The Generalized Transfer Theorem {GTTm}

Abstract

We generalize the transfer theorem for global fields proved in [FrJ08, Chap. 20] to a transfer theorem for finitely generated extensions of global fields. The main tool used in the proof is the Artin-Chebotarev density theorem for the latter fields, due to Serre [Ser65].

Introduction

The transfer theorem [FrJ08, p. 447, Thm. 20.9.3] considers the ring of integers O_K of a global field K and a sentence θ of the language $\mathcal{L}(\mathrm{ring}, O_K)$ of rings with each element of O_K being a constant symbol. It says that the set $\Sigma_{\tilde{K}/K}(\theta)$ of all σ in the absolute Galois group $\mathrm{Gal}(K)$ of K for which θ holds in the fixed field $\tilde{K}(\sigma)$ of σ in the algebraic closure $\tilde{K}$ of K is measurable and its Haar measure $\mu_K(\Sigma_{\tilde{K}/K}(\theta))$ is equal to the Dirichlet density $\delta(A_{O_K}(\theta))$ of the set $A_{O_K}(\theta)$ of all maximal ideals $\mathfrak{p}$ of O_K for which θ holds in $\bar{K}_\mathfrak{p} := O_K/\mathfrak{p}$.

The aim of this work is to generalize the transfer theorem to integrally closed integral domains R that are finitely generated as $\mathbb{Z}$-algebras or finitely generated as $\mathbb{F}_p$-algebras and are infinite.

To this end we recall that an element $\mathfrak{p}$ of $\mathrm{Spec}(R)$ is closed if and only if $\mathfrak{p}$ is a maximal ideal of R, so $R/\mathfrak{p}$ is a finite field (Lemma 1.7). Then we use the "Dirichlet density" δ on the set $\mathrm{Max}(R)$ of all maximal ideals of R introduced by Serre in [Ser65, p. 91] and the Artin-Chebotarev density theorem [Ser65, p. 91, Thm. 7]. Let $K = \mathrm{Quot}(R)$. The generalized transfer theorem says that $\mu_K(\Sigma_{\tilde{K}/K}(\theta)) = \delta(A_R(\theta))$ for every sentence θ of $\mathcal{L}(\mathrm{ring}, R)$ (Theorem 4.4).

Moreover, combining the latter theorem with [FrJ08, p. 440, Lemma 20.6.1], we get that $\delta(A_R(\theta))$ is a rational number for each sentence θ of $\mathcal{L}(\mathrm{ring}, R)$. Furthermore, if R and θ are "explicitly given", then $\delta(A_R(\theta))$ can be recursively (and even primitive recursively) computed (Theorem 4.7).

Acknowledgement: The authors are indebted to the anonymous referee for many useful comments.

1 Preliminaries

{PRL}

The classical Chebotarev density theorem deals with a finite Galois extension L/K of global fields and with the corresponding extension O_L/O_K of their rings of integers. The generalized density theorem replaces O_L/O_K by a "finite Galois cover $X \to Y$" of irreducible schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. Thus, there is a finite group G that acts on X such that $Y = X/G$ and the associated action of G on the function field of X being faithful (Remark 1.9). Subsets of closed points of Y which are contained in closed subsets of Y of lower dimension replace the finite exceptional sets that appear in the classical case.

The generalized theorem is due to Jean-Pierre Serre [Ser65, p. 91, Thm. 7], who calls it the **Artin-Chebotarev density theorem**. The missing proofs in [Ser65] can be found in the master thesis [Hol04] of Armin Holschbach.

A central geometric concept that enters the density theorem is a "morphism of finite type".

{FGN}

Remark 1.1 (Finitely generated)**.** Let $\varphi\colon R_0 \to R$ be a homomorphism of integral domains and set $\bar{r}_0 = \varphi(r_0)$ for $r_0 \in R_0$ and $\bar{R}_0 = \varphi(R_0)$. Then, the rule $r_0 \cdot r = \varphi(r_0)r$ for $r_0 \in R_0$ and $r \in R$ makes R into an R_0-algebra. Assume that as such R is **finitely generated**. Thus, there exist $x_1, \ldots, x_m \in R$ such that every element r in R is a polynomial in $x_{1,0} \cdot r = \varphi(r_0)r$ for $r_0 \in R_0$ and $r \in R$ makes R into an R_0-algebra. Assume that as such R is **finitely generated**. Thus, there exist $x_1, \ldots, x_m \in R$ such that every element r in R is a polynomial in $x_1, \ldots, x_m$ with coefficients in $\bar{R}_0$, so $R = \bar{R}_0[\mathbf{x}]$ with $\mathbf{x} = (x_1, \ldots, x_m)$. [1]

Let $\varphi^*\colon \mathrm{Spec}(R) \to \mathrm{Spec}(R_0)$ be the morphism induced by φ. In particular, $\varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Since both $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R_0)$ are affine, "φ^* is of finite type".

To this end recall that a morphism $f\colon X \to Y$ of schemes is of **finite type** if there exists a covering of Y by open affine subsets $V_i := \mathrm{Spec}(B_i)$ such that for each i, $f^{-1}(V_i)$ can be covered by finitely many open affine subsets $U_{ij} := \mathrm{Spec}(A_{ij})$, where each A_{ij} is a finitely generated B_i-algebra [Har77, p. 84, 1st Definition].

By [Har77, p. 91, Exer. 3.3(a)], every morphism of schemes f that is of finite type is quasi-compact. Hence, our definition (taken from [Har77]) coincides with other definitions that demand f to be quasi-compact (e.g. [GoW10, p. 243, Def. 10.6]).

In the special case where R/R_0 is an extension of integral domains such that R is a finitely generated R_0-algebra, we take $\varphi\colon R_0 \to R$ to be the inclusion map. In particular, $\varphi^*(\mathfrak{p}) = \mathfrak{p} \cap R_0$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Again, φ^* is of finite type.

If, in addition, $R' := R[y_1, \ldots, y_n]$ is an integral domain extension of R, then $R' = R_0[\mathbf{x}, y_1, \ldots, y_n]$ is a finitely generated R_0-algebra, so the corresponding map $\mathrm{Spec}(R') \to \mathrm{Spec}(R_0)$ is also of finite type. Replacing R by R' turns out to be useful in the sequel.

{H1N}

Example 1.2. The case where Y in Remark 1.1 is $\mathrm{Spec}(\mathbb{Z})$: Let L be a finitely generated field extension of $\mathbb{Q}$ and let $\mathbf{t} := (t_1, \ldots, t_r)$ be a transcendence basis for $L/\mathbb{Q}$. Then, by the Hilbert basis theorem, $\mathbb{Z}[\mathbf{t}]$ is a Noetherian domain [Eis95, p. 27, Thm. 1.2]. Hence, a theorem of Emmy Noether, [Eis95, p. 127, Thm. 4.14], gives $u_1, \ldots, u_s \in L$ such that $S := \sum_{j=1}^{s} \mathbb{Z}[\mathbf{t}]u_j$ is the integral closure of $\mathbb{Z}[\mathbf{t}]$ in L. In particular, $S = \mathbb{Z}[\mathbf{t}, \mathbf{u}]$ is a finitely generated

[1] One could also write $R = R_0[\mathbf{x}]$, where each element in R is a polynomial over R_0 interpreted in R via φ [Eis95, p. 13, 2nd paragraph]. However, in this note we will write $R = R_0[\mathbf{x}]$ only if $R_0 \subseteq R$ and φ is the inclusion map $R_0 \to R$.

$\mathbb{Z}$-algebra. Hence, by the second paragraph of the current example, the epimorphism $\mathrm{Spec}(S) \to \mathrm{Spec}(\mathbb{Z})$ attached to the inclusion $\mathbb{Z} \to S$ is of finite type.

If L is a finitely generated field extension of $\mathbb{F}_p$, then one may choose a finitely generated integral domain extension S of $\mathbb{F}_p$ with $\mathrm{Quot}(S) = L$. Then, the reduction $\mathbb{Z} \to \mathbb{F}_p$ modulo p combined with the inclusion map $\mathbb{F}_p \to S$ give a homomorphism $\varphi\colon \mathbb{Z} \to S$ that makes S a finitely generated $\mathbb{Z}$-algebra. As above, the corresponding morphism $\varphi^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(\mathbb{Z})$ is of finite type.

Whenever needed in the comming proofs we may replace S by $S[u_1, \ldots, u_r]$, where $u_1, \ldots, u_r$ are arbitrary elements of L, in particular, when we need S to be integrally closed.

{GRL}

Definition 1.3 (Generalized ring of integers)**.** Recall that a field K is said to be **global** if K is either a finite extension of $\mathbb{Q}$ or K is a finite extension of $\mathbb{F}_p(t)$ for some prime number p and a transcendental element t over $\mathbb{F}_p$. The **ring of integers** of K is the integral closure of $\mathbb{Z}$ in K, in the first case, and the integral closure of $\mathbb{F}_p[t]$ in K, in the second case. Note that in the second case, the ring of integers depends on t.

We say that K is a **generalized global field** if K is either a finitely generated field extension of $\mathbb{Q}$ or a finitely generated infinite field extension of $\mathbb{F}_p$ for some prime number p.

Likewise, we say that an integral domain R is a **generalized subring of integers** if $R = \mathbb{Z}[x_1, \ldots, x_n]$ is a finitely generated ring extension of $\mathbb{Z}$ or $R = \mathbb{F}_p[x_1, \ldots, x_n]$ is an infinite finitely generated ring extension of $\mathbb{F}_p$ for some prime number p.

Note that in both cases the scheme $\mathrm{Spec}(R)$ is integral [Liu06, p. 65, Prop. 4.17] (hence, irreducible), Noetherian, and of finite type over $\mathrm{Spec}(\mathbb{Z})$. Moreover, the quotient field of R is a generalized global field.

If in addition R is integrally closed, we say that R is a **generalized ring of integers**. In this case $\mathrm{Spec}(R)$ is also normal.

{GPD}

Definition 1.4 (Dimension)**.** Let X be an irreducible scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$. Then, the dimension of X is the maximum length of a chain $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n$ of closed irreducible subschemes of X wih $X_i \neq X_{i+1}$ for $i = 0, \ldots, n-1$ [Liu06, p. 68, Def. 5.1]. Moreover, $\dim(X)$ is also the **Kronecker dimension** of the function field F of X. This means that $\dim(X) = \mathrm{trans.deg}(F/\mathbb{Q}) + 1$ if $\mathrm{char}(F) = 0$ and $\dim(X) = \mathrm{trans.deg}(F/\mathbb{F}_p)$ if $\mathrm{char}(F) = p > 0$ [Ser65, p. 83, (1)].

Here and in the sequel, the adjectives open, closed, dense and alike for subsets of a scheme are meant in the Zariski topology of the scheme.

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{CLP}

Definition 1.5 (Closed points)**.** A point $\mathbf{x}$ of a scheme X is **closed** if the set $\{\mathbf{x}\}$ is closed in X [GoW10, p. 43, Def. 2.8(1)].

If $X = \mathrm{Spec}(S)$ for an integral domain S, then $\mathbf{x}$ is a prime ideal $\mathfrak{p}$ of S, and $\mathbf{x}$ is a closed point of X if and only if $\mathfrak{p}$ is a maximal ideal of S [GoW10, p. 44, Example 2.9(1)], that is if and only if the quotient ring $S/\mathfrak{p}$ is a field.

{SCP}

Remark 1.6 (Sets of closed points)**.** Given a scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$, [Ser65, p. 83, 1.2] denotes the set of closed points of X by $\bar{X}$. We find this notation somewhat misleading, because the bar notation could be confused with the closure operation. Therefore, we denote the set of closed points of X by $\mathrm{CLP}(X)$. Given a subset A of X we write $\mathrm{CLP}(A) = A \cap \mathrm{CLP}(X)$ for the set of closed points of A. This notation satisfies the obvious rule: {disj}

(1) If A and A' are disjoint subsets of X, then so are $\mathrm{CLP}(A)$ and $\mathrm{CLP}(A')$ and $\mathrm{CLP}(A \cup A') = \mathrm{CLP}(A) \cup \mathrm{CLP}(A')$.

By Definition 1.5, if $X = \mathrm{Spec}(S)$ for an integral domain S, then $\mathrm{CLP}(X)$ is the set $\mathrm{Max}(S)$ of all maximal ideals of S.

{RFF}

Lemma 1.7. *Let S be a generalized subring of integers. Then, a point $\mathbf{x}$ of $\mathrm{Spec}(S)$ is closed if and only if the residue field of $\mathbf{x}$ is a finite field.*

Proof. (See also [Hol04, Lemma 3.1.1].) By Definition 1.3, $X = \mathrm{Spec}(S)$ where $S = \mathbb{Z}[x_1, \ldots, x_n]$ is a finitely generated extension of $\mathbb{Z}$, in the characteristic 0 case, and $S = \mathbb{F}_p[x_1, \ldots, x_n]$ in the characteristic p case. Let $\mathfrak{p}$ be a closed point of $\mathrm{Spec}(S)$. By Definition 1.5, $\mathfrak{p}$ is a maximal ideal of S, so $S/\mathfrak{p}$ is a field.

Note that both $\mathbb{Z}$ and $\mathbb{F}_p$ are **Jacobson rings**, i.e. rings in which every prime ideal is the intersection of maximal ideals [Eis95, p. 131]. Hence, by a general form of the Nullstellensatz [Eis95, p. 132, Thm. 4.19], in the characteristic 0 case, $\mathfrak{p} \cap \mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, that is $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some prime number p. Similarly $\mathfrak{p} \cap \mathbb{F}_p = \mathbf{0}$ in the characteristic p case.

In both cases, $S/\mathfrak{p}$ is a finite field extension of $\mathbb{F}_p$ [Eis95, p. 132, Thm. 4.19]. Therefore, $S/\mathfrak{p}$ is a finite field, as claimed. □

In the notation of the proof of Lemma 1.7 we denote the order of $S/\mathfrak{p}$ by $N(\mathfrak{p})$ and also by $N(\mathbf{x})$ when we consider $\mathfrak{p}$ as a point $\mathbf{x}$ of X.

{NPT}

Lemma 1.8 (Number of Points)**.** *Let X be a scheme of finite type over* $\mathrm{Spec}(\mathbb{Z})$ *and F a finite field. Then*

(a) *X has only finitely many closed points with residue field isomorphic to F. Moreover, let m be a positive integer. Then, X has only finitely many closed points with resdue fields of cardinality at most m.*

(b) *The set of all closed points of X is countable.*

Proof. (See also [Hol04, Lemma 3.1.4].) By [Har77, p. 90, Exer. 3.1], X is a union of finitely many open affine subsets, each of them is of finite type over $\mathrm{Spec}(\mathbb{Z})$. Hence, we may assume that $X = \mathrm{Spec}(S)$, where, in the characteristic 0 case, $S = \mathbb{Z}[x_1, \ldots, x_n]$ is a finitely generated algebra over $\mathbb{Z}$. In the characteristic p case, $S = \mathbb{F}_p[x_1, \ldots, x_n]$ is a finitely generated algebra over $\mathbb{F}_p$.

Statement (b) is now a consequence of statement (a) and Lemma 1.7, so it suffices now to prove statement (a).

Note that the set of closed points of X with residue field isomorphic to F is contained in the set $X(F)$ of F-rational points of X. By definition, each F-rational point of X corresponds to a homomorphism $h\colon S \to F$. In the characteristic 0 case, the restriction of h to $\mathbb{Z}$ is the reduction $\mathbb{Z} \to \mathbb{F}_p$ modulo

$p := \mathrm{char}(F)$. In the characteristic p case, $S = \mathbb{F}_p[x_1, \dots, x_n]$ and the restriction of h to $\mathbb{F}_p$ is the identity map. Thus, in both cases, the F-rational point corresponds to the n-tuple $(h(x_1), \dots, h(x_n))$. Since F^n is a finite set, there are up to isomorphism only finitely many such n-tuples, so X has only finitely many F-rational points.

Let m be a positive integer. Since $F = \mathbb{F}_{p^k}$ for some prime number p and a positive integer k, there are up to isomorphism only finitely many fields with cardinality at most m. Hence, by the preceding paragraph, X has only finitely many closed points with residue fields, up to isomorphism, of cardinality at most m. $\square$

{AFG}

Remark 1.9 (Action of a finite group on a scheme)**.** Let S be an integral domain and G a finite group of automorphisms of S. Write $R := S^G := \{a \in S \mid \sigma a = a \text{ for all } \sigma \in G\}$ for the fixed integral domain of S under the action of G. Then, the action of G on S extends to an action of G on the field $L := \mathrm{Quot}(S)$ and L is a Galois extension of the fixed field $K := L^G = \mathrm{Quot}(R)$ of L under G with $\mathrm{Gal}(L/K) = G$ [Lan02, p. 264, Thm. 1.8]. By definition, the action of G on S, hence the action of G on L, is **faithful**. Thus, $\sigma a = a$ for all $a \in S$ implies that σ is the identity element of G.

The action of G on S naturally induces an action of G on $X := \mathrm{Spec}(S)$. Let $Y = \mathrm{Spec}(R)$ and $\rho\colon X \to Y$ be the restriction morphism, in particular, $\rho(\mathfrak{p}) = \mathfrak{p} \cap R$ for each $\mathfrak{p} \in \mathrm{Spec}(S)$.

It turns out that Y is then the **quotient scheme of X under G**, also denoted by X/G [GoW10, p. 44, (2.3) and Prop. 2.10]. This means that $\rho \circ \sigma = \rho$ for each $\sigma \in G$, and for every morphism $f\colon X \to Y'$ of schemes with $f \circ \sigma = f$ for all $\sigma \in G$ there exists a unique morphism $\bar{\rho}\colon Y \to Y'$ with $\bar{\rho} \circ \rho = f$ [GoW10, p. 331, (12.7)].

By [GoW10, p. 331, Prop. 12.27(2)], for all $\mathbf{x}, \mathbf{x}' \in X$ the equality $\rho(\mathbf{x}) = \rho(\mathbf{x}')$ holds if and only if there exists $\sigma \in G$ with $\sigma(\mathbf{x}) = \mathbf{x}'$. Moreover, the morphism ρ is integral and surjective [GoW10, p. 331, Prop. 12.27(3)]. In particular, ρ is closed [GoW10, p. 325, Prop. 12.12]. If X is of finite type over a Noetherian ring S_0 and G acts on X by S_0-automorphisms, then the morphism $\rho\colon X \to Y$ is **finite** [Har77, p. 84, second definition] and X/G is of finite type over $\mathrm{Spec}(S_0)$ [GoW10, p. 331, Prop. 12.27(4)].

Although we won't use the following remark, it is still interesting to note the ring theoretic analogue of the geometric one appearing in Remark 1.9.

{INC}

Remark 1.10. If S in Remark 1.9 is integrally closed, then so is R and S is the integral closure of R in L. Morever, if R is Noetherian, then S is a finitely generated module [GoW10, p. 331, Prop. 12.27(4)].

Indeed, if $r \in K$ is integral over R, then $r \in L$ and r is integral over S. Since S is integrally closed, $r \in S$. In addition, $\sigma r = r$ for each $\sigma \in G$, so $r \in S^G = R$. Therefore, R is also integrally closed.

Further, each $s \in S$ is a root of the monic polynomial $\prod_{\sigma \in G}(T - \sigma s)$ with coefficients in R, so s is integral over R.

Conversely, if $s \in L$ is integral over R, then s is integral over S, so $s \in S$. Therefore, S is the integral closure of R in L, as claimed.

Now assume that S in Remark 1.9 is a generalized ring of integers. Then, $L = \mathrm{Quot}(S)$ is a generalized global field. Moreover, $R = S^G$ is a generalized ring of integers of the generalized global field K.

{DIG}

Definition 1.11 (Decomposition and inertia groups)**.** Let $\rho\colon X \to Y$ and G be as in Remark 1.9. Then G acts on $\mathrm{CLP}(X)$, so $\mathrm{CLP}(Y)$ may be identified with $\mathrm{CLP}(X)/G$.

Indeed, let $\mathbf{x} \in \mathrm{CLP}(X)$. Since ρ is closed, $\mathbf{y} := \rho(\mathbf{x})$ lies in $\mathrm{CLP}(Y)$. Let $\mathfrak{p} \in \mathrm{Max}(S)$ and $\mathfrak{q} = \mathfrak{p} \cap R \in \mathrm{Max}(R)$ be the maximal ideals corresponding to $\mathbf{x}$ and $\mathbf{y}$, respectively. Then, the field $\bar{K}_{\mathbf{y}} := R/\mathfrak{q}$ naturally embeds in $\bar{L}_{\mathbf{x}} := S/\mathfrak{p}$.

Assume that S is integrally closed. Then $D_{\mathbf{x}} := \{\sigma \in G \mid \sigma\mathbf{x} = \mathbf{x}\}$ is the **decomposition group** of $\mathbf{x}$, the extension of residue fields $\bar{L}_{\mathbf{x}}/\bar{K}_{\mathbf{y}}$ is normal, and there is a natural epimorphism

$$D_{\mathbf{x}} \to \mathrm{Aut}(\bar{L}_{\mathbf{x}}/\bar{K}_{\mathbf{y}})$$

mapping each $\sigma \in D_{\mathbf{x}}$ onto the unique automorphism $\bar{\sigma} \in \mathrm{Aut}(\bar{L}_{\mathbf{x}}/\bar{K}_{\mathbf{y}})$ satisfying $\bar{\sigma}\bar{x} = \overline{\sigma x}$ for each $x \in S$, where the reduction is modulo the maximal ideal $\mathfrak{p}$ [FrJ08, p. 108, Lemma 6.1.1(a)]. The kernel $I_{\mathbf{x}}$ of that homomorphism is the **inertia group** of $\mathbf{x}$. When $I_{\mathbf{x}}$ is the trivial subgroup $\mathbf{1}$ of G and $\bar{L}_{\mathbf{x}}/\bar{K}_{\mathbf{y}}$ is separable, the morphism $X \to Y$ is **unramified** at $\mathbf{x}$. In this case, $D_{\mathbf{x}}$ is canonically isomorphic to $\mathrm{Gal}(\bar{L}_{\mathbf{x}}/\bar{K}_{\mathbf{y}})$.

In the notation of Remark 1.9, we may choose an element z of S with $K(z) = L$. Then, consider the discriminant $u := \mathrm{discr}(f)$ of the irreducible polynomial f of z over K. Note that f is monic, irreducible, with coefficients in R, and separable (because L/K is Galois), so $u \in R$ and $u \neq 0$. Then, in the language of [FrJ08, p. 109, Def. 6.1.3], $S[u^{-1}]/R[u^{-1}]$ is a **ring cover**. In particular $X' := \mathrm{Spec}(S[u^{-1}])$ is **unramified** over $Y' := \mathrm{Spec}(R[u^{-1}])$, i.e. $X' \to Y'$ is unramified at each $\mathbf{x}' \in X'$ [FrJ08, p. 109, Lemma 6.1.4]. Indeed, that map is even standard étale at each point $\mathbf{x}' \in X'$ [Mil80, p. 26, 3rd paragraph].

Assume that S is a generalized ring of integers and the morphism $X \to Y$ is unramified at $\mathbf{x}$. Since, by Lemma 1.7, $\bar{L}_{\mathbf{x}}/\bar{K}_{\mathbf{y}}$ is a finite extension of finite fields, this extension is Galois and $D_{\mathbf{x}}$ is generated by the unique element $\mathrm{Frob}_{\mathbf{x}}$ that corresponds to the **Frobenius element** $\mathrm{Frob}_{\mathbf{x}}$ of $\mathrm{Gal}(\bar{L}_{\mathbf{x}}/\bar{K}_{\mathbf{y}})$ defined by $\mathrm{Frob}_{\mathbf{x}}(c) = c^{\mathrm{card}(\bar{K}_y)}$ for each $c \in \bar{L}_{\mathbf{x}}$.

2 The Artin-Chebotarev Density Theorem

{ACD}

START HERE

One of the main tools in the proof of the transfer theorem for generalized global fields mentioned in Definition 1.3 is the **Chebotarev density theorem**: Let L/K be a finite Galois extension of global fields, O_L/O_K the corresponding extension of its rings of integers, and $\mathcal{C}$ a conjugacy class of $\mathrm{Gal}(L/K)$. Then, the Dirichlet density of the set of prime ideals $\mathfrak{p}$ of O_K for which the "Artin symbol" $\left(\frac{L/K}{\mathfrak{p}}\right)$ is contained in $\mathcal{C}$ is $\frac{\mathrm{card}(\mathcal{C})}{[L:K]}$ (e.g. [FrJ08, Sections 6.2 and 6.3]).

One of the features of the Dirichlet density is that it is zero on finite sets. Hence, one may assume that the prime ideals that appear in the theorem are unramified, so that the corresponding Artin symbols are well defined.

Following [Ser65], we generalize the density theorem to irreducible schemes of dimension ≥ 1 and of finite type over $\mathrm{Spec}(\mathbb{Z})$.

{ZFN}

Remark 2.1 (The zeta function)**.** Let Y be a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$. The zeta function of Y is defined by the formal Euler product

$$\zeta(Y,s) = \prod_{\mathbf{y}\in \mathrm{CLP}(Y)} \frac{1}{1-\frac{1}{N(\mathbf{y})^s}} \tag{2}$$ {ztfn}

for a complex variable s, where $\mathrm{CLP}(Y)$ is, as above, the set of all closed points of Y [Ser65, p. 83, (2)].

By [Ser65, p. 83, Thm. 1], $\zeta(Y,s)$ converges absolutely (meaning, "the product on the right hand side of (2) converges absolutely") on the right half complex plane $\mathrm{Re}(s) > \dim(Y)$. Thus, $\zeta(Y,s)$ is an analytic function on that right half plane. By [Ser65, p. 84, Thm. 2], $\zeta(Y,s)$ can be continued as a meromorphic function to the half-plane $\mathrm{Re}(s) > \dim(Y) - \frac{1}{2}$.

Assume that Y is irreducible and let E be the function field of Y. Then, [Ser65, p. 84, Thm. 3] supplies the following information:

If $\mathrm{char}(E) = 0$, then the only pole of $\zeta(Y,s)$ in the half plane $\mathrm{Re}(s) > \dim(Y) - \frac{1}{2}$ is $s = \dim(Y)$ and it is a simple pole. In particular, the domain of convergence of the zeta function $\zeta(Y,s)$ is the half plane $\mathrm{Re}(s) > \dim(Y)$. Moreover, if $Y = \mathrm{Spec}(\mathbb{Z})$, then $\zeta(Y,s)$ coincides with the classical Riemann zeta function [FrJ08, p. 80, Prop. 4.2.2].

If $\mathrm{char}(E) = p > 0$, let q be the highest power of p with $\mathbb{F}_q \subseteq E$. Then, the only poles of $\zeta(Y,s)$ in the half plane $\mathrm{Re}(s) > \dim(Y) - \frac{1}{2}$ are the points $s = \dim(Y) + \frac{2\pi i\cdot n}{\log(q)}$ with $n \in \mathbb{Z}$ and they are simple.

Remark 2.2 (Dirichlet's density)**.** Let Y be an irreducible scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ of dimension ≥ 1. Using the fact that $\zeta(Y,s)$ has a simple pole $s = \dim(Y)$ one proves that {DDN}

$$\sum_{\mathbf{y}\in \mathrm{CLP}(Y)} \frac{1}{N(\mathbf{y})^s} \sim \log \frac{1}{s-\dim(Y)} \qquad \text{as } s \to \dim(Y)^+ \tag{3}$$ {simp}

[Ser65, p. 91, (18)].

Here and in the sequel "$s \to r^+$" for a real number r and a complex variable s means, as usual, that s approaches r on the real axis from the right.

A subset A of $\mathrm{CLP}(Y)$ has **Dirichlet density** $\delta(A)$ if

$$\delta(A) = \lim_{s\to \dim(Y)^+} \Big(\sum_{\mathbf{y}\in A} \frac{1}{N(\mathbf{y})^s}\Big) \Big/ \log \frac{1}{s-\dim(Y)} \tag{4}$$ {drdn}

[Ser65, p. 91, first paragraph of Section 2.7].

Whenever we write $\delta(A)$ for a subset A of $\mathrm{CLP}(Y)$, we assume that A has a Dirichlet density, given by (4). In particular this means that the limit on the right hand side of (4) exists.

{UPR}

Lemma 2.3. *The Dirichlet density has the following useful properties:*

(a) $\delta(\mathrm{CLP}(Y)) = 1$.

(b) *If $A, A' \subseteq \mathrm{CLP}(Y)$, both A and A' have a Dirichlet density, and $A \subseteq A'$, then $\delta(A) \leq \delta(A')$.*

(c) *If $A \subseteq \mathrm{CLP}(Y)$ has density 0 and $A_0 \subseteq A$, then $\delta(A_0) = 0$.*

(d) *If $A, A' \subseteq \mathrm{CLP}(Y)$, $A \cap A' = \emptyset$, and both A and A' have a Dirichlet density, then $\delta(A \cup A') = \delta(A) + \delta(A')$.*

(e) *If $A, A', B \subseteq \mathrm{CLP}(Y)$, $A \cap A' = \emptyset$, $A \cup A' = B$, and both A and B have a Dirichlet density, then $\delta(A') = \delta(B) - \delta(A)$. In particular, $\delta(\mathrm{CLP}(Y) \smallsetminus A) = 1 - \delta(A)$.*

(f) *If A is a finite subset of $\mathrm{CLP}(Y)$, then $\delta(A) = 0$. If B, B' are subsets of $\mathrm{CLP}(Y)$ with $\delta(B) = \delta(B') = 0$, then $\delta(B \cup B') = 0$.*

(g) *If Y_0 is a closed subset of Y with $\dim(Y_0) < \dim(Y)$, then $\delta(\mathrm{CLP}(Y_0)) = 0$.*

(h) *If a subset A of $\mathrm{CLP}(Y)$ has a Dirichlet density and U is a nonempty open subset of Y, then $\delta(A \cap U) = \delta(A)$. In particular, by (a), $\delta(\mathrm{CLP}(U)) = 1$.*

(i) *Let U be a nonempty open subset of Y. Then, $\delta(\mathrm{CLP}(Y \smallsetminus U)) = 0$.*

(j) *Let A be a subset of $\mathrm{CLP}(Y)$ and U a nonempty open subset of Y. If $A \cap U$ has a Dirichlet density, then $\delta(A) = \delta(A \cap U)$.*

(k) *If R is a ring of integers of a global field and $Y = \mathrm{Spec}(R)$, then (4) is the usual definition of the Dirichlet density of a set of prime ideals of R.*

Proof of (a). Use (3).

Proof of (b). Use (4).

Proof of (c). Indeed, $\sum_{\mathbf{y} \in A_0} \frac{1}{N(\mathbf{y})^s} \leq \sum_{\mathbf{y} \in A} \frac{1}{N(\mathbf{y})^s}$ for every real number $s > \dim(Y)$. Thus, our claim follows from (4).

Proof of (d). Use (4).

Proof of (e). Consider the continuous real valued functions

$$g(s) = \Big(\sum_{\mathbf{y} \in B} 1/N(\mathbf{y})^s\Big) \Big/ \log \frac{1}{s - \dim(Y)} \quad \text{and}$$

$$h(s) = \Big(\sum_{\mathbf{y} \in A} 1/N(\mathbf{y})^s\Big) \Big/ \log \frac{1}{s - \dim(Y)}$$

defined for $s > \dim(Y)^+$. Then,

$$\begin{aligned} \lim_{s \to \dim(Y)^+} g(s) - \lim_{s \to \dim(Y)^+} h(s) &= \lim_{s \to \dim(Y)^+} (g(s) - h(s)) \\ &= \lim_{s \to \dim(Y)^+} \Big(\sum_{\mathbf{y} \in A'} \frac{1}{N(\mathbf{y})^s}\Big) \Big/ \log \frac{1}{s - \dim(Y)}. \end{aligned}$$

Hence, by (4), $\delta(B) - \delta(A) = \delta(A')$, as claimed.

Proof of (f). Both statements follow from (4).

Proof of (g). By definition (4), $\delta(\mathrm{CLP}(Y_0))$ is equal to the expression

$$\lim_{s \to \dim(Y)^+} \sum_{\mathbf{y} \in \mathrm{CLP}(Y_0)} \frac{1}{N(\mathbf{y})^s} \Big/ \log \frac{1}{s - \dim(Y)}, \tag{5}$$ {nzsl}

if the limit exists.

We endow Y_0 with the induced reduced subscheme structure [GoW10, p. 88, Prop. 3.52]. Let $\iota\colon Y_0 \to Y$ be the corresponding closed immersion [GoW10, p. 84, first paragraph after Def. 3.41]. By [GoW10, p. 243, Prop. 10.7(1)], the morphism ι is of finite type. Hence, by [GoW10, p. 244, Prop. 10.7(2)], the combined morphism $Y_0 \to Y \to \mathrm{Spec}(\mathbb{Z})$ is of finite type.

By Remark 2.1, applied to Y_0 rather than to Y, we get that $\zeta(Y_0, s)$ converges at $s = \dim(Y) > \dim(Y_0)$. Hence, $\zeta(Y_0, \dim(Y)) < \infty$. Therefore, the numerator in (5) converges at $s = \dim(Y)$.

Indeed, by Lemma 1.8(b), the set $\mathrm{CLP}(Y_0)$ is countable. For each $\mathbf{y} \in \mathrm{CLP}(Y_0)$ let $a_\mathbf{y} = \frac{1}{N(\mathbf{y})^{\dim(Y)}}$. Then $0 < a_\mathbf{y} < 1$ and

$$\begin{aligned}\sum_{\mathbf{y}\in\mathrm{CLP}(Y_0)} \frac{1}{N(\mathbf{y})^{\dim(Y)}} &= \sum_{\mathbf{y}\in\mathrm{CLP}(Y_0)} a_\mathbf{y} \le \prod_{\mathbf{y}\in\mathrm{CLP}(Y_0)} (1+a_\mathbf{y}) \\ &\le \prod_{\mathbf{y}\in\mathrm{CLP}(Y_0)} \frac{1}{1-a_\mathbf{y}} \overset{(2)}{=} \zeta(Y_0, \dim(Y)) < \infty.\end{aligned}$$

On the other hand, the denominator of the right hand side of (5) diverges at $s = \dim(Y)$. Hence, $\delta(\mathrm{CLP}(Y_0)) = 0$, as claimed.

Proof of (h). Indeed, $Y \smallsetminus U$ is a proper closed subset of Y. Since Y is irreducible, we have $\dim(Y \smallsetminus U) < \dim(Y)$. Hence, by (g), $\delta(\mathrm{CLP}(Y \smallsetminus U)) = 0$. Taking into account that $A \smallsetminus U \subseteq \mathrm{CLP}(Y \smallsetminus U)$, we have by (c) that $\delta(A \smallsetminus U) = 0$. Therefore,

$$\delta(A \cap U) \overset{(e)}{=} \delta(A) - \delta(A \smallsetminus U) = \delta(A),$$

as claimed.

Proof of (i). By assumption, $Y \smallsetminus U$ is a proper closed subset of Y. Since Y is irreducible, $\dim(Y \smallsetminus U) < \dim(Y)$. Hence, by (g), $\delta(\mathrm{CLP}(Y \smallsetminus U)) = 0$, as claimed.

Proof of (j). Since A is a subset of $\mathrm{CLP}(Y)$, so are $A \smallsetminus U$ and $A \cap U$. By (i) and (c), $\delta(A \smallsetminus U) = 0$. Hence,

$$\delta(A) \overset{(d)}{=} \delta(A \smallsetminus U) + \delta(A \cap U) = \delta(A \cap U),$$

as claimed.

Proof of (k). See for example, [FrJ08, p. 113, Sec. 6.3]. $\square$

{DNT}

Theorem 2.4 (The Density Theorem). *Let X be an irreducible scheme of finite type over* $\mathrm{Spec}(\mathbb{Z})$ *with* $\dim(X) \geq 1$ *and let F be the function field of X. Let G be a finite group that acts on X such that G acts faithfully on F and suppose that $Y := X/G$ exists. Assume that the inertia group of each $\mathbf{x} \in \mathrm{CLP}(X)$ is trivial. Finally, consider a G-conjugacy domain C in G. Then, the set of elements $\mathbf{y} \in \mathrm{CLP}(Y)$ such that* $\mathrm{Frob}_\mathbf{y} \subseteq C$ *has a Dirichlet density equal to* $\mathrm{card}(C)/\mathrm{card}(G)$.

This theorem is stated as [Ser65, p. 91, Thm. 7], using the analytic theory of L-functions. Detailed proofs of the statements included in [Ser65] are given in [Hol04, p. 55, Thm. 3.7.2]. The case where X is the spectrum of the ring of integers of a global field appears for example in [FrJ08, p. 114, Thm. 6.3.1].

We improve Theorem 2.4 in the special case where $X = \mathrm{Spec}(S)$ as introduced in Remark 1.9, with $S_0 = \mathbb{Z}$, by getting rid of the ramification condition:

{RAC}

Corollary 2.5. *Let S be a generalized ring of integers (Definition 1.3). Let G be a finite group that acts faithfully on S, let C be a conjugacy domain of G, and set $R = S^G$. Put $L = \mathrm{Quot}(S)$, $K = \mathrm{Quot}(R)$, $X = \mathrm{Spec}(S)$, and $Y = \mathrm{Spec}(R)$.*

Then, the set $\mathrm{CLP}(Y)_C$ of points $\mathbf{y} \in \mathrm{CLP}(Y)$ unramified under the morphism $X \to Y$ with $\mathrm{Frob}_{\mathbf{y}} \subseteq C$ has a Dirichlet density equal to $\mathrm{card}(C)/\mathrm{card}(G)$.

Proof. As in Definition 1.11, we choose a nonzero element $u \in R$ such that $X' := \mathrm{Spec}(S[u^{-1}])$ is unramified over $Y' := \mathrm{Spec}(R[u^{-1}])$. Then, G acts on X'.

Since S is a finitely generated $\mathbb{Z}$-algebra, so is R [Lan02, p. 147, Cor. 7.2]. Hence, by Example 1.1, both morphisms $X' \to \mathrm{Spec}(\mathbb{Z})$ and $Y' \to \mathrm{Spec}(\mathbb{Z})$ are of finite type. In addition $L = \mathrm{Quot}(S[u^{-1}])$ and $K = \mathrm{Quot}(R[u^{-1}])$.

Note that, by Remark 1.9, G acts faithfully on L and $Y' = X'/G$ (see also [Tak69, p. 325, Thm. 1.7]). Hence, by Theorem 2.4, the Dirichlet density of the set

$$\mathrm{CLP}(Y')_C := \{\mathbf{y} \in \mathrm{CLP}(Y') \mid \mathrm{Frob}_{\mathbf{y}} \in C\}$$

is $\mathrm{card}(C)/\mathrm{card}(G)$. Since Y is irreducible and Y' is a nonempty open subset of Y, it follows from Lemma 2.3(j) that the density of $\mathrm{CLP}(Y)_C$ is the same as that of $\mathrm{CLP}(Y')_C$, that is, $\mathrm{card}(C)/\mathrm{card}(G)$, as claimed. □

3 Test Sentences

{GTT}

This and the next section generalize the transfer theorem from rings of integers of global fields [FrJ08, p. 447, Thm. 20.9.3] to generalized rings of integers (whose quotient fields are generalized global fields), as introduced in Definition 1.3. By [FrJ08, p. 242, Thm. 13.4.2],

{prc}

(6) every generalized global field is Hilbertian [FrJ08, p. 219, Section 12.1].

We choose a generalized ring of integers R with $\mathrm{Quot}(R) = K$ and a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$ of finite type as in Example 1.1, "The case where $Y = \mathrm{Spec}(\mathbb{Z})$".

As in Definition 1.11, we denote the residue field of a closed point $\mathfrak{p} \in \mathrm{Spec}(R)$ by $\bar{K}_{\mathfrak{p}}$. By Lemma 1.7 and the paragraph that follows that lemma, $\bar{K}_{\mathfrak{p}}$ is a finite field with $N(\mathfrak{p}) = \mathrm{card}(\bar{K}_{\mathfrak{p}})$. In addition we denote the first order language of rings whose constant symbols are the elements of R by $\mathcal{L}(\mathrm{ring}, R)$ [FrJ08, p. 135, Example 7.3.1] and consider $\mathcal{L}(\mathrm{ring}, R)$-structures which are either field extensions of K or one of the residue fields $\bar{K}_{\mathfrak{p}}$ with $\mathfrak{p} \in \mathrm{Max}(R)$.

{SMS}

Remark 3.1 (Small sets)**.** Let $\mathcal{S}$ be the family of all subsets A of $\mathrm{Max}(R)$ which are contained in a proper closed subset of $\mathrm{Spec}(R)$. By Lemma 2.3(g), $\delta(A) = 0$ for each $A \in \mathcal{S}$. The family $\mathcal{S}$ has the following properties: {sms1}

(7a) $\mathrm{Max}(R) \notin \mathcal{S}$, {sms2}

(7b) $A, B \in \mathcal{S}$ implies $A \cup B \in \mathcal{S}$, {sms3}

(7c) $B \in \mathcal{S}$ and $A \subseteq B$ imply $A \in \mathcal{S}$, and {sms4}

(7d) $A \in \mathcal{S}$ for every finite subset A of $\mathrm{Max}(R)$.

Proof of (7a)**.** Let W be a proper closed subset of $\mathrm{Spec}(R)$. We endow W with the induced reduced subscheme structure [GoW10, p. 88, Prop. 3.52]. Thus, $W = V(\mathfrak{a})$ for some nonzero ideal $\mathfrak{a}$ of R [GoW10, p. 84, Thm. 3.42], where its underlying topological space is $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$. By assumption, R is finitely generated over $\mathbb{Z}$. Hence, since $\mathbb{Z}$ is a Jacobson ring, so is R [Eis95, p. 132, Thm. 4.19]. In particular, the intersection of all maximal ideals of R is the zero ideal. Thus, there is a maximal ideal $\mathfrak{q}$ of R not containing $\mathfrak{a}$. It follows that $\mathrm{Max}(R) \not\subseteq W$. Conclude that $\mathrm{Max}(R) \notin \mathcal{S}$.

Proof of (7b)**.** By assumption, there exist proper closed subsets V, W of $\mathrm{Spec}(R)$ such that $A \subseteq V$ and $B \subseteq W$. Hence, $A \cup B \subseteq V \cup W \subset \mathrm{Spec}(R)$, again, because $\mathrm{Spec}(R)$ is irreducible.

Proof of (7c)**.** By assumption $\mathrm{Spec}(R)$ has a proper closed subset V such that $A \subseteq B \subseteq V \subset \mathrm{Spec}(R)$. Hence $A \in \mathcal{S}$.

Proof of (7d)**.** As a finite subset of $\mathrm{Max}(R)$, the finite set A is closed. Since $\dim(\mathrm{Spec}(R)) \geq 1$, A is a proper subset of $\mathrm{Spec}(R)$. Hence, $A \in \mathcal{S}$.

Thus, in the terminology of [FrJ08, p. 139, Sec. 7.6], $\mathcal{S}$ is a family of small sets that contains every finite subset of $\mathrm{Max}(R)$. Henceforth, we will say that a subset A of $\mathrm{Max}(R)$ is **small** if $A \in \mathcal{S}$. Then, we say that a subset B of $\mathrm{Max}(R)$ is **large** if $\mathrm{Max}(R) \smallsetminus B$ is small.

{pisq}

Example 3.2. Consider the case where K is a global field and O_K is the ring of integers of K. Then, O_K is a Dedekind ring and $\dim(\mathrm{Spec}(O_K)) = 1$. Thus, small subsets of $\mathrm{Spec}(O_K)$ are just finite sets of nonzero prime ideals of O_K, in agreement with the convention of [FrJ08, p. 446, Section 20.9 and p. 147, Section 7.9].

This convention is stronger than taking the small sets to be those with density zero. Consider, for example, the case where $K = \mathbb{Q}$ and $O_K = \mathbb{Z}$. In this case there exist infinite sets of prime numbers of Dirichlet density zero.

{LRS}

Remark 3.3 (Filter)**.** Taking complements of subsets of $\mathrm{Max}(R)$ and using Remark 3.1, we find that the family $\mathcal{S}' := \{A \subseteq \mathrm{Max}(R) \mid A' \in \mathcal{S}\}$ with $A' := \mathrm{Max}(R) \smallsetminus A$ satisfies the following rules: {lrg1}

(8a) $\emptyset \notin \mathcal{S}'$, {lrg2}

(8b) $A, B \in \mathcal{S}'$ implies $A \cap B \in \mathcal{S}'$, {lrg3}

(8c) $A \in \mathcal{S}'$ and $A \subseteq B \subseteq \mathrm{Max}(R)$ imply $B \in \mathcal{S}'$, and

{lrg4}

(8d) $A \in \mathcal{S}'$ if A is cofinite in $\mathrm{Max}(R)$.

Conditions (8a)–(8c) say that $\mathcal{S}'$ is a **filter** on $\mathrm{Max}(R)$ [FrJ08, p. 138, Sec. 7.5].

By [FrJ08, p. 139, Cor. 7.5.3], $\mathcal{S}'$ is contained in an **ultrafilter** $\mathcal{D}$ of subsets of $\mathrm{Max}(R)$. Thus, in addition to conditions (8a)–(8c) for $\mathcal{D}$ rather that for $\mathcal{S}'$, the family $\mathcal{D}$ satisfies the following one: {ulf}

(9) $A, B \subseteq \mathrm{Max}(R)$ and $A \cup B \in \mathcal{D}$ imply $A \in \mathcal{D}$ or $B \in \mathcal{D}$.

In addition, $\mathcal{D}$ contains no finite subset. Thus, $\mathcal{D}$ is a **nonprincipal ultrafilter** [FrJ08, p. 139, Example 5.1(b)].

{TRS}

Definition 3.4. Given a sentence θ of $\mathcal{L}(\mathrm{ring}, R)$ [FrJ08, p. 133], we set

$$A_R(\theta) = \{\mathfrak{p} \in \mathrm{Max}(R) \mid \bar{K}_{\mathfrak{p}} \models \theta\},$$

where "$\bar{K}_{\mathfrak{p}} \models \theta$" means that "$\theta$ is true in $\bar{K}_{\mathfrak{p}}$" [FrJ08, p. 134]. We call $A_R(\theta)$ the **truth set of** θ **along** $\mathrm{Max}(R)$ and say that θ **is true in** $\bar{K}_{\mathfrak{p}}$ **for almost all** $\mathfrak{p} \in \mathrm{Max}(R)$ if $A_R(\theta)$ is a large subset of $\mathrm{Max}(R)$.

{PSG}

Remark 3.5 (The probability space $\mathrm{Gal}(K)$)**.** We denote the maximal separable extension of K in $\tilde{K}$ by K_{sep} and let $\mathrm{Gal}(K)$ be the absolute Galois group $\mathrm{Gal}(K_{\mathrm{sep}}/K)$ of K. Then we denote the fixed field in K_{sep} of an element $\sigma \in \mathrm{Gal}(K)$ by $K_{\mathrm{sep}}(\sigma)$ and write $\tilde{K}(\sigma)$ for the maximal purely inseparable extension of $K_{\mathrm{sep}}(\sigma)$ in $\tilde{K}$.

Being a profinite group, $\mathrm{Gal}(K)$ is equipped with a unique Haar measure μ_K with $\mu_K(\mathrm{Gal}(K)) = 1$ [FrJ08, p. 366, Prop. 18.2.1]. In this case, a **small subset of** $\mathrm{Gal}(K)$ is just a subset of measure 0 and a **large subset of** $\mathrm{Gal}(K)$ is a subset of measure 1. Each of the perfect fields $\tilde{K}(\sigma)$ is an $\mathcal{L}(\mathrm{ring}, R)$-structure. Then, "a sentence θ of $\mathcal{L}(\mathrm{ring}, R)$ is true in $\tilde{K}(\sigma)$ **for almost all** σ" means that θ is true in $\tilde{K}(\sigma)$ for a large set of σ's in $\mathrm{Gal}(K)$.

We define the **truth set** of θ **along** $\mathrm{Gal}(K)$ by

$$\Sigma_{\tilde{K}/K}(\theta) := \{\sigma \in \mathrm{Gal}(K) \mid \tilde{K}(\sigma) \models \theta\},$$

(observe the change of notation from [FrJ08, p. 440, (1)]).

{BPL}

Remark 3.6 (Boolean polynomials)**.** Following [FrJ08, p. 140], we define **Boolean polynomials** in the variables $Z_1, \ldots, Z_m$ recursively: $Z_1, \ldots, Z_m$ are Boolean polynomials, and if U, U_1, U_2 are Boolean polynomials, then U', $U_1 \cup U_2$, and $U_1 \cap U_2$ are Boolean polynomials.

Evaluate a Boolean polynomial $\mathcal{P}(Z_1, \ldots, Z_m)$ at subsets $A_1, \ldots, A_m$ of a set by interpreting the symbols $\cup, \cap$, and $'$ as a union, an intersection, and taking the complement, respectively. Likewise, evaluate $\mathcal{P}(Z_1, \ldots, Z_m)$ at sentences $\theta_1, \ldots, \theta_m$ of a first order language [FrJ08, p. 132, Sec. 7.1] by interpreting $\cup, \cap$, and $'$ as disjunction, conjunction, and negation, respectively.

Thus, in order to prove a property P of $\mathcal{P}(A_1, \ldots, A_n)$ for subsets (resp. $\mathcal{P}(\theta_1, \ldots, \theta_n)$ for sentences) it suffices to prove:

(10a) P holds for each A_i (resp. θ_i),

(10b) P holds for a subset A (resp. a sentence θ) implies that P holds for the complement of A (resp. for the negation of θ), and

(10c) P holds for sets A_1, A_2 (resp. sentences θ_1, θ_2) implies that P holds for the union $A_1 \cup A_2$ (resp. the disjunction $\theta_1 \vee \theta_2$).

This procedure is called an **induction on structure** [FrJ08, p. 133].

In particular, induction on structure shows for sentences $\theta_1, \dots, \theta_m$ of $\mathcal{L}(\mathrm{ring}, R)$ and a Boolean polynomial $\mathcal{P}(Z_1, \dots, Z_m)$ that

$$\begin{aligned} &A_R(\mathcal{P}(\theta_1, \dots, \theta_m)) = \mathcal{P}(A_R(\theta_1), \dots, A_R(\theta_m)) \text{ and} \\ &\Sigma_{\tilde{K}/K}(\mathcal{P}(\theta_1, \dots, \theta_m)) = \mathcal{P}(\Sigma_{\tilde{K}/K}(\theta_1), \dots, \Sigma_{\tilde{K}/K}(\theta_m)) \end{aligned}$$

in the notation of Definition 3.4 and Remark 3.5.

{TSN}

Remark 3.7 (Test sentences)**.** We call a sentence λ of $\mathcal{L}(\mathrm{ring}, R)$ of the form

$$\mathcal{P}((\exists T)[f_1(T) = 0], \dots, (\exists T)[f_m(T) = 0]) \tag{11}$$ {tss}

with $f_1, \dots, f_m \in R[T]$ separable polynomials and $\mathcal{P}$ a boolean polynomial a **test sentence**. It is also a test sentence of $\mathcal{L}(\mathrm{ring}, K)$ in the sense of [FrJ08, p. 440, Sect. 20.6]. Let L be the splitting field of $f_1 f_2 \cdots f_m$ over K. Denote the set of all $\tau \in \mathrm{Gal}(L/K)$ with $L(\tau) \models \lambda$ (i.e. λ is true in the fixed field $L(\tau)$ of τ in L) by $\Sigma_{L/K}(\lambda)$. Then, by [FrJ08, p. 440, (3)],

$$\Sigma_{\tilde{K}/K}(\lambda) = \{\sigma \in \mathrm{Gal}(K) \mid \mathrm{res}_L \sigma \in \Sigma_{L/K}(\lambda)\}. \tag{12}$$ {srl}

Therefore, by [FrJ08, p. 370, Example 18.2.3],

$$\mu_K(\Sigma_{\tilde{K}/K}(\lambda)) = \frac{\mathrm{card}(\Sigma_{L/K}(\lambda))}{[L : K]}. \tag{13}$$ {mus}

{DPS}

Definition 3.8 (Decomposition and inertia groups)**.** We give here the ring theoretic analog of the notions introduced in Definition 1.11. These notions will be used in the next lemma.

Recall that if R is an integrally closed domain with quotient field K, L is a finite Galois extension of K, S is the integral closure of R in L, $\mathfrak{p} \in \mathrm{Spec}(R)$, and $\mathfrak{q} \in \mathrm{Spec}(S)$ lies over $\mathfrak{p}$ (i.e. $\mathfrak{q} \cap R = \mathfrak{p}$), then $D_\mathfrak{q} := \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma\mathfrak{q} = \mathfrak{q}\}$ is the **decomposition group of $\mathfrak{q}$ over** K (alternatively, **over** $\mathfrak{p}$). Let $\bar{K}_\mathfrak{p} := \mathrm{Quot}(R/\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $\bar{L}_\mathfrak{q} := \mathrm{Quot}(S/\mathfrak{q}) = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ be the respective residue fields. For each $x \in S$ let $\bar{x} := x + \mathfrak{q}$ be the equivalence class of x modulo $\mathfrak{q}$. Each $\sigma \in D_\mathfrak{q}$ induces a unique automorphism $\bar{\sigma}$ of $\bar{L}_\mathfrak{q}$ over $\bar{K}_\mathfrak{p}$ satisfying $\bar{\sigma}\bar{x} = \overline{\sigma x}$ for each $x \in S$. By [FrJ08, p. 108, Lemma 6.1.1(a)], the field extension $\bar{L}_\mathfrak{q}/\bar{K}_\mathfrak{p}$ is normal and the map $\sigma \mapsto \bar{\sigma}$ is an epimorphism from $D_\mathfrak{q}$ onto $\mathrm{Aut}(\bar{L}_\mathfrak{q}/\bar{K}_\mathfrak{p})$.

The **inertia group of $\mathfrak{q}$ over** K is

$$\begin{aligned} I_\mathfrak{q} :&= \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma x \in x + \mathfrak{q} \text{ for each } x \in S\} \\ &= \{\sigma \in \mathrm{Gal}(L/K) \mid \bar{\sigma} = 1\}. \end{aligned}$$

If $\bar{K}_\mathfrak{p}$ is a finite field, then $\bar{L}_\mathfrak{q}/\bar{K}_\mathfrak{p}$ is a Galois extension. If, in addition, $I_\mathfrak{q}$ is trivial, then the map $\sigma \mapsto \bar{\sigma}$ is an isomorphism of $D_\mathfrak{q}$ onto $\mathrm{Gal}(\bar{L}_\mathfrak{q}/\bar{K}_\mathfrak{q})$ [FrJ08, p. 108, Lemma 6.1.1(b)]. Again, in this case, we say that $\mathfrak{q}$ is unramified over K and $\mathfrak{p}$ is unramified in L.

The following result generalizes [FrJ08, p. 446, Lemma 20.9.2].

{TSS}

Lemma 3.9. *Let λ be the test sentence* (11) *of* $\mathcal{L}(\mathrm{ring}, R)$. *Let B be the set of all $\mathfrak{p} \in \mathrm{Spec}(R)$ such that the leading coefficients and the discriminants of the f_i's are units of $R_\mathfrak{p}$, $i = 1, \ldots, m$. Denote the splitting field of $f_1 \cdots f_m$ over K by L and let S be the integral closure of R in L. Then:*

(a) *For each $\mathfrak{p} \in \mathrm{CLP}(B)$, every $\mathfrak{q} \in \mathrm{Spec}(S)$ over $\mathfrak{p}$, every $\sigma \in D_\mathfrak{q}$, and every field extension F of $\bar{K}_\mathfrak{p}$ satisfying $\bar{L}_\mathfrak{q} \cap F = \bar{L}_\mathfrak{q}(\bar{\sigma})$, where $\bar{\sigma}$ is the image of σ under the map $D_\mathfrak{q} \to \mathrm{Gal}(\bar{L}_\mathfrak{q}/\bar{K}_\mathfrak{p})$ induced by $\mathfrak{q}$, we have $L(\sigma) \models \lambda$ if and only if $F \models \lambda$.*

(b) $B \cap A_R(\lambda) = \{\mathfrak{p} \in B \cap \mathrm{Max}(R) \mid \mathrm{Frob}_\mathfrak{p} \subseteq \Sigma_{L/K}(\lambda)\}$.

(c) $\delta(A_R(\lambda)) = \frac{\mathrm{card}(\Sigma_{L/K}(\lambda))}{[L:K]}$.

Proof of (a). If $m = 1$, then λ is $(\exists T)[f_1(T) = 0]$ and statement (a) is a consequence of [FrJ08, p. 111, Lemma 6.1.8(a)]. The general case follows by induction on the structure of λ.

Proof of (b). Each $\mathfrak{p}$ that belongs to either of the sides of (b) lies in $\mathrm{Max}(R)$ (Definition 3.4).

Consider $\mathfrak{p}$ that belongs to the right hand side of (b). Let $\mathfrak{q}$ be a prime ideal of S lying over $\mathfrak{p}$, so $\mathfrak{q} \in \mathrm{Max}(S)$. Let $\bar{L}_\mathfrak{q} = S/\mathfrak{q}$ and $\bar{K}_\mathfrak{p} = R/\mathfrak{p}$. Then, $\bar{L}_\mathfrak{q}/\bar{K}_\mathfrak{p}$ is a finite Galois extension of finite fields and there is a generator σ of $D_\mathfrak{q}$ which is mapped onto a generator $\bar{\sigma}$ of $\mathrm{Gal}(\bar{L}_\mathfrak{q}/\bar{K}_\mathfrak{p})$. Since $L(\sigma) \models \lambda$, we get, by (a), that $\bar{K}_\mathfrak{p} \models \lambda$. Thus, $\mathfrak{p} \in B \cap A_R(\lambda)$.

The other direction of (b) follows similarly.

Proof of (c). Note that B is a nonempty open subset of $\mathrm{Spec}(R)$ with $\dim(B) = \dim(\mathrm{Spec}(R))$, because each f_i is separable. By [FrJ08, p. 111, Lemma 6.1.8(b)], the inertia group in $\mathrm{Gal}(L/K)$ of each closed point of B is trivial. By Lemma 2.3(h), $\delta(B \cap A_R(\lambda)) = \delta(A_R(\lambda))$.

By Corollary 2.5, with $G = \mathrm{Gal}(L/K)$,

$$\delta(\{\mathfrak{p} \in B \cap \mathrm{Max}(R) \mid \mathrm{Frob}_\mathfrak{p} \subseteq \Sigma_{L/K}(\lambda)\}) = \frac{\mathrm{card}(\Sigma_{L/K}(\lambda))}{[L : K]}.$$

Therefore, by (b), $\delta(A_R(\lambda)) = \delta(B \cap A_R(\lambda)) = \frac{\mathrm{card}(\Sigma_{L/K}(\lambda))}{[L:K]}$, as claimed. $\square$

4 Ultraproducts

{ULP}

We refer to Remark 3.3 and [FrJ08, p. 141, Sec. 7.7], respectively, for the concepts "ultrafilter of a family $\mathcal{D}$ of subsets of a set S" and "ultraproduct $\prod \mathcal{A}_s/\mathcal{D}$ of models $\mathcal{A}_s$ of an elementary theory with indices $s \in S$ modulo $\mathcal{D}$". If S is equipped with a family of small sets (hence, also a family of large sets), then an ultrafilter $\mathcal{D}$ is **regular** if it contains no small set of S, equivalently if "each large set of S belongs to $\mathcal{D}$". In this case we say that $\prod \mathcal{A}_s/\mathcal{D}$ is a **regular ultraproduct**.

As in Section 3, K is a generalized global field and R is a generalized ring of integers with $\mathrm{Quot}(R) = K$ equipped with a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$ of finite type, as in Definition 1.3.

The following result is a special case of [FrJ08, p. 437, Prop. 20.4.4]. To this end note that since every element of K is a quotient of two elements of R, the language $\mathcal{L}(\mathrm{ring}, K)$ used in the latter proposition can be interpreted in $\mathcal{L}(\mathrm{ring}, R)$.

{AxR}

Lemma 4.1. *There exists a set* $\mathrm{Ax}(R)$ *of axioms in the language* $\mathcal{L}(\mathrm{ring}, R)$ *such that a field extension F of K satisfies those axioms if and only if F is perfect, PAC, and* $\mathrm{Gal}(F)$ *is procyclic.*

These axioms are sentences that interpret the field axioms [FrJ08, p. 135, Example 7.3.1], perfectness axioms, $[p \neq 0] \vee (\forall X)(\exists Y)[Y^p = X]$*, as p ranges over the prime numbers, the positive diagram of R [FrJ08, p. 135, Example 7.3.1], and the following axioms:*

(a) *PAC axioms: Every absolutely irreducible polynomial $f(X,Y)$ of degree d has a zero,* $d = 1, 2, 3, \ldots$.

(b) *Procyclic axioms: The finite groups which appear as Galois groups over F are all cyclic. Thus,* $\mathrm{Gal}(F)$ *is procyclic [FrJ08, p. 16, Exer. 6].*

The following result connects the fields $\tilde{K}(\sigma)$ with $\sigma \in \mathrm{Gal}(K)$ to the residue fields $\bar{K}_\mathfrak{p}$ where $\mathfrak{p} \in \mathrm{Max}(R)$. It generalizes [FrJ08, p. 446, Lemma 20.9.1].

{SAL}

Lemma 4.2. *If a sentence θ of* $\mathcal{L}(\mathrm{ring}, R)$ *is true in $\tilde{K}(\sigma)$ for almost all* $\sigma \in \mathrm{Gal}(K)$*, then θ is true in $\bar{K}_\mathfrak{p}$ for almost all* $\mathfrak{p} \in \mathrm{Max}(R)$.

Proof. By [FrJ08, p. 146, Prop. 7.8.1(a)], a sentence $\theta \in \mathcal{L}(\mathrm{ring}, R)$ is true in $\bar{K}_\mathfrak{p}$ for almost all $\mathfrak{p} \in \mathrm{Max}(R)$ if and only if θ is true in every regular ultraproduct of the $\bar{K}_\mathfrak{p}$'s. By Remark 3.3, every regular ultrafilter $\mathcal{D}$ on $\mathrm{Max}(R)$ is nonprincipal, hence the map $a \mapsto \bar{a}_\mathfrak{p}$, with $\mathfrak{p}$ ranging on $\mathrm{Max}(R)$, embeds R into the ultraproduct $F := \prod \bar{K}_\mathfrak{p}/\mathcal{D}$. In particular, if $a \in R$ and $a \neq 0$, then $\bar{a}_\mathfrak{p} \in \bar{K}_\mathfrak{p}^\times$ for almost all $\mathfrak{p} \in \mathrm{Max}(R)$.

By Lemma 1.8(a), {ulp1}

(14) for each positive integer m there are only finitely many $\mathfrak{p} \in \mathrm{Max}(R)$ such that $\mathrm{card}(\bar{K}_\mathfrak{p}) \leq m$.

Using that every finite subset of $\mathrm{Max}(R)$ is small, Lemma 4.1, and Łoš' theorem [FrJ08, p. 142, Prop. 7.7.1], we have that F is perfect, $\mathrm{Gal}(F) \cong \hat{\mathbb{Z}}$, and F is a PAC field.

Indeed, every finite field is perfect. By [FrJ08, p. 15, Section 1.5], $\mathrm{Gal}(\bar{K}_\mathfrak{p}) \cong \hat{\mathbb{Z}}$ for each $\mathfrak{p} \in \mathrm{Max}(R)$. Also, by [FrJ08, p. 105, Cor. 5.4.2] and (14), the following statement is true for all but finitely many $\mathfrak{p} \in \mathrm{Max}(R)$: {ulp2}

(15) For every absolutely irreducible polynomial $f \in \bar{K}_\mathfrak{p}[X,Y]$ of degree d there is a point $(x,y) \in \bar{K}_\mathfrak{p} \times \bar{K}_\mathfrak{p}$ with $f(x,y) = 0$.

Moreover, by (6), K is Hilbertian. Hence, by [FrJ08, p. 439, Thm. 20.5.4], θ is true in $\bar{K}_\mathfrak{p}$ for almost all $\mathfrak{p} \in \mathrm{Max}(R)$, as claimed. $\square$

We write $\mathrm{Almost}(K)$ for the theory of all $\theta \in \mathcal{L}(\mathrm{ring}, R)$ that are true in $\tilde{K}(\sigma)$ for almost all $\sigma \in \mathrm{Gal}(K)$. Likewise, we write $\mathrm{Almost}(R)$ for the theory of all $\theta \in \mathcal{L}(\mathrm{ring}, R)$ that are true in $\bar{K}_\mathfrak{p}$ for almost all $\mathfrak{p} \in \mathrm{Max}(R)$.

{TST}

Lemma 4.3. *For each sentence θ of $\mathcal{L}(\mathrm{ring}, R)$ there exists a test sentence λ satisfying the following condition:*

The sets $\Sigma_{\tilde{K}/K}(\theta)$ and $\Sigma_{\tilde{K}/K}(\lambda)$ differ only by a small subset of $\mathrm{Gal}(K)$, i.e. by a set of measure zero. Thus, the sentence $\theta \leftrightarrow \lambda$ belongs to $\mathrm{Almost}(K)$.

Proof. By (6), K is Hilbertian. Hence, [FrJ08, p. 442, Prop. 20.6.6] gives a test sentence λ in $\mathcal{L}(\mathrm{ring}, K)$ that satisfies the requirement of the lemma. Multiplying all of the polynomials that appear in λ by an appropriate nonzero element of R, we may assume that the coefficients of those polynomials are in R, so λ is a test sentence in $\mathcal{L}(\mathrm{ring}, R)$, as required. $\square$

Here is our main result:

{TRT}

Theorem 4.4 (The Generalized Transfer Theorem)**.** *Let θ be a sentence of $\mathcal{L}(\mathrm{ring}, R)$. Then, $\Sigma_{\tilde{K}/K}(\theta)$ is measurable, $A_R(\theta)$ has a Dirichlet density, and*

$$\delta(A_R(\theta)) = \mu_K(\Sigma_{\tilde{K}/K}(\theta)). \quad (16) \quad \{\texttt{mukd}\}$$

Moreover:
(a) $\delta(A_R(\theta))$ *is a rational number.*
(b) $\delta(A_R(\theta)) = 0$ *if and only if* $A_R(\theta)$ *is a small set.*
(c) $\delta(A_R(\theta))$ *depends only on* K.

Proof. Lemma 4.3 provides a test sentence λ in $\mathcal{L}(\mathrm{ring}, R)$ of the form (11) such that $\theta \leftrightarrow \lambda$ is true in $\tilde{K}(\sigma)$ for almost all $\sigma \in \mathrm{Gal}(K)$. By Lemma 4.2, $\theta \leftrightarrow \lambda$ is also true in $\bar{K}_{\mathfrak{p}}$ for almost all $\mathfrak{p} \in \mathrm{Max}(R)$. Hence, $\Sigma_{\tilde{K}/K}(\theta)$ and $\Sigma_{\tilde{K}/K}(\lambda)$ differ only by a subset of $\mathrm{Gal}(K)$ of measure 0 and $A_R(\theta)$ differs from $A_R(\lambda)$ only by a small set, i.e. a subset of $\mathrm{Max}(R)$ which is contained in a proper closed subset of $\mathrm{Spec}(R)$, so its Dirichlet density is zero (Remark 3.1). Therefore, it suffices to prove the theorem for λ rather than for θ.

Let L be the splitting field over K of the polynomial $f_1 \cdots f_m$, with $f_1, \ldots, f_m$ being the polynomials occurring in the definition (11) of the test sentence λ. Then L is a finite Galois extension of K, $\Sigma_{L/K}(\lambda) := \{\tau \in \mathrm{Gal}(L/K) \mid L(\tau) \models \lambda\}$ is a conjugacy domain of $\mathrm{Gal}(L/K)$, and $\Sigma_{\tilde{K}/K}(\lambda) = \{\sigma \in \mathrm{Gal}(K) \mid \mathrm{res}_L \sigma \in \Sigma_{L/K}(\lambda)\}$ (by (12)). By (13), $\mu_K(\Sigma_{\tilde{K}/K}(\lambda)) = \frac{\mathrm{card}(\Sigma_{L/K}(\lambda))}{[L:K]}$. By Lemma 3.9(c), $\delta(A_R(\lambda)) = \frac{\mathrm{card}(\Sigma_{L/K}(\lambda))}{[L:K]}$. Consequently, $\delta(A_R(\lambda))$ is a rational number, so (a) holds. Also, $\mu_K(\Sigma_{\tilde{K}/K}(\lambda)) = \delta(A_R(\lambda))$. This proves (16).

If $A_R(\theta)$ is a small set, then by Remark 3.1, $\delta(A_R(\theta)) = 0$. Conversely, if $\mu_K(\Sigma_{\tilde{K}/K}(\lambda)) = \delta(A_R(\lambda)) = 0$, then $\Sigma_{L/K}(\lambda) = \emptyset$. Hence, by Lemma 3.9(b), $B \cap A_R(\lambda) = \emptyset$, where B is the nonempty open subset of $\mathrm{Spec}(R)$ from Lemma 3.9. Thus, $A_R(\lambda)$ is a subset of $\mathrm{Max}(R)$ which is contained in the proper closed subset $\mathrm{Spec}(R) \smallsetminus B$ of $\mathrm{Spec}(R)$, so $A_R(\lambda)$ is a small set, as required. This proves (b).

Finally, $\mu_K(\Sigma_{\tilde{K}/K}(\theta))$ depends only on the quotient field K of R, hence so does $\delta(A_R(\theta))$, as stated in (c). $\square$

Here is a generalization of Theorem 4.4(c).

{IND}

Proposition 4.5. *Let $R \subseteq R'$ be generalized rings of integers and θ a sentence of $\mathcal{L}(\mathrm{ring}, R)$, viewed also as a sentence of $\mathcal{L}(\mathrm{ring}, R')$. Suppose that $K' := \mathrm{Quot}(R')$ is a regular extension of $K := \mathrm{Quot}(R)$. Then, $\delta(A_R(\theta)) = \delta(A_{R'}(\theta))$.*

Proof. Both K and K' are generalized global fields and K' is a regular extension of K. In particular, by (6), both K and K' are Hilbertian, so $(K, 1)$ and $(K', 1)$ are "Hilbertian pairs" in the sense of [FrJ08, p. 439].

By Theorem 4.4, $\delta(A_R(\theta)) = \mu_K(\Sigma_{\tilde{K}/K}(\theta))$ and $\delta(A_{R'}(\theta)) = \mu_{K'}(\Sigma_{\widetilde{K'}/K'}(\theta))$. By [FrJ08, p. 443, Thm. 20.7.1(c)], $\mu_K(\Sigma_{\tilde{K}/K}(\theta)) = \mu_{K'}(\sum_{\widetilde{K'}/K'}(\theta))$. Hence, $\delta(A_R(\theta)) = \delta(A_{R'}(\theta))$. □

{RGL}

Example 4.6. The regularity condition in Proposition 4.5 is essential. For example, let $R = \mathbb{Z}$ and $R' = \mathbb{Z}[\sqrt{2}]$. Then $\mathrm{Quot}(R) = \mathbb{Q}$ and $\mathrm{Quot}(R') = \mathbb{Q}(\sqrt{2})$. By [Lan70, p. 76, Thm. 5], R' is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{2})$.

Let θ be the sentence $(\exists X)[X^2 = 2]$. Then, $\mu_{\mathbb{Q}}(\Sigma_{\tilde{\mathbb{Q}}/\mathbb{Q}}(\theta)) = \frac{1}{2}$ but

$$\mu_{\mathbb{Q}(\sqrt{2})}\big(\Sigma_{\tilde{\mathbb{Q}}/\mathbb{Q}(\sqrt{2})}(\theta)\big) = 1.$$

The following result generalizes [FrJ08, p. 442, Thm. 20.6.7]. In this result we refer to the field K, its subring R introduced in the second paragraph of this section, and a sentence θ of $\mathcal{L}(\mathrm{ring}, R)$. As in [FrJ08, p. 440, Sec. 20.6], we speak about the **explicit case**, when all of these objects are "presented" in the sense of [FrJ08, p. 403–406, Sec. 19.1] and K has an **elimination theory** as defined in [FrJ08, p. 410, Def. 19.2.8].

{DCT}

Theorem 4.7 (Decidability Theorem)**.** *Let R be a generalized ring of integers, and $K := \mathrm{Quot}(R)$ the corresponding generalized global field. Let θ be a sentence of $\mathcal{L}(\mathrm{ring}, R)$. Then, in the explicit case, the rational number $\delta(A_R(\theta))$ can be recursively computed. Indeed, it can be even primitive recursively computed.*

Thus, $\mathrm{Th}(\mathrm{Almost}(R))$ *is recursive and even primitive recursive.*

Proof. By [FrJ08, p. 442, Thm. 20.6.7], the rational number $\mu_K(\Sigma_{\tilde{K}/K}(\theta))$ can be recursively computed. Since, by Theorem 4.4, $\delta(A_R(\theta)) = \mu_K(\Sigma_{\tilde{K}/K}(\theta))$, also $\delta(A_R(\theta))$ can be recursively computed. In particular, one can recursively decide whether $\mu_K(\Sigma_{\tilde{K}/K}(\theta)) = 1$. Hence, one can recursively decide whether $\delta(A_R(\theta)) = 1$, which by Theorem 4.4(b) happens if and only if $A_R(\theta)$ is large. Therefore, one can recursively decide whether θ holds in $\bar{K}_{\mathfrak{p}}$ for almost all $\mathfrak{p} \in \mathrm{Max}(R)$. Thus, $\mathrm{Th}(\mathrm{Almost}(R))$ is recursive.

Finally, by [FrJ08, p. 726, Thm. 30.7.2], again in the explicit case, the function $\mu_K(\Sigma_{\tilde{K}/K}(\theta))$ from sentences of $\mathcal{L}(\mathrm{ring}, K)$ to rational numbers is primitive recursive. Hence, as in the previous paragraph, $\delta(A_R(\theta))$ can be primitive recursively computed and $\mathrm{Th}(\mathrm{Almost}(R))$ is even primitive recursive. □