TRANSFER PRINCIPLES FOR FINITE AND P-ADIC FIELDS

M. JARDEN

1. INTRODUCTION

The aim of this work is to give a new proof to the transfer principle for finite fields proved in [9] and also to add an analogous principle for the p-adic fields based on Ax-Kochen theorems [2 and 13]. The new proof is taken from the point of view of the space of superprimes developed in [11] by Jehne and Klingen and enriches the principle by a third point. In the process of proof it is shown that the generalized Artin map proved by Jehne and Klingen to be continuous and surjective, is also measurable.

The basic objects we consider are a global field K and its ring of integers R (e.g. $K = \mathbb{Q}$ and $R = \mathbb{Z}$). We attach to them two families of fields,

$$\{\bar{K}_{\mathfrak{p}} \mid \mathfrak{p} \in P(K)\} \text{ and } \{\tilde{K}(\sigma) \mid \sigma \in G(K)\}.$$

Here P(K) is the set of finite primes of K and $\bar{K}_{\mathfrak{p}} = R/\mathfrak{P}$ are the corresponding finite residue fields. The notation G(K) stands for the absolute Galois group $\mathcal{G}(K_s/K)$ of the field K, and $\tilde{K}(\sigma)$ is the fixed field of σ in the algebraic closure $\tilde{K}$ of K. The set P(K) is equipped with the Dirichlet density d. It is defined for a subset A of P(K) as the limit (if it exists)

$$d(A) = \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in A} |\bar{K}_{\mathfrak{p}}|^{-s}}{-\log(s-1)} .$$

The compact group G(K) is equipped with the normalized Haar measure

μ. It is defined for every Borel subset of $G(K)$ and is invariant under translations, i.e. $\mu(\sigma S) = \mu(S)$ for every $\sigma \in G(K)$ and a measurable set S. In particular we have $\mu(G(L)) = [L : K]^{-1}$, if L is a finite separable extension of K. We speak about these fields with the language $\bar{L}(R)$ that consists of the usual language of the theory of fields augmented by constants for the elements of R. Every sentence of $\bar{L}(R)$ is equivalent to a sentence of the form

$$(Q_1 X_1)\dots(Q_n X_n) \bigvee_i \bigwedge_j f_{ij}(\underline{X}) = 0 \wedge g_{ij}(\underline{X}) \neq 0,$$

where each of the Q_i is either the existential quantifier $\exists$ or the universal quantifier $\forall$, and $f_{ij}, g_{ij} \in R[X_1,\dots,X_n]$. In $\bar{K}_{\mathcal{P}}$ this sentence is interpreted by reducing the coefficients of f_{ij} and g_{ij} modulo $\mathcal{P}$. Therefore the following definitions make sense

$$\bar{A}(\theta) = \{\mathcal{P} \in P(K) \mid \bar{K}_{\mathcal{P}} \models \theta\} \text{ and } \bar{S}(\theta) = \{\sigma \in G(\tilde{K}) \mid \tilde{K}(\sigma) \models \theta\}.$$

Here '$\bar{K}_{\mathcal{P}} \models \theta$' means that "$\theta$ is true in $\bar{K}_{\mathcal{P}}$". The transfer principle for finite fields says that $d(\bar{A}(\theta)) = \mu(\bar{S}(\theta))$. For example, if θ is the sentence $(\exists X)[X^2 = 2]$ and $R = \mathbb{Z}$, then

$$\bar{A}(\theta) = \{p \in P(\mathbb{Q}) \mid p \equiv 1,7 \pmod 8\} \cup \{2\} \text{ and}$$
$$\bar{S}(\theta) = G(\mathbb{Q}(\sqrt{2})).$$

Hence $d(\bar{A}(\theta)) = \frac{1}{2}$, by Dirichlet's theorem and $\mu(\bar{S}(\theta)) = \frac{1}{2}$ by the above formula. The general case was reduced in [9] to the case where θ is a boolean combination of sentences similar to the one appearing in the example, and have the form $(\exists X)[f(X) = 0]$, where $f \in R[X]$. This method goes back to Ax [1], where he used it to establish the decidability of the elementary theory of finite fields.

In this work we go in another direction and use first non-standard methods in order to extend the Dirichlet density d of $P(K)$ to a finitely additive function d' on the family of *all* subsets of $P(K)$. It is proved that if A is a subset of $P(K)$ having no density, then there are two extensions d_1' and d_2' of d such that $d_1'(A) \neq d_2'(A)$.

Next we consider the compact space Ω_K of superprimes of $P(K)$

consisting of all non-principal ultrafilters of P(K) and the corresponding ultraproducts $\bar{K}_{\mathcal{D}} = \Pi\bar{K}_p/\mathcal{D}$. It should be noted that ultraproducts play a central role in our work as well as in its references (e.g. [1], [9], [10], [11] etc.) Therefore it deserves a suitable definition.

Let I be a set and let $\mathcal{D}$ be a family of subsets of I. Then $\mathcal{D}$ is said to be an *ultrafilter* of I if it satisfies the following conditions: a) $I \in \mathcal{D}$ and $\phi \notin \mathcal{D}$; b) if $A,B \in \mathcal{D}$; then $A \cap B \in \mathcal{D}$; c) if $A \in \mathcal{D}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{D}$; d) if $A \subseteq I$, then either $A \in D$ or $I - A \in \mathcal{D}$. An example of an ultrafilter is obtained by considering an element $a \in I$ and defining $\mathcal{D}$ to be the family of all subsets of I that contain a. An ultrafilter of this type is said to be *principal*. Other examples of ultra-filters are unfortunately not as constructible as the principal ones. They are obtained by using Zorn's Lemma. Indeed, suppose that $\mathcal{D}_0$ is a nonempty family of subsets of I with the property: if $A,B \in \mathcal{D}_0$, then there exists a $C \subseteq I$ such that $C \subseteq A \cap B$. Then, by Zorn's Lemma there exist maximal families $\mathcal{D}$ that contain $\mathcal{D}_0$ and have this property. Each one of them is an ultra-filter of I.

Let now $\mathcal{D}$ be an ultrafilter of I and suppose that for each $i \in I$ we are given a field F_i. We introduce an equivalence relation $\sim$ to the cartesian product $\prod_{i \in I} F_i$ in the following way:

$$f \sim g \iff \{i \in I \mid f(i) = g(i)\} \in \mathcal{D}.$$

The set of all equivalence classes of ΠF_i modulo this relation is called, *the ultraproduct of the* F_i's *modulo* $\mathcal{D}$, and is denoted by $F = \Pi F_i/\mathcal{D}$. Addition and multiplication are defined in F component-wise and it turns out that F becomes a field.

In particular if $I = P(K)$, and $\mathcal{D} \in \Omega_K$ is a non-principal ultrafilter of P(K), then $\bar{K}_{\mathcal{D}} = \Pi\bar{K}_p/\mathcal{D}$ is a field that contains K. The fundamental theorem of ultraproducts would say that if θ is a sentence of $\bar{L}(R)$, then $\bar{K}_{\mathcal{D}} \models \theta$ if and only if $A(\theta) \in \mathcal{D}$.

Now, Jehne and Klingen considered in [9] the compact topology on Ω_K whose basis was the family of all subsets $\Omega(A) = \{\mathcal{D} \in \Omega_K \mid A \in \mathcal{D}\}$, where $A \subseteq P(K)$. By the preceeding paragraph we have $\Omega(\theta) = \Omega(\bar{A}(\theta)) = \{\mathcal{D} \in \Omega_K \mid \bar{K}_{\mathcal{D}} \models \theta\}$. Then we show that every extension d' of the Dirichlet density d can be uniquely lifted to a regular Borel measure δ of Ω_K such that $\delta(\bar{\Omega}(A)) = d'(A)$. In particular we have $d'(A(\theta)) =$

$\delta(\bar{\Omega}(A))$.

As $\bar{K}_{\mathcal{D}}$ is an extension of K we can intersect it with the algebraic closure $\tilde{K}$ of K. It turns out that there exists a $\sigma \in G(K)$ such that $\tilde{K} \cap \bar{K}_{\mathcal{D}} \cong_K \tilde{K}(\sigma)$. This σ is uniquely defined in G(K) up to a conjugation. Jehne and Klingen defined therefore σ (more accurately, its conjugacy class) as the value of $\mathcal{D}$ under the generalized Artin map $\phi : \Omega_K \to G(K)$. This name is justified by the fact that if L is a finite Galois extension of K and

$$A = \{\mathfrak{p} \in P(K) \mid \mathrm{Res}_L \sigma \in (\frac{L/K}{\mathfrak{p}})\},$$

where $(\frac{L/K}{\mathfrak{p}})$ is the usual Artin symbol, then $A \in \mathcal{D}$. Ax proved in [1] that ϕ is surjective and that $\phi^{-1}(\bar{S}(\theta)) = \bar{\Omega}(\theta)$. Jehne and Klingen proved that ϕ is continuous. In this work we show that ϕ is measurable and that $\bar{S}(\theta)$ is a Borel subset of G(K). Hence $d'(A(\theta)) = \delta(\bar{\Omega}(\theta)) = \mu(\bar{S}(\theta))$. The right-hand-side does not depend on the extension d'. Hence $\bar{A}(\theta)$ has a Dirichlet density and we have

$$d(\bar{A}(\theta)) = \delta(\bar{\Omega}(\theta)) = \mu(\bar{S}(\theta)).$$

This is our strengthened transfer principle for finite fields.

The new theorem proved in this work concerns the language $L_v(R)$ of the theory of valued fields with constant symbols for the elements of R. This language can be used in order to speak about properties of valued fields, concerning zeros of polynomials and elements of the field having certain values. For example, a special case of Hensel's Lemma can be written as the following sentence of $L_v(R)$.

$$(\exists X)[v(X) \geq 0 \wedge v(f(X)) > 0 \wedge v(f'(X)) = 0] \to$$
$$(\exists Y)[f(Y) = 0 \wedge v(X-Y) > 0],$$

here $f \in R[X]$ and it is assumed that the elements of R have non-negative values.

The basic result about valued fields we are going to use is a theorem of Ax and Kochem that can be reformulated as follows: (see [2], [13] and [15]:

Let E and F be two Hensel valued fields with residue fields $\bar{E}$ and $\bar{F}$, and with value groups Γ and Δ. Suppose that $\mathrm{char}(\bar{E}) = 0$. If $\bar{E}$ is elementarily equivalent to $\bar{F}$, as fields, and Γ is elementarily equivalent to Δ as ordered groups, then E is elementarily equivalent to F as valued fields.

In view of this theorem we assume that K is now a number field. We introduce the following notation: If $p \in P(K)$, then K_p denotes the completion of K at p. If $\mathcal{D} \in \Omega_K$, then $K_{\mathcal{D}} = \Pi K_p/\mathcal{D}$. It is a Hensel valued field with $\bar{K}_{\mathcal{D}}$ as the residue field and hence, the above theorem can be applied. We also denote by K((t)) the field of power series with coefficients in K. Every element x of its algebraic closure $\widetilde{K((t))}$ has a Puiseux expansion $\sum_{i=m}^{\infty} a_i t^{i/n}$ with $a_i \in \tilde{K}$ and n a positive integer. If $\sigma \in G(K)$, then σ is extended to $\widetilde{K((t))}$ by acting on the coefficients of the Puiseux expansion. Then, for $L = K(t)$ we have that $G(\tilde{L} \cap \tilde{K}(\sigma)((t)))$ is generated by σ and an additional element, say τ (see Fried [5, p.0.9]), thus $\tilde{L} \cap \tilde{K}(\sigma)((t)) = \tilde{L}(\sigma,\tau)$. Using methods of Kochen [13] and Kiehne [12] we show that for almost all $\sigma \in G(K)$, the field $\tilde{L}(\sigma,\tau)$ is an elementary valued-subfield of $K(\sigma)((t))$ and if σ corresponds to $\mathcal{D}$ under the generalized Artin map, then $\tilde{K}(\sigma)((t))$ is an elementary valued subfield of $K_{\mathcal{D}}$. If θ is a sentence of $\mathcal{L}_v(R)$ we denote

$$A(\theta) = \{p \in P(K) \mid K_p \models \theta\} \qquad \Omega(\theta) = \{\mathcal{D} \in \Omega_K \mid K_{\mathcal{D}} \models \theta\}$$

$$S(\theta) = \{\sigma \in G(K) \mid \tilde{K}(\sigma)((t)) \models \theta\} \qquad T(\theta) = \{\sigma \in G(K) \mid \tilde{L}(\sigma,\tau) \models \theta\}$$

Then, using the method of the proof of the transfer principle for finite fields we prove:

$$d(A(\theta)) = \delta(\Omega(\theta)) = \mu(S(\theta)) = \mu(T(\theta)).$$

This is the transfer principle for p-adic fields.

Acknowledgement: The author is indebted to Jonatan Stavi for bringing the results of Section 1 to his attention.

1. AN EXTENSION OF A DENSITY FUNCTION

Let E be a set. Denote by $\mathcal{P} = \mathcal{P}(E)$ its powers set, i.e. the collection of all subsets of E. Suppose that for every positive integer n we have a finitely additive measure on E, i.e. a function $d_n: \mathcal{P} \to \mathbb{R}$ such that

a) $d_n(\phi) = 0$; $d_n(E) = 1$

b) $A \cap B = \phi \Rightarrow d_n(A \cup B) = d_n(A) + d_n(B)$

c) $0 \le d_n(A) \le 1$

Suppose also that

d) if A is a finite set, then $\lim_{n\to\infty} d_n(A) = 0$.

Define a *density function* d on E by the limit

$$d(A) = \lim_{n\to\infty} d_n(A)$$

(if it exists). Then d has the following properties

A) $d(A) = 0$ for every finite set A; $d(E) = 1$.

B) If A and B are disjoint sets having density, then $A \cup B$ has density as well and we have $d(A \cup B) = d(A) + d(B)$

C) $0 \le d(A) \le 1$ for every set A with a density.

Let now $\mathcal{D}$ be a non-principal ultrafilter of $\mathbb{N}$ and consider the ultrapowers and ultraproducts ${}^*\mathcal{P} = \mathcal{P}^N/\mathcal{D}$ and ${}^*\mathbb{R} = \mathbb{R}^N/\mathcal{D}$ and ${}^*d = \prod d_n/\mathcal{D}$. Then ${}^*\mathcal{P}$ is a boolean algebra that extends $\mathcal{P}$, the field ${}^*\mathbb{R}$ extends $\mathbb{R}$ and ${}^*d: {}^*\mathcal{P} \to {}^*\mathbb{R}$ is a finitely additive function.

Let ${}^*\mathbb{R}_{fin} = \{\alpha \in {}^*\mathbb{R} \mid \exists\, n \in N: |\alpha| < n\}$ and ${}^*\mathbb{R}_0 = \{\alpha \in {}^*\mathbb{R} \mid |\alpha| < \varepsilon$ for every positive $\varepsilon \in \mathbb{R}\}$ be the finite and the infinitesimal part of ${}^*\mathbb{R}$. Then ${}^*\mathbb{R}_{fin}$ is a subring of ${}^*\mathbb{R}$ having ${}^*\mathbb{R}_0$ as an ideal that happens to be the kernel of the surjective map $\theta: {}^*\mathbb{R}_{fin} \to \mathbb{R}$ that maps every $\alpha \in {}^*\mathbb{R}_{fin}$ to the corresponding Dedekind cut. Combining ${}^{\star}d$ with θ we have a finitely additive function d': $d': {}^{\star}\mathcal{P} \to \mathbb{R}$;

CLAIM: If $A \subseteq E$ has a density, then $d'(A) = d(A)$.

Indeed, for every positive $\varepsilon \in \mathbb{R}$ the set $\{n \in \mathbb{N} \mid |d_n(A)-d(A)|<\varepsilon\}$ is cofinite, hence it belongs to $\mathcal{D}$. It follows that $|{}^{\star}d(A)-d(A)|<\varepsilon$. Since this inequality holds for every $\varepsilon > 0$ we have ${}^{\star}d(A) - d(A) \in \operatorname{Ker} \theta$. Hence $d'(A) = d(A)$.

If on the other hand, a subset B of E has no density, then the sequence $\{\hat{d}_n(B)\}_{n=1}^{\infty}$ does not converge. It contains therefore two subsequences $\{d_n(B) \mid n \in M_i\}$, that converge to β_i, $i = 1,2$, respectively, such that $\beta_1 \neq \beta_2$. For each i there exists a nonprincipal ultrafilter $\mathcal{D}_i$ that contains M_i. Let d_i' be the corresponding extension of d. Then one proves as before that $d_i'(B) = \beta_i$. Hence $d_1'(B) \neq d_2'(B)$.

All these results can be combined in the following

THEOREM 1.1. *Let* E *be a set, let* $\{d_n \mid n \in \alpha\}$ *be a sequence of finitely additive measure on* E *and let* d *be the density function on* E *defined by this sequence. Then* d *can be extended to a finitely additive measure,* $d': \mathcal{P}(E) \to \mathbb{R}$, *on* E. *If* B *is a subset of* E *without a density, then there are two extensions* d_1', d_2' *of* d *such that* $d_1'(B) \neq d_2'(B)$.

Clearly Theorem 1.1 implies an analogous theorem for a family of finitely additive measures indexed by a real parameter:

THEOREM 1.2. *Let* E *be a set and suppose that for every real* $s = 1$ $d_s: \mathcal{P}(E) \to \mathbb{R}$ *is a finitely additive measure such that* $\lim_{s\to 1^+} d_s(A) = 0$ *for every finite subset* A *of* E. *Then passing to the limit,* $d(A) = \lim_{s\to 1^+} d_s(A)$ *defines a density function on* E. *It can be extended to a finitely additive measure* $d': \mathcal{P}(E) \to \mathbb{R}$. *If* R *is a subset of* E *without a density, then there exist two extensions* d_1' *and* d_2' *of* d *such that* $d_1'(B) \neq d_2'(B)$.

2. THE SPACE OF NON-PRINCIPAL ULTRAFILTERS OF A SET

Let E be a set. Denote by Ω_E the set of all non-principal ultrafilters of E. For every subset D of E let $\Omega(D) = \Omega_E(D) = \{\mathcal{D} \in \Omega_E \mid D \in \mathcal{D}\}$.

Then the family $\{\Omega(D) | D \subseteq E\}$ forms a basis for a Hausdorff, compact and totally disconnected topology on Ω_E (cf. Jehne and Klingen [11, p. 210]). Note that the map $D \to \Omega_E(D)$ preserves the operations of a union, an intersection and taking a complement.

Let now d' be a finitely additive measure on E in the sense of Section 1 with the additional assumption that $d'(A) = 0$ for every finite subset $A \subseteq E$. Define a function δ on the class of all compact (= closed) subsets of Ω_E by

$$\delta(\Gamma) = \mathrm{Inf}\{d'(D) | D \subseteq E \text{ and } \Gamma \subseteq \Omega(D)\}$$

then δ has the following properties

1) $0 \leq \delta(\Gamma) \leq 1$

2) $\Gamma_1 \subseteq \Gamma_2 \Rightarrow \delta(\Gamma_1) \leq \delta(\Gamma_2)$

3) $\delta(\Gamma_1 \cap \Gamma_2) \leq \delta(\Gamma_1) + \delta(\Gamma_2)$

4) $\Gamma_1 \cap \Gamma_2 = \emptyset \Rightarrow \delta(\Gamma_1 \cup \Gamma_2) = \delta(\Gamma_1) + \delta(\Gamma_2)$.

PROOF. We have only to prove the inverse inequality to 3) in the case $\Gamma_1 \cap \Gamma_2 = \emptyset$. Indeed, Ω_E as a compact Hausdorff space is normal, hence there exist two disjoint open subsets Δ_1 and Δ_2 such that $\Gamma_i \subseteq \Delta_i$ for $i = 1,2$. Further $\Delta_i = \bigcup_\alpha \Omega_E(D_{i\alpha})$ for a family $\{D_{i\alpha}\}$ of subsets of E. Since Γ_1 is compact it can be covered by a finite subfamily $\Gamma_i \subseteq \bigcup_{\alpha=1}^{n_i} \Omega_E(D_{i\alpha})$ hence $\Gamma_i \subseteq \Omega_E(D_i)$, where $D_i = \bigcup_{\alpha=1}^{n_i} D_{i\alpha}$. Clearly $D_1 \cap D_2$ is a finite set. If $D \subseteq E$ and $\Gamma_1 \cup \Gamma_2 \subseteq \Omega(D)$, then

$$\delta(\Gamma_1) + \delta(\Gamma_2) \leq d'(D \cap D_1) + d'(D \cap D_2) =$$
$$d'(D \cap (D_1 \cup D_2)) \leq d'(D)$$

Hence

$$\delta(\Gamma_1) + \delta(\Gamma_2) \leq \delta(\Gamma_1 \cup \Gamma_2).$$

All this means that δ is a *content* function in the sense of Halmos [7,p.23]. It has the additional properties

5) $D \subseteq E \Rightarrow \delta(\Omega(D)) = d'(D)$.

This follows since $\Omega(D) \subseteq \Omega(D')$ is equivalent to $D \underset{\sim}{\subseteq} D'$, i.e. D is contained in D up to a finite set. In particular it follows that

6) $\delta(\phi) = 0$ and $\delta(\Omega_E) = 1$.

7) The function δ is a *regular* content, i.e. for every compact subset Γ of Ω_E we have $\delta(\Gamma) = \mathrm{Inf}\{\delta(\Delta) \mid \Gamma \subseteq \Delta^o$ and Δ is compact in $\Omega_E\}$ where Δ^o denotes the interior of Δ.

This follows from 5), since the sets $\Omega(D)$ are both compact and open in Ω_E.

We conclude, by Halmos [7, p.234, Thm. E and p.237 Thms. A and B] that:

THEOREM 2.1. *Let* E *be a set, let* d' *be a finitely additive measure on* E *that vanishes on finite sets. Then there exists a unique regular Borel measure* δ *on* Ω_E *such that* $\delta(\Omega_E(D)) = d'(D)$ *for every subset* D *of* E.

3. A MEASURE ON THE SPACE OF THE SUPERPRIMES OF A GLOBAL FIELD

Let K be a global field and denote by P = P(K) the set of finite primes of K. For every $p \in P$ let $\bar{K}_p$ and K_p be the residue field and the completion of K with respect to p. Denote further by $\Omega_K = \Omega_{P(K)}$ the compact space of the non-principal ultrafilters of P(K). (They are also known as the *superprimes* of K, cf. [11, p.209]). For every real s > 1 the series $\sum_{p \in P} (Np)^{-s}$ converges, hence $d_s(A) = \sum_{p \in A} (Np)^{-s} / \sum_{p \in P} (Np)^{-s}$ is a finitely additive measure on P in the sense of Theorem 1.2. the limit $d(A) = \lim_{s \to 1^+} d_s(A)$ defines a density function d of P, known as the Dirichlet Density. By Theorem 1.2. d can be extended to a finitely additive measure $d': \mathcal{P}(P) \to \mathbb{R}$ that vanishes on finite sets. By Theorem 2.1, d' can be uniquely lifted to a regular Borel measure δ on Ω_K such that $\delta(\Omega(D)) = d'(D)$ for every subset D of P. We call δ *an extension of the Dirichlet density*.

It is known (cf. [11, section 3]) that to every $\mathcal{D} \in \Omega_k$ and for every Galios extension L of K there corresponds a *generalized Artin symbol* $(\frac{L/K}{\mathcal{D}})$, which is a conjugacy class in the Galois group $G(L/K)$. For each embedding of L in the algebraic closure of $\bar{K}_{\mathcal{D}} = \Pi\bar{K}_p/\mathcal{D}$, the group $G(L/L \cap \bar{K}_{\mathcal{D}})$ is procyclic and generated by an element of $(\frac{L/K}{\mathcal{D}})$. If K'/K is a sub-Galois extension of L/K, then the restriction of

$(\frac{L/K}{\mathcal{D}})$ to K' coincides with $(\frac{K'/K}{\mathcal{D}})$. If in particular K' is finite over K, then $(\frac{K'/K}{\mathcal{D}})$ is the conjugacy class characterized by the condition

(1) $\{p \in P \mid p$ is unramified in K' and $(\frac{K'/K}{p}) = (\frac{K'/K}{\mathcal{D}})\} \in \mathcal{D}$,

where $(\frac{K'/K}{p})$ is the usual Artin symbol. Finally denote by $\phi = \phi_{L/K}$ the map of Ω_K into the set Con $G(L/K)$ of all conjugacy classes of $G(L/K)$ that assigns to each $\mathcal{D}$ in Ω_K its generalized Artin symbol $(\frac{L/K}{\mathcal{D}})$. Then ϕ is surjective and also continuous with respect to the Krull topology of Con $G(L/K)$.

Consider now the Haar measure μ of $G(L/K)$ with respect to the Krull topology, normalized by the condition $\mu(G(L/K)) = 1$. It induces a measure on Con $G(L/K)$ such that the map Con $G(L/K) \to$ Con $G(L/K)$ that maps every $\sigma \in G(L/K)$ onto its conjugacy class, con σ, is measurable. This measure on Con $G(L/K)$ will also be denoted by μ. Thus, if S is a measurable subset of $G(L/K)$ which is closed under conjugation, then $\mu(S) = \mu(\text{Con } S)$. Also, if K'/K is a finite Galois subextension of L/K, $\bar{S}$ is a *conjugacy domain* in $G(K'/K)$, (i.e., a subset of $G(K'/K)$ which is closed under conjugation) and S is its lifting to $G(L/K)$, then $\mu(\text{Con } S) = [K':K]^{-1}|\bar{S}|$.

The most important observation of this section is contained in

LEMMA 3.1. *The map* $\phi:\Omega_K \to$ Con $G(L/K)$ *is measurable with respect to every extension* δ of the Dirichlet Density d *of* P(K).

PROOF. Let K'/K be a finite Galois subextension of L/K, let $\bar{S}$ be a conjugacy domain in $G(K'/K)$ and let S be its lifting to $G(L/K)$. Then

(2) $\phi^{-1}(\text{Con } S) = \Omega(C)$, where

$C = \{p \in P(K) \mid p$ is unramified in K' and $(\frac{K'/K}{p}) \subseteq S\}$.

Indeed, let $p \in \phi^{-1}(\text{Con } S)$. Then $(\frac{L/K}{p}) \subseteq S$ and hence $(\frac{K'/K}{p}) \subseteq \bar{S}$. It follows that C contains the set

$$\{p \in P(K) \mid p \text{ is unramified in K' and } (\tfrac{K'/K}{p}) = (\tfrac{K'/K}{\mathcal{D}})\},$$

which is contained in $\mathcal{D}$, by (1). Hence $C \in \mathcal{D}$ and $\mathcal{D} \in \Omega(C)$. Conversely, suppose that $\mathcal{D} \in \Omega(C)$. Then the set $\{p \in P(K) \mid p$ is unramified in K' and $(\frac{K'/K}{p}) = (\frac{K'/K}{\mathcal{D}})$ and $(\frac{K'/K}{p}) \subseteq \bar{S}\}$ belongs to $\mathcal{D}$ in particular it is not empty. Hence $(\frac{K'/K}{\mathcal{D}}) \subseteq \bar{S}$, hence $\mathcal{D} \in \phi^{-1}(\text{Con } S)$.

The set C has a Dirichlet density equal to $[K':K]^{-1}|\bar{S}|$, by

Čebotarev Density Theorem (cf. Cassels a Fröhlich [3, p.165]). Hence, by (2), by Theorem 1.1 and by Theorem 2.1, we have

$$\delta(\phi^{-1}(\mathrm{Con}\ S)) = \delta(\Omega(C)) = d(C) = [K':K]^{-1}|\bar{S}| = \mu(\mathrm{Con}\ S).$$

Every subset of Con $\mathcal{G}(L/K)$ that belongs to the Boolean algebra generated by the sets of the form Con S can be represented as a disjoint union of them. Hence $\delta \circ \phi^{-1}$ and μ coincide on this field. Since both $\delta \circ \phi^{-1}$ and μ are σ-additive measures they coincide also on the σ-field generated by the sets Con S (by Alexandroff Theorem, cf. Halmos [7, p.54]). This σ-field is the Borel field of Con $\mathcal{G}(L/K)$. It follows that $\delta \circ \phi^{-1}$ and μ coincide also on the completion of the Borel field.

REMARK. If K is a number field, then the set C in the last proof has also a natural density and it is equal to the Dirichlet density of C (cf. Goldstein [6, p.256] plus Deuring-Macclauer Argument in [6, p. 169]). Theorem 3.1. remains therefore true, if δ is the natural density of P(K).

4. A TRANSFER PRINCIPLE FOR FINITE FIELDS.

Let K be a field. Denote by K_s and $\tilde{K}$ its separable and algebraic closure, respectively. Let $G(K) = \mathcal{G}(K_s/K)$ be the absolute Galois group of K and let μ be its normalized Haar measure. Every σ in G(K) is uniquely extendable to $\tilde{K}$; we denote by $\tilde{K}(\sigma)$ its fixed field in $\tilde{K}$.

Denote by $\mathcal{L}(K)$ the first order language of the theory of fields augmented by constant symbols for the elements of K. Suppose that $\lambda(X_1,\ldots,X_n)$ is a formula of $\mathcal{L}(K)$ all of its free variables are among $\{X_1,\ldots,X_n\}$. For every $x_1,\ldots,x_n$ in $\tilde{K}$ we define $\bar{S}(\lambda(\underline{x})) = \{\sigma \in G(K) \mid \sigma x_i = x_i \text{ for } i = 1,\ldots,n \text{ and } \lambda(\underline{x}) \text{ is true in } \tilde{K}(\sigma)\}$.

LEMMA 4.1. *If* K *is a countable field, then* $\bar{S}(\lambda(\underline{x}))$ *is a Borel set of* G(K).

PROOF. We prove the Lemma by a structure induction on λ. Suppose first that $\lambda(\underline{x})$ is an atomic formula. Then $\lambda(x)$ has the form $f(\underline{X}) = 0$, where $f \in K[\underline{X}]$. In this case $\bar{S}(\lambda(\underline{x}))$ is empty if $f(x) \neq 0$ and

$\bar{S}(\lambda(\underline{x})) = G(K_s \cap K(\underline{x}))$ if $f(\underline{x}) = 0$.

The induction steps follow from the identities

$$\bar{S}(-\lambda(\underline{x})) = G(K_s \cap K(\underline{x})) - \bar{S}(\lambda(\underline{x})),$$
$$\bar{S}(\lambda_1(\underline{x}) \cup \lambda_2(\underline{x})) = \bar{S}(\lambda_1(\underline{x})) \cup \bar{S}(\lambda_2(\underline{x}))$$
$$\bar{S}((\exists Y)\lambda(\underline{x},Y)) = \bigcup_{y \in \tilde{K}} \bar{S}(\lambda(\underline{x},y))$$

and from the fact that $\tilde{K}$ is countable.

COROLLARY 4.2. *Let* K *be a countable field and let* θ *be a sentence of* $L(K)$. *Then*

$$\bar{S}(\theta) = \{\sigma \in G(K) \,|\, \theta \text{ is true in } \tilde{K}(\sigma)\}$$

is a Borel set of G(K) *which is invariant under conjugation.*

Return now to the case where K is a global field with a ring of integers R. If θ is a sentence of $L(R)$ and p is a prime of R, then θ has an interpretation in $\bar{K}_p$, by reducing the elements of R modulo p. Therefore, the following definitions make sense

$$A(\theta) = \{p \in P(K) \,|\, \bar{K}_p \models \theta\},$$
$$\bar{\Omega}(\theta) = \{\mathcal{D} \in \Omega_K \,|\, \bar{K}_{\mathcal{D}} \models \theta\}.$$

Recall also that a perfect field F is said to be *pseudo-finite* if $G(F) \cong \hat{\mathbb{Z}}$ and if every non-void absolutely irreducible variety defined over F has an F-valued point. It was proved in [9, Cor. 2.6] that the set S of all $\sigma \in G(K)$, such that $\tilde{K}(\sigma)$ is pseudo-finite is of measure 1. It was further proved in [9, Thm. 3.9] and also in [10, Thm. 4.4], that $\mathcal{D} \in \Omega_K$, if $\sigma \in S$ and if $\tilde{K}(\sigma) \cong_K \tilde{K} \cap \bar{K}_{\mathcal{D}}$, then $\bar{K}_{\mathcal{D}}$ is an elementary extension of $\tilde{K}(\sigma)$. In particular if θ is a sentence of $L(R)$, then $\tilde{K}(\sigma) \models \theta$ if and only if $\bar{K}_{\mathcal{D}} \models \theta$.

THEOREM 4.3. *Let* K *be a global field with a ring of integers* R *and let* θ *be a sentence of* $L(R)$. *Then*

a) $\bar{\Omega}(\theta) = \Omega(\bar{A}(\theta))$

b) *If* δ *is any extension of the Dirichlet density* d *of* P(K), *then* $\bar{\Omega}(\theta)$ *is* δ*-measurable.*

c) *If* $\phi\colon \Omega_K \to \mathrm{Con}\, G(K)$ *is the generalized Artin symbol map,* $\phi(\mathcal{D}) = \left(\frac{K_S/K}{\mathcal{D}}\right)$, *then* $\phi^{-1}(\mathrm{Con}(\bar{S}(\theta) \cap S)) = \bar{\Omega}(\theta) \cap \phi^{-1}(\mathrm{Con}\, S)$.

d) $\bar{S}(\theta)$ *is a conjugacy domain and it is* μ*-measurable*

e) $\bar{A}(\theta)$ *has a Dirichlet density (and even a natural density if* K *is a number field).*

f) $\mu(\bar{S}(\theta)) = \delta(\Omega(\theta)) = d(\bar{A}(\theta))$.

PROOF. a) follows from the basic property of ultraproducts:

$$\Pi\, \bar{K}_p/\mathcal{D} \models \theta \text{ if and only if } \bar{A}(\theta) \in \mathcal{D}.$$

b) It follows from a) that $\bar{\Omega}(\theta)$ is a compact open subset of Ω_K, hence it is δ-measurable.

c) follows from the remarks preceeding the proof, since if

$$\sigma \in \left(\frac{K_s/K}{\mathcal{D}}\right), \text{ then } \tilde{K}(\sigma) \cong_K \tilde{K} \cap \bar{K}_{\mathcal{D}}, \text{ by section 3.}$$

d) is a special case of Corollary 4.2.

e),f) let d' be an extension of the Dirichlet density d of P(K) and let δ be its lifting to Ω_K. Then $d'(\bar{A}(\theta)) = \delta(\Omega(\bar{A}(\theta))) = \delta(\bar{\Omega}(\theta))$, by Section 3 and by a). Also $\delta(\bar{\Omega}(\theta)) = \mu(\bar{S}(\theta))$, by c), d) and Lemma 3.1, since $\mu(S) = 1$. It follows that $d'(\bar{A}(\theta)) = \mu(\bar{S}(\theta))$. The right hand-side of this equality does not depend on the specific extension d' of d that has been chosen. Hence, by Theorem 1.2, $\bar{A}(\theta)$ has a density and $d'(\bar{A}(\theta)) = d(\bar{A}(\theta))$. Our assertion is therefore valid.

COROLLARY 4.4. *Let* K *be a global field with a ring of integers* R *and let* θ *be a statement which belongs to the* σ*-field generated by the sentences of* L(R). *Let* $\bar{\Omega}(\theta)$ *and* $\bar{S}(\theta)$ *be as above. Then* $\delta(\bar{\Omega}(\theta)) = \mu(\bar{S}(\theta))$ *for every Borel measure* δ *that extends the Dirichlet density* d *of* P(K).

PROOF. The Corollary follows from Theorem 4.3, since the operators S and Ω commute with countable unions, countable intersections and taking complements. Also, both μ and δ are σ-additive, hence Alexandroff Theorem [7, p.54] can be applied.

5. A TRANSFER PRINCIPLE FOR p-ADIC FIELDS

We restrict ourselves in this section to the case where K is a number field. If $p \in P(K)$, then K_p denotes the completion of K at p. We also denote by U_p and T_p the maximal unramified and tamely ramified extension, respectively, of K_p. Let $\mathcal{D} \in \Omega_K$, let $K_\mathcal{D} = \Pi\, K_p/\mathcal{D}$ and let $U_\mathcal{D} = \tilde{K}_\mathcal{D} \cap \Pi\, U_p/\mathcal{D}$.

CLAIM: $\tilde{K}_\mathcal{D} \cap \Pi\, T_p/\mathcal{D} = \tilde{K}_\mathcal{D}$. Indeed, let $x \in \tilde{K}_\mathcal{D}$ and denote by x_p the p-th coordinate of a representative of x modulo $\mathcal{D}$. Then there are only finitely many p's such that $\mathrm{char}(\bar{K}_p) \le [K_\mathcal{D}(x):K_\mathcal{D}]$. All the other p's satisfy $x_p \in T_p$, hence $x \in \Pi\, T_p/\mathcal{D}$.

Now, for every $p \in P(K)$, the group $G(U_p/K_p)$ is isomorphic to $\hat{\mathbb{Z}}$, and the group $G(T_p/K_p)$ is generated by two elements, σ_p and τ_p. The restriction of σ_p to U_p is the Frobenius automorphism and τ_p generates $G(T_p/U_p)$. This group is isomorphic to $\Pi\, \mathbb{Z}_p$, where p runs over all rational primes other than $\mathrm{char}(\bar{K}_p)$. The generators σ_p and τ_p satisfy the defining relation $\sigma_p\tau_p\sigma_p^{-1} = \tau_p^{q_p}$, where $q_p = |\bar{K}_p|$. (cf. Iwasawa [8, Thm. 2]). It follows that if we denote by $\sigma = \sigma_\mathcal{D}$ and $\tau = \tau_\mathcal{D}$ the restrictions of $\Pi\, \sigma_p/\mathcal{D}$ and $\Pi\, \tau_p/\mathcal{D}$ to $\tilde{K}_\mathcal{D}$, then σ and τ generate $G(K_\mathcal{D})$. The restriction of σ to $U_\mathcal{D}$ generates $G(U_\mathcal{D}/K_\mathcal{D})$, which is isomorphic to $\hat{\mathbb{Z}}$. Together σ and τ satisfy the profinite defining relation $\sigma\tau\sigma^{-1} = \tau^q$, where q is an element of $\hat{\mathbb{Z}}$, not in $\mathbb{Z}$, which is the "finite" part of $\Pi\, q_p/\mathcal{D}$.

For every $p \in P(K)$ denote by $v_p\colon K_p^\times \to \mathbb{Z}$ the normalized valuation of K_p. Then ${}^*v = {}^*v_\mathcal{D} = \Pi\, v_p/\mathcal{D}$ is a valuation of $K_\mathcal{D}$ with ${}^*\mathbb{Z}_\mathcal{D} = \mathbb{Z}^{P(K)}/\mathcal{D}$ ${}^*\mathbb{Z}_\mathcal{D} = \mathbb{Z}^{P(K)}/\mathcal{D}$ as the value group. Thus $K_\mathcal{D}$ is a valued field, which is Henselian and $U_\mathcal{D}$ is its maximal unramified extension. The residue field of $K_\mathcal{D}$ is $\bar{K}_\mathcal{D} = \Pi\, \bar{K}_p/\mathcal{D}$. It is a field of characteristic 0. The group $\mathbb{Z}$ is an isolated subgroup of ${}^*\mathbb{Z}$. One can therefore consider the canonical projection ${}^*\mathbb{Z} \to {}^*\mathbb{Z}/\mathbb{Z}$ and combine it with ${}^*v$ to get a valuation $\dot{v}\colon K_\mathcal{D} \to {}^*\mathbb{Z}/\mathbb{Z}$, which is obviously trivial on $\mathbb{Q}$. Consider therefore a maximal subfield K_b of $K_\mathcal{D}$ on which $\dot{v}$ is trivial. This subfield is isomorphic to the residue field of $K_\mathcal{D}$ with respect to $\dot{v}$

(cf. Endler [4, p.35]), and it is called the *bounded product of the* K_p's *modulo* $\mathcal{D}$ (see Kiehne [12]). The restriction v of $^*v$ to K_b^x maps K_b^x onto $\mathbb{Z}$ and the residue fields of K_b with respect to v coincides with the residue field of $K_{\mathcal{D}}$ with respect to $^*v$, hence it is isomorphic to $\bar{K}_{\mathcal{D}}$. The maximality of K_b implies that K_b is algebraically closed in $K_{\mathcal{D}}$, and this, together with the Henselian property of $K_{\mathcal{D}}$ implies that K_b is Henselian too. It follows by Ax - Kochen Theorem that K_b is an elementary valued subfield of K (cf. Kiehne [12, Thm. 2.2]). In addition, $K_{\mathcal{D}}$ is pseudo-complete, hence K_b is a complete valued field. It follows that K_b can be identified with the power series field $\bar{K}_{\mathcal{D}}((t))$ with the canonical valuation.

Now, the restriction of σ to $\bar{K}_{\mathcal{D}}$ generates $G(\bar{K}_{\mathcal{D}})$, which is still isomorphic to $\hat{\mathbb{Z}}$. The restriction of σ to $\tilde{\mathbb{Q}}$ belongs to the Artin class $\left(\frac{\tilde{K}/K}{\mathcal{D}}\right)$. Hence $\tilde{K}(\sigma)((t))$ is a valued subfield of $\bar{K}_{\mathcal{D}}((t))$ which is also complete and has $\tilde{K}(\sigma)$ as a residue subfield. If σ belongs to S, i.e. if σ is pseudofinite, then $\tilde{K}(\sigma)$ is an elementary subfield of $\bar{K}_{\mathcal{D}}$, as we mentioned in section 4. Using the Ax-Kochen Theorem again we find that $\tilde{K}(\sigma)((t))$ is an elementary valued subfield of $\bar{K}_{\mathcal{D}}((t))$ hence also of $K_{\mathcal{D}}$.

$\tilde{K}$ is an unramified field extension of $\tilde{K}(\sigma)$, hence $\tilde{K}$ is contained in $U_{\mathcal{D}}$. The restriction map $G(U_{\mathcal{D}}/K_{\mathcal{D}}) \to G(\tilde{K}(\sigma))$ is therefore surjective, hence it is an isomorphism, since $G(\tilde{K}(\sigma)) \cong \hat{\mathbb{Z}}$ by assumption (cf. Ribes [14, p.76]). Also $\tilde{K}(\sigma)$ is algebraically closed in $K(\sigma)((t))$.

Hence $\tilde{K}.K_{\mathcal{D}} = U_{\mathcal{D}}$ and the restriction map $G(U_{\mathcal{D}}/K_{\mathcal{D}}) \to G(\tilde{K}((t))/\tilde{K}(\sigma)((t)))$ is also an isomorphism. Another application of the Ax-Kochen Theorem implies that $\tilde{K}((t))$ is an elementary subfield of $U_{\mathcal{D}}$. In particular it follows that $\tilde{K}((t))$ is algebraically closed in U. Further, it is well-known that $G(\tilde{K}((t)) \cong \hat{\mathbb{Z}}$ hence the restriction map $G(U_{\mathcal{D}}) \to G(\tilde{K}((t)))$ is an isomorphism. Every element x of $\widetilde{K((t))}$ has a Puiseux expansion $x = \sum_{j=m}^{\infty} a_j t^{j/n}$ with $a_j \in \tilde{K}$ and $m \in \mathbb{Z}$, and $n \in \mathbb{N}$. For a fixed n, this expansion is unique. The action of τ on $\widetilde{K((t))}$ is given by

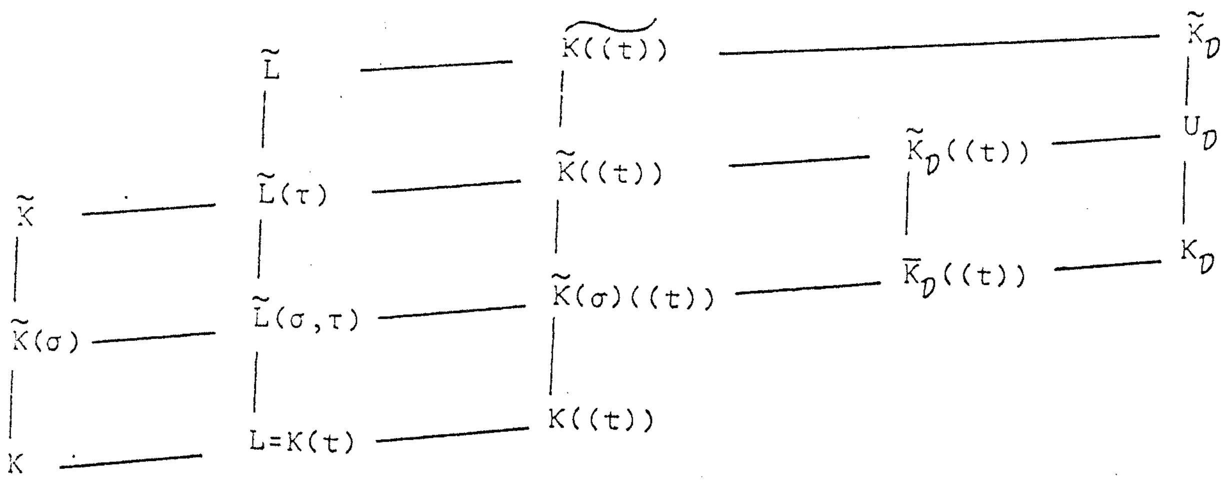
$$(1) \qquad \tau x = \sum_{j=m}^{\infty} a_j \zeta_n^j t^{j/n}$$

where $\{\zeta_n \mid n \in \mathbb{N}\}$ is a compatible sequence of roots of unity, i.e. $\zeta_n^n = 1$ and $\zeta_{kn}^k = \zeta_n$. The action of σ on $\tilde{K}_{\mathcal{D}}$ is continuous, hence the action of σ on $\widetilde{K((t))}$ is given by

$$(2) \qquad \sigma x = \sum_{j=m}^{\infty} (\sigma a_j) t^{j/n} .$$

In particular $\sigma x = x$ if and only if $\sigma a_j = a_j$ for every $j \geq m$.

To establish our last step downwards we consider the fields $L = K(t)$ and $\tilde{L}(\sigma,\tau) = \tilde{L} \cap K_{\mathcal{D}}$ and $\tilde{L}(\tau) = \tilde{L} \cap U_{\mathcal{D}}$. Then $\tilde{L}(\sigma,\tau)$ is a valued subfield of $\tilde{K}(\sigma)((t))$ with $\tilde{K}(\sigma)$ as a residue field and as the value group. Also $\tilde{L}(\sigma,\tau)$ is Henselian, since it is algebraically closed in $\tilde{K}(\sigma)((t))$. It follows, by Ax-Kochen Theorem, that $\tilde{L}(\sigma,\tau)$ is an elementary valued subfield of $\tilde{K}(\sigma)((t))$, hence also of $K_{\mathcal{D}}$.



Note that if we start with a $\sigma \in G(K)$, then (2) defines a canonical extension of σ to $\tilde{L}$. The automorphism τ depends on the ζ_n's but its fixed field does not. Therefore $\tilde{L}(\sigma,\tau)$ is uniquely determined by σ.

We now imitate the procedure of Section 4 and denote by $L_v(R)$ the first order language of the theory of valued field with constant symbols for the elements of R. The models of $L_v(R)$ that we consider in the next Lemma are all submodules of $\tilde{K}((t))$ that contain L, where $\tilde{K}((t))$ is equipped with the canonical power series valuation.

<u>LEMMA 5.1</u>. *Let $\lambda(X_1,\ldots,X_n)$ be a formula of $L_v(R)$ whose free variables are among the $\{X_1,\ldots,X_n\}$. Let $x_1,\ldots,x_n$ be elements of $\tilde{K}((t))$. Then*

$S(\lambda(x)) = \{\sigma \in G(K) \mid \sigma x_i = x_i$ *for* $i = 1,\dots,n$ *and* $\tilde{L}(\sigma,\tau) \models \lambda(x)\}$ *is a Borel subset of* $G(K)$.

PROOF. Suppose first that $\lambda(X_1,\dots,X_n)$ is an atomic formula, i.e. a formula of the form $f(X_1,\dots,X_n) = 0$ or of the form $\mathrm{ord}(f(X_1,\dots,X_n)) \le m$, where $f \in L[X_1,\dots,X_n]$ and $m \in \mathbb{Z}$. Each one of the x_i can be written in the form $x_i = \sum_{j=m}^{\infty} a_{ij}t^j$, where $a_{ij} \in \tilde{K}$. Denote by K' the field generated over K by the a_{ij}, for $i = 1,\dots,m$. Then $S(\lambda(x)) = G(K')$ if $\lambda(x)$ is true in $\tilde{K}((t))$, and $S(\lambda(\underline{x}))$ is empty otherwise. In both cases $S(\lambda(\underline{x}))$ is a closed subset of $G(K)$, hence a Borel set.

The rest of the proof is carried out exactly as in Lemma 4.1.

Let now θ be a sentence of $L_v(R)$. Consider the following sets:

$$A(\theta) = \{\mathfrak{p} \in P(K) \mid K_{\mathfrak{p}} \models \theta\} \qquad B(\theta) = \{\mathfrak{p} \in P(K) \mid \bar{K}_{\mathfrak{p}}((t)) \models \theta\}$$

$$\Omega(\theta) = \{\mathcal{D} \in \Omega_K \mid K_{\mathcal{D}} \models \theta\} \qquad \Lambda(\theta) = \{\mathcal{D} \in \Omega_K \mid \bar{K}_{\mathcal{D}}((t)) \models \theta\}$$

$$S(\theta) = \{\sigma \in G(K) \mid \tilde{K}(\sigma)((t)) \models \theta\} \qquad T(\theta) = \{\sigma \in G(K) \mid \tilde{L}(\sigma,\tau) \models \theta\}$$

Then we have:

THEOREM 5.2. *Let* K *be a number field with a ring of integers* R *and let* θ *be a sentence of* $L_v(R(t))$. *Then*

a) $A(\theta)$ *and* $B(\theta)$ *differ only by a finite set.*

b) $\Lambda(\theta) = \Omega(\theta) = \Omega(A(\theta)) = \Omega(B(\theta))$

c) $S(\theta)$ *and* $T(\theta)$ *differ only by a set of measure zero.*

d) *If* δ *is any extension of the natural density* d *of* $P(K)$, *then* $\hat{\Omega}(\theta)$ *is* δ*-measurable.*

e) *If* $\phi: \Omega_K \to \mathrm{Con}\, G(K)$ *is the generalized Artin map, then* $\phi^{-1}(\mathrm{Con}(S(\theta) \cap S)) = \Omega(\theta) \cap \phi^{-1}(\mathrm{Con}\ S)$.

f) $S(\theta)$ *is a conjugacy domain and it is* μ*-measurable.*

g) $A(\theta)$ *has a natural density.*

h) $\mu(T(\theta)) = \mu(S(\theta)) = \delta(\hat{\Lambda}(\theta)) = \delta(\Omega(\theta)) = d(B(\theta)) = d(A(\theta))$.

PROOF. a) Assume for example that $A(\theta) = B(\theta)$ were an infinite set. Then there would exist a $\mathcal{D} \in \Omega_K$ that contains $A(\theta)$ but not $B(\theta)$. This would lead to a contradiction, since by Ax-Kochen $\Pi \bar{K}_p((t))/\mathcal{D}$ is elementarily equivalent to $K_{\mathcal{D}} = \Pi K_p/\mathcal{D}$.

g) As was mentioned above, the difference between $S(\theta)$ and $T(\theta)$ is contained in the zero set $G(K) - S$.

The rest of the Theorem is proved as Theorem 4.3., using Lemma 5.1. and the discussion that preceeded it.

6. ON ONE VARIABLE STATEMENTS AND A RECURSIVE COMPUTATION PROCEDURE

If one wishes to ignore the middle terms in Theorem 5.2. h) that involve the measure of Ω_K, one can take the following short cut. According to Ax [1, p.266] there exists a primitive recursive set Π of sentences in the language $L_v(R)$ such that a valued field F is a model of Π if and only if F contains K, F is Henselian, its value group is a $\mathbb{Z}$-group and its residue field is pseudo finite. Ax proves that Π is a set of axioms for the theory of sentences of $L_v(R)$ that are true in K_p, for almost all $p \in P(K)$ (i.e. for all but finitely many p's). He proves further [1, Thm. 16] that for every sentence θ of $L_v(R)$ there exists a one variable sentence λ of over R such that

(1) $\qquad A(\theta) \approx A(\lambda)$ i.e. such that $\Pi \models \theta \leftrightarrow \lambda$.

Here a one variable sentence over R is a boolean combination of sentences of the form $(\exists x)\ f(x) = 0$, where $f \in R[X]$. Also we write $A \approx A'$ for two subsets A and A' of P(K) that differ from each other only by a finite set.

If σ belongs to S, i.e. if $\tilde{K}(\sigma)$ is pseudofinite, then $\tilde{K}(\sigma)((t))$ is evidently a model of Π. Hence

(2) $\qquad S(\theta) \approx S(\lambda)$

where here $\approx$ means equality up to a set of measure zero. The fact that $\tilde{K}(\sigma)$ is algebraically closed in $\tilde{K}(\sigma)((t))$ implies that

(3) $\qquad S(\lambda) = \bar{S}(\lambda)$.

Also, it follows from Hensel's Lemma that

$$(4) \qquad A(\lambda) \approx A(\lambda).$$

From the Translation Theorem for one variable sentences [9, Lemma 3.15] we know that $\bar{A}(\lambda)$ has a natural density which is a rational number, $\bar{S}(\lambda)$ is measurable and $\delta(\bar{A}(\lambda)) = \mu(\bar{S}(\lambda))$. Hence, using (1), (2), (3) and (4), we conclude that the same is true for $A(\theta)$ and $S(\theta)$, i.e. $A(\theta)$ has a natural density which is a rational number, $S(\theta)$ is measurable and $\delta(A(\theta)) = \mu(S(\theta))$.

As in [9, p.263] one can use Godel's completeness Theorem in order to establish a recursive procedure to find for a given sentence θ of $L_v(R)$ a one variable sentence such that (1) holds. Using (4) one can proceed and obtain also a recursive procedure to compute the rational number $\delta(A(\theta))$.

REFERENCES

[1] AX, JAMES, *The elementary theory of finite fields*. Annals of Mathematics 88 (1968), 239-271.

[2] AX, J. and KOCHEN, S., *Diophantine problems over local fields* III. Decidable fields, Annals of Mathematics 83 (1966), 437-456.

[3] CASSELS, J.W.S. and FRÖHLICH, A., *Algebraic Number theory,* Academic Press, London/New York, 1967.

[4] ENDLER, O., *Valuation theory,* Springer Verlag, Berlin/Heidelberg/New York, 1972.

[5] FRIED, M., *Toward a general theory of dophantine problems with application to P-adic fields and fields of finite corank,* A Manuscript.

[6] GOLDSTEIN, L., *Analytic Number Theory,* Prentice-Hall, Englewood Cliffs, N.J. 1971.

[7] HALMOS, P.R., *Measure Theory,* Van Nostrand, Princeton 1964.

[8] IWASAWA, K., *On the Galois group of local fields,* Transactions of AMS 80 (1955), 448-469.

[9] JARDEN, M., *Elementary statements over large algebraic fields*, Transactions of AMS 164 (1972), 67-91.

[10] JARDEN, M. and KIEHNE, U., *The elementary theory of algebraic fields of finite corank*, Inventiones Mathematicae 30 (1975), 275-294.

[11] JEHNE, W. and KLINGEN, N., *Superprimes and a generalized Frobenius symbol*, Acta Arithmetica 32 (1977), 209-232.

[12] KIEHNE, U., *Bounded products in the theory of valued fields*, To appear.

[13] KOCHEN, S., *The model theory of local fields*, Logic Conference Kiel 1974, Springer Lecture Notes 499, Heidelberg (1975).

[14] RIBES, L., *Introduction to profinite groups and Galois cohomology*, Queen's papers in pure and applied Mathematics 24, Queen's University, Kingston 1970.

[15] ZIEGLER M., Die elementare Theorie der henselschen Körper, Dissertation, Köln, 1972.

(Received, March 3, 1980).