop. 2.3.25].

Part C: *Equality of coordinate rings.* We choose non-zero homogeneous elements $a_0 \in KI$ and $b_0 \in KB$ of $K[\mathbf{t}]$. Then, both Zariski-closed subsets $V_+(a_0K[\mathbf{t}])$ and $V_+(b_0K[\mathbf{t}])$ of $\bar{X}_K$ are of dimension 0. Therefore, $(X_K \smallsetminus Z_{B,K}) \cap (V_+(a_0K[\mathbf{t}]) \cup V_+(b_0K[\mathbf{t}]))$ is a finite set, say $\{P_1, \ldots, P_m\}$. For each i between 1 and m we choose non-zero homogeneous elements $a_i \in KI \smallsetminus P_i$ and $b_i \in KB \smallsetminus P_i$ of $K[\mathbf{t}]$. Note that $P_i \notin Z_K$, because X_K and Z_K are disjoint.

Now we assume, in addition to the conditions we have put so far on k, that

$$
k \geq k' + \max_{0 \leq i \leq m} (\deg_{K[\mathbf{t}]}(a_i) + q \deg_{K[\mathbf{t}]}(b_i)), \tag{10}
$$

where $k' = e'k_0$ (Part A).

We consider $P \in X_K \smallsetminus Z_{B,K}$. If $P \notin V_+(a_0K[\mathbf{t}]) \cup V_+(b_0K[\mathbf{t}])$, we set $a_P = a_0$ and $b_P = b_0$. Otherwise $P = P_i$ for some i between 1 and m and we set $a_P = a_i$ and $b_P = b_i$. In each case

(11) $a_P \in (KI \smallsetminus P) \cap \{a_0, \ldots, a_m\}$ and $b_P \in (KB \smallsetminus P) \cap \{b_0, \ldots, b_m\}$.

By Lemma 9.6(b), there exists i' between $e+1$ and $l(k_0)$ such that $s_{i'}^{(k_0)} \notin P$. By Lemma 9.5(f), $s_{i'}^{(k_0)} \in K[\mathbf{t}]_{k_0} \cap KI \cap KB^q$. We set $s' = (s_{i'}^{(k_0)})^{e'}$. Then,

(12) $s' \in K[\mathbf{t}]_{k'} \cap KI \cap KB^q$ and $s' \notin P$.

For each $1 \le j \le n'$ we consider the element $x'_j = x_j s'$ of $K(\mathbf{t})$. Since $x_j \in F$,

(13) $\deg_{K[\mathbf{t}]}(x'_j) = \deg_{K[\mathbf{t}]}(s') = k'$ (second and third paragraphs of Example 1.6).

Since $X_K \smallsetminus Z_{B,K} = \mathrm{Spec}(K[x_1,\dots,x_{n'}])$, we have $\mathrm{ord}_Q(x_j) \ge 0$ for each $Q \in X_K \smallsetminus Z_{B,K}$ and every $1 \le j \le n'$. We choose $u_1 \in K[\mathbf{t}]_1 \smallsetminus Q$ (e.g. one of the elements $t_0,\dots,t_r$) and write

(14) $$\frac{x'_j}{u_1^{k'}} = x_j \cdot \left(\frac{s_{i'}^{(k_0)})}{u_1^{k_0}}\right)^{e'}.$$

By Example 1.6(b), $\mathrm{ord}_Q(\frac{s_{i'}^{(k_0)}}{u_1^{k_0}}) \ge 0$. Hence, by (14),

(15a) $$\mathrm{ord}_Q\left(\frac{x'_j}{u_1^{k'}}\right) = \mathrm{ord}_Q(x_j) + e' \cdot \mathrm{ord}_Q\left(\frac{s_{i'}^{(k_0)}}{u_1^{k_0}}\right) \ge 0.$$

Given an i between 1 and $d(Z)$, we choose $u_2 \in K[\mathbf{t}]_1 \smallsetminus KI_i$ (e.g. one of the elements $t_0,\dots,t_r$). By Lemma 9.5(c), $s_{i'}^{(k_0)} \in I_i$, hence by Example 1.6(c), $\mathrm{ord}_{KI_i}(\frac{s_{i'}^{(k_0)}}{u_2^{k_0}}) \ge 1$. Therefore, by (14) (with u_2 replacing u_1) and (1),

(15b) $$\mathrm{ord}_{KI_i}\left(\frac{x'_j}{u_2^{k'}}\right) = \mathrm{ord}_{KI_i}(x_j) + e' \cdot \mathrm{ord}_{KI_i}\left(\frac{s_{i'}^{(k_0)}}{u_2^{k_0}}\right) \ge \mathrm{ord}_{KI_i}(x_j) + e' \ge 0.$$

Finally, we choose u_3 in $K[\mathbf{t}]_1 \smallsetminus KB$. Since $s_{i'}^{(k_0)} \in (KB)^q$ (Lemma 9.5(f)), we have by Example 1.6(e) that $\mathrm{ord}_{KB}(\frac{s_{i'}^{(k_0)}}{u_3^{k_0}}) \ge q$. Hence, by (14) (with u_3 replacing u_1) and (1),

(15c) $$\mathrm{ord}_{KB}\left(\frac{x'_j}{u_3^{k'}}\right) = \mathrm{ord}_{KB}(x_j) + e' \cdot \mathrm{ord}_{KB}\left(\frac{s_{i'}^{(k_0)}}{u_3^{k_0}}\right) \ge \mathrm{ord}_{KB}(x_j) + e'q \ge 0.$$

By (13), $\deg_{K[\mathbf{t}]}(x'_j) = k'$. It follows from (15) and Example 1.6(d) that $x'_j \in \mathcal{O}_{\bar{X}_K}(k')_Q$ for each $Q \in \bar{X}_K$. Hence, by Example 1.6(a), $x'_j \in K[\mathbf{t}]_{k'}$ (note that, by the last paragraph of Proposition 7.6, each positive multiple of k_0 satisfies Diagrams (2) and (3) of Subsection 7.2). Now we choose $0 \le j' \le r$ such that $t_{j'} \notin P$. We use (10), (11), (12), and (13) to set $x''_j = t_{j'}^{k-\deg_{K[\mathbf{t}]}(a_P)-q\deg_{K[\mathbf{t}]}(b_P)-k'} a_P b_P^q x'_j \in K[\mathbf{t}]_k \cap KI \cap (KB)^q$ and $s = t_{j'}^{k-\deg_{K[\mathbf{t}]}(a_P)-q\deg_{K[\mathbf{t}]}(b_P)-k'} a_P b_P^q s' \in K[\mathbf{t}]_k \cap KI \cap (KB)^q \smallsetminus P$. By Lemma 9.5(f),

(16) $x''_j, s \in \sum_{i=e+1}^{l(k)} K s_i^{(k)}$.

By (7), $\varphi(P) \in Y_0$. Since $Y = \mathrm{Proj}(K[s_0^{(k)},\dots,s_{l(k)}^{(k)}])$, we have by the definition of φ that $\varphi(P) = P \cap K[s_0^{(k)},\dots,s_{l(k)}^{(k)}]$. Since $s \in K[s_0^{(k)},\dots,s_{l(k)}^{(k)}] \smallsetminus P$, we conclude that $s \notin \varphi(P)$. Hence, by (16), $x_j = \frac{x'_j}{s'} = \frac{x''_j}{s} \in \mathcal{O}_{Y,\varphi(P)} = \mathcal{O}_{Y_0,\varphi(P)}$.

It follows from (8) that each x_j with $1 \le j \le n'$ lies in $\mathcal{O}_{Y_0,P_0}$ for each $P_0 \in Y_0$, so $x_j \in K[Y_0]$ [Lan58, p. 31, Thm. 6]. Thus, $K[X_K \smallsetminus Z_{B,K}] = K[x_1,\dots,x_{n'}] \subseteq K[Y_0]$. We conclude from (9) that $K[Y_0] = K[X_K \smallsetminus Z_{B,K}]$.

Part D: *End of proof.* By (8) and by the conclusion of Part C, φ maps the affine curve $X_K \smallsetminus Z_{B,K}$ isomorphically onto Y_0. Since X_K is smooth (Statement (14) in Subsection 5.11),

(17) Y_0 is smooth

and the morphism $\varphi\colon \bar{X}_K \to Y$ is birational. We know that $\bar{X}(\tilde{K})$ is the disjoint union of $(X \smallsetminus Z_B)(\tilde{K})$, $\{\mathbf{b}_1\},\dots,\{\mathbf{b}_e\}$, and $Z(\tilde{K})$. By (5), $\tilde{\varphi}(\mathbf{b}_j) = \mathbf{y}_j$ for $j = 1,\dots,e$ and, by (3), $\tilde{\varphi}(Z(\tilde{K})) = \{\mathbf{y}_0\}$. We conclude that

(18) $\varphi^{-1}(\mathbf{y}_0) = Z_K$ and $\tilde{\varphi}^{-1}(\mathbf{y}_j) = \mathbf{b}_j$, $j = 1,\dots,e$.

This settles all of the statements of the lemma. ∎

Lemma 10.4: *Suppose $q \geq 2$. Then, for each $1 \leq j \leq e$, the point $\mathbf{y}_j$ of $Y(\tilde{K})$ is a cusp of multiplicity q.*

Proof: We consider $1 \leq j \leq e$. By Lemma 10.3(c), the simple point $\tilde{B}_j$ of $X_{\tilde{K}}$ is the unique point of $\bar{X}_{\tilde{K}}$ that $\tilde{\varphi}$ maps onto $\mathbf{y}_j$. Thus, $\mathcal{O}_{\bar{X}_{\tilde{K}},\tilde{B}_j}$ is the unique valuation ring of $\tilde{K}F/\tilde{K}$ that contains the local ring $O_{Y_{\tilde{K}},\mathbf{y}_j}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{Y_{\tilde{K}},\mathbf{y}_j}$. Since $q \geq 2$, it suffices to prove that $q = \min_{m\in\mathfrak{m}} \mathrm{ord}_{\tilde{B}_j}(m)$ (Subsection 8.4).

Part A: *Lower bound.* By Lemma 9.5(b), $s_0^{(k)} \notin \tilde{B}_j$. Hence, $\tilde{K}\big[\frac{s_1^{(k)}}{s_0^{(k)}},\dots,\frac{s_{l(k)}^{(k)}}{s_0^{(k)}}\big]$ is the coordinate ring of an open affine neighborhood of $\mathbf{y}_j$ in $Y_{\tilde{K}}$. Therefore, by Lemma 9.5(d),(e),

(19) $\mathfrak{m}$ is generated by the elements $\frac{s_1^{(k)}}{s_0^{(k)}} - w_1^{\sigma_j},\dots,\frac{s_e^{(k)}}{s_0^{(k)}} - w_e^{\sigma_j}, \frac{s_{e+1}^{(k)}}{s_0^{(k)}},\dots,\frac{s_{l(k)}^{(k)}}{s_0^{(k)}}$.

Moreover, by Lemma 9.5(d),(e), $\mathrm{ord}_{\tilde{B}_j}\big(\frac{s_i^{(k)}}{s_0^{(k)}} - w_i^{\sigma_j}\big) \geq q$ for $i = 1,\dots,e$ and $\mathrm{ord}_{\tilde{B}_j}\big(\frac{s_i^{(k)}}{s_0^{(k)}}\big) \geq q$ for $i = e+1,\dots,l(k)$. Hence, $\mathrm{ord}_{\tilde{B}_j}(m) \geq q$ for all $m \in \mathfrak{m}$.

Part B: *Vector spaces.* The proof of the lemma will be complete, once we produce an element of $\mathcal{O}_{Y_{\tilde{K}},\mathbf{y}_j}$ whose $\mathrm{ord}_{\tilde{B}_j}$-value is q. To this end we consider the K-vector-spaces

$$V_j = \{(a_1,\dots,a_r) \in K^r \mid \mathrm{ord}_{\tilde{B}_j}\big(\sum_{i=1}^r a_i(\frac{t_i}{t_0} - b_i^{\sigma_j})\big) \geq 1\}$$

$$V_j^{(2)} = \{(a_1,\dots,a_r) \in K^r \mid \mathrm{ord}_{\tilde{B}_j}\big(\sum_{i=1}^r a_i(\frac{t_i}{t_0} - b_i^{\sigma_j})\big) \geq 2\},$$

We also consider for each $j' \neq j$ the K-vector-space

$$V_{j'} = \{(a_1,\dots,a_r) \in K^r \mid \mathrm{ord}_{\tilde{B}_j}\big(\sum_{i=1}^r a_i(\frac{t_i}{t_0} - b_i^{\sigma_{j'}})\big) \geq 1\}.$$

Claim B1: $V_j \not\subseteq V_j^{(2)}$. Indeed, since $\tilde{B}_j$ is a simple point of the curve $\bar{X}_{\tilde{K}}$ (Lemma 6.5), $\mathfrak{m}_{\bar{X}_{\tilde{K}},\tilde{B}_j}$ is the maximal ideal of the discrete valuation ring $\mathcal{O}_{\bar{X}_{\tilde{K}},\tilde{B}_j}$. By Notation 6.3, $\tilde{B}_j = \sum_{i=1}^r \tilde{K}[\mathbf{t}](t_i - b_i^{\sigma_j}t_0)$. Since $t_0 \notin \tilde{B}_j$ (Corollary 6.6), we have $\mathfrak{m}_{\bar{X}_{\tilde{K}},\tilde{B}_j} = \sum_{i=1}^r \mathcal{O}_{\bar{X}_{\tilde{K}},\tilde{B}_j}\big(\frac{t_i}{t_0} - b_i^{\sigma_j}\big)$. Hence, there exists $1 \leq i \leq r$ such that $\mathrm{ord}_{\tilde{B}_j}(\frac{t_i}{t_0} - b_i^{\sigma_j}) = 1$. By definition, $(0,\dots,0,1,0,\dots,0) \in V_j \smallsetminus V_j^{(2)}$, where 1 stands in the ith place, as desired.

Claim B2: *$V_j \not\subseteq V_{j'}$ for each $j' \neq j$.* Assume toward contradiction that $V_j \subseteq V_{j'}$ for some $j' \neq j$. Then, for each $(a_1,\dots,a_r) \in K^r$ we have

$$\mathrm{ord}_{\tilde{B}_j}\big(\sum_{i=1}^r a_i(\frac{t_i}{t_0} - b_i^{\sigma_j})\big) \geq 1 \text{ implies } \mathrm{ord}_{\tilde{B}_j}\big(\sum_{i=1}^r a_i(\frac{t_i}{t_0} - b_i^{\sigma_{j'}})\big) \geq 1. \tag{20}$$

For each $1 \leq i \leq r$ we have $t_i - b_i^{\sigma_j}t_0 \in \tilde{B}_j$, so by Example 1.6(c), $\mathrm{ord}_{\tilde{B}_j}(\frac{t_i}{t_0} - b_i^{\sigma_j}) \geq 1$. By (20), $\mathrm{ord}_{\tilde{B}_j}(\frac{t_i}{t_0} - b_i^{\sigma_{j'}}) \geq 1$. Since $\tilde{B}_{j'} = \sum_{i=1}^r \tilde{K}[\mathbf{t}](t_i - b_i^{\sigma_{j'}}t_0)$, we get $\tilde{B}_{j'} \subseteq \tilde{B}_j$ in contrast to Lemma 6.5.

It follows from Claims B1 and B2 that $V_j^{(2)}$ and $V_j \cap V_{j'}$ for $j' \neq j$ are proper subspaces of V_j. Since K is an infinite field, there exists $(a_1,\dots,a_r) \in V_j \smallsetminus (V_j^{(2)} \cup \bigcup_{j'\neq j} V_{j'})$. In other words,

$$\mathrm{ord}_{\tilde{B}_j}\big(\sum_{i=1}^r a_i(\frac{t_i}{t_0} - b_i^{\sigma_j})\big) = 1 \text{ and } \mathrm{ord}_{\tilde{B}_j}\big(\sum_{i=1}^r a_i(\frac{t_i}{t_0} - b_i^{\sigma_{j'}})\big) = 0 \text{ for } j' \neq j. \tag{21}$$

We multiply $a_1,\dots,a_r$ by a non-zero element of R to assume that $a_1,\dots,a_r$ are in R.

PART C: *An element of* $R[\mathbf{t}]_k \cap I \cap B^q$. We consider the element

$$\tilde{s} = \prod_{j'=1}^{e} \big(a_1(t_1 - b_1^{\sigma_{j'}} t_0) + \cdots + a_r(t_r - b_r^{\sigma_{j'}} t_0)\big) \tag{22}$$

of $\tilde{K}[\mathbf{t}]_e$. By Lemma 9.4, $\tilde{s} \in R[\mathbf{t}]$, hence $\tilde{s} \in R[\mathbf{t}]_e$. By the first statement of (21) and by Example 1.6(c), $\sum_{i=1}^r a_i(t_i - b_i^{\sigma_j} t_0) \in \tilde{B}_j$. Hence, by (22), $\tilde{s} \in R[\mathbf{t}] \cap \tilde{B}_j = B$ (Lemma 6.5).

Assuming that $k \ge k_I + eq$, we set $s = t_0^{k-k_I-eq} s_I \tilde{s}^q$, where s_I is the homogeneous element of $I \smallsetminus B$ chosen in Setup 9.3(4d) and $k_I = \deg_{K[\mathbf{t}]}(s_I)$. Then,

(23) $s \in R[\mathbf{t}]_k \cap I \cap B^q$.

PART D: *The* $\mathrm{ord}_{\tilde{B}_j}$*-value of* $\frac{s}{s_0^{(k)}}$. By (21) and (22), the $\mathrm{ord}_{\tilde{B}_j}$-value of the j-factor of the product on the right hand side of

$$\frac{s}{t_0^k} = \frac{s_I}{t_0^{k_I}} \prod_{j'=1}^{e} \Big(a_1\big(\frac{t_1}{t_0} - b_1^{\sigma_{j'}}\big) + \cdots + a_r\big(\frac{t_r}{t_0} - b_r^{\sigma_{j'}}\big)\Big)^q \tag{24}$$

is q and the $\mathrm{ord}_{\tilde{B}_j}$-value of the j'th factor is 0 for each $j' \ne j$. Since $s_I, t_0 \notin \tilde{B}_j$ (because $s_I \in R[\mathbf{t}] \smallsetminus B$), we have $\mathrm{ord}_{\tilde{B}_j}\big(\frac{s_I}{t_0^{k_I}}\big) = 0$ (Example 1.6(b)). Therefore, by (24), $\mathrm{ord}_{\tilde{B}_j}\big(\frac{s}{t_0^k}\big) = q$. Finally, since $t_0, s_0^{(k)} \notin \tilde{B}_j$, we have $\mathrm{ord}_{\tilde{B}_j}\big(\frac{s}{s_0^{(k)}}\big) = q$.

END OF PROOF: By Lemma 9.5(f), $s_{e+1}^{(k)}, \ldots, s_{l(k)}^{(k)}$ generate $R[\mathbf{t}]_k \cap I \cap B^q$ over R. Hence, by (23), there exist $a'_{e+1}, \ldots, a'_{l(k)} \in R$ such that $s = \sum_{i=e+1}^{l(k)} a'_i s_i^{(k)}$. It follows from (19) that $\frac{s}{s_0^{(k)}} = \sum_{i=e+1}^{l(k)} a'_i \frac{s_i^{(k)}}{s_0^{(k)}} \in \mathfrak{m}$, as desired. ∎

PROPOSITION 10.5: *Let q be a large positive integer. Then, for each large positive multiple k of the integer k_0 introduced in Proposition 7.6, there exists a birational morphism φ of $\bar{X}_K$ onto an absolutely integral projective curve Y in $\mathbb{P}_K^{l(k)}$ such that $Y_{\tilde{K}}$ is a q-curve (Definition 8.5).*

Proof: In the notation of Subsection 5.9, let $Z_K(\tilde{K}) = \{\mathbf{z}_1, \ldots, \mathbf{z}_{\tilde{d}}\}$. Since each point of $\bar{X}_K$ and in particular each point of Z_K is normal (Subsection 5.11), each $\mathbf{z}_\delta$ with $1 \le \delta \le \tilde{d}$ is simple or a cusp of $\bar{X}_{\tilde{K}}$ [Neu98, p. 234, Lemma 2.14]. In each case $\mathbf{z}_\delta$ lies under a unique prime divisor $\tilde{Z}_\delta$ of $\tilde{K}F/\tilde{K}$.

In the other direction, $\mathbf{z}_\delta$ lies over the point $Z_{i(\delta),K}$ of Z_K for a unique $i(\delta)$ between 1 and $d(Z)$ (Subsection 5.9). Since $KI_{i(\delta)}$ is a normal point of $\bar{X}_K$, we may identify $Z_{i(\delta),K}$ with the restriction of $\tilde{Z}_\delta$ to F. Let z_δ be a generator of $\mathfrak{m}_{\bar{X}_K, Z_{i(\delta),K}}$. Then, $\mathrm{ord}_{\tilde{Z}_\delta}(z_\delta)$ is the ramification index $e_{\tilde{Z}_\delta/Z_{i(\delta),K}}$ of $\tilde{Z}_\delta$ over $Z_{i(\delta),K}$. We consider an integer

(25) $q \ge \sum_{\delta=1}^{\tilde{d}} e_{\tilde{Z}_\delta/Z_{i(\delta),K}} = \sum_{\delta=1}^{\tilde{d}} \mathrm{ord}_{\tilde{Z}_\delta}(z_\delta)$.

Next we choose a large positive multiple k of k_0 that satisfies the conditions of the preceding lemmas of this section. In particular, $Y = \mathrm{Proj}(K[s_0^{(k)}, \ldots, s_{l(k)}^{(k)}])$ is the integral projective curve in $\mathbb{P}_K^{l(k)}$ and $\varphi\colon \bar{X}_K \to Y$ is the morphism with $\varphi(\mathbf{t}) = \mathbf{s}^{(k)}$ introduced in Setup 10.1.

By Lemma 10.3(a), φ is a birational morphism. Since $\bar{X}_K$ is absolutely integral, so is Y. By Lemma 9.5(h), Y is a characteristic-0-like curve.

By Lemma 10.3, each of the points of $Y(\tilde{K})$ except possibly $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_e$ is simple, hence of multiplicity 1 in $Y_{\tilde{K}}$ (Subsection 8.4). By Lemma 10.4, each of the points $\mathbf{y}_1, \ldots, \mathbf{y}_e$ is a cusp of $Y_{\tilde{K}}$ of multiplicity q. Therefore, it suffices to prove that $\mathrm{mult}(Y_{\tilde{K}}, \mathbf{y}_0) \le q$ (Definition 8.5).

By Lemma 10.3(b), $\varphi^{-1}(\mathbf{y}_0) = Z_K$. Hence, by Subsection 8.4, $\mathrm{mult}(Y_{\tilde{K}}, \mathbf{y}_0) = \sum_{\delta=1}^{\tilde{d}} \mathrm{mult}(Y_{\tilde{K}}, \tilde{Z}_\delta)$, so if for each $1 \le \delta \le \tilde{d}$ we produce

(26) $y_\delta \in \mathfrak{m}_{Y,\mathbf{y}_0}$ with $\mathrm{ord}_{\tilde{Z}_\delta}(y_\delta) = \mathrm{ord}_{\tilde{Z}_\delta}(z_\delta)$,

then, by (25),

$$\operatorname{mult}(Y_{\tilde K}, \mathbf{y}_0) = \sum_{\delta=1}^{\tilde d} \operatorname{mult}(Y_{\tilde K}, \tilde Z_\delta) \le \sum_{\delta=1}^{\tilde d} \operatorname{ord}_{\tilde Z_\delta}(y_\delta) = \sum_{\delta=1}^{\tilde d} \operatorname{ord}_{\tilde Z_\delta}(z_\delta) \le q,$$

and we will be done.

In order to produce y_δ as in (26), we recall that $Z_{B,K}$ and Z_K are disjoint (Subsection 6.10), in particular $KB^q \not\subseteq KI_{i(\delta)}$. Thus, we may choose a positive integer k' and an element $\nu \in (K[\mathbf{t}]_{k'} \cap KB^q) \smallsetminus KI_{i(\delta)}$.

By Subsection 5.9, the point $Z_{i(\delta),K}$ of Z_K corresponds to the homogeneous prime ideal $KI_{i(\delta)}$ of $K[\mathbf{t}]$ that contains KI. Since $z_\delta \in \mathfrak{m}_{\bar X_K, Z_{i(\delta),K}}$, we may write

(27) $z_\delta = \frac{\mu''}{\lambda}$, where $\mu'', \lambda \in K[\mathbf{t}]_{k''}$ for some positive integer k'' such that $\mu'' \in KI_{i(\delta)}$ and $\lambda \notin KI_{i(\delta)}$ (Example 1.6(c)).

Next we choose a homogeneous element $\rho' \in (\bigcap_{j\ne i(\delta)} KI_j) \smallsetminus KI_{i(\delta)}$ (Subsection 5.9) and an $0 \le i' \le r$ with $t_{i'} \notin KI_{i(\delta)}$.

Observe that k', k'', and ρ' depend on $\bar X_K$ but not on Y, so we may assume that $k > k' + k'' + \deg_{K[\mathbf{t}]}(\rho')$. This assumption allows us to set $\rho = t_{i'}^{k-k'-k''-\deg_{K[\mathbf{t}]}(\rho')}\rho'$. Then, $\rho \in K[\mathbf{t}]_{k-k'-k''} \cap (\bigcap_{j\ne i(\delta)} KI_j) \smallsetminus KI_{i(\delta)}$, so $\mu''\rho \in \bigcap_{j=1}^{d(Z)} KI_j = KI$ (Subsection 5.9) and $\deg_{K[\mathbf{t}]}(\mu''\rho) = k - k'$.

It follows that $\mu = \mu''\nu\rho \in K[\mathbf{t}]_k \cap KI \cap KB^q$. By Lemma 9.5(f), μ is a linear combination of $s_{e+1}^{(k)}, \ldots, s_{l(k)}^{(k)}$ with coefficients in K. Since μ belongs to KI, it vanishes on Z_K, hence also at $\mathbf{y}_0$. By Lemma 9.5(a), $s_0^{(k)}$ does not vanish on $Z(\tilde K)$, hence $s_0^{(k)}$ does not vanish at $\mathbf{y}_0$ (which is the image of $Z(\tilde K)$ under $\tilde\varphi$, by Lemma 10.3(b)). Therefore, $y_\delta = \frac{\mu}{s_0^{(k)}} \in \mathfrak{m}_{Y,\mathbf{y}_0}$ (Example 1.6(c)).

In order to compute $\operatorname{ord}_{\tilde Z_\delta}(y_\delta)$, we choose $0 \le j \le r$ with $t_j(\mathbf{z}_\delta) \ne 0$. Since $\nu, \rho \in K[\mathbf{t}] \smallsetminus KI_{i(\delta)}$, we also have $\nu(\mathbf{z}_\delta) \ne 0$, $\rho(\mathbf{z}_\delta) \ne 0$, and $s_0^{(k)}(\mathbf{z}_\delta) \ne 0$. Hence, each of the elements $\frac{\nu}{t_j^{k'}}$, $\frac{\rho}{t_j^{k-k'-k''}}$, and $\frac{\lambda t_j^{k-k''}}{s_0^{(k)}}$ of $\mathcal{O}_{\bar X_{\tilde K}, \mathbf{z}_\delta}$ is invertible. Therefore, the $\operatorname{ord}_{\tilde Z_\delta}$-value of these elements is 0. Writing

$$y_\delta = \frac{\mu''}{\lambda} \cdot \frac{\nu}{t_j^{k'}} \cdot \frac{\rho}{t_j^{k-k'-k''}} \cdot \frac{\lambda t_j^{k-k''}}{s_0^{(k)}}$$

we get from (27) that

$$\begin{aligned} \operatorname{ord}_{\tilde Z_\delta}(y_\delta) &= \operatorname{ord}_{\tilde Z_\delta}\left(\frac{\mu''}{\lambda}\right) + \operatorname{ord}_{\tilde Z_\delta}\left(\frac{\nu}{t_j^{k'}}\right) + \operatorname{ord}_{\tilde Z_\delta}\left(\frac{\rho}{t_j^{k-k'-k''}}\right) + \operatorname{ord}_{\tilde Z_\delta}\left(\frac{\lambda t_j^{k-k''}}{s_0^{(k)}}\right) \\ &= \operatorname{ord}_{\tilde Z_\delta}(z_\delta), \end{aligned}$$

as desired. ∎

Having established in Proposition 10.5 that the absolutely integral projective curve $Y = \operatorname{Proj}(K[s_0^{(k)}, \ldots, s_{l(k)}^{(k)}])$ is a q-curve with function field F for a large positive multiple k of k_0 and a large positive integer q, we choose q as a large prime number and apply Proposition 8.6 with Y replacing Δ to deduce the following mile stone of the work:

Proposition 10.6: *Under Setup 10.1 and in the notation of Subsection 8.1, the following statement holds for every large positive multiple k of k_0:*

There exists a non-empty Zariski-open subset U_i of $\mathbb{P}_K^i$, $i = 2, 3, \ldots, l(k)$, such that with $U = U_2 \times U_3 \times \cdots \times U_{l(k)}$, for each $\mathbf{A} \in (\psi^{(k)})^{-1}(U(K))$ and with $\binom{\mathbf{a}}{\mathbf{b}} = \mu^{(k)}(\mathbf{A})$, the element $t = \sum_{i=0}^{l(k)} a_i s_i^{(k)} / \sum_{i=0}^{l(k)} b_i s_i^{(k)}$ $[F : K(t)]$-symmetrically stabilizes F/K.

Remark 10.7: The case where K is a number field is much simpler. In this case $\tilde{K}$ is a separable extension of K. Hence, the normal absolutely integral curve $\bar{X}_K$ remains normal under the base change from K to $\tilde{K}$. Thus, in this case $\bar{X}_{\tilde{K}}$ is a smooth projective curve. This allows us to forget about the special separable point B of X constructed in Section 6. The birational morphism $\varphi\colon \bar{X}_K \to Y$ now maps X_K isomorphically onto Y_0. However, we have to take extra care of the point $\mathbf{y}_0 = \varphi(Z_K)$. Over $\tilde{K}$, $\mathbf{y}_0$ is a higher ordinary point of $Y_{\tilde{K}}$. In other words, the tangents to $\bar{X}_{\tilde{K}}$ at points that lie over $\mathbf{y}_0$ are distinct. Then, we may use a much simple version of Proposition 8.6 that makes a big part of the paper [GJR17] redundant. ∎

11. A Normalized Stabilizing Element

Proposition 11.2 below allows us to choose the stabilizing element more carefully. We prove that t can be chosen in Proposition 10.6 such that $a_0 = 1$, $b_0 = 1$, $b_1 = a_1 + 1$, and $(a_1, \ldots, a_{l(k)})$ and $(b_2, \ldots, b_{l(k)})$ respectively belong to given $\mathcal{T}$-open subsets of $R^{l(k)}$ and $R^{l(k)-1}$, where the $\mathcal{T}$-topologies on powers of R are the product $\mathcal{T}$-topologies.

Lemma 11.1: *Let m be a positive integer and $\mathcal{C}$ a non-empty $\mathcal{T}$-open subset of R^m. Then, $\mathcal{C}$ is Zariski-dense in $\mathbb{A}^m_K$.*

Proof: It suffices to prove that if $f \in K[X_1, \ldots, X_m]$ is non-zero, then there exists $\mathbf{x} \in \mathcal{C}$ such that $f(\mathbf{x}) \neq 0$. In order to do it we first choose a point $\mathbf{c} = (c_1, \ldots, c_m) \in \mathcal{C}$ and a positive real number ε such that if $\mathbf{x} \in R^m$ satisfies $|\mathbf{x} - \mathbf{c}|_\mathfrak{p} < \varepsilon$ for all $\mathfrak{p} \in \mathcal{T}$, then $\mathbf{x} \in \mathcal{C}$. Using induction, we may assume that $m = 1$. Then, we use the strong approximation theorem of algebraic number theorey [CaF67, p. 67] to choose $a \in R$ such that $|a|_\mathfrak{p} < \varepsilon$ for all $\mathfrak{p} \in \mathcal{T}$. Then $x = c_1 + ay \in \mathcal{C}$ for each $y \in R$. Hence, $f(x) \neq 0$ for all but finitely many $x \in \mathcal{C}$. ∎

Proposition 11.2: *Under Setup 10.1, let k be a large positive multiple of k_0 such that Proposition 10.6 holds. Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty $\mathcal{T}$-open subsets of $R^{l(k)}$ and $R^{l(k)-1}$, respectively. Set $s_i = s_i^{(k)}$ for $i = 0, \ldots, l(k)$. Then, there exist $(a_1, \ldots, a_{l(k)}) \in \mathcal{A}$ and $(b_2, \ldots, b_{l(k)}) \in \mathcal{B}$ such that with $b_1 = a_1 + 1$ the quotient $t = \frac{s_0 + a_1 s_1 + \cdots + a_{l(k)} s_{l(k)}}{s_0 + b_1 s_1 + \cdots + b_{l(k)} s_{l(k)}}$ symmetrically stabilizes F/K.*

Proof: We write $l = l(k)$ and simplify the notation introduced in Subsection 8.1 by setting $\mathbb{M} = \mathbb{M}^{(l)}$, $\mu = \mu^{(l)}$, $\mathbb{P} = \mathbb{P}^{(l)}$, and $\psi = \psi^{(k)}$. Then, (1) of that subsection simplifies to the row

$$\mathbb{P} \xleftarrow{\psi} \mathbb{M} \xrightarrow{\mu} \mathbb{M}^*_{2,l+1}. \tag{1}$$

For each $2 \le i \le l$ let U_i be the non-empty Zariski-open subset of $\mathbb{P}^i_K$ that Proposition 10.6 supplies. Shrink U_i, if necessary, to assume that

(2) each $(a_1 : \cdots : a_i : a_{i+1}) \in U_i(\tilde{K})$ satisfies $a_{i+1} \neq 0$.

Let $U = U_2 \times \cdots \times U_l$. By Proposition 10.6,

(3) for each $\mathbf{A} \in \psi^{-1}(U(K))$ and with $\mu(\mathbf{A}) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ b_0 & b_1 & \cdots & b_l \end{pmatrix}$, the element $t = \frac{a_0 s_0 + \cdots + a_l s_l}{b_0 s_0 + \cdots + b_l s_l}$ symmetrically stabilizes F/K.

We are going to extend row (1) to a commutative diagram:

$$\begin{tikzcd} \mathbb{P} = \mathbb{P}^2 \times \cdots \times \mathbb{P}^l & \mathbb{M} = \mathbb{M}^*_2 \times \cdots \times \mathbb{M}^*_l \arrow[l, "\psi"'] \arrow[r, "\mu"] & \mathbb{M}^*_{2,l+1} \\ \mathbb{A} = \mathbb{A}^2 \times \cdots \times \mathbb{A}^l \arrow[u, "\rho"] & \mathbb{M}' = \mathbb{M}'_2 \times \cdots \times \mathbb{M}'_l \arrow[l, "\psi'"'] \arrow[r, "\mu'"] \arrow[u, hook] & \mathbb{A}^{2l-1} \arrow[u, "\theta'"'] \end{tikzcd} \tag{4}$$

THE SUBSET $\mathbb{M}'$ OF $\mathbb{M}$: Let $\mathbb{M}'_2$ be the Zariski-closed subset of $\mathbb{M}^*_2$ such that $\mathbb{M}'_2(\mho)$ consists of all matrices of the form

$$A_2 = \begin{pmatrix} 1 & a_{11} & a_{12} \\ 1 & a_{11}+1 & a_{22} \end{pmatrix}. \tag{5}$$

For each $3 \le i \le l$ let $\mathbb{M}'_i$ be the Zariski-closed subset of $\mathbb{M}^*_i$ such that $\mathbb{M}'_i(\mho)$ consists of all matrices of the form

$$A_i = \begin{pmatrix} 1 & a_{11} & \cdots & \cdot & a_{1i} \\ 0 & 1 & \cdots & \cdot & a_{2i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{ii} \end{pmatrix}. \tag{6}$$

Then, for each $2 \le i \le l$, $\mathbb{M}'_i$ is naturally isomorphic to the affine space $\mathbb{A}^{i(i+1)/2}$. We define a closed immersion $\theta'\colon \mathbb{A}^{2l-1} \to \mathbb{M}^*_{2,l+1}$ by

$$\theta'(a_1, \ldots, a_l, b_2, \ldots, b_l) = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_l \\ 1 & a_1+1 & b_2 & \cdots & b_l \end{pmatrix}. \tag{7}$$

Now we set $\mathbb{M}' = \mathbb{M}'_2 \times \cdots \times \mathbb{M}'_l$ and observe by induction on l that $\mu(\mathbb{M}') \subseteq \theta'(\mathbb{A}^{2l-1})$. Hence, there exists a unique morphism $\mu'\colon \mathbb{M}' \to \mathbb{A}^{2l-1}$ such that $\theta' \circ \mu' = \mu|_{\mathbb{M}'}$.

THE MORPHISM ρ: For each $2 \le i \le l$ we define an embedding $\rho_i\colon \mathbb{A}^i \to \mathbb{P}^i$ by

$$\rho_i(a_1, \ldots, a_i) = (a_1{:}\cdots{:}a_i{:}1).$$

Let $\mathbb{A} = \mathbb{A}^2 \times \cdots \times \mathbb{A}^l$ and consider the morphism $\rho = \rho_2 \times \cdots \times \rho_l$ from $\mathbb{A}$ to $\mathbb{P}$.

THE MORPHISM $\psi'_i\colon \mathbb{M}'_i \to \mathbb{A}^i$: In the notation of (5) and by Subsection 8.1, $\psi_2(A_2) = (y_0{:}y_1{:}y_2)$ is the unique element of $\mathbb{P}^2$ that satisfies

$$\begin{pmatrix} 1 & a_{11} \\ 1 & a_{11}+1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} y_2 = \begin{pmatrix} 1 & a_{11} & a_{12} \\ 1 & a_{11}+1 & a_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = 0. \tag{8}$$

Let $A'_2 = \begin{pmatrix} 1 & a_{11} \\ 1 & a_{11}+1 \end{pmatrix}$. Since $y_i \neq 0$ for at least one i and $\det(A'_2) = 1$, we have $y_2 \neq 0$. Hence, we may assume that $y_2 = 1$ and conclude that $\psi_2(A_2) = (y_0{:}y_1{:}1) = \rho_2(y_0, y_1)$.

Similarly, for $i = 3, \ldots, l$ we consider a matrix A_i as in (6). Then $\psi_i(A_i) = (y_0{:}\cdots{:}y_i)$ is the unique element of $\mathbb{P}^i$ that satisfies

$$\begin{pmatrix} 1 & a_{11} & \cdots & a_{1,i-1} \\ 0 & 1 & \cdots & a_{2,i-1} \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{i-1} \end{pmatrix} + \begin{pmatrix} a_{1i} \\ a_{2i} \\ \cdots \\ a_{ii} \end{pmatrix} y_i = 0. \tag{9}$$

Again, the determinant of the $i \times i$ matrix A'_i on the left hand side of (9) is 1, hence $y_i \neq 0$, so we may assume that $y_i = 1$. As in the previous case, we conclude that

$$\psi_i(A_i) = (y_0{:}\cdots{:}y_{i-1}{:}1) = \rho_i(y_0, \ldots, y_{i-1}). \tag{10}$$

Let $\tilde{\mathbf{y}}_i$ and $\tilde{\mathbf{a}}_i$ be the first and the second columns of height i that appear in (8) if $i = 2$ and in (9) if $3 \le i \le l$. Then, $A'_i\tilde{\mathbf{y}}_i + \tilde{\mathbf{a}}_i = 0$ and we define the morphism $\psi'_i\colon \mathbb{M}'_i \to \mathbb{A}^i$ by the formula

$$\psi'_i(A_i) = \tilde{\mathbf{y}}_i = -(A'_i)^{-1}\tilde{\mathbf{a}}_i \tag{11}$$

and consider $\psi'_i(A_i)$ in the sequel as a row. It follows from (10) and (11) that $\rho_i \circ \psi'_i = \psi_i|_{\mathbb{M}'_i}$. Writing $\psi' = \psi'_2 \times \cdots \times \psi'_l$, this establishes the left part of Diagram (4).

Claim A: *For each* $2 \le i \le l$ *we have* $\psi_i'(\mathbb{M}_i') = \mathbb{A}^i$. Indeed, let $y_0, y_1, \dots, y_{i-1} \in \mho$. For $i = 2$ we set $a_{11} = 0$, $a_{12} = -y_0$, and $a_{22} = -y_0 - y_1$ in A_2. Then, (8) holds for $y_2 = 1$, so by (11), $\psi_2'(A_2) = (y_0, y_1)$.

When $l > 2$, we set for each $3 \le i \le l$,

$$A_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & -y_0 \\ 0 & 1 & \cdots & 0 & -y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -y_{i-1} \end{pmatrix} \in \mathbb{M}_i'.$$

Substituting the corresponding values for the parameters appearing in (9) and setting $y_i = 1$, we get that $\psi_i'(A_i) = (y_0, \dots, y_{i-1})$, as desired.

Claim B: $\mu'(\mathbb{M}'(R)) = \mathbb{A}^{2l-1}(R)$. First observe that if $\mathbf{A} \in \mathbb{M}'(R)$, then $\mu(\mathbf{A}) \in \theta'(\mathbf{A}^{2l-1}(R))$, hence by (4) and (7), $\mu'(\mathbf{A}) = (\theta')^{-1}(\mu(\mathbf{A})) \in \mathbb{A}^{2l-1}(R)$.

To prove the inclusion in the other direction, we consider $(a_1, \dots, a_l, b_2, \dots, b_l) \in \mathbb{A}^{2l-1}(R)$. If $l = 2$, let $A_2 = \begin{pmatrix} 1 & a_1 & a_2 \\ 1 & a_1 + 1 & b_2 \end{pmatrix} \in \mathbb{M}_2'(R)$. If $l > 2$, let

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in \mathbb{M}_2'(R), \quad A_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbb{M}_i'(R)$$

for $i = 3, \dots, l-1$, and

$$A_l = \begin{pmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_{l-1} & a_l \\ 0 & 1 & b_2 - a_2 & b_3 - a_3 & \cdots & b_{l-1} - a_{l-1} & b_l - a_l \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbb{M}_l'(R).$$

Then,

$$A_2 A_3 \cdots A_{l-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{M}_{2,l}(R).$$

Thus, in both cases,

$$\mu(\mathbf{A}) = A_2 \cdots A_{l-1} \cdot A_l = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_l \\ 1 & a_1 + 1 & b_2 & \cdots & b_l \end{pmatrix} = \theta'(a_1, \dots, a_l, b_2, \dots, b_l).$$

Hence, by the commutativity of (4), $(a_1, \dots, a_l, b_2, \dots, b_l) = (\theta')^{-1}(\mu(\mathbf{A})) = \mu'(\mathbf{A}) \in \mu'(\mathbb{M}'(R))$, as desired.

Conclusion of the proof: The product $\mathcal{A} \times \mathcal{B}$ is a non-empty $\mathcal{T}$-open subset of $\mathbb{A}^{2l-1}(R)$. By Claim B, $\mu'(\mathbb{M}'(R)) = \mathbb{A}^{2l-1}(R)$. Hence, the $\mathcal{T}$-open subset $(\mu')^{-1}(\mathcal{A} \times \mathcal{B})$ of $\mathbb{M}'(R)$ is non-empty. By definition, $\mathbb{M}'$ is isomorphic to an affine space. Hence, by Lemma 11.1, $(\mu')^{-1}(\mathcal{A} \times \mathcal{B})$ is Zariski-dense in $\mathbb{M}'$.

Since $a_{i+1} \neq 0$ for each $2 \le i \le l$ and every $(a_1{:}\cdots{:}a_i{:}a_{i+1}) \in U_i$ (by (2)), we have $U_i \subseteq \rho_i(\mathbb{A}^i)$, hence $U \subseteq \rho(\mathbb{A})$. Therefore, $U' = \rho^{-1}(U)$ is a non-empty Zariski-open subset of $\mathbb{A}$.

By Claim A, $\psi'(\mathbb{M}') = \mathbb{A}$, hence $(\psi')^{-1}(U')$ is a non-empty Zariski-open subset of $\mathbb{M}'$. Therefore, there exists $\mathbf{A} \in (\mu')^{-1}(\mathcal{A} \times \mathcal{B}) \cap (\psi')^{-1}(U')$. Let $(a_1, \dots, a_l, b_2, \dots, b_l) = \mu'(\mathbf{A})$. Then, $\mathbf{a} = (a_1, \dots, a_l) \in \mathcal{A}$, $\mathbf{b} = (b_2, \dots, b_l) \in \mathcal{B}$, $\mu(\mathbf{A}) = \theta'(\mu'(\mathbf{A})) = \begin{pmatrix} 1 & a_1 & \cdots & a_l \\ 1 & b_1 & \cdots & b_l \end{pmatrix}$ with $b_1 = a_1 + 1$, and $\psi(\mathbf{A}) = \rho(\psi'(\mathbf{A})) \in U(K)$. By (3), the element $t = \frac{s_0 + a_1 s_1 + \cdots + a_l s_l}{s_0 + b_1 s_1 + \cdots + b_l s_l}$ symmetrically stabilizes F/K, as desired. ∎

12. M-points on Varieties Defined over K

Using the notation of Subsection 4.8, we fix a global field K, a proper subset $\mathcal{V}$ of the set $\mathbf{P}_K$ of all primes of K, and a finite subset $\mathcal{S}$ of $\mathcal{V}$. We also consider a finite subset $\mathcal{T}$ of $\mathcal{V}$ that contains $\mathcal{S}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$. The following definition puts together those properties of the fields $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ that are used in the proof of Theorem C. Then, Proposition 12.3 restates Theorem C for curves for algebraic extensions of K having those properties.

Definition 12.1: [GJR00, Def. 1.10]. Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$ and let $\mathcal{O}$ be a subset of M. We say that M is **weakly** (resp. **weakly symmetrically**) **K-stably P$\mathcal{S}$C over $\mathcal{O}$** if for every absolutely irreducible polynomial $h \in K[T,Y]$ monic in Y with $d = \deg_Y(h)$ and every polynomial $g \in K[T]$ satisfying
(1a) $h(0,Y)$ has d distinct roots in $K_{\mathrm{tot},\mathcal{S}}$, $g(0) \neq 0$, and
(1b) $\mathrm{Gal}(h(T,Y),K(T)) \cong \mathrm{Gal}(h(T,Y),\tilde{K}(T))$ (resp. and is isomorphic to the symmetric group $\mathfrak{S}_d$).
there exists $(a,b) \in \mathcal{O} \times M$ such that $h(a,b) = 0$ and $g(a) \neq 0$.

Note that in that case, if $M \subseteq M' \subseteq K_{\mathrm{tot},\mathcal{S}}$, then M' is also weakly K-stably P$\mathcal{S}$C over $\mathcal{O}$. Also note that if M is weakly K-stably P$\mathcal{S}$C over $\mathcal{O}$, then M is also weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}$. ∎

Setup 12.2: Proposition 7.6 introduces a positive integer k_0, for each positive multiple k of k_0 an isomorphism $\alpha^{(k)}\colon \mathcal{O}_Z \to \mathcal{O}_Z(k)$ of sheaves and an element $s_0^{(k)} \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k))$ such that the isomorphism $\alpha^{(k)}(Z)\colon \Gamma(Z,\mathcal{O}_Z) \to \Gamma(Z,\mathcal{O}_Z(k))$ of $\Gamma(Z,\mathcal{O}_Z)$-modules induced by $\alpha^{(k)}$ satisfies $\rho^{(k)}_{\bar{X},Z}(s_0^{(k)}) = \alpha^{(k)}(Z)(1)$, where 1 is the unit element of the ring $\Gamma(Z,\mathcal{O}_Z)$. We choose k sufficiently large such that Proposition 10.6 holds. Then, we consider the elements $s_1^{(k)},\ldots,s_{l(k)}^{(k)}$ of $\mathrm{Ker}(\rho^{(k)}_{\bar{X},Z})$ that appear in Proposition 10.6 and set $\mathbf{s} = (s_0^{(k)}, s_1^{(k)}, \ldots, s_{l(k)}^{(k)})$.

As in Subsection 7.9, for each algebraic extension K' of K and every $\mathfrak{p} \in \mathcal{T}$ let $\Gamma^{(k)}_{\mathbf{s},\mathfrak{p},K'}$ be the set of all $s \in \Gamma(\bar{X}_{\hat{K}_\mathfrak{p}K'}, \mathcal{O}_{\bar{X}_{\hat{K}_\mathfrak{p}K'}}(k))$ of the form $s = s_0^{(k)} + \sum_{i=1}^{l(k)} a_i s_i^{(k)}$ with $a_1,\ldots,a_{l(k)} \in \hat{K}_\mathfrak{p}K'$ such that $\mathrm{div}(s) \in \hat{\Omega}^{[d_k]}_{\mathfrak{p},K'}$, where $d_k = \deg(\mathcal{O}_{\bar{X}_K}(k))$. In particular, $\mathrm{div}(s)$ totally splits in $F\hat{L}_\mathfrak{p}K'$ into d_k distinct components each of which is a point that belongs to $\Omega_\mathfrak{p}(\hat{L}_\mathfrak{p}K')$ (Subsection 7.4). ∎

Proposition 12.3: *Let $K,\mathcal{S},\mathcal{T},\mathcal{V}$ be as in the first paragraph of this section. Let C be an absolutely integral affine curve over K and let $(L_\mathfrak{p},\Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}}$ be approximation data for $K,\mathcal{S},\mathcal{T},\mathcal{V},C$ (as in Subsection 4.7). Let M be a subfield of $K_{\mathrm{tot},\mathcal{S}}$ that contains K. Suppose M is weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$ (resp. $\mathcal{O}_{M,\mathcal{V}}$). Then, there exists $\mathbf{z} \in C(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau \in \bigcap_{\mathfrak{p}\in\mathcal{T}} \Omega_\mathfrak{p}$ (resp. $\mathbf{z}^\tau \in \bigcap_{\mathfrak{p}\in\mathcal{S}} \Omega_\mathfrak{p} \cap \bigcap_{\mathfrak{p}\in\mathcal{T}\smallsetminus\mathcal{S}} \Omega_\mathfrak{p}(L_\mathfrak{p}K_{\mathrm{tot},\mathcal{S}})$) for each $\tau \in \mathrm{Gal}(K)$.*

Proof: We let X and $\bar{X}$ be as in Subsection 5.5 and Lemma 5.6, respectively, and write F for the common function field of X and $\bar{X}$. Following Lemma 4.10, we change C and $\mathcal{T}$, if necessary, to meet all of the assumptions of Sections 5 and 6. We also simplify our notation by setting $l = l(k)$ and $s_i = s_i^{(k)}$ for $i = 0,\ldots,l$. We set $\mathbf{s} = (s_0, s_1, \ldots, s_l)$.

The rest of the proof naturally breaks up into six parts.

Part A: *The subset $\mathcal{A}$ of R^l.* Lemma 7.10 supplies a $\mathcal{T}$-open neighborhood $\mathcal{A}$ of $(0,\ldots,0)$ in R^l such that if L is an algebraic extension of K, if R_L is the integral closure of R in L, and if $(a_1,\ldots,a_l)$ belongs to the $\mathcal{T}_L$-open neighborhood $\mathcal{A}(R_L)$ of $(0,\ldots,0)$ in R_L^l induced by $\mathcal{A}$, then, in the notation of Setup 12.2, $(s_0 + \sum_{i=1}^l a_i s_i)_\mathfrak{p} \in \Gamma^{(k)}_{\mathbf{s},\mathfrak{p},L}$ for each $\mathfrak{p} \in \mathcal{T}$, where for $s \in \Gamma(\bar{X}_{R_L}, \mathcal{O}_{\bar{X}_{R_L}}(k))$, $s_\mathfrak{p}$ is the section in $\Gamma(\bar{X}_{\hat{K}_\mathfrak{p}L}, \mathcal{O}_{\bar{X}_{\hat{K}_\mathfrak{p}L}}(k))$ obtained from s by base change from R_L to $\hat{K}_\mathfrak{p}L$. We set $\mathcal{B} = R^{l-1}$.

Proposition 11.2 gives $\mathbf{a} = (a_1,\ldots,a_l) \in \mathcal{A}$ and $(b_2,\ldots,b_l) \in \mathcal{B}$ such that, with $b_1 = a_1 + 1$, $s = s_0 + \sum_{i=1}^l a_i s_i$ and $s^* = s_0 + \sum_{i=1}^l b_i s_i$, the element $t = \frac{s}{s^*}$ symmetrically stabilizes F/K.

Part B: *$K_{\mathrm{tot},\mathcal{S}}$-rational points of X.* By Subsection 7.9, s and s^* are elements of $\Gamma_\mathbf{s}^{(k)}$, hence they belong to $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(k), \alpha^{(k)})$ (Setup 12.2). Moreover, by Part A, $s_\mathfrak{p} \in \Gamma^{(k)}_{\mathbf{s},\mathfrak{p},K}$ for each $\mathfrak{p} \in \mathcal{T}$. Following

Subsection 2.6, we consider $\mathrm{div}(s)$ as an effective Weil divisor on $\bar{X}$. By Lemma 4.10, we may assume that $\mathcal{T}$ is non-empty. By Setup 12.2, for each $\mathfrak{p} \in \mathcal{T}$, $\mathrm{div}(s_\mathfrak{p})$ belongs to $\hat{\Omega}^{[d_k]}_{\mathfrak{p},K}$, where $d_k = \deg(\mathcal{O}_{\bar{X}_K}(k))$, hence $\mathrm{div}(s_\mathfrak{p})$ totally splits in $F\hat{L}_\mathfrak{p}$. The components of $\mathrm{div}(s_\mathfrak{p})$ are points in $X(\hat{L}_\mathfrak{p})$ and there are exactly d_k of them (Subsection 7.4). When $\mathfrak{p} \in \mathcal{S}$, we have $\hat{L}_\mathfrak{p} = \hat{K}_\mathfrak{p}$, so the components of $\mathrm{div}(s_\mathfrak{p})$ are in this case $(\tilde{K} \cap \hat{K}_\mathfrak{p})$-rational points of X. By Subsection 4.4, they are $K_\mathfrak{p}$-rational points. Since $\mathrm{div}(s)$ is invariant under the action of $\mathrm{Gal}(K)$, each of those components is $K_\mathfrak{p}^\tau$-rational for all $\tau \in \mathrm{Gal}(K)$ and $\mathfrak{p} \in \mathcal{S}$. Therefore, with $N = K_{\mathrm{tot},\mathcal{S}}$ and R_N the integral closure of R in N, we have

(2) $\mathrm{div}(s)_{R_N} = \mathrm{div}(s) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_N)$ is a formal sum of d_k $K_{\mathrm{tot},\mathcal{S}}$-rational points of X, each with multiplicity 1.

Note that if S is empty, then $K_{\mathrm{tot},\mathcal{S}} = K_{\mathrm{sep}}$, so (2) also holds in this case.

Part C: *Choosing y.* The homogeneous element $s^* \in K[t_0, \dots, t_r]$ gives rise to the Zariski-open affine subscheme $C_0 = D_+(s^*)$ of $\bar{X}_K$ [Liu06, p. 51, Lemma 3.36(a)]. Thus, $C_0 = \mathrm{Spec}(A)$, where A is an integrally closed domain (because $\bar{X}_K$ is normal) with quotient field F. Therefore,

$$A = \bigcap_{\substack{\mathbf{p} \in \bar{X}_K \\ s^* \notin \mathbf{p}}} \mathcal{O}_{\bar{X}_K,\mathbf{p}}.$$

In particular, $t = \frac{s}{s^*} \in A$ and A is integral over $K[t]$.

By (4) in Subsection 2.7, $\mathrm{div}(t) = \mathrm{div}(s) - \mathrm{div}(s^*)$. Hence, since $\mathrm{div}(s)$ and $\mathrm{div}(s^*)$ are effective Weil divisors (Subsection 2.4), $\mathrm{div}_0(t) \le \mathrm{div}(s)$, so each zero of t is also a zero of s. It follows from (2) that t has at most d_k zeros, each with multiplicity 1.

We choose $y \in A$ such that $F = K(t, y)$ and let $h_0 \in K[T, Y]$ be the absolutely irreducible polynomial, monic in Y, such that $h_0(t, y) = 0$. Let $d = [F : K(t)]$, let $y_1, \dots, y_d$ be the roots of $h_0(t, Y)$ in $K(t)_{\mathrm{sep}}$ with $y_1 = y$, and let $\Delta(t) = \prod_{i \ne j}(y_i - y_j) \in K[t]$. Since h_0 is separable in Y, $\Delta(t) \ne 0$. We write $h_0(T, Y) = Y^d + f_{d-1}(T)Y^{d-1} + \dots + f_0(T)$ with $f_0, \dots, f_{d-1} \in K[T]$. Since the roots of $h_0(0, Y)$ bijectively correspond to the zeros of t, it follows from the preceding paragraph, that $h_0(0, Y)$ has d distinct roots, $\bar{y}_1, \dots, \bar{y}_d$ in K_{sep} and $d \le d_k$, so $\Delta(0) = \prod_{i \ne j}(\bar{y}_i - \bar{y}_j) \ne 0$.

Part D: *Another stabilizing element.* For each $1 \ne a_0 \in K$ we have

$$\frac{\sum_{i=1}^l (a_i - a_0 b_i)s_i}{1 - a_0} = \sum_{i=1}^l \big(a_i + (a_i - b_i)\frac{a_0}{1 - a_0}\big)s_i,$$

hence

$$t_0 = \frac{t - a_0}{1 - a_0} = \frac{s_0 + \sum_{i=1}^l (a_i + (a_i - b_i)\frac{a_0}{1-a_0})s_i}{s_0 + \sum_{i=1}^l b_i s_i}. \tag{3}$$

Note that $K(t_0) = K(t)$, so also t_0 symmetrically stabilizes F/K. Since $\mathcal{A}$ is $\mathcal{T}$-open, there exists a positive real number $\gamma_\mathbf{a}$ such that

(4) if $\mathbf{c} \in R_N^l$ satisfies $|\mathbf{c} - \mathbf{a}|_\mathfrak{p} < \gamma_\mathbf{a}$ for each $\mathfrak{p} \in \mathcal{T}_N$, then $\mathbf{c} \in \mathcal{A}(R_N)$.

We use the strong approximation theorem for K [CaF67, p. 67] to choose a non-zero $m \in R = \mathcal{O}_{K,\mathcal{V} \smallsetminus \mathcal{T}}$ such that

$$|(a_i - b_i)m|_\mathfrak{p} < \gamma_\mathbf{a} \tag{5}$$

for all $1 \le i \le l$ and $\mathfrak{p} \in \mathcal{T}$. In particular, for $i = 1$, we get $|m|_\mathfrak{p} < \gamma_\mathbf{a}$ for all $\mathfrak{p} \in \mathcal{T}$.

Let $t' = \frac{t}{m(1-t)}$ and note that $t = \frac{mt'}{1+mt'}$. We let j' be a positive integer such that

$$h_1(t', Y) = \big(\frac{1 + mt'}{m}\big)^{j'} \cdot h_0\big(\frac{mt'}{1 + mt'}, Y\big) \in K[t', Y]$$

and write $h_1(t',Y) = f_d^*(t')Y^d + f_{d-1}^*(t')Y^{d-1} + \cdots + f_0^*(t')$ with $f_0^*,\ldots,f_d^* \in K[T']$ and $f_d^*(T') = \left(\frac{1+mT'}{m}\right)^{j'}$. In particular, $f_d^*(0) = m^{-j'} \neq 0$. We set $Y' = f_d^*(t')Y$ and $h(T',Y') = f_d^*(T')^{d-1}\cdot h_1(T',Y)$. Then, $h \in K[T',Y']$ is monic of degree d in Y' and $y_1^* = f_d^*(t')y_1,\ldots,y_d^* = f_d^*(t')y_d$ are the roots of $h(t',Y')$. Let $\Delta^*(t') = \prod_{i\neq j}(y_i^* - y_j^*) \in K[t']$. Then,

$$\Delta^*(t') = f_d^*(t')^{d(d-1)} \cdot \prod_{i\neq j}(y_i - y_j) = f_d^*(t')^{d(d-1)} \cdot \Delta\big(\frac{mt'}{1+mt'}\big)\,.$$

In particular, by Part C, $\Delta^*(0) = f_d^*(0)^{d(d-1)}\Delta(0) \neq 0$, so $h(0,Y')$ has d distinct roots.

Since $K(t') = K(t) \subseteq F$, we may consider a prime divisor Q of $F\tilde{K}/\tilde{K}$ such that $t'(Q) = 0$. Then, $t(Q) = \frac{mt'(Q)}{1+mt'(Q)} = 0$. Let $\mathbf{q}$ be the point of $\bar{X}(\tilde{K})$ that lies under Q. Then, $\mathbf{q}$ is a zero of t, hence of s, so by (2), $\mathbf{q} \in X(K_{\mathrm{tot},\mathcal{S}})$.

Since $f_d^*(t') \neq 0$, we have $K(t',y_1^*) = K(t', f_d^*(t')y) = K(t',y) = K(t,y) = F$. Also, since $h(T',Y')$ is absolutely irreducible, the d distinct roots of $h(0,Y')$ are the images of y_1^* at the distinct prime divisors of $F\tilde{K}/\tilde{K}$ which are zeros of t' [Lan58, p. 10, Thm. 2]. By the preceding paragraph each of these roots lies in $K_{\mathrm{tot},\mathcal{S}}$. Thus, $h(T',Y')$ satisfies Condition (1a) (with (T',Y') replacing (T,Y)). Since t is a symmetrically stabilizing element for F/K, so is t'. Hence, $h(T',Y')$ also satisfies Condition (1b), with $\mathrm{Gal}(h(T',Y'),K(T')) \cong \mathfrak{S}_d$.

Part E: *A prime divisor of FM/M of degree 1.* By the assumption on M, there exists $(\bar{t},\bar{y}) \in \mathcal{O}_{K,\mathcal{V}} \times M$ (resp. $(\bar{t},\bar{y}) \in \mathcal{O}_{M,\mathcal{V}} \times M$) such that $h(\bar{t},\bar{y}) = 0$, $h(\bar{t},Y)$ is separable, $m\bar{t}+1 \neq 0$, and $x_1,\ldots,x_n$ (introduced in Subsection 5.1) belong to the local ring of $M[t',y_1^*]$ at $(\bar{t},\bar{y})$. Since C is a smooth curve (Statement (1) of Section 5), there exists a prime divisor P of FM/M of degree 1 such that $t'(P) = \bar{t}$ is in $\mathcal{O}_{K,\mathcal{V}}$ (resp. in $\mathcal{O}_{M,\mathcal{V}}$), $1 + mt'(P) \neq 0$, and $\mathbf{z} = \mathbf{x}(P) \in C(M)$. Hence, $t = \frac{mt'}{1+mt'}$ is defined at P, $a_0 = t(P) \neq 1$, P is a zero of $\frac{t-a_0}{1-a_0}$, and

(6) $\frac{a_0}{1-a_0} = mt'(P)$ is in $m\mathcal{O}_{K,\mathcal{V}}$ (resp. in $m\mathcal{O}_{M,\mathcal{V}}$).

Let $\mathbf{a}' = \mathbf{a} + (\mathbf{a}-\mathbf{b})\frac{a_0}{1-a_0}$ and set $s' = s_0 + \sum_{i=1}^l a_i's_i$. Since P is a zero of the left hand side of (3), P is also a zero of the right hand side of (3). The latter is $\frac{s'}{s^*}$. Again, since $\mathrm{div}(s')$ and $\mathrm{div}(s^*)$ are effective divisors, P is a zero of s'.

By the properties of $t'(P)$ mentioned in the preceding paragraph, by (6), and by (5), $|a_i' - a_i|_{\mathfrak{p}} = \left|(a_i - b_i)\frac{a_0}{1-a_0}\right|_{\mathfrak{p}} = |(a_i-b_i)mt'(P)|_{\mathfrak{p}} < \gamma_{\mathbf{a}}$ for all $1 \leq i \leq l$ and $\mathfrak{p} \in \mathcal{T}$ (resp. $\mathfrak{p} \in \mathcal{T}_M$). By (4), $\mathbf{a}' \in \mathcal{A}(R_N) \cap K^l$ (resp. $\mathbf{a}' \in \mathcal{A}(R_N) \cap M^l$), hence $\mathbf{a}' \in \mathcal{A}$ (resp. $\mathbf{a}' \in \mathcal{A}(R_M)$). Therefore, by Part A, for each $\mathfrak{p} \in \mathcal{T}$ the section $s_{\mathfrak{p}}'$ lies in $\Gamma_{\mathbf{s},\mathfrak{p},K}^{(k)}$ (resp. in $\Gamma_{\mathbf{s},\mathfrak{p},M}^{(k)}$). In particular, $\mathrm{div}(s_{\mathfrak{p}}')$ has no multiple components (Setup 12.2), so $\mathrm{div}(s')_{R_N}$ has no multiple components.

Part F: *The irreducible components of* $\mathrm{div}(s')_{R_N}$. Let $\mathbf{p}$ be an irreducible component of $\mathrm{div}(s')_{R_N}$. By Lemma 7.8(b) (for s' replacing s), the restriction of the morphism $f_{R_N}\colon X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_N) \to \mathrm{Spec}(R_N)$ (induced from the morphism f which is introduced in Subsection 5.5) to $\mathbf{p}$ is finite and surjective over $\mathrm{Spec}(R_N)$. Since $\mathbf{p}$ is not a multiple component of $\mathrm{div}(s')_{R_N}$, we may consider $\mathbf{p}$ as a prime ideal of $R_N[\mathbf{x}]$. If $\mathfrak{p}_0 = \mathbf{p} \cap R_N \neq 0$, then the image of $\mathbf{p}$ considered as an irreducible component of $\mathrm{div}(s')_{R_N}$ in $\mathrm{Spec}(R_N)$ contains exactly one element, namely $\mathfrak{p}_0$, in contrast to the surjectivity of f_{R_N} on $\mathbf{p}$. Thus, $\mathbf{p} \cap R_N = 0$, so the coordinates $z_1',\ldots,z_n'$ of $\mathbf{z}' = (x_1 + \mathbf{p},\ldots,x_n+\mathbf{p})$ are algebraic over K. Since $\mathbf{p}$ is finite over $\mathrm{Spec}(R_N)$, the ring $R_N[z_1',\ldots,z_n']$ is a finitely generated R_N-module. Hence, $z_1',\ldots,z_n'$ are integral over R_N (hence, over R). In addition, by Setup 12.2, $\mathbf{z}' \in \Omega_{\mathfrak{p}}$ (resp. $\mathbf{z}' \in \Omega_{\mathfrak{p}}(L_{\mathfrak{p}}M)$) for each $\mathfrak{p} \in \mathcal{T}$. Since $\mathbf{z}'$ is algebraic over K, we have $\mathbf{z}' \in \Omega_{\mathfrak{p}}$ (resp. $\mathbf{z}' \in \Omega_{\mathfrak{p}}(L_{\mathfrak{p}}M)$) for each $\mathfrak{p} \in \mathcal{T}$.

If $\mathbf{p}$ is the irreducible component of $\mathrm{div}(s')_{R_N}$ that corresponds to P, then by Part E, $\mathbf{z} = \mathbf{x}(P) \in C(M)$. Since $z_1,\ldots,z_n$ are integral over R, we have $\mathbf{z} \in C(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$.

Next observe that for each $\tau \in \mathrm{Gal}(K)$ (resp. $\tau \in \mathrm{Gal}(M)$) we have $\mathrm{div}(s')^\tau = \mathrm{div}(s')$, because $a_1',\ldots,a_l' \in K$ (resp. because $a_1',\ldots,a_l' \in M$). Hence, $\mathbf{p}^\tau$ is also an irreducible component of $\mathrm{div}(s')_{R_N}$. Therefore, by the paragraph preceding the latter one, $\mathbf{z}^\tau \in \Omega_{\mathfrak{p}}$ (resp. $\mathbf{z}^\tau \in \Omega_{\mathfrak{p}}(L_{\mathfrak{p}}M)$) for each $\mathfrak{p} \in \mathcal{T}$.

In the alternative case (i.e. when M is weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$), we note that if $\mathfrak{p} \in \mathcal{S}$, then $M \subseteq K_{\mathrm{tot},\mathcal{S}} \subseteq K_{\mathfrak{p}} = L_{\mathfrak{p}}$. Hence, $\Omega_{\mathfrak{p}}(L_{\mathfrak{p}}M) = \Omega_{\mathfrak{p}}$, so by the preceding paragraph,

$\mathbf{z}^\tau \in \Omega_\mathfrak{p}$. If $\mathfrak{p} \in \mathcal{T} \smallsetminus \mathcal{S}$, then by the preceding paragraph, $\mathbf{z}^\tau \in \Omega_\mathfrak{p}(L_\mathfrak{p} M) \subseteq \Omega_\mathfrak{p}(L_\mathfrak{p} K_{\mathrm{tot},\mathcal{S}})$, as desired. ∎

PROPOSITION 12.4: *Let $K, \mathcal{S}, \mathcal{V}$ be as in Subsection 4.8. Let M be a subfield of $K_{\mathrm{tot},\mathcal{S}}$ that contains K and is weakly symmetrically K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. Then, $(M, K, \mathcal{S}, \mathcal{V}) \models$* SAT *(Subsection 4.7).*

Proof: By Proposition 12.3, $(M, K, \mathcal{S}, \mathcal{V}, C) \models$ SAT for every absolutely integral affine curve C over K. Hence, by Lemma 4.12, $(M, K, \mathcal{S}, \mathcal{V}) \models$ SAT, as claimed. ∎

13. Varieties over M

We use the strong approximation theorem for varieties defined over K to prove the strong approximation theorem for varieties defined over M. The first step is to remove the adverb "symmetrically K-stably" from the condition "M is weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$" that appears in Proposition 12.4 and allow instead the polynomial h that appears in Definition 12.1 to have coefficients in M (and not only in K). This is done via Weil's descent.

Definition 13.1: [GJR00, Def. 1.10]. Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$ and let $\mathcal{O}$ be a subset of M. We say that M is **weakly P$\mathcal{S}$C over $\mathcal{O}$** if for every absolutely irreducible polynomial $h \in M[T,Y]$ monic in Y such that $h(0,Y)$ decomposes into distinct monic linear factors over $K_{\mathrm{tot},\mathcal{S}}$ and every polynomial $g \in M[T]$ with $g(0) \neq 0$ there exists $(a,b) \in \mathcal{O} \times M$ such that $h(a,b) = 0$ and $g(a) \neq 0$. ∎

LEMMA 13.2: *Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$ which is weakly symmetrically K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. Then, M is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{M,\mathcal{V}}$.*

Proof: Let $h \in M[T,Y]$ and $g \in M[T]$ be as in Definition 13.1 We prove that there exists $(a,b) \in \mathcal{O}_{M,\mathcal{V}} \times M$ such that $h(a,b) = 0$ and $g(a) \neq 0$.

PART A: *Weil's descent.* Let L be a finite extension of K in M with $h \in L[T,Y]$ and $g \in L[T]$. Let V be the absolutely integral affine curve in $\mathbb{A}^3_L$ defined by $h(T,Y) = 0$ and $g(T)Z - 1 = 0$.

Let $d = [L : K]$ and let $\sigma_1, \ldots, \sigma_d$ with $\sigma_1 = 1$ be elements of $\mathrm{Gal}(K)$ whose restrictions to L are all of the K-embeddings of L into $\tilde{K}$. Let $\omega_1, \ldots, \omega_d \in \mathcal{O}_L$ be a basis for L/K, where $\mathcal{O}_L$ is the ring of integers of the global field L (Subsection 4.6).

Consider the linear morphism $\lambda\colon \mathbb{A}^{3d}_L \to \mathbb{A}^3_L$ defined by

$$\lambda(\mathbf{a},\mathbf{b},\mathbf{c}) = \Big(\sum_{i=1}^d \omega_i a_i, \sum_{i=1}^d \omega_i b_i, \sum_{i=1}^d \omega_i c_i\Big),$$

where $\mathbf{a} = (a_1, \ldots, a_d)$, $\mathbf{b} = (b_1, \ldots, b_d)$, and $\mathbf{c} = (c_1, \ldots, c_d)$. By Weil's descent [FrJ08, p. 183, Prop. 10.6.2], there exists an absolutely integral affine variety W in $\mathbb{A}^{3d}_K$ such that the restriction of $\lambda^{\sigma_1}_{\tilde{K}} \times \cdots \times \lambda^{\sigma_d}_{\tilde{K}}$ to $W_{\tilde{K}}$ is an isomorphism $\Lambda\colon W_{\tilde{K}} \to V^{\sigma_1}_{\tilde{K}} \times \cdots \times V^{\sigma_d}_{\tilde{K}}$ which is defined by

$$\Lambda(\mathbf{a},\mathbf{b},\mathbf{c}) = \Big(\sum_{i=1}^d \omega_i^{\sigma_1} a_i, \sum_{i=1}^d \omega_i^{\sigma_1} b_i, \sum_{i=1}^d \omega_i^{\sigma_1} c_i, \ldots, \sum_{i=1}^d \omega_i^{\sigma_d} a_i, \sum_{i=1}^d \omega_i^{\sigma_d} b_i, \sum_{i=1}^d \omega_i^{\sigma_d} c_i\Big). \tag{1}$$

PART B: *Approximation data.* Let $t_0 \in K_{\mathrm{sep}}$ be a root of $h(0,Y)$. By assumption $\mathbf{x}_0 = (0, t_0, g(0)^{-1}) \in V_{\mathrm{simp}}(K_{\mathrm{tot},\mathcal{S}})$. Let L' be a finite Galois extension of K in $K_{\mathrm{tot},\mathcal{S}}$ that contains $L(t_0)$. Then, $\mathbf{x}_0 \in V_{\mathrm{simp}}(L')$, so $\mathbf{x}_0^{\sigma_i} \in V^{\sigma_i}_{\mathrm{simp}}(L')$, $i = 1, \ldots, d$. Hence, since Λ is defined over L',

$$\mathbf{z}_0 = \Lambda^{-1}\big(\mathbf{x}_0^{\sigma_1}, \ldots, \mathbf{x}_0^{\sigma_d}\big) \in W_{\mathrm{simp}}(L'). \tag{2}$$

Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ such that $\mathcal{S} \subseteq \mathcal{T}$, $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$, and $\mathbf{z}_0 \in W(\mathcal{O}_{L',\mathcal{V} \smallsetminus \mathcal{T}})$.

For each $\mathfrak{p} \in \mathcal{T}$ let $L_\mathfrak{p} = K_\mathfrak{p} L'$ and

$$\begin{aligned}\Omega_\mathfrak{p} = \{(\mathbf{a},\mathbf{b},\mathbf{c}) \in W_{\mathrm{simp}}(L_\mathfrak{p}) \mid\ & |\mathbf{a}|_\mathfrak{p} \le 1 \text{ if } \mathfrak{p} \in \mathbf{P}_{K,\mathrm{fin}} \text{ and} \\ & |\mathbf{a}|_\mathfrak{p} < \delta_\mathfrak{p} \text{ if } \mathfrak{p} \in \mathbf{P}_{K,\mathrm{inf}}\},\end{aligned} \tag{3}$$

where $\delta_{\mathfrak{p}} = \big(d \cdot \max_{1\le i,j\le d}|\omega_i^{\sigma_j}|_{\mathfrak{p}}\big)^{-1}$ if $\mathfrak{p}\in\mathbf{P}_{K,\mathrm{inf}}$. If $\mathfrak{p}\in\mathcal{S}$, then $L_{\mathfrak{p}} = K_{\mathfrak{p}}$, because $L'\subset K_{\mathrm{tot},\mathcal{S}}\subset K_{\mathfrak{p}}$.

Let $\mathbf{a}_0,\mathbf{b}_0,\mathbf{c}_0$ be the points of $(L')^d$ such that $\mathbf{z}_0 = (\mathbf{a}_0,\mathbf{b}_0,\mathbf{c}_0)$. By (1) and (2),

$$\begin{aligned}(0,t_0^{\sigma_1},&(g(0)^{-1})^{\sigma_1},\ldots,0,t_0^{\sigma_d},(g(0)^{-1})^{\sigma_d}) = (\mathbf{x}_0^{\sigma_1},\ldots,\mathbf{x}_0^{\sigma_d}) = \Lambda(\mathbf{z}_0) = \Lambda(\mathbf{a}_0,\mathbf{b}_0,\mathbf{c}_0)\\ &= \Big(\sum_{i=1}^d\omega_i^{\sigma_1}a_{0,i},\sum_{i=1}^d\omega_i^{\sigma_1}b_{0,i},\sum_{i=1}^d\omega_i^{\sigma_1}c_{0,i},\ldots,\sum_{i=1}^d\omega_i^{\sigma_d}a_{0,i},\sum_{i=1}^d\omega_i^{\sigma_d}b_{0,i},\sum_{i=1}^d\omega_i^{\sigma_d}c_{0,i}\Big).\end{aligned}$$

Let $Q = \big(\omega_i^{\sigma_j}\big)_{1\le i,j\le d}\in\mathrm{GL}_d(L')$ [Lan93, p. 286, consequence of Cor. 5.4]. Then, $Q\mathbf{a}_0 = \mathbf{0}$ (where $\mathbf{a}_0$ is now considered as a column), so $\mathbf{a}_0 = \mathbf{0}$. Hence, by (3), $\mathbf{z}_0\in\Omega_{\mathfrak{p}}$ for each $\mathfrak{p}\in\mathcal{T}$. Therefore, $\Omega_{\mathfrak{p}}$ is a non-empty $\mathfrak{p}$-open subset of $W_{\mathrm{simp}}(L_{\mathfrak{p}})$, invariant under $\mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}})$, for each $\mathfrak{p}\in\mathcal{T}$. Since $\mathbf{z}_0\in W(\mathcal{O}_{L',\mathcal{V}\smallsetminus\mathcal{T}})$, we have $\mathbf{z}_0\in W(\mathcal{O}_{\tilde{K},\mathfrak{p}})$ for each $\mathfrak{p}\in\mathcal{V}\smallsetminus\mathcal{T}$. It follows that $(L_{\mathfrak{p}},\Omega_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}}$ is approximation data for $K,\mathcal{S},\mathcal{T},\mathcal{V},W$.

Part C: *Conclusion of the proof.* By Proposition 12.4,

$$(M,K,\mathcal{S},\mathcal{V},W,\mathcal{T},(L_{\mathfrak{p}},\Omega_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}})\models\mathrm{SAT}.$$

Hence, there exists $\mathbf{z} = (\mathbf{a},\mathbf{b},\mathbf{c})\in W(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau\in\Omega_{\mathfrak{p}}$ for each $\mathfrak{p}\in\mathcal{T}$ and each $\tau\in\mathrm{Gal}(K)$. Let $(a,b,c) = \lambda(\mathbf{z}) = \big(\sum_{i=1}^d\omega_ia_i,\sum_{i=1}^d\omega_ib_i,\sum_{i=1}^d\omega_ic_i\big)$. Since $\omega_1,\ldots,\omega_d\in L\subseteq M$, we have $(a,b,c)\in V(M)$. Hence, $a,b,c\in M$, $h(a,b) = 0$, and $g(a)c = 1$, so $g(a)\neq 0$. Moreover, $a = \sum_{i=1}^d\omega_ia_i\in\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}}$ (because $\omega_1,\ldots,\omega_d\in\mathcal{O}_L$) and, by (3), $|a^\tau|_{\mathfrak{p}}\le 1$ for each $\mathfrak{p}\in\mathcal{T}$ and each $\tau\in\mathrm{Gal}(K)$. (Note that if $\mathfrak{p}\in\mathcal{T}\cap\mathbf{P}_{K,\mathrm{inf}}$, then $|a^\tau|_{\mathfrak{p}}\le\sum_{i=1}^d|\omega_i^\tau|_{\mathfrak{p}}|a_i^\tau|_{\mathfrak{p}} < d\cdot\max_{1\le i,j\le d}|\omega_i^{\sigma_j}|_{\mathfrak{p}}\cdot\delta_{\mathfrak{p}} = 1$.) Hence, $a\in\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}}\cap\mathcal{O}_{M,\mathcal{T}} = \mathcal{O}_{M,\mathcal{V}}$, as desired. ∎

Lemma 13.2 makes it possible to generalize the strong approximation theorem from varieties V defined over K to varieties V defined over finite extensions of K in $K_{\mathrm{tot},\mathcal{S}}$.

To this end we choose for each finite extension K' of K in $K_{\mathrm{tot},\mathcal{S}}$ and for each $\mathfrak{p}\in\mathbf{P}_{K'}$ a completion $\hat{K}'_{\mathfrak{p}}$ of K' at $\mathfrak{p}$ and an embedding of $\tilde{K}$ into the algebraic closure of $\hat{K}'_{\mathfrak{p}}$, as we do in Subsection 4.1. Then the notions defined with respect to K are also well defined for K'. In particular, $\mathcal{S}_{K'}$, $\mathcal{T}_{K'}$, and $\mathcal{V}_{K'}$ are the sets of all $\mathfrak{p}\in\mathbf{P}_{K'}$ that lie over $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{V}$, respectively. Note that $\mathcal{S}_{K'}$ and $\mathcal{T}_{K'}$ are finite sets, $\mathcal{V}_{K'}$ is a proper subset of $\mathbf{P}_{K'}$, $\mathcal{S}_{K'}\subseteq\mathcal{T}_{K'}\subseteq\mathcal{V}_{K'}$, and $\mathcal{V}_{K'}\smallsetminus\mathcal{T}_{K'}\subseteq\mathbf{P}_{K',\mathrm{fin}}$. Moreover, $K'_{\mathfrak{p}} = \hat{K}'_{\mathfrak{p}}\cap\tilde{K}$, for all $\mathfrak{p}\in\mathcal{T}_{K'}$. Finally, observe that $K'_{\mathrm{tot},\mathcal{S}_{K'}} = K_{\mathrm{tot},\mathcal{S}}$.

Proposition 13.3: *Let $K,\mathcal{S},\mathcal{T},\mathcal{V}$ be as in Subsection 4.8, let K' be a finite extension of K in $K_{\mathrm{tot},\mathcal{S}}$. Let M be an extension of K' in $K_{\mathrm{tot},\mathcal{S}}$ which is weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$. Consider an absolutely integral affine variety V in $\mathbb{A}^n_{K'}$ for some positive integer n. Let $(L_{\mathfrak{p}},\Omega_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{T}_{K'}}$ be approximation data for $K',\mathcal{S}_{K'},\mathcal{T}_{K'},\mathcal{V}_{K'},V$. Then there exists $\mathbf{z}\in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau\in\Omega_{\mathfrak{p}}(L_{\mathfrak{p}}K_{\mathrm{tot},\mathcal{S}})$ for all $\mathfrak{p}\in\mathcal{T}_{K'}$ and $\tau\in\mathrm{Gal}(K')$.*

Proof: First we assume that V is a curve. By Lemma 13.2, M is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$. By definition, $\mathcal{O}_{M,\mathcal{V}_{K'}\smallsetminus\mathcal{T}_{K'}} = \mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}}$. Moreover, M is also weakly symmetrically K'-stably P$\mathcal{S}_{K'}$C over $\mathcal{O}_{M,\mathcal{V}_{K'}}$. Hence, we may apply Proposition 12.3 to K' rather than to K and find $\mathbf{z}\in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau\in\Omega_{\mathfrak{p}}(L_{\mathfrak{p}}K_{\mathrm{tot},\mathcal{S}})$ for all $\mathfrak{p}\in\mathcal{T}_{K'}$ and $\tau\in\mathrm{Gal}(K')$.

Finally, the reduction lemmas 4.10 and 4.12 work if we replace K by K' and the condition "$\mathbf{z}^\tau\in\Omega_{\mathfrak{p}}$ for all $\tau\in\mathrm{Gal}(K)$ and $\mathfrak{p}\in\mathcal{T}$" by the condition "$\mathbf{z}^\tau\in\Omega_{\mathfrak{p}}(L_{\mathfrak{p}}K_{\mathrm{tot},\mathcal{S}})$ for all $\tau\in\mathrm{Gal}(K')$ and $\mathfrak{p}\in\mathcal{T}_{K'}$". Hence, the case where V is a curve implies the general case. ∎

An interesting special case of Proposition 13.3 is the local-global principle stated in Proposition 13.4 below. It is a consequence of Lemma 13.2 and [JaR08, Thm. 2.5]. However, since the latter theorem is one of the main results of [JaR08] and its proof extends over all of that paper, we prefer to give a proof that relies on the results of the present work.

Given a field $K\subseteq M\subseteq K_{\mathrm{tot},\mathcal{S}}$ and a prime $\mathfrak{q}\in\mathcal{V}_M$ we set

$$D_{M,\mathfrak{q}} = \{x\in M \mid\ |x|_{\mathfrak{q}}\le 1 \text{ if } \mathfrak{q}|_K\in\mathbf{P}_{K,\mathrm{fin}} \text{ and } |x|_{\mathfrak{q}} < 1 \text{ if } \mathfrak{q}|_K\in\mathbf{P}_{K,\mathrm{inf}}\}.$$

We also let $D_{M,\mathcal{V}} = \bigcap_{\mathfrak{q}\in\mathcal{V}_M} D_{M,\mathfrak{q}}$. Given $\mathfrak{p} \in \mathcal{V}_{K'}$ for some extension K' of K in $K_{\mathrm{tot},\mathcal{S}}$ and a field $K'_\mathfrak{p} \subseteq L \subseteq \tilde{K}$, we set

$$D(L) = D_\mathfrak{p}(L) = \{x \in L \mid |x|_\mathfrak{p} \le 1 \text{ if } \mathfrak{p}|_K \in \mathbf{P}_{K,\mathrm{fin}} \text{ and } |x|_\mathfrak{p} < 1 \text{ if } \mathfrak{p}|_K \in \mathbf{P}_{K,\mathrm{inf}}\}\,.$$

PROPOSITION 13.4 (Local-global principle): *Let K be a global field, $\mathcal{V}$ a proper subset of $\mathbf{P}_K$, and $\mathcal{S}$ a finite subset of $\mathcal{V}$. Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$. Suppose M is weakly symmetrically K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. Let V be an absolutely integral affine variety in $\mathbb{A}^n_M$ for some positive integer n such that $V_{\mathrm{simp}}(D(M_\mathfrak{q})) \neq \emptyset$ for each $\mathfrak{q} \in \mathcal{S}_M$ and $V(D(M_\mathfrak{q})) \neq \emptyset$ for all $\mathfrak{q} \in \mathcal{V}_M \smallsetminus \mathcal{S}_M$. Then, $V(D_{M,\mathcal{V}}) \neq \emptyset$.*

Proof: We choose a finite extension K' of K in M over which V is defined [Lan58, Sec. III.5, p. 74]. For each $\mathfrak{p} \in \mathcal{S}_{K'}$ the $\mathfrak{p}$-closure $K'_\mathfrak{p}$ of K' that we have chosen contains $K_{\mathrm{tot},\mathcal{S}}$, hence also M. Thus, $K'_\mathfrak{p} = M_\mathfrak{q}$, where $\mathfrak{q}$ is the prime of M induced by $K'_\mathfrak{p}$. By assumption, $\Omega_\mathfrak{p} = V_{\mathrm{simp}}(D(K'_\mathfrak{p}))$ is non-empty. We set $L_\mathfrak{p} = K'_\mathfrak{p}$.

Next we choose a finite subset $\mathcal{T}$ of $\mathcal{V}$ that contains $\mathcal{S} \cup (\mathcal{V} \cap \mathbf{P}_{K,\mathrm{inf}})$. For each $\mathfrak{p} \in \mathcal{T}_{K'} \smallsetminus \mathcal{S}_{K'}$ the $\mathfrak{p}$-adic topology on $MK'_\mathfrak{p}$ (which is actually K_{sep}, by [GJR00, p. 220, Prop. 1.15]) induces a prime $\mathfrak{q} \in \mathcal{T}_M \smallsetminus \mathcal{S}_M$, so $MK'_\mathfrak{p}$ contains $M_\mathfrak{q}$. Since $V(D(M_\mathfrak{q})) \neq \emptyset$, there exists $\mathbf{z}_\mathfrak{p} \in V(D(MK'_\mathfrak{p}))$. We choose a finite Galois extension $L_\mathfrak{p}$ of $K'_\mathfrak{p}$ such that $\mathbf{z}_\mathfrak{p} \in V(L_\mathfrak{p})$ and set $\Omega_\mathfrak{p} = V(D(L_\mathfrak{p}))$. Then, $\mathbf{z}_\mathfrak{p} \in \Omega_\mathfrak{p}$.

The collection $(L_\mathfrak{p}, \Omega_\mathfrak{p})_{\mathfrak{p}\in\mathcal{T}_{K'}}$ obtained in this way is approximation data for $K', \mathcal{S}_{K'}, \mathcal{T}_{K'}, \mathcal{V}_{K'}, V$. By Proposition 13.3, there exists $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau \in \Omega_\mathfrak{p}(L_\mathfrak{p}K_{\mathrm{tot},\mathcal{S}})$ for all $\mathfrak{p} \in \mathcal{T}_{K'}$ and all $\tau \in \mathrm{Gal}(K')$. The latter condition implies that $z \in D_{M,\mathfrak{q}}$ for every coordinate z of $\mathbf{z}$ and every $\mathfrak{q} \in \mathcal{T}_M$. Combining this conclusion with the former condition, we conclude that $\mathbf{z} \in V(D_{M,\mathcal{V}})$, as desired. ∎

Definition 13.5: We say that a field M_0 is **PAC over** a subset O if for every absolutely irreducible polynomial $f \in M_0[X,Y]$ which is separable in Y there exist infinitely many points $(a,b) \in O \times M_0$ such that $f(a,b) = 0$. ∎

The next two results contain notation introduced in the second paragraph of the introduction.

LEMMA 13.6 ([GJR00, p. 218, Lemma 1.12]): *Let M_0 be an algebraic extension of K, $M = M_0 \cap K_{\mathrm{tot},\mathcal{S}}$, and e a positive integer. Suppose that M_0 is PAC over $\mathcal{O}_{K,\mathcal{V}}$. Then:*

(a) *M is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. In particular, $K_{\mathrm{tot},\mathcal{S}}$ is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$ and $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.*

(b) *Let M' be the maximal Galois extension of K inside M. Then M' is weakly K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. In particular, $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ is weakly K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.*

We conclude our work with the main result.

THEOREM 13.7: *Let K be a global field, e a non-negative integer, $\mathcal{V}$ a proper subset of the set of all primes of K, and $\mathcal{S}$ a finite subset of $\mathcal{V}$. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ and for every subfield M of $K_{\mathrm{tot},\mathcal{S}}$ that contains $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$, we have:*

(a) *M is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{M,\mathcal{V}}$.*

(b) $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$.

(c) *M satisfies the local-global principle 13.4.*

Proof: For almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, Lemma 13.6 assures that $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ is weakly K-stably P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$. Hence, by Definition 12.1, M is also weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$. It follows from Proposition 12.4 that $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$. Moreover, M satisfies the local-global principle 13.4. Finally, by Lemma 13.2, M is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$. ∎

Remark 13.8:

(a) Statements (a) and (c) of Theorem 13.7 settle a question posed in [Jar06, p. 376, Remark 6] when $K = \mathbb{Q}$ and $\mathcal{S} = \emptyset$.

(b) Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$. It is possible to prove Proposition 13.3 under the assumption that M is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$ (rather that M is weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$). Conversely, one may use the arguments of the proof of Lemma 13.2 to prove that if M satisfies the conclusion of Proposition 13.3, then M is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$.

(c) The local global principle mentioned in the abstract is a quick consequence of Theorem 13.7(c). ∎

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