Strong Approximation Theorem
for Absolutely Irreducible Varieties
over the Compositum of all Symmetric Extensions of a Global Field
by
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Abstract: Let $K$ be a global field, $\mathcal{V}$ a proper subset of the set of all primes of $K$, $\mathcal{S}$ a finite subset of $\mathcal{V}$, and $\bar{K}$ (resp. $K_{\text{sep}}$) a fixed algebraic (resp. separable algebraic) closure of $K$ with $K_{\text{sep}} \subseteq \bar{K}$. Let $\text{Gal}(K) = \text{Gal}(K_{\text{sep}}/K)$ be the absolute Galois group of $K$. For each $p \in \mathcal{V}$ we choose a Henselian (respectively, a real or algebraic) closure $K_p$ of $K$ at $p$ in $\bar{K}$ if $p$ is non-archimedean (respectively, archimedean). Then, $K_{\text{tot},\mathcal{S}} = \bigcap_{p \in \mathcal{S}} \bigcap_{\tau \in \text{Gal}(K)} K_p^\tau$ is the maximal Galois extension of $K$ in $K_{\text{sep}}$ in which each $p \in \mathcal{S}$ totally splits. For each $p \in \mathcal{V}$ we choose a $p$-adic absolute value $| |_p$ of $K_p$ and extend it in the unique possible way to $\bar{K}$. Finally, we denote the compositum of all symmetric extensions of $K$ by $K_{\text{symm}}$.

We consider an affine absolutely integral variety $V$ in $\mathbb{A}^n_K$. Suppose that for each $p \in \mathcal{S}$ there exists a simple $K_p$-rational point $z_p$ of $V$ and for each $p \in \mathcal{V} \setminus \mathcal{S}$ there exists $z_p \in V(\bar{K})$ such that in both cases $|z_p|_p \leq 1$ if $p$ is non-archimedean and $|z_p|_p < 1$ if $p$ is archimedean. Then, there exists $z \in V(K_{\text{tot},\mathcal{S}} \cap K_{\text{symm}})$ such that for all $p \in \mathcal{V}$ and for all $\tau \in \text{Gal}(K)$ we have: $|z^\tau|_p \leq 1$ if $p$ is archimedean and $|z^\tau|_p < 1$ if $p$ is non-archimedean. For $\mathcal{S} = \emptyset$, we get as a corollary that the ring of integers of $K_{\text{symm}}$ is Hilbertian and Bezout.
Introduction

The strong approximation theorem for a global field $K$ gives an $x \in K$ that lies in given $p$-adically open discs for finitely many given primes $p$ of $K$ such that the absolute $p$-adic value of $x$ is at most 1 for all other primes $p$ except possibly one [?, p. 67]. A possible generalization of that theorem to an arbitrary absolutely integral affine variety $V$ over $K$ fails, because in general, $V(K)$ is a small set. For example, if $V$ is a curve of genus at least 2, then $V(K)$ is finite (by Faltings). This obstruction disappears as soon as we switch to appropriate “large Galois extensions” of $K$.

Extensions of $K$ of this type occur in our work [?]. In that work we fix an algebraic closure $\hat{K}$ of $K$, set $K_{\text{sep}}$ to be the separable closure of $K$ in $\hat{K}$, and consider a non-negative integer $e$. We equip $\text{Gal}(K)^e$ with the normalized Haar measure [?, Section 18.5] and use the expression “for almost all $\sigma \in \text{Gal}(K)^e$” to mean “for all $\sigma$ in $\text{Gal}(K)^e$ outside a set of measure 0”. For each $\sigma = (\sigma_1, \ldots, \sigma_e) \in \text{Gal}(K)^e$ let $K_{\text{sep}}(\sigma) = \{x \in K_{\text{sep}} \mid x^{\sigma_1} = x \text{ for } i = 1, \ldots, e\}$ and let $K_{\text{sep}}[\sigma]$ be the maximal Galois extension of $K$ in $K_{\text{sep}}(\sigma)$.

Further, let $\mathbb{P}_K$ be the set of all primes of $K$, let $\mathbb{P}_{K, \text{fin}}$ be the set of all non-archimedean primes, and let $\mathbb{P}_{K, \text{inf}}$ be the set of all archimedean primes. We fix a proper subset $\mathcal{V}$ of $\mathbb{P}_K$, a finite subset $\mathcal{T}$ of $\mathcal{V}$, and a subset $\mathcal{S}$ of $\mathcal{T}$ such that $\mathcal{V} \setminus \mathcal{T} \subseteq \mathbb{P}_{K, \text{fin}}$. For each $p$ we fix a completion $\hat{K}_p$ of $K$ at $p$ and embed $\hat{K}$ in an algebraic closure $\tilde{\hat{K}}_p$ of $\hat{K}_p$. Then, we extend a normalized absolute value $|\cdot|_p$ of $\hat{K}_p$ to $\tilde{\hat{K}}_p$ in the unique possible way. In particular, this defines $|x|_p$ for each $x \in \hat{K}$.

Next, we set $K_p = \hat{K} \cap \hat{K}_p$, and note that $K_p$ is a Henselian closure of $K$ at $p$ if $p \in \mathbb{P}_{K, \text{fin}}$ and a real or the algebraic closure of $K$ at $p$ if $p \in \mathbb{P}_{K, \text{inf}}$. Thus,

$$K_{\text{tot}, \mathcal{S}} = \bigcap_{p \in \mathcal{S}} \bigcap_{\sigma \in \text{Gal}(K)} K_p^{\sigma}$$

is the maximal Galois extension of $K$ in which each $p \in \mathcal{S}$ totally splits. For each $\sigma \in \text{Gal}(K)^e$ we set $K_{\text{tot}, \mathcal{S}}[\sigma] = K_{\text{sep}}[\sigma] \cap K_{\text{tot}, \mathcal{S}}$.

For each extension $M$ of $K$ in $\hat{K}$ and every $p \in \mathbb{P}_{\text{fin}} \cap \mathcal{V}$ we consider the valuation ring $\mathcal{O}_{M, p} = \{x \in M \mid |x|_p \leq 1\}$ of $M$ at $p$. For each subset $\mathcal{U}$ of $\mathcal{V}$ we let

$$\mathcal{O}_{M, \mathcal{U}} = \{x \in M \mid |x^\tau|_p \leq 1 \text{ for all } p \in \mathcal{U} \text{ and } \tau \in \text{Gal}(K)\}.$$  

Then, the main result of [?] is the following theorem:

**Theorem A:** Let $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, e$ be as above. Then, for almost all $\sigma \in \text{Gal}(K)^e$ the field $M = K_{\text{tot}, \mathcal{S}}[\sigma]$ satisfies the strong approximation theorem:

Let $V$ be an absolutely integral affine variety over $K$ in $\mathbb{A}_K^n$ for some positive integer $n$. For each $p \in \mathcal{S}$ let $\Omega_p$ be a non-empty $p$-open subset of $V_{\text{simp}}(K_p)$. For each $p \in \mathcal{T} \setminus \mathcal{S}$ let $\Omega_p$ be a non-empty $p$-open subset of $V(\hat{K})$, for each
invariant under the action of $\text{Gal}(K_p)$. Finally, for each $p \in \mathcal{V} \setminus \mathcal{T}$ we assume that $V(O_{K, p}) \neq \emptyset$. Then, $V(O_{M, \mathcal{V} \setminus \mathcal{T}}) \cap \bigcap_{p \in \mathcal{T}} \cap \tau \in \text{Gal}(K) \Omega^{\tau}_{p} \neq \emptyset$.

The main result of the present work establishes the strong approximation theorem for much smaller fields. To this end we call a Galois extension $L$ of $K$ symmetric if $\text{Gal}(L/K)$ is isomorphic to the symmetric group $S_n$ for some positive integer $n$. We denote the compositum of all symmetric extensions of $K$ by $K_{\text{symm}}$.

**Theorem B:** Let $K, S, T, \mathcal{V}, e$ be as above. Then, for almost all $\sigma \in \text{Gal}(K)^e$ the field $M = K_{\text{symm}} \cap K_{\text{tot}, S}[\sigma]$ satisfies the strong approximation theorem (as in Theorem A). In particular, $K_{\text{symm}} \cap K_{\text{tot}, S}$ satisfies the strong approximation theorem.

Additional interesting information about the fields mentioned in Theorem B and their rings of integers is contained in the following result.

**Theorem C:** Let $K$ be a global field and $e$ a non-negative integer. Then, for almost all $\sigma \in \text{Gal}(K)^e$ the field $M = K_{\text{symm}} \cap K_{\text{sep}}[\sigma]$ is PAC (Definition ??) and Hilbertian, hence $\text{Gal}(M) \cong \hat{F}_\omega$. Moreover, the ring of integers of $M$ is Hilbertian and Bezout (Definition ??).

Note that the statement about the Hilbertianity of $M$ in Theorem C is due to [?]. See also the proof of Proposition ???. The authors are indebted to the anonymous referee for pointing out that proposition and its proof.
1 Weakly Symmetrically $K$-Stably PSC Fields over Holomorphy Domains

Let $K$ be a global field, that is $K$ is either a number field or an algebraic function field of one variable over a finite field. Throughout this work, we use the notation $\mathbb{P}_K$, $\hat{K}$, $K_{\text{sep}}$, $\text{Gal}(K)$, $K_p$ and $|\cdot|_p$ for $p \in \mathbb{P}_K$, introduced in the introduction. For each $p \in \mathbb{P}_K$ and every subfield $M$ of $\hat{K}$ we consider the closed disc

$$O_{M,p} = \{x \in M \mid |x|_p \leq 1\}$$

of $M$ at $p$. If $p$ is non-archimedean, then $O_{M,p}$ is a valuation ring of rank 1 of $M$.

Next we consider a subset $\mathcal{U}$ of $\mathbb{P}_K$ and a field $K \subseteq M \subseteq \hat{K}$. A prime of $M$ is an equivalence class of absolute values of $M$, where two absolute values on $M$ are equivalent if they define the same topology on $M$. Let $\mathcal{U}_M$ be the set of all primes of $M$ that lie over $\mathcal{U}$. If $q \in \mathcal{U}_M$ lies over $p \in \mathcal{U}$, then we denote the unique absolute value of $M$ that extends $|\cdot|_p$ to $M$ and represents $q$ by $|\cdot|_q$.

In this case there exists $\tau \in \text{Gal}(K)$ such that $|x|_q = |x^\tau|_p$ for each $x \in M$. Conversely, the latter condition defines $q$. We set

$$O_{M,\mathcal{U}} = \bigcap_{q \in \mathcal{U}_M} \{x \in M \mid |x|_q \leq 1\}$$

for the $\mathcal{U}$-holomorphy domain of $M$. If $\mathcal{U}$ consists of non-archimedean primes, then $O_{M,\mathcal{U}}$ is the integral closure of $O_{K,\mathcal{U}}$ in $M$. If $\mathcal{U}$ is arbitrary but $M$ is Galois over $K$, then

$$O_{M,\mathcal{U}} = \bigcap_{p \in \mathcal{U}} \bigcap_{\tau \in \text{Gal}(K)} O_{M,p}^\tau.$$

In the number field case (i.e. $\text{char}(K) = 0$), we denote the (cofinite) set of all non-archimedean primes of $K$ by $\mathbb{P}_{K,\text{fin}}$. In the function field case, where $p = \text{char}(K) > 0$, we fix a separating transcendence element $t_K$ for $K/\mathbb{F}_p$ and let $\mathbb{F}_{K,\text{fin}} = \{p \in \mathbb{P}_K \mid |t_K|_p \leq 1\}$. In both cases we set

$$O_K = O_{K,\mathbb{P}_{K,\text{fin}}} = \{x \in K \mid |x|_p \leq 1 \text{ for all } p \in \mathbb{P}_{K,\text{fin}}\}.$$

If $K$ is a number field, then $O_K$ is the integral closure of $\mathbb{Z}$ in $K$. In the function field case $O_K$ is the integral closure of $\mathbb{F}_p[t_K]$ in $K$. In both cases $O_K$ is a Dedekind domain. Following the convention in algebraic number theory, we call $O_K$ the ring of integers of $K$.

Next we consider a finite (possibly empty) subset $\mathcal{S}$ of $\mathbb{P}_K$. We set

$$K_{\text{tot},\mathcal{S}} = \bigcap_{p \in \mathcal{S}} \bigcap_{\tau \in \text{Gal}(K)} K_p^\tau$$

as in the introduction. If $\mathcal{S} = \emptyset$, then $K_{\text{tot},\mathcal{S}} = K_{\text{sep}}$.

We also choose a non-empty proper subset $\mathcal{V}$ of $\mathbb{P}_K$ that contains $\mathcal{S}$. 

1 WEAKLY SYMMETRIC
**Definition 1.1.** [?, Def. 12.1] Let $M$ be an extension of $K$ in $K_{tot,S}$ and let $\mathcal{O}$ be a subset of $M$. We say that $M$ is **weakly symmetrically $K$-stably PSC over $\mathcal{O}$** if for every polynomial $g \in K[T]$ with $g(0) \neq 0$ and for every absolutely irreducible polynomial $h \in K[T, Y]$ monic in $Y$ with $d = \deg_Y(h)$ satisfying

1. $h(0, Y)$ has $d$ distinct roots in $K_{tot,S}$, and
2. $\Gal(h(T, Y), K(T)) \cong \Gal(h(T, Y), K(T))$ and is isomorphic to the symmetric group $S_d$,

there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$.

Note that in that case, if $M \subseteq M' \subseteq K_{tot,S}$, then $M'$ is also weakly symmetrically $K$-stably PSC over $\mathcal{O}$.

If $\mathcal{S} = \emptyset$, we say that $M$ is **weakly symmetrically $K$-stably PAC over $\mathcal{O}$**.

**Definition 1.2.** [?, Def. 13.1] Let $M$ be an extension of $K$ in $K_{tot,S}$ and let $\mathcal{O}$ be a subset of $M$. We say that $M$ is **weakly PAC over $\mathcal{O}$** if for every absolutely irreducible polynomial $h \in M[T, Y]$ monic in $Y$ such that $h(0, Y)$ decomposes into distinct monic linear factors over $K_{tot,S}$ and every polynomial $g \in M[T]$ with $g(0) \neq 0$ there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$. In particular, $\mathcal{O}$ is infinite.

If $\mathcal{S} = \emptyset$, then $M$ is **PAC over $\mathcal{O}$** [?, Def. 13.5], i.e. for every absolutely irreducible polynomial $f \in M[T, X]$ which is separable in $X$ there exist infinitely many points $(a, b) \in \mathcal{O} \times M$ such that $f(a, b) = 0$.

Indeed, let $f \in M[T, X]$ be an absolutely irreducible polynomial which is separable in $X$. Let $\Delta \in M[T]$ be the discriminant of $f$, let $g \in M[T]$ be the leading coefficient of $f$, and let $d = \deg_X(f)$. Since $\mathcal{O}$ is infinite, we can choose $c \in \mathcal{O}$ with $\Delta(c)g(c) \neq 0$. Let $Y = g(T)X$, let $h'(T, Y) = g(T)^{d-1}f(T, g(T)^{-1}Y)$, and let $h(T, Y) = h'(T + c, Y)$. Then, $h \in M[T, Y]$ is an absolutely irreducible polynomial, monic in $Y$, such that $h(0, Y)$ decomposes into distinct monic linear factors over $K_{sep}$. By assumption, there exist infinitely many $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$, hence $f(a + c, g(a)^{-1}b) = 0$.

Note that in that case, $M$ is a **PAC field**, i.e. every absolutely integral variety over $M$ has an $M$-rational point [?, Lemma 1.3].

**Lemma 1.3.** Let $M_0$ be an extension of $K$ in $K_{sep}$, let $M = M_0 \cap K_{tot,S}$, and let $\mathcal{O}$ be a subset of $\mathcal{O}_{M,S}$ such that $\mathcal{O}_{K,Y} \cdot \mathcal{O} \subseteq \mathcal{O}$. Suppose that $M_0$ is weakly symmetrically $K$-stably PAC over $\mathcal{O}$. Then, $M$ is weakly symmetrically $K$-stably PSC over $\mathcal{O}$.

**Proof:** Let $g$ be a polynomial in $K[T]$ with $g(0) \neq 0$ and let $h$ be an absolutely irreducible polynomial in $K[T, Y]$, monic in $Y$, with $d = \deg_Y(h)$ satisfying (1). By [?, Lemma 1.9], there exists $c \in \mathcal{O}_{K,Y}$ which is sufficiently $S$-close to 0 such that for each $a \in \mathcal{O}_{K_{tot},S}$ all the roots of $h(ac, Y)$ are simple and belong to $K_{tot,S}$. Consider the polynomial $h(cT, Y) \in K[T, Y]$. Then, since $M_0$ is weakly symmetrically $K$-stably PAC over $\mathcal{O}$, there exists $a \in \mathcal{O}$ and $b \in M_0$ such that
h(ac, b) = 0 and g(a) ≠ 0. Then, ac ∈ O and b ∈ M₀ ∩ K_{tot,S} = M, as desired.

Lemma 1.4. [?, Lemma 13.2] Let M be an extension of K in K_{tot,S} which is weakly symmetrically K-stably PSC over O_{K,V}. Then, M is weakly PSC over O_{M,V}.  

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2 Composita of Symmetric Extensions of a Global Field

A symmetric extension of $K$ is a finite Galois extension of $K$ with Galois group isomorphic to $S_m$ for some positive integer $m$. Let $K_{\text{symm}}$ be the compositum of all symmetric extensions of $K$.

Using the notation introduced in the introduction, we prove that for almost all $\sigma \in \text{Gal}(K)^e$, the field $K_{\text{symm}}[\sigma]$ is PAC and Hilbertian, so $\text{Gal}(K_{\text{symm}}[\sigma]) \cong \tilde{F}_e$. Moreover, if $\mathcal{V}$ contains only non-archimedean primes, then the ring $O_{K_{\text{symm}}[\sigma],\mathcal{V}}$ is Hilbertian and Bezout. Finally, the field $M = K_{\text{tot,}\mathcal{S}} \cap K_{\text{symm}}[\sigma]$ is weakly PSC over $O_{M,\mathcal{V}}$. This leads in Section ?? to a strong approximation theorem for $M$.

**Definition 2.1.** Let $O$ be an integral domain with quotient field $F$. We consider variables $T_1,\ldots,T_r,X$ over $F$ and abbreviate $(T_1,\ldots,T_r)$ to $T$. Let $f_1,\ldots,f_m$ be irreducible and separable polynomials in $F[T][X]$ and let $g$ be a non-zero polynomial in $F[T]$. Following [FrJ08, Sec. 12.1], we write $H_F(f_1,\ldots,f_n;g)$ for the set of all $a \in F^r$ such that $f_1(a,X),\ldots,f_m(a,X)$ are defined, irreducible, and separable in $F[X]$ with $g(a) \neq 0$. Then, we call $H_F(f_1,\ldots,f_m;g)$ a separable Hilbert subset of $F^r$. We say that the ring $O$ is Hilbertian if for every positive integer $r$ every separable Hilbert subset of $F^r$ has a point with coordinates in $O$. Finally, we say that $O$ is Bezout if every finitely generated ideal of $O$ is principal.

**Example 2.2.** Taking $q_0 \in \mathbb{P}_K \setminus \mathcal{V}$ in [?, p. 241, Thm. 13.3.5(b)], we find that $H \cap O_{K,\mathcal{V}} \neq \emptyset$ for each $r \geq 1$ and every separable Hilbert subset $H$ of $K^r$. In particular, if $\mathcal{V}$ contains only non-archimedean primes, then $O_{K,\mathcal{V}}$ is a Hilbertian domain.

Let $d$ be a positive integer. Denote the set of all absolutely irreducible polynomials $h \in K[T,Y]$, monic in $Y$ with $d = \deg_Y(h)$, that satisfy (1) of Section 1 with $\mathcal{S} = \emptyset$, i.e.

1. $h(0,Y)$ has $d$ distinct roots in $K_{\text{sep}}$, and
2. $\text{Gal}(h(T,Y),K(T)) \cong \text{Gal}(h(T,Y),\tilde{K}(T)) \cong S_d$

by $H_d$. Let $\mathcal{H} = \bigcup_{d=1}^\infty H_d$.

**Lemma 2.3.** Let $e$ be a non-negative integer. Then, for almost all $\sigma \in \text{Gal}(K)^e$ every separable algebraic extension $M$ of $K_{\text{symm}}[\sigma]$ is weakly symmetrically $K$-stably PAC over $O_{K,\mathcal{V}}$.

In particular, the field $K_{\text{symm}}$ is weakly symmetrically $K$-stably PAC over $O_{K,\mathcal{V}}$.

**Proof:** By Definition ??, it suffices to consider the case $e \geq 1$ and to prove that for almost all $\sigma \in \text{Gal}(K)^e$ the field $K_{\text{symm}}[\sigma]$ is weakly symmetrically $K$-stably PAC over $O_{K,\mathcal{V}}$. Moreover, since the set $\mathcal{H}$ is countable, it suffices to
consider a positive integer $d$, a polynomial $h \in \mathcal{H}_d$, and a non-zero polynomial $g \in K[T]$, and to prove that for almost all $\sigma \in \text{Gal}(K)^e$ there exists $(a, b) \in \mathcal{O}_{K, V} \times K_{\text{symm}}[\sigma]$ such that $h(a, b) = 0$ and $g(a) \neq 0$.

By Borel-Cantelli [7, p. 378, Lemma 18.5.3(b)], it suffices to construct a sequence of pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ that satisfies for each $n \geq 1$ the following conditions:

1. $a_n \in \mathcal{O}_{K, V}$ and $h(a_n, X)$ is separable,
2. the splitting field $K_n$ of $h(a_n, X)$ over $K$ has Galois group $S_d$,
3. $h(a_n, b_n) = 0$, in particular $b_n \in K_n$, and $g(a_n) \neq 0$,
4. $K_1, K_2, \ldots, K_n$ are linearly disjoint over $K$.

Indeed, inductively suppose that $n$ is a positive integer and $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})$ satisfy Condition (2) (for $n - 1$ rather than for $n$). Let $L = K_1K_2 \cdots K_{n-1}$. By [7, p. 294, Prop. 16.1.5] and [7, p. 224, Cor. 12.2.3], $K$ has a separable Hilbert subset $H$ such that for each $a \in H$ the polynomial $h(a, X)$ is separable, $\text{Gal}(h(a, X), K) \cong \text{Gal}(h(a, X), L) \cong S_d$, and $g(a) \neq 0$. Using Example ??, we choose an element $a_n \in H \cap \mathcal{O}_{K, V}$ and a root $b_n \in K_{\text{sep}}$ of $h(a_n, X)$. Then, $b_n$ lies in the splitting field $K_n$ of $h(a_n, X)$, so all of the statements (2a) – (2d) are satisfied.

By Lemmas ?? and ??, we get the following corollary:

**Corollary 2.4.** Let $e$ be a non-negative integer. Then, for almost all $\sigma \in \text{Gal}(K)^e$ each extension $M$ of $K_{\text{tot}, \mathcal{S}} \cap K_{\text{symm}}[\sigma]$ in $K_{\text{tot}, \mathcal{S}}$ is weakly symmetrically $K$-stably PSC over $\mathcal{O}_{K, V}$. Hence, $M$ is weakly PSC over $\mathcal{O}_{M, V}$.

In particular, the field $M = K_{\text{tot}, \mathcal{S}} \cap K_{\text{symm}}$ is weakly symmetrically $K$-stably PSC over $\mathcal{O}_{K, V}$, so it is also weakly PSC over $\mathcal{O}_{M, V}$.

When $\mathcal{S} = \emptyset$, we get by Definition ??:

**Corollary 2.5.** Let $e$ be a non-negative integer. Then, for almost all $\sigma \in \text{Gal}(K)^e$ each separable algebraic extension $M$ of the field $K_{\text{symm}}[\sigma]$ is PAC over $\mathcal{O}_{M, V}$.

In particular, the field $M = K_{\text{symm}}$ is PAC over $\mathcal{O}_{M, V}$.

**Proposition 2.6.** Let $L$ be a Hilbertian field and $M$ an extension of $L$ in $L_{\text{symm}}$. Then, $M$ is Hilbertian.

**Proof:** Following [7, Sec. 2.1], we say that a profinite group $G$ has abelian-simple length $n$ if there is a finite series $1 = G^{(n)} \triangleleft \cdots \triangleleft G^{(1)} \triangleleft G^{(0)} = G$ of closed subgroups, where for $i = 0, \ldots, n - 1$, the group $G^{(i+1)}$ is the intersection of all open normal subgroups $N$ of $G^{(i)}$ such that $G^{(i)}/N$ is abelian or simple.

As mentioned in the proof of [7, Thm. 5.5], the abelian-simple length of each symmetric group $S_n$ is at most 3. Hence, by [7, Prop. 2.8], the abelian-simple length of $\text{Gal}(L_{\text{symm}}/L)$ is at most 3. Therefore, by [7, Thm. 3.2], every field $M$ between $L$ and $L_{\text{symm}}$ is Hilbertian.
Corollary 2.7. Let $e$ be a positive integer. Suppose that $V$ contains only non-Archimedean primes. Then, for almost all $\sigma \in \Gal(K)^e$ the rings $\mathcal{O}_{K_{\text{symm}}}[\sigma], V$ and $\mathcal{O}_{K_{\text{symm}}}[\sigma], V$ are Hilbertian. In addition, the ring $\mathcal{O}_{K_{\text{symm}}}, V$ is Hilbertian.

Proof: By Proposition ??, for all $\sigma \in \Gal(K)^e$ the field $K_{\text{symm}}[\sigma]$ is Hilbertian. By ??, Thm. 27.4.8, for almost all $\sigma \in \Gal(K)^e$ the field $K_{\text{sep}}[\sigma]$ is Hilbertian. It follows from Corollary ?? that for almost all $\sigma \in \Gal(K)^e$ the rings $\mathcal{O}_{K_{\text{symm}}}[\sigma], V$ and $\mathcal{O}_{K_{\text{symm}}}[\sigma], V$ are Hilbertian.

Finally, by Proposition ??, the field $K_{\text{symm}}$ is also Hilbertian. By Corollary ??, $K_{\text{symm}}$ is PAC over $\mathcal{O}_{K_{\text{symm}}}, V$. Hence, by the preceding paragraph, the ring $\mathcal{O}_{K_{\text{symm}}}, V$ is Hilbertian. □

Corollary 2.8. Let $e$ be a non-negative integer. Then, for almost all $\sigma \in \Gal(K)^e$ the field $K_{\text{symm}}[\sigma]$ is PAC, Hilbertian, and $\Gal(K_{\text{symm}}[\sigma]) \cong \hat{F}_\omega$.

Proof: By Corollary ??, Definition ??, and Corollary ??, for almost all $\sigma \in \Gal(K)^e$ the field $M = K_{\text{symm}}[\sigma]$ is PAC and Hilbertian. Hence, by [?, p. 90, Thm. 5.10.3], $\Gal(M) \cong \hat{F}_\omega$, as claimed. □

Remark 2.9. (a) It is not true that $K_{\text{symm}}[\sigma]$ is PAC for every $\sigma \in \Gal(K)^e$.

For example, [FrJ08, p. 381, Remark 18.6.2] gives $\sigma \in \Gal(Q)$ such that $\hat{Q}(\sigma)$ is not a PAC field. Hence, by [FrJ08, p. 196, Cor. 11.2.5] also the subfield $Q_{\text{symm}}[\sigma]$ of $\hat{Q}(\sigma)$ is not PAC.

(b) In a forthcoming note, we make some mild changes in the proof of Theorem 1.1 of [?] and in some lemmas on which it depends in order to prove in the setup of Proposition ?? that if $L$ is the quotient field of a Hilbertian domain $R$ and $S$ is the integral closure of $R$ in $M$, then $S$ is also a Hilbertian domain.

In particular, in view of the proof of Proposition ??, the latter result applies to every extension $M$ of $L$ in $L_{\text{symm}}$. It will follow, in the notation of Corollary ??, that each of the rings $\mathcal{O}_{K_{\text{symm}}}[\sigma], V$ is Hilbertian. □

By [?, Lemma 4.6], if $M$ is an algebraic extension of $K$ which is PAC over its ring of integers $\mathcal{O}_M = \mathcal{O}_M, \mathfrak{P}_{K, \text{fin}}$, then $\mathcal{O}_M$ is a Bezout domain. Thus, Corollary ??, applied to $V = \mathfrak{P}_{K, \text{fin}}$, yields the following result:

Corollary 2.10. Let $e$ be a non-negative integer. Then, for almost all $\sigma \in \Gal(K)^e$ the ring of integers of each separable extension of $K_{\text{symm}}[\sigma]$ is Bezout.

In particular, the ring $\mathcal{O}_{K_{\text{symm}}}$ is Bezout.
3 Strong Approximation Theorem

In the notation of Section 1, we prove that for almost all $\sigma \in \text{Gal}(K)^e$ the field $K_{\text{tot}, S} \cap K_{\text{symm}}[\sigma]$ satisfies the strong approximation theorem for absolutely integral affine varieties.

Given a variety $V$ we write $V_{\text{simp}}$ for the Zariski-open subset of $V$ that consists of all simple (= non-singular) points of $V$. We cite two results from [?]. The first one is Proposition 12.4 of [?]:

**Proposition 3.1** (Strong approximation theorem). Let $M$ be a subfield of $K_{\text{tot}, S}$ that contains $K$ and is weakly symmetrically $K$-stably PSC over $O_{K, V}$. Then, $(M, K, S, V)$ satisfies the following condition, abbreviated as $(M, K, S, V) \models \text{SAT}$:

Let $\mathcal{T}$ be a finite subset of $V$ that contains $S$ such that $V \setminus \mathcal{T} \subseteq \mathbb{P}_{K, \text{fin}}$. Let $V$ be an absolutely integral affine variety over $K$ in $\mathbb{A}_K^n$ for some positive integer $n$. For each $p \in \mathcal{T}$ let $L_p$ be a finite Galois extension of $K_p$ such that $L_p = K_p$ if $p \in S$ and let $\Omega_p$ be a non-empty $p$-open subset of $V_{\text{simp}}(L_p)$, invariant under the action of $\text{Gal}(L_p/K_p)$. Assume that $V(O_{K_p}) \neq \emptyset$, for each $p \in V \setminus \mathcal{T}$. Then, there exists $z \in V(O_{M,V \setminus \mathcal{T}})$ such that $z^p \in \Omega_p$ for all $p \in \mathcal{T}$ and all $\tau \in \text{Gal}(K)$.

The second result is Proposition 13.4 of [?], applied (for simplicity) to the case where $S$ consists only of finite primes of $K$ and $V = \mathbb{P}_{K, \text{fin}}$:

**Proposition 3.2** (Local-global principle). Let $M$ be a subfield of $K_{\text{tot}, S}$ that contains $K$ and is weakly symmetrically $K$-stably PSC over $O_{K, V}$. Then, $(M, S)$ satisfies the following condition, abbreviated as $(M, S) \models \text{LGP}$:

Let $V$ be an absolutely integral affine variety over $M$ in $\mathbb{A}_M^n$ for some positive integer $n$ such that $V_{\text{simp}}(O_{M,q}) \neq \emptyset$ for each $q \in S_M$ and $V(O_{M,q}) \neq \emptyset$ for each $q \in \mathbb{P}_{M, \text{fin}} \setminus S_M$. Then, $V(O_M) \neq \emptyset$.

Recall that an extension $M$ of $K$ in $K_{\text{tot}, S}$ is said to be PSC (= pseudo-S-closed) if every absolutely integral variety $V$ over $M$ with a simple $K^*_\tau$-rational point for each $p \in S$ and every $\tau \in \text{Gal}(K)$ has an $M$-rational point [?, Def. 1.3]. Also, a field $M$ is **ample** if the existence of an $M$-rational simple point on $V$ implies that $V(M)$ is Zariski-dense in $V$ [?, p. 67, Lemma 5.3.1]. In particular, every PSC field is ample.

The next lemma is observed in [?, Cor. 2.7].

**Lemma 3.3**. Let $M$ be an extension of $K$ in $K_{\text{tot}, S}$. Suppose that $(M, K, S, S) \models \text{SAT}$. Then, $M$ is a PSC field, hence ample.

**Proof:** Consider an absolutely integral variety $V$ over $M$ with a simple $K^*_\tau$-rational point for each $p \in S$ and every $\tau \in \text{Gal}(K)$. Replacing $K$ by a finite extension $K'$ in $K_{\text{tot}, S}$ and $S$ by $S_{K'}$, we may assume that $V$ is defined over $K$.

\footnote{The work [?] uses the adjective “large” rather than “ample.”}
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and has a simple $K_p$-rational point for each $p \in \mathcal{S}$. Moreover, we may assume that $V$ is affine. Thus, we may apply Proposition ?? to the case $V = \mathcal{T} = \mathcal{S}$ and $\Omega_p = V_{\text{simp}}(K_p)$ for each $p \in \mathcal{S}$. Observe that in this case $\mathcal{O}_{M,V,\mathcal{T}} = M$.

Corollary ??, Lemma ??, Proposition ??, and Proposition ?? yield the following result:

{\textbf{Theorem 3.4.}} Let $e$ be a non-negative integer. Then, for almost all $\sigma \in \text{Gal}(K)^e$, every extension $M$ of $K_{\text{tot},\mathcal{S}} \cap K_{\text{symm}}[\sigma]$ in $K_{\text{tot},\mathcal{S}}$ has the following properties.

(a) $(M, K, \mathcal{S}, V) \models \text{SAT}$.

(b) $M$ is PSC, hence ample.

(c) If $\mathcal{S}$ consists only of finite primes of $K$, then $(M, \mathcal{S}) \models \text{LGP}$.

In particular, $M = K_{\text{tot},\mathcal{S}} \cap K_{\text{symm}}$ satisfies (a), (b), and (c).

{\textbf{Proof:}} By Corollary ??, for almost all $\sigma \in \text{Gal}(K)^e$ every extension $M$ of the field $K_{\text{tot},\mathcal{S}} \cap K_{\text{symm}}[\sigma]$ in $K_{\text{tot},\mathcal{S}}$ is weakly symmetrically $K$-stably PSC over $\mathcal{O}_{K,V}$. Hence, by Proposition ??, $(M, K, \mathcal{S}, V) \models \text{SAT}$, so (a) holds. It follows from Lemma ?? that $M$ is PSC, as (b) states. Finally, if in addition, $\mathcal{S}$ consists only of finite primes, then by Proposition ??, $(M, \mathcal{S}) \models \text{LGP}$, which establishes (c).
References


