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An Analogue of Artin-Schreier Theorem

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Introduction

Let v be an absolute value of $\mathbb{Q}$, let $\hat{\mathbb{Q}}_v$ be the completion of $\mathbb{Q}$ with respect to v and let $\mathbb{Q}_v = \tilde{\mathbb{Q}} \cap \hat{\mathbb{Q}}_v$ be the algebraic subfield of $\hat{\mathbb{Q}}_v$. Then $\mathbb{Q}_v$ has the following two properties:

(A) $\mathbb{Q}_v$ *has no non-trivial automorphisms, in particular* $\mathbb{Q}_v$ *is a Galois extension of no proper subfield.*

(B) $\mathbb{Q}_v$ *has no proper subfield of a finite co-degree.*

(The *co-degree* of a subfield E of F is simply the degree of F over E.)

If v is archimedean, then $\mathbb{Q}_v = \tilde{\mathbb{Q}} \cap \mathbb{R}$, and (B) is a consequence of the famous (cf. Jacobson [5, p. 316]).

Artin-Schreier Theorem. *If a proper subfield of an algebraically closed field has a finite co-degree, then this co-degree is equal to* 2.

If v is non-archimedean, then $\mathbb{Q}_v = \tilde{\mathbb{Q}} \cap \hat{\mathbb{Q}}_p$, and (B) is a consequence of

F. K. Schmidt Theorem. *A field F which is not separably closed can be Henselian with respect to at most one rank-1 valuation (see* [14]*).*

Note that the Galois group $G(\mathbb{Q}_v) = \mathcal{G}(\tilde{\mathbb{Q}}_v/\mathbb{Q}_v)$ is finitely generated (cf. Jakovlev [6] and Zel'venskii [18]). Hence $\mathbb{Q}_v$ is of the form $\tilde{\mathbb{Q}}(\sigma)$, where $(\sigma) = (\sigma_1, \dots, \sigma_e) \in G(\mathbb{Q})^e$ and $\tilde{\mathbb{Q}}(\sigma)$ is the fixed field in $\tilde{\mathbb{Q}}$ of $\sigma_1, \dots, \sigma_e$. The e-tuples that appear as generators of $G(\mathbb{Q}_v)$, for all v, are "special", because, as was shown in [7, Theorem 2.5 and Lemma 2.9], they form only a zero set in $G(\mathbb{Q})^e$ with respect to the Haar measure μ (see [8, Sect. 4] for more details on the Haar measure of $G(\mathbb{Q})^e$). If we replace those special (σ)'s with arbitrary ones, then (A) and (B) may become false. Indeed take $E = \tilde{\mathbb{Q}}(\tau_1, \dots, \tau_d)$ for some $\tau_1, \dots, \tau_d \in G(\mathbb{Q})$ and let F be a proper finite Galois extension of E. Then $F = \tilde{\mathbb{Q}}(\sigma_1, \dots, \sigma_e)$ for some $\sigma_1, \dots, \sigma_e \in G(\mathbb{Q})$ and F has certainly none of the properties (A) and (B). By choosing $\tau_1, \dots, \tau_d$ and F appropriately, one can actually achieve every e in this way.

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In [8] it was however conjectured that those counter examples are exceptional. More precisely, the followsing conjecture was made.

If K is a Hilbertian field, then for almost all $(\sigma)\in G(K)^e$ *we have:*

(C) $K_s(\sigma)$ *is a Galois extension of no proper subfield that contains* K.

(D) $K_s(\sigma)$ *has no proper subfield of finite co-degree that contains* K.

In establishing this conjecture three weaker theorems were proved:

(E) *Let K be a global field. Then for almost all* $(\sigma)\in G(K)^e$, *the centralizer of* $\langle\sigma\rangle$ *in* $G(K)$ *is equal to* $\langle\sigma\rangle$ *if* $e=1$, *and is trivial if* $e\geqq 2$ [8, Theorem 14.1].

(F) *Let K be a Hilbertian field, then for almost all* $(\sigma)\in G(K)^e$, *the field* $K_s(\sigma)$ *contains no formally real subfield of finite co-degree that contains* K [8, Theorem 12.2].

(G) (D) *is true for* $e=1$ [8, Theorem 13.1].

In this work we make a further major step in proving the conjecture and prove:

(H) *Let K be a Hilbertian field. Then for almost all* $(\sigma)\in G(K)^e$ *the field* $K_s(\sigma)$ *is a Galois extension of no proper subfield of a finite co-degree that contains* K.

If K is a global field, then for almost all $(\sigma)\in G(K)^e$ *we have:*

(I) $K_s(\sigma)$ *is a Galois extension of no proper subfield that contains* K.

(J) *If E is a subfield of* $K_s(\sigma)$ *that contains K such that* $K_s(\sigma)/E$ *is a finite separable extension, then* $[K_s(\sigma):E]$ *divides* $e-1$.

(K) *If* $1\leqq e\leqq 5$, *then* $K_s(\sigma)$ *is a separable extension of no proper subfield of a finite co-degree that contains* K.

Note that (J) is an analogue of Artin-Schreier theorem. As a consequence of (I) we supply in Sect. 9 a counter example to an infinite analogue of Iwasawa-Uchida's theorem.

Notation

$\mathbb{Q}$	= the field of rational numbers.
$\mathbb{R}$	= the field of real numbers.
$\mathbb{C}$	= the field of complex numbers.
$\hat{\mathbb{Q}}_p$	= the field of p-adic numbers.
K_s	= the separable closure of a field K.
$\tilde{K}$	= the algebraic closure of K.
K_{ab}	= the maximal abelian of K.
$K^{(p)}$	= the maximal p-extension of K.
$N(\sigma)$	= the fixed field of e automorphisms $\sigma_1, \dots. \sigma_e$ of a field N.
$\mathrm{rank}(G)\leqq e$	= the pro-finite group G is generated by e elements.
$e_p(K)$	= $\mathrm{rank}(\mathcal{G}(K^{(p)}/K))$.
$\langle\sigma_1, \dots, \sigma_e\rangle$	= the closed subgroup generated by elements $\sigma_1, \dots, \sigma_e$ of G.
$\hat{F}_e$	= the free pro-finite group generated by e elements.
ζ_n	= a primitive n-th root of unity.

1. The Maximal p-Extension of a Field

For a field E and a prime p we denote by $E^{(p)}$ the maximal p-extension of E. The rank of $\mathcal{G}(E^{(p)}/E)$ is denoted by $e_p(E)$. If $E\subsetneqq F\subseteqq E^{(p)}$ is an intermediate field, then

$E^{(p)} = F^{(p)}$, since $E^{(p)}$ has no proper p-extensions. If E' is an algebraic extension of E which is linearly disjoint from $E^{(p)}$, then there is an epimorphism $\mathcal{G}(E'^{(p)}/E') \to \mathcal{G}(E^{(p)}/E)$ and hence $e_p(E') \geqq e_p(E)$.

Lemma 1.1. *Let p be a prime and let E be a field, which is not formally real if $p=2$. Then $\mathcal{G}(E^{(p)}/E)$ is a torsion free group.*

Proof. If $\mathcal{G}(E^{(p)}/E)$ contains an element of a finite order, then there exists a field $E \subsetneqq F \subset E^{(p)}$ such that $[E^{(p)}:F]=p$. This however contradicts Theorem 2 of Whaples[1] [17], which implies that $[F^{(p)}:F]=\infty$.

Lemma 1.1 will be used in order to satisfy one of the conditions of the following

Theorem of Serre. *If a torsion free pro-p group G contains an open free pro-p subgroup, then G is free too* (*see* [15, p. 413]).

A complement to Serre's theorem is:

Nielsen-Schreier Formula. *Let C be either the category of pro-finite groups or the category of pro-p groups. Let G be a free C-group of a finite rank and let H be an open subgroup of G. Then H is also free and*

$$\operatorname{rank}(H) - 1 = (G:H)(\operatorname{rank}(G) - 1)$$

(*see* [1, p. 108]).

2. Abelian Extensions of Hilbertian Fields

In [8, p. 286] it was proved that if K is a Hilbertian field, then $\langle \sigma_1, \ldots, \sigma_e \rangle \cong \hat{F}_e$ for almost all $(\sigma) \in G(K)^e$ (this is the "Free Generators Theorem"). In this section we prove the analogous result for the group $\mathcal{G}(K_{ab}/K)$.

Lemma 2.1.
Let K be a field and let t be a transcendental element over K. Then every finite Abelian group can be realized over $K(t)$.

Proof. If $\operatorname{char}(K) = p \neq 0$, then, by a result of Lenstra, every finite abelian group A can be realized over a finitely generated, purely transcendental extension of K, since the field generated over K by a root of unity is a cyclic extension of K (see [12, p. 322, Corollary 7.5]). By Hilbert irreducibility theorem, which $K(t)$ satisfies, we get that A can be realized also over $K(t)$.

For $\operatorname{char}(K)=0$, the lemma can be found in Frey [3].

Corollary 2.2. *If K is a Hilbertian field, then every finite abelian group can be realized over K.*

The corollary can be strengthened in the following way:

Theorem 2.3. *Let K be a Hilbertian field and let A be a finite abelian group. Then there exists a linearly disjoint sequence $K_1, K_2, K_3, \ldots$ of Galois extensions of K such that $\mathcal{G}(K_i/K) \cong A$ for every $i \geqq 1$.*

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1 The author is indebted to Wulf-Dieter Geyer for calling his attention to Whaples result

Theorem 2.3 is a consequence of Corollary 1.2 and the following general

Lemma 2.4. *Let K be a field and let G be a finite group. Suppose that the direct power G^n is realizable over K for every n. Then there exists a linearly disjoint sequence $K_1, K_2, K_3, \ldots$ of Galois extensions of K such that $\mathcal{G}(K_i/K) \cong G$ for every $i \geqq 1$.*

Proof. Suppose, by induction, that m linearly disjoint Galois extensions $K_1, \ldots, K_m$ of K have been constructed, such that $\mathcal{G}(K_i/K) \cong G$ for $i=1, \ldots, m$. The field $L = K_1 \ldots K_m$ has only a finite number, say n, of subfields that contain K. By assumption, there exists a Galois extension M of K such that $\mathcal{G}(M/K) \cong G^{n+1}$. M is therefore the composition of $n+1$ linearly disjoint Galois extensions $M_1, \ldots, M_{n+1}$ of K such that $\mathcal{G}(M_j/K) \cong G$. Among the M_j's there must be one such that $M_j \cap L = K$, since otherwise $M_1 \cap L, \ldots, M_{n+1} \cap L$ are $n+1$ distinct subfields of L that contain K, which is a contradiction to the definition of n. Define therefore K_{m+1} to be one of the M_j's for which $M_j \cap L = K$. Then $K_1, \ldots, K_m, K_{m+1}$ are linearly disjoint over K.

The sequence $K_1, K_2, K_3, \ldots$ thus constructed satisfies the conclusion of the lemma.

3. The Normalizer of $\langle \sigma_1, \ldots, \sigma_e \rangle$ in $G(K)$, for K Hilbertian

Lemma 3.1. *If K is a Hilbertian field, then for almost all $(\sigma) \in G(K)^e$, for all fields $K \subseteqq E \subsetneqq K_s(\sigma)$ and for all primes p we have $e_p(E) \geqq e$.*

Proof. It suffices to prove that for a given prime p, for almost all $(\sigma) \in G(K)^e$ and for all fields $K \subseteqq E \subsetneqq K_s(\sigma)$ we have $e_p(E) \geqq e$.

Indeed, by Theorem 2.3, there exists a linearly disjoint sequence $K_1, K_2, K_3, \ldots$ of Galois extensions of K such that $\mathcal{G}(K_i/K) \cong (\mathbb{Z}/p\mathbb{Z})^e$ for every $i \geqq 1$. Let $\sigma_{i1}, \ldots, \sigma_{ie}$ be generators of $\mathcal{G}(K_i/K)$ and let

$$S = \bigcup_{i=1}^{\infty} \{(\sigma) \in G(K)^e \mid \sigma_j | K_i = \sigma_{ij}, \quad \text{for} \quad j = 1, \ldots, e\}.$$

Then $\mu(S) = 1$, by [8, Lemma 4.1].

Suppose that $(\sigma) \in S$ and let $K \subseteqq E \subsetneqq K_s(\sigma)$ be an intermediate field. Let i be a positive integer such that $\sigma_j | K_i = \sigma_{ij}$, for $j = 1, \ldots, e$. Then $K_i \cap K_s(\sigma) = K$, hence $K_i \cap E = K$ too. It follows that $(\mathbb{Z}/p\mathbb{Z})^e$ is a homomorphic image of $\mathcal{G}(E^{(p)}/E)$. Hence $e_p(E) \geqq e$.

Theorem 3.2. *Let K be a Hilbertian field. Then for almost all $(\sigma) \in G(K)^e$ the field $K_s(\sigma)$ is a Galois extension of no proper subfield of a finite co-degree that contains K.*

Proof. Denote by S the set of all $(\sigma) \in G(K)^e$ that satisfy a) $\langle \sigma \rangle \cong \hat{F}_e$.

b) For all primes p and for all fields E between K and $K_s(\sigma)$ we have $e_p(E) \geqq e$.

c) $K_s(\sigma)$ contains no formally real subfield of a finite co-degree that contains K.

By the free generators theorem, by Lemma 3.1 and by (F) of the introduction, S has the measure 1.

Let $(\sigma) \in S$ and let $F = K_s(\sigma)$. Assume that there exists a field $K \subseteqq E \subsetneqq F$ such that F/E is a finite non trivial Galois extension. Let p be a prime divisor of $[F:E]$. By

Sylow's theorem there exists a field $E \subsetneqq E_1 \subset F$ such that F/E_1 is a Galois extension of degree p. Without loss of generality we can assume that $E_1 = E$.

The group $\mathscr{G}(E^{(p)}/E)$ is a torsion-free p-group, by Lemma 1.1. It contains the free pro-p group $\mathscr{G}(E^{(p)}/F)$ of rank e and of index p as a closed subgroup. Hence, by the theorem of Serre $\mathscr{G}(E^{(p)}/E)$ is a free pro-p group. The rank $e_p(E)$ satisfies the Nielsen-Schreier Formula $(e-1)=p(e_p(E)-1)$. Hence $e>e_p(E)$, which is a contradiction to b).

4. The Maximal p-Extension of Fields Underneath $K_s(\sigma)$

Having proved theorem (H) for arbitrary Hilbertian fields, we turn now to the proofs of Theorems (I), (J), and (K) for global fields K. In this section we consider fields $K \subseteqq E \subseteqq K_s(\sigma)$ and give sufficient conditions for $\mathscr{G}(E^{(p)}/E)$ to be free. We shall use results from local class field theory. They are incorporated in the following lemma, which is a combination of Theorem 9.1, 9.3, and 9.7 of Koch [9].

Lemma 4.1. *Let E be an algebraic extension of a global field K. Suppose that for every non archimedean absolute value v of E and for every prime number l, the degree $[E\hat{K}_v : \hat{K}_v]$ is divisible by l^∞. Suppose further that E is not formally real. Then $\mathscr{G}(E^{(p)}/E)$ is a free pro-p group for every prime p.*

The condition "$l^\infty | [E\hat{K}_v : \hat{K}_v]$" is certainly satisfied if $\hat{E}_v$ is algebraically closed, because then $E\hat{K}_v$ contains the separable closure of $\hat{K}_v$. We give here an additional sufficient condition for the condition to be true.

Lemma 4.2. *Let M be a non-archimedean local field and let $\tau \in G(M)$. Then l^∞ divides $[M_s(\tau):M]$ for every prime l.*

Proof (Neukirch). Let l be a prime such that l^∞ does not divide $[M_s(\tau):M]$. Without loss of generality we can assume that $\zeta_l \in M$. Our assumption implies that $N = M_s(\tau) \cap M^{(l)}$ is a finite extension of M. Further $M^{(l)} = N^{(l)}$ and $\mathscr{G}(N^{(l)}/N)$ is a pro-cyclic group. This however contradicts Theorems 10.3 and 10.4 of Koch [9], according to which the rank of $\mathscr{G}(L^{(l)}/L)$ is at least 2.

Lemma 4.3. *Let K be a global field. Then for almost all $(\sigma) \in G(K)^e$, the field $K_s(\sigma)$ has the following property: Suppose that $K_s(\sigma)$ is an algebraic separable extension of a field E that contains K such that $K_s(\sigma)/E$ is either finite or a pro-cyclic extension. Then for every algebraic extension E' of E and every prime p, the group $\mathscr{G}(E'^{(p)}/E')$ is pro-p free.*

Proof. Denote by S the set of all $(\sigma) \in G(K)^e$ with the following properties:

a) The completion of $K_s(\sigma)$ under every absolute value is algebraically closed.

b) There does not exist a field $K \subseteqq E \subsetneqq K_s(\sigma)$ of finite co-degree which is formally real.

By [2, Lemma 5.3] and by (F), S has measure 1.

Let $(\sigma) \in S$ and let $F = K_s(\sigma)$. Let E, E' be fields such that $K \subseteqq E \subseteqq F$, such that F/E is either finite or a pro-cyclic extension and such that E' is an algebraic extension of E. Then E and hence E' satisfies the conditions of Lemma 4.1 by a), b), by Artin-Schreier theorem and by Lemma 4.2. It follows that $\mathscr{G}(E'^{(p)}/E')$ is free for every prime p.

5. The Trivial Normalizer Theorem, for K global

We recall that a non trivial pro-p group G is free if and only if $\mathrm{cd}(G)=1$ (cf. Ribes [13, p. 235]).

Lemma 5.1. *Let G be a pro-p group and let H be a normal closed subgroup of G. Suppose that both H and G/H are non trivial free pro-p groups and H is finitely generated. Then G is not free.*

Proof. Our assumptions imply that $\mathrm{cd}(H)=\mathrm{cd}(G/H)=1$. Further $H^1(H, F_p)$ is finite. It follows that

$$\mathrm{cd}(G)=\mathrm{cd}(H)+\mathrm{cd}(G/H)=2$$

(cf. Ribes [13, p. 221]). Hence G is not free.

Theorem 5.2. *If K is a global field, then for almost all $(\sigma)\in G(K)^e$ the field $K_s(\sigma)$ is a Galois extension of no proper subfield that contains K, i.e. $\langle\sigma\rangle$ is its own normalizer in $G(K)$.*

Proof. Denote by S the set of all $(\sigma)\in G(K)^e$ such that: a) $\langle\sigma\rangle\cong\hat{F}_e$.

b) If $E_1\subseteq K_s(\sigma)$ and $K_s(\sigma)/E_1$ is a pro-cyclic extension, then $\mathscr{G}(E_1^{(p)}/E_1)$ is a free pro-p group for every prime p.

c) $K_s(\sigma)$ is a Galois extension of no proper subfield of a finite co-degree that contains K.

By the free generators theorem, by Lemma 4.3 and by Theorem 3.2, S is of measure 1.

Let $(\sigma)\in S$ and let $F=K_s(\sigma)$. Assume that there exists a proper subfield E of F such that $\mathscr{G}(F/E)$ is Galois. By c) the group $\mathscr{G}(F/E)$ is torsion free. Hence, by Sylow theorem for pro-finite groups there exists a field $E\subseteq E_1\subset F$ such that $\mathscr{G}(F/E_1)\cong\hat{\mathbb{Z}}_p$ for some prime p. By a), $\mathscr{G}(F^{(p)}/F)$ is a non trivial free pro-p group. Hence $\mathscr{G}(E_1^{(p)}/E_1)$ cannot be free, by Lemma 5.1. This is however a contradiction to b).

6. On the Bottom Conjecture

We come now to the analogue of Artin-Schreier theorem.

Theorem 6.1. *Let K be a global field and let $e\geqq 2$. Then for almost all $(\sigma)\in G(K)^e$, the field $F=K_s(\sigma)$ has the following property:*

If F is a finite separable extension of a field E that contains K, then $[F:E]$ divides $e-1$.

Moreover, let F' be the Galois closure of F/E, let p be a prime and let q be the largest power of p that divides $[F':E]$. Then $q\leqq[F':F]$.

Proof. Denote by S the set of all $(\sigma)\in G(K)^e$ such that:

a) $\langle\sigma\rangle\cong\hat{F}_e$.

b) For every prime p and for every field $E\subseteq K_s(\sigma)$ such that $K_s(\sigma)/E$ is separable algebraic, $e_p(E)\geqq e$.

c) For every prime p and fields $E\subseteq E'$ such that $K_s(\sigma)$ is a finite separable extension of E and E' is an algebraic extension of E, the group $\mathscr{G}(E'^{(p)}/E')$ is free.

Then S has measure 1, by the free generators theorem, by Lemma 3.1, and by Lemma 4.3.

Let $(\sigma)\in S$ and let F, E, and F' be as in the theorem. Let further p be a prime and let $E_1=F\cap E^{(p)}$. Then $e_p(E_1)\leqq e$, since $\mathcal{G}(E^{(p)}/E_1)$ is a homomorphic image of $\mathcal{G}(F^{(p)}/F)$. On the other hand we have, by b), that $E_p(E_1)\geqq e$. Hence $e_p(E_1)=e$. Now $\mathcal{G}(E^{(p)}/E_1)$ is a closed subgroup of the free pro-p group $\mathcal{G}(E^{(p)}/E)$, hence $e-1=[E_1:E](e_p(E)-1)$, by Nielsen-Schreier formula. An additional use of b) implies $e_p(E)\geqq e$. It follows that $e_p(E)=e$ and that $[E_1:E]=1$, i.e. $F\cap E^{(p)}=E$.

Denote by p^i and p^j the largest powers of p that divide $[F:E]$ and $[F':F]$, respectively, and let $q=p^{i+j}$. By Sylow's theorem there exists a field $E\subseteqq E'\subseteqq F'$ such that $[F':E']=q$. The degree $[E':E]$ is prime to p, hence E' is linearly disjoint from $E^{(p)}$ over E, hence

$$e_p(E')\geqq e. \tag{1}$$

Also F'/E' is a p-extension and $\mathcal{G}(E'^{(p)}/E')$ is a free pro-p group, by c). Hence

$$e_p(F')-1=q(e_p(E')-1). \tag{2}$$

Another application of Nielsen-Schreier formula gives

$$\operatorname{rank}(G(F'))-1=[F':F](e-1),$$

and since $e_p(F')=\operatorname{rank}(G(F'))$ we obtain

$$e_p(F')-1=[F':F](e-1). \tag{3}$$

Using (1)–(3) we get that $q\leqq[F':F]$ and that p^i divides $e-1$. Since this is true for every p we have that $[F:E]$ divides $e-1$.

Our last main result is the proof of the bottom conjecture in some cases.

Corollary 6.2. *Let K be a global field and let $1\leqq e\leqq 5$. Then for almost all $(\sigma)\in G(K)^e$ the field $K_s(\sigma)$ is a separable extension of no proper subfield E of a finite co-degree that contains K.*

Proof. The corollary is true for $e=1$, by (G) and suppose therefore that $e\geqq 2$. Use the notation of Theorem 6.1, let $(\sigma)\in S$ and $F=K_s(\sigma)$. Assume that there exists a field $E\subsetneqq F$ such that F/E is a finite proper separable extension.

If $e=2$, then $[F:E]$ divides 1, which is a contradiction.

If $e=3$, then $[F:E]=2$. Hence F/E is Galois, hence $F'=F$, $q=2$ and $1\geqq 2$, a contradiction.

If $e=4$, then $[F:E]=3$ and $[F':F]$ divides 2. Hence $q=3$ and $2\geqq 3$, a contradiction.

Suppose that $e=5$. Then $[F:E]$ equals 2 or 4. The case $[F:E]=2$ gives a contradiction as in the case $e=3$. Suppose therefore that $[F:E]=4$. Then $[F':F]$ divides 6. If 2 divides $[F':F]$ then for $p=2$ we have that $q=8$, hence $6\geqq 8$, a contradiction. Otherwise $q=4$, hence $3\geqq 4$, again a contradiction.

7. Infinite Counter Examples to Iwasawa-Uchida's Theorem

Iwasawa and Uchida independently proved in [4] and [16] the following

Theorem. *Let K and L be two number fields and let $\alpha: G(K)\to G(L)$ be an isomorphism of their Galois groups. Then α is induced by an inner automorphism of $G(\mathbb{Q})$. In particular K is isomorphic to L.*

Our first counter example shows that the theorem does not remain true if the condition "K and L are number fields" is replaced by "K and L are algebraic extensions of $\mathbb{Q}$". Indeed, by Corollary 7.2 of [8] there exists a subset S of $G(\mathbb{Q})^e$ of cardinality $2^{\aleph_0}$ such that $\langle\sigma\rangle \cong \hat{F}_e$ for every $(\sigma)\in S$, but $\tilde{\mathbb{Q}}(\sigma) \not\cong \tilde{\mathbb{Q}}(\sigma')$ for every two distinct e-tuples (σ) and (σ') in S.

A consequence of Iwasawa-Uchida's theorem is the following

Corollary. *Let K be a number field. Then $G(K)$ is a complete group if and only if* $\operatorname{Aut} K$ *is a trivial group.*

Now, Theorem 5.2 can be rephrased for $\mathbb{Q}$ as:

Theorem 7.1. *The group* $\operatorname{Aut}\tilde{\mathbb{Q}}(\sigma)$ *is trivial for almost all* $(\sigma)\in G(\mathbb{Q})^e$.

Consider therefore, for $e \geqq 2$, a $(\sigma)\in G(\mathbb{Q})^e$ such that $\langle\sigma\rangle \cong \hat{F}_e$ and such that $\operatorname{Aut}\tilde{\mathbb{Q}}(\sigma)$ is trivial. It is known that $\hat{F}_e$ has a trivial center (cf. [8, Theorem 16.1]), but $\hat{F}_e$ is not a complete group, since it has automorphisms which are not inner. For example, if $z_1, \ldots, z_e$ are generators of $\hat{F}_e$, then the automorphism induced by the map $(z_1, \ldots, z_e) \mapsto (z_1^{-1}, \ldots, z_e^{-1})$ is not inner. It follows that the corollary is false if K is replaced by $\tilde{\mathbb{Q}}(\sigma)$.

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