On the model companion of the theory of e-fold ordered fields

by

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0. Introduction

The present work is inspired by three papers, [11] of Van den Dries, [9] of Prestel and [5]. Van den Dries considers structures of the form $(K, P_1, \ldots, P_e)$, where K is a field and $P_1, \ldots, P_e$ are e orderings of the K. They are called, *e-fold ordered fields*. The appropriate first ordered language is denoted by $\mathscr{L}_e$. He proves that the theory of e-fold ordered fields in $\mathscr{L}_e$ has a model companion $\overline{OF}_e$. The models $(K, P_1, \ldots, P_e)$ of $\overline{OF}_e$ are characterized on one hand by being existentially closed in the family of e-fold, ordered fields, and by satisfying certain axioms of $\mathscr{L}_e$ on the other hand.

In particular Van den Dries proves that the absolute Galois group $G(K)$ of K is a pro-2-group generated by e involutions. If K is algebraic over $\mathbf{Q}$ and R is a real closure of $\mathbf{Q}$, this implies that there exist $\sigma_1, \ldots, \sigma_e \in G(\mathbf{Q})$ such that $K = R^{\sigma_1} \cap \ldots \cap R^{\sigma_e}$. In general, if $\sigma_1, \ldots, \sigma_e \in G(\mathbf{Q})$, we write $\mathbf{Q}_\sigma = R^{\sigma_1} \cap \ldots \cap R^{\sigma_e}$ and denote by $P_{\sigma i}$ the ordering of $\mathbf{Q}$ induced by the unique ordering of the real closed field R^{σ_i}. In this way we attain a family of e-fold ordered fields, $\mathscr{Q}_\sigma = (\mathbf{Q}_\sigma, P_{\sigma 1}, \ldots, P_{\sigma e})$, indexed by $G(\mathbf{Q})^e$.

Geyer proves in [4] that for almost all $\sigma \in G(\mathbf{Q})^e$ (in the sense of the Haar measure of $G(\mathbf{Q})^e$), the group $G(\mathbf{Q}_\sigma)$ is isomorphic to the free product, $\hat{D}_e$, of e copies of $\mathbf{Z}/2\mathbf{Z}$, in the category of profinite groups. This takes us away from the models of $\overline{OF}_e$ and leads us in [5] to make the following

Definition. An e-fold ordered field $(K, P_1, \ldots, P_e)$ is said to be a *Geyer-field of corank e* if the following conditions hold:

(α) If V is an absolutely irreducible variety defined over K and if each of the orderings P_i extends to the function field of V, then V has a K-rational point.

(β) The orderings $P_1, \ldots, P_e$ induce distinct topologies on K.

(γ) We have $G(K) \cong \hat{D}_e$.

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The main result of [5] is that $\mathcal{Q}_\sigma$ is a Geyer-field of corank e for almost all $\sigma \in G(K)^e$. It is also proved in [5] that the theory of Geyer-fields of corank e coincides with the theory of all sentences Θ of $\mathcal{L}_e$ which are true in $\mathcal{Q}_\sigma$ for almost all $\sigma \in G(\mathbf{Q})^e$. Finally, a (recursive) decision procedure is given for the theory of Geyer-fields of corank e.

As an atempt to return to the models of $\overline{OF}_e$, we call an e-fold ordered field $(K, P_1, \ldots, P_e)$ by the name *Van den Dries-field* if it satisfies (α) and (β) above and also:

(γ') The group $G(K)$ is isomorphic to the free product, $\hat{D}_e(2)$, of e copies of $\mathbf{Z}/2\mathbf{Z}$, in the category of pro-2-groups.

The group $\hat{D}_e(2)$ is obviously the maximal pro-2 quotient of $\hat{D}_e$. Using this observation, it is not difficult to show that every Geyer-field $\mathcal{K}=(K, P_1, \ldots, P_e)$ has an algebraic extension $\mathcal{K}'=(K', P_1', \ldots, P_e')$ such that $G(K')\cong\hat{D}_e(2)$. One may therefore wonder whether $\mathcal{K}'$ is a Van den Dries-field. The first obvious attempt to solve this problem fails, since as McKenna [8, p. 5.13] and Prestel [9, p. 2] point out, it is not true that if an e-fold ordered field $(K, P_1, \ldots, P_e)$ satisfies (α), then every algebraic extension $(L, Q_1, \ldots, Q_e)$ satisfies (α) too. The problem is that (α) implies, among others, that $P_1, \ldots, P_e$ are the only orderings of K, and it may happen that L has more than e orderings.

Prestel overcomes this difficulty by making the right definition. He calls a field K PRC if it satisfies the following modificaiton of condition (α):

(α') If V is an absolutely irreducible variety defined over K and if every ordering of K extends to the function field of V, the V has a K-rational point.

Then he proves that every algebraic extension of a PRC field is a PRC field (Theorem 3.1 of [9]). Coming back to $\mathcal{K}'$ we prove that $P_1', \ldots, P_e'$ are all the orderings of K' and therefore $\mathcal{K}'$ is indeed a Van den Dries-field.

This result implies that for almost all $\sigma \in G(\mathbf{Q})^e$ we may choose an algebraic extension $\mathcal{Q}_\sigma'$ of $\mathcal{Q}_\sigma$ which is a Van den Dries-field of corank e. Then we prove that the following three theories coincide.

(a) The theory of all sentences Θ of $\mathcal{L}_e$ that hold in $\mathcal{Q}_\sigma'$ for almost all $\sigma \in G(\mathbf{Q})^e$.
(b) The theory of all Van den Dries-fields of corank e.
(c) The theory $\overline{OF}_e$.

In particular it follows that if $(K, P_1, \ldots, P_e)$ is a model of $\overline{OF}_e$, the $G(K)\cong\hat{D}_e(2)$.

1. PRC fields

Let $<$ be an ordering of a field K and let $P=\{x\in K|x>0\}$ be the positive cone of $<$. We abuse our language and speak about P as "the ordering of K". The real closure of K with respect to P is denoted by $\bar{K}_P$. Our intention is to consider the family of all orderings of K. A. Prestel proves with this respect the following proposition in [9, Theorem 1.2]:

PROPOSITION 1.1. *The following two conditions on a field K are equivalent.*

(a) *If F is a regular extension of K and if every ordering of K extends to F, then K is existentially closed in F.*

(b) *If V is an absolutely irreducible variety defined over K and if V has a $\bar{K}_P$-rational simple point, for every ordering P of K, then V has a K-rational point.*

A field that satisfies the conditions of Proposition 1.1 is said to be pseudo-real-closed (abbreviated PRC). Note that this definition makes sense even if K has no orderings. In this case K turns out to be a PAC field (cf. Frey [2, p. 204]).

Prestel goes on and proves in [9, Proposition 1.4], the following properties of PRC fields:

PROPOSITION 1.2. *Let K be a* PRC *field.*

(a) *If P is an ordering of K, then K is P-dense in $\bar{K}_P$.*
(b) *Distinct orderings of K induce distinct topologies on K.*
(c) *If L is an algebraic extension of K the L is also a* PRC *field.*

We are mainly interested here in the case where K has only finitely many orderings. Thus we consider systems $\mathcal{K}=(K,P_1,\ldots,P_e)$ consisting of a field K and e orderings $P_1,\ldots,P_e$ and denote by $\bar{K}_i$ the real closure of K with respect to P_i. The corresponding language is denoted by $\mathcal{L}_e(K)$. It consists of the usual first order language for the theory of fields augmented by e predicate symbols for $P_1,\ldots,P_e$ and by constant symbols for the elements of K.

PROPOSITION 1.3. *Let $\mathcal{K}=(K,P_1,\ldots,P_e)$ be a field with e distinct orderings. The the following conditions are equivalent.*

(a) *The field K is* PRC *and $P_1,\ldots,P_e$ are all of its orderings.*

(b) *If C is an absolutely irreducible curve defined over K and C has a $\bar{K}_i$-rational simple point, for $i=1,\ldots,e$, then C has a K-rational point.*

(c) *If* $\mathscr{F}=(F, Q_1, \ldots, Q_e)$ *is an extension of* $\mathscr{K}=(K, P_1, \ldots, P_e)$ *such that F is regular over K, then* $\mathscr{K}$ *is existentially closed in* $\mathscr{F}$ *in the language* $\mathscr{L}_e(K)$.

(d) (i) *If* $f \in K[T_1, \ldots, T_r, X]$ *is an absolutely irreducible polynomial for which there exist an* $\mathbf{a}_0 \in K^r$ *such that* $f(\mathbf{a}_0, X)$ *changes sign on K with respect to each of the* P_i*'s and if* U_i *is a* P_i*-neighbourhood of* $\mathbf{a}_0$, *for* $i=1, \ldots, e$, *then there exists an* $(\mathbf{a}, b) \in K^{r+1}$ *such that* $\mathbf{a} \in U_1 \cap \ldots \cap U_e$ *and* $f(\mathbf{a}, b)=0$.

(ii) *The orderings* $P_1, \ldots, P_e$ *induce distinct topologies on K.*

Proof. The equivalence (a)$\Leftrightarrow$(b) is just a rephrasing of Theorem 2.1 of Prestel [9]. Similarly (a)$\Leftrightarrow$(*c*) is a repetition of Theorem 1.7 of [9]. Finally, the equivalence (a)$\Leftrightarrow$(*d*) follows from Proposition 1.2 (b) and from Lemmas 2.2 and 2.3 of [5]. Note that we have to use here the well-known fact that if V is an absolutely irreducible variety defined over a field K with an ordering P, then P extends to the function field of V if and only if V has a $\bar{K}_P$-rational simple point. Q.E.D.

Proposition 1.3 implies that the present definition of a PRC field coincides with those that appear in [5] for e orderings, in McKenna [8] and in Basarab [1] for one ordering. An e-fold ordered field $(K, P_1, \ldots, P_e)$ is said to be PRCe if it satisfies the conditions of Proposition 1.3.

As an application we generalize Theorem 2.1 of McKenna [8] from PRC1 fields to arbitrary PRCe fields. In the proof of this generalization we use the following argument about a real closed field R. If a polynomial $f \in R[X]$ changes sign in an interval (a, b) of R, then it has a zero in (a, b). Therefore if a polynomial $g \in R[X]$ is close enough to f, it also changes sign in (a, b) and therefore has a zero in (a, b).

PROPOSITION 1.4. *Let* $(K, P_1, \ldots, P_e)$ *be a* PRCe *field and let V be an absolutely irreducible variety defined over K. For every* $1 \leq i \leq e$ *let* $q_i \in V(\bar{K}_i)$ *be a simple point. Then V has a K-rational point q, arbitrary* P_i*-close to* q_i, *for* $i=1, \ldots, e$.

Proof. The assumption that the q_i are simple implies that there exists a hypersurface W and a birational map $\varphi: V \to W$, defined over K, such that φ is biregular at $q_1, \ldots, q_e$ (cf. [3, Lemma 5.1]). We may therefore assume that V is defined by an absolutely irreducible polynomial $f \in K[T_1, \ldots, T_r, X]$ and that $q_i=(a_{i1}, \ldots, a_{ir}, b_i)$, for $i=1, \ldots, e$. We may also assume that $\partial f/\partial x \neq 0$, since one of the partial derivatives of f is not zero.

The assumption that q_i is a simple point of V means that $f(\mathbf{a}_i, b_i)=0$ and at least one of the partial derivatives of f does not vanish at q_i. If it is not $\partial f/\partial x$, then we may assume without loss that $(\partial f/\partial T_1)(\mathbf{a}_i, b_i) \neq 0$. In particular the polynomial

$f(T_1, a_{i2}, \dots, a_{ir}, b_i)$ changes sign on $\bar{K}_i$ in a neighbourhood of a_{i1}. As $\partial f/\partial x \neq 0$, it is relatively prime to f in the ring $K(T_2, \dots, T_r, X)[T_1]$. Therefore there exist polynomials $h_1, h_2 \in K[T_1, \dots, T_r, X]$ and $0 \neq g \in K[T_2, \dots, T_r, X]$ such that

$$h_1 f + h_2 \frac{\partial f}{\partial x} = g. \tag{1}$$

There exist now elements $a'_{i2}, \dots, a'_{ir}, b'_i \in \bar{K}_i$ which are P_i-close to $a_{i2}, \dots, a_{ir}, b_i$ such that $g(a'_{i2}, \dots, a'_{ir}, b'_i) \neq 0$. Then $f(T_1, a'_{i2}, \dots, a'_{ir}, b'_i)$ changes sign on $\bar{K}_i$ in the neighbourhood of a_{i1} and therefore it has a zero $a'_{i1} \in \bar{K}_i$ which is P_i-close to a_{i1}. Then (1) implies that $(\partial f/\partial x)(\mathbf{a}'_i, b'_i) \neq 0$.

Thus, replacing $(\mathbf{a}_i, b_i)$ by $(\mathbf{a}'_i, b'_i)$, if necessary, we may assume that

$$f(\mathbf{a}_i, b_i) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(\mathbf{a}_i, b_i) \neq 0 \quad \text{for } i = 1, \dots, e.$$

Let $t_1, \dots, t_r$ be algebraically independent elements over K. For each $1 \leq i \leq e$ extend P_i to an ordering of $\bar{K}_i(t_1, \dots, t_r)$ such that $t_1, \dots, t_r$ are P_i-infinitesimally close to $a_{i1}, \dots, a_{ir}$. Let R_i be a real closure of $\bar{K}_i(t_1, \dots, t_r)$. Then the polynomial $f(\mathbf{t}, X)$ changes sign on R_i in the neighbourhood of b_i and therefore has a root $x_i \in R_i$ in this neighbourhood. Take also a root x of $f(\mathbf{t}, X)$ and let $F = K(\mathbf{t}, x)$. Then K is algebraically closed in F and the map $x \mapsto x_i$ can be extended to a $K(t)$-isomorphism of F into R_i. This isomorphism defines an extension of the ordering P_i to F such that $(\mathbf{t}, x)$ is P_i-close to $(\mathbf{a}_i, b_i)$.

Note that all the above neighbourhoods are already defined by elements of K, since, by Proposition 1.2, K is P_i-dense in $\bar{K}_i$, for $i = 1, \dots, e$. It follows from Proposition 1.3(c) that there exists a point $(\mathbf{a}, b) \in K^{r+1}$ which is P_i-close to $(\mathbf{a}_i, b_i)$ for $i = 1, \dots, e$ such that $f(\mathbf{a}, b) = 0$. Q.E.D.

2. Van den Dries-fields

Denote by D_e the free product of e copies of $\mathbf{Z}/2\mathbf{Z}$ in the category of groups. Consider its completion $\hat{D}_e = \varprojlim D_e/N$, where N runs over all normal subgroups of finite index. The maximal pro-2-quotient $\hat{D}_e(2)$ of $\hat{D}_e$ can also be described as the inverse limit $\hat{D}_e(2) = \varprojlim D_e/N$, where N runs now over all normal subgroups of D_e such that D_e/N are 2-groups. The group $\hat{D}_e$ (and also $\hat{D}_e(2)$) has a system of e-generators $\varepsilon_1, \dots, \varepsilon_e$ satisfying $\varepsilon_1^2 = \dots = \varepsilon_e^2 = 1$. If $x_1, \dots, x_e$ are involutions in a profinite (resp. pro-2) group, then the map $\varepsilon_i \mapsto x_i$, $i = 1, \dots, e$ can be extended to a homomorphism of $\hat{D}_e$ (resp. of $\hat{D}_e(2)$) into G. Indeed, every system of e involutions that generate $\hat{D}_e$ (resp. $\hat{D}_e(2)$) has this property. Thus $\hat{D}_e$ (resp. $\hat{D}_e(2)$) is the free product in the category of profinite (resp. pro-2) groups of e copies of $\mathbb{Z}/2\mathbb{Z}$.

The group $\hat{D}_e$ plays a central role in [5]. This role is now shifted to the group $\hat{D}_e(2)$. Analogously to $\hat{D}_e$, if $\varepsilon_1, \dots, \varepsilon_e$ are involutions that generate $\hat{D}_e(2)$, then no two of them are conjugate, since a map of $\varepsilon_1, \dots, \varepsilon_e$ onto a basis of $(\mathbb{Z}/2\mathbb{Z})^e$ can be extended to an epimorphism of $\hat{D}_e$ onto $(\mathbb{Z}/2\mathbb{Z})^e$. We have the following characterization of $\hat{D}_e(2)$, similar to that of $\hat{D}_e$:

LEMMA 2.1. *A pro-2-group G is isomorphic to $\hat{D}_e(2)$ if and only if its finite quotients are exactly the 2-groups which are generated by e involutions.*

Proof. See e.g. Schuppar [10, Satz 2.1]. Q.E.D.

A PRCe field $(K, P_1, \dots, P_e)$ for which $G(K) \cong \hat{D}_e$ is called in [5] a Geyer field of corank e. Similarly we say that a PRCe field $(L, Q_1, \dots, Q_e)$ is a *Van den Dries-field of corank e* if $G(L) \cong \hat{D}_e(2)$. The condition on the absolute Galois group of L is responsible for the unique feature of the Van den Dries-fields among n the PRCe fields.

For example we have the following:

LEMMA 2.2. *If L is a* PRC *field and $G(L) \cong \hat{D}_e(2)$, then L has exactly e orderings $Q_1, \dots, Q_e$. They satisfy $Q_1 \cap \dots \cap Q_e = L^{*2}$ and $(L, Q_1, \dots, Q_e)$ is a Van den Dries-field.*

Proof. By assumption $G(L)$ is generated by e involutions $\varepsilon_1, \dots, \varepsilon_e$ which are not conjugate to each other. Hence they induce e distinct orderings $Q_1, \dots, Q_e$ on L. By Proposition 1.2, $Q_1, \dots, Q_e$ induce distinct topologies on L. If $x \in Q_1 \cap \dots \cap Q_e$, then $\sqrt{x} \in \tilde{L}_1 \cap \dots \cap \tilde{L}_e = \tilde{L}(\varepsilon_1, \dots, \varepsilon_e) = L$. Hence $Q_1 \cap \dots \cap Q_e = L^{*2}$ and consequently every ordering of L contains $Q_1 \cap \dots \cap Q_e$. It follows from Van den Dries [11, p. 90] that $Q_1, \dots, Q_e$ are the only orderings of L. Q.E.D.

The above lemma is also true for Geyer-fields if we replace $\hat{D}_e(2)$ by $\hat{D}_e$. However its following converse holds only for Van den Dries-fields.

LEMMA 2.3. *Let $\mathscr{L} = (L, Q_1, \dots, Q_e)$ be a Van den Dries-field. Then:*

(a) *The structure $\mathscr{L}$ has no proper algebraic extensions.*

(b) *If $\varepsilon_1, \dots, \varepsilon_e$ are involutions of $G(L)$ that induce $Q_1, \dots, Q_e$ on L, then they generate $G(L)$.*

(c) *Conversely, if $\varepsilon_1, \dots, \varepsilon_e$ are involutions that generate $G(L)$, then they induce $Q_1, \dots, Q_e$ on L (possibly after re-enumeration).*

Proof. (a) (Van den Dries [11, p. 77].) Let $\mathscr{L}'=(L', Q_1', \dots, Q_e')$ be a proper algebraic extension of $\mathscr{L}$. Without loss of generality we may assume that $[L':L]<\infty$. Let N be a finite normal extension of L that contains L'. Then $\mathscr{G}(N/L)$ is a 2-group. Hence L has a quadratic extension $L(\sqrt{x})$ which is contained in L'. It follows, by Lemma 2.2, that $x=(\sqrt{x})^2=Q_1\cap\dots\cap Q_e=L^{*2}$, a contradiction.

(b) $Q_1, \dots, Q_e$ can be extended to $\tilde{L}(\varepsilon_1, \dots, \varepsilon_e)$. Hence, by (a), $\tilde{L}(\varepsilon_1, \dots, \varepsilon_e)=L$.

(c) The involutions $\varepsilon_1, \dots, \varepsilon_e$ are "free" generators of $G(L)$ and as such they are not conjugate to each other. Hence they induce e distinct orderings on L, which are exactly $Q_1, \dots, Q_e$, by Lemma 2.2. Q.E.D.

3. The elementary equivalence theorem for Van den Dries-fields

Proposition 1.3 (d) gives an elementary characterisation of PRCe fields in the language $\mathscr{L}_e$ of e-fold ordered fields. Lemma 2.1 provides an elementary characterisation of fields L with $G(L)\cong\hat{D}_e(2)$. Together we have:

LEMMA 3.1. *There is an explicit (primitive recursive) set Δ_e of sentences of $\mathscr{L}_e$ such that an e-fold ordered field $(E, P_1, \dots, P_e)$ is a Van den Dries-field if and only if it satisfies Δ_e.*

If $\mathscr{E}=(E, P_1, \dots, P_e)$ is an e-fold ordered field and L is a subfield of E, then $\tilde{L}\cap\mathscr{E}=(\tilde{L}\cap E, \tilde{L}\cap P_1, \dots, \tilde{L}\cap P_e)$ is a substructure of E. Similar to Geyer-fields we have the following theorem for Van den Dries-fields.

THEOREM 3.2. *Let $\mathscr{E}=(E, P_1, \dots, P_e)$ and $\mathscr{F}=(F, Q_1, \dots, Q_e)$ be two Van den Dries-fields and let L be a common subfield of E and F. If $\tilde{L}\cap\mathscr{E}\cong_L\tilde{L}\cap\mathscr{F}$, then $\mathscr{E}\equiv_L\mathscr{F}$.*

Proof. Without loss of generality we may assume that $\tilde{L}\cap\mathscr{E}=\tilde{L}\cap\mathscr{F}=(M, S_1, \dots, S_e)$. By Lemma 2.3 there exist involutions $\varepsilon_1, \dots, \varepsilon_e$ that generate $G(E)$ and induce $P_1, \dots, P_e$ on E, respectively. Let $\gamma_i=\mathrm{Res}_{\tilde{M}}\varepsilon_i$, for $i=1, \dots, e$. Then $\gamma_1, \dots, \gamma_e$ are involutions that generate $G(M)$ and induce $S_1, \dots, S_e$ on M, respectively. For each $1\leq i\leq e$, the fields $\tilde{M}(\sigma_i)$ and F are linearly disjoint over M, hence Q_i can be extended to an ordering of $\tilde{M}(\sigma_i)F$. Choose an involution ζ_i that induces this ordering. Then $\mathrm{Res}_{\tilde{M}}\zeta_i=\gamma_i$. By Lemma 2.3, $\zeta_1, \dots, \zeta_e$ generate $G(F)$. The map $\zeta_i\mapsto\varepsilon_i$ for $i=1, \dots, e$ can be extended to an isomorphism $\varphi: G(F)\to G(E)$ such that $\mathrm{Res}_{\tilde{M}}\varphi(\sigma)=\mathrm{Res}_{\tilde{M}}\sigma$, since both $G(E)$ and $G(F)$ are isomorphic to $\hat{D}_e(2)$. If follows from Theorem 3.2 of [5] that $\mathscr{E}\equiv_M\mathscr{F}$. Q.E.D.

Remark. Note that the proof of Theorem 3.2 is easier than the proof of the corresponding theorem for Geyer-fields (Theorem 5.4 of [5]), since Lemma 2.3 makes the use of the Gaschütz-type Lemma 5.3 of [5] redundant.

COROLLARY 3.3. *If* $(E, P_1, \ldots, P_e) \subseteq (F, Q_1, \ldots, Q_e)$ *are two Van den Dries-fields, then* $(E, P_1, \ldots, P_e) \prec (F, Q_1, \ldots, Q_e)$; *in other words, the theory of Van den Dries-fields of corank e is model complete.*

4. On the existence of Van den Dries-fields

We have the following connection between the two types of fields.

PROPOSITION 4.1. *Every Geyer-field* $(K, P_1, \ldots, P_e)$ *has an algebraic extension* $(K', P'_1, \ldots, P'_e)$ *which is a Van den Dries-field.*

Proof. By Lemma 4.3 of [5], the group $G(K)$ is generated by e involutions $\varepsilon_1, \ldots, \varepsilon_e$ that induce $P_1, \ldots, P_e$ on K. Denote by N the maximal 2-extension of K. Then $\mathcal{G}(N/K)$ is the maximal 2-quotient of $G(K)$. Therefore $\mathcal{G}(N/K) \cong \hat{D}_e(2)$ and $\bar{\varepsilon}_i = \mathrm{Res}_N \varepsilon_i$, $i=1, \ldots, e$, generate $\mathcal{G}(N/K)$. Denote by $\bar{Q}_i$ the ordering of $N(\bar{\varepsilon}_i) = N \cap \tilde{K}(\varepsilon_i)$ which is induced by ε_i. Let D be a 2-sylow subgroup of $G(K)$. Its fixed field $\tilde{K}(D)$ has an odd degree over K and therefore it is linearly disjoint from N, hence from $N(\bar{\varepsilon}_i)$, over K. It follows that $N(\bar{\varepsilon}_i)\tilde{K}(D)$ has an odd degree over $N(\bar{\varepsilon}_i)$, hence $\bar{Q}_i$ extends to an ordering $\bar{Q}'_i$ of $N(\varepsilon_i)\tilde{K}(D)$. Let ε'_i be an involution of D that induces $\bar{Q}'_i$ on $N(\varepsilon_i)\tilde{K}(D)$. The map $\bar{\varepsilon}_i \mapsto \varepsilon'_i$, for $i=1, \ldots, e$, can be extended to a homomorphism of $\mathcal{G}(N/K)$ into D and the map Res: $\langle \varepsilon'_1, \ldots, \varepsilon'_e \rangle \to \mathcal{G}(N/K)$ is its inverse. It follows that $\langle \varepsilon'_1, \ldots, \varepsilon'_e \rangle \cong D_e(2)$.

If we write $K' = \tilde{K}(\varepsilon'_1, \ldots, \varepsilon'_e)$ and denote by P'_i the ordering of K' induced by ε'_i, then $K' = (K', P'_1, \ldots, P'_e)$ is an e-fold ordered field that extends $\mathcal{K} = (K, P_1, \ldots, P_e)$ and $G(K) \cong \hat{D}_e(2)$. By Proposition 1.2(c), K' is a PRC field. Hence, by Lemma 2.2, $\mathcal{K}'$ is a Van den Dries field. Q.E.D.

5. The identification of Van den Dries-fields

Van den Dries considers in his thesis [11] the theory OF_e of e-fold ordered fields in the language $\mathcal{L}_e$. He proves that OF_e has a unique model companion $\overline{OF}_e$, which is, by definition, a theory in $\mathcal{L}_e$ such that (i) each model of $\overline{OF}_e$, is a model of OF_e, (ii) each model of OF_e can be embedded in a model of $\overline{OF}_e$, and (iii) $\overline{OF}_e$ is model complete. He shows that an e-fold ordered field $\mathscr{E}=(E, P_1, \ldots, P_e)$ is a model of $\overline{OF}_e$ if and only if it has the following two properties:

(α) For every irreducible polynomial $f \in E[T, X]$ and every $a_0 \in E$ such that $f(a_0, X)$ changes sign on E with respect to each of the P_i, there exist $a, b \in E$ such that $f(a, b)=0$.

(β) $P_1, \ldots, P_e$ are independent.

A. Prestel identifies the models of $\overline{OF}_e$ as those PRCe fields which have no proper algebraic extensions (see [9, Theorem 2.4]).

While we are unable to prove directly that (α) and (β) are equivalent to our axioms of Van den Dries-fields, nor can we do it for Prestel's characterization, we can still prove it using a model theoretic criterion.

THEOREM 5.1. *The theory of Van den Dries-fields is the model companion of OF_e. In other words, an e-fold ordered field $\mathscr{E}$ is a model of $\overline{OF}_e$ if and only if it is a Van den Dries-field.*

Proof. The theory of Van den Dries-fields is model complete, by Corollary 3.3. Hence it suffices to prove that every e-fold ordered field $\mathscr{L}=(L, Q_1, \ldots, Q_e)$ is contained in a Van den Dries-field. Using the diagram of $\mathscr{L}$, and a compactness argument one sees that it suffices to consider the case where L is countable. Let t be a trancendental element over L, and extend $Q_1, \ldots, Q_e$ to orderings $Q_1', \ldots, Q_e'$ of $L(t)$. Note that $L(t)$ is a Hilbertian field (cf. Lang [7, p. 155]). Hence, by Theorem 6.7 of [5], $(L(t), Q_1', \ldots, Q_e')$ has an extension $\mathscr{E}$ which is a Geyer-field. By Proposition 4.1, $\mathscr{E}$ has an extension $\mathscr{E}'$ which is a Van den Dries-field. $\mathscr{E}'$ is the desired extension of $\mathscr{L}$. Q.E.D.

COROLLARY 5.2. *If $(E, P_1, \ldots, P_e)$ is a model of $\overline{OF}_e$, then $G(E) \cong \hat{D}_e(2)$.*

Remark. Corollary 5.2 is a special case of Theorem 3.13 stated in [11] without a proof.

6. The theory of almost all $\mathscr{K}_\sigma'$

Suppose now that K is a countable Hilbertian field equipped with e orderings $P_1, \ldots, P_e$. Let $\bar{K}_1, \ldots, \bar{K}_e$ be some fixed real closures of K that induce the orderings $P_1, \ldots, P_e$, respectively. For every $\sigma_1, \ldots, \sigma_e \in G(K)$ let $K_\sigma=\bar{K}_1^{\sigma_1} \cap \ldots \cap \bar{K}_e^{\sigma_e}$ and denote by $P_{\sigma 1}, \ldots, P_{\sigma e}$ the orderings of K induced by $\bar{K}_1^{\sigma_1}, \ldots, \bar{K}_e^{\sigma_e}$, respectively. Then $\mathscr{K}_\sigma=(K_\sigma, P_{\sigma 1}, \ldots, P_{\sigma e})$ is an e-fold ordered field that extend $\mathscr{K}=(K, P_1, \ldots, P_e)$.

The group $G(K)^e$ is equipped with a unique normalized Harr measure. With respect to this measure we have proved in [5, Theorem 6.7] that $\mathcal{K}_0$ is a Geyer-field for almost all $\sigma \in G(K)^e$. Let N_σ be the maximal 2-extension of K_σ. By proposition 4.1, $\mathcal{K}_\sigma$ has an algebraic Van den Dries extension $\mathcal{K}'_\sigma=(K'_\sigma, P'_{\sigma 1}, \dots, P'_{\sigma e})$ such that $N_\sigma \cap \bar{K}_{\sigma i}=N_\sigma \cap \bar{K}'_{\sigma i}$, for $i=1, \dots, e$. For those σ's such that $\mathcal{K}_\sigma$ is not a Geyer-field we let $\mathcal{K}'_\sigma=\mathcal{K}_\sigma$.

Recall that an ultrafilter $\mathcal{D}$ of $G(K)^e$ is said to be regular if $\mathcal{D}$ contains all subsets of $G(K)^e$ of measure 1 (cf. [6, p. 288]).

LEMMA 6.1. *Let* $\boldsymbol{\tau} \in G(K)^e$ *be an e-tuple such that* $G(K_\tau)$ *is a pro-2-group. Then* $G(K)^e$ *has a regular ultrafilter* $\mathcal{D}$ *such that* $\mathcal{K}_\tau=\tilde{K} \cap \prod K_\sigma/\mathcal{D}$.

Proof. Let L be a finite Galois extension of K and consider the non-empty open subset of $G(K)^e$,

$$S(L)=\{\sigma \in G(K)^e \mid \mathrm{Res}_L \sigma_i=\mathrm{Res}_L \tau_i \quad \text{for } i=1, \dots, e\}.$$

If L' is a finite Galois extension of K that contains L, then $S(L') \subseteq S(L)$. It follows that the intersection of finitely many sets of the form $S(L)$ is a non-empty open set. By [6, Lemma 6.1] there exists a regular ultrafilter $\mathcal{D}$ of $G(K)^e$ that contains all sets $S(L)$.

Let $F=K'_\sigma/\mathcal{D}$, $Q_i=\prod P'_{\sigma i}/\mathcal{D}$ and $\bar{F}_i=F \cap \prod \bar{K}'_{\sigma i}/\mathcal{D}$. Then $\mathcal{F}=(F, Q_1, \dots, Q_n)=\prod K'_\sigma/\mathcal{D}$ and $\bar{F}_i$ is the real closure of F with respect to Q_i.

Consider a finite Galois extension L of K. If $\sigma \in S(L)$, then $L \cap \bar{K}_{\sigma i}=L \cap \bar{K}_{\tau i}$ and $L \cap K_\sigma=L \cap K_\tau$. The group $G(L/L \cap K_\tau)$ is a 2-group. Therefore LK_σ is a 2-extension of K_σ, hence it is contained in the maximal 2-extension N_σ of K_σ. It follows that

$$L \cap \bar{K}'_{\sigma i}=L \cap N_\sigma \cap \bar{K}'_{\sigma i}=L \cap N_\sigma \cap \bar{K}_{\sigma i}=L \cap \bar{K}_{\sigma i}=L \cap \bar{K}_{\tau i}.$$

As $S(L) \in \mathcal{D}$, we conclude that $L \cap \bar{F}_i=L \cap \bar{K}_{\tau i}$ for $i=1, \dots, e$.

We let L run over all finite Galois extensions of K and have that $\tilde{K} \cap \bar{F}_i=\bar{K}_{\tau i}$ for $i=1, \dots, e$. Hence $\tilde{K} \cap \mathcal{F}=\mathcal{K}_\tau$. Q.E.D.

THEOREM 6.2. *Let* K *be a countable and Hilbertian field, and let* $\mathcal{K}=(K, P_1, \dots, P_e)$ *be an e-fold ordered field. Then a sentence* Θ *of* $\mathcal{L}_e(K)$ *is true in all Van den Dries-fields of corank e that extends* $\mathcal{K}$ *if and only if* Θ *is true in* $\mathcal{K}'_\sigma$ *for almost all* $\sigma \in G(K)^e$.

Proof. Almost all the structures $\mathcal{K}'$ are Van den Dries-fields of corank e. This provides one direction of the theorem.

Suppose in the other direction that Θ is true in $\mathcal{K}'_\sigma$ for almost all $\sigma \in G(K)^e$ and let $\mathcal{E}=(E, Q_1, \ldots, Q_e)$ be a Van den Dries-field that extends $\mathcal{K}$. Then $\tilde{K} \cap \mathcal{E}=\mathcal{K}_\tau$ for some $\tau \in G(K)^e$ and $G(K_\tau)$ is a pro-2-group. By Lemma 6.1, there exists a regular ultrafilter $\mathcal{D}$ of $G(K)^e$ such that $\tilde{K} \cap \Pi \mathcal{K}'_\sigma/\mathcal{D}=\mathcal{K}_\tau$. It follows from Theorem 3.2, that $\Pi \mathcal{K}'/\mathcal{D} \equiv_K \mathcal{E}$. The sentence Θ is true in $\Pi \mathcal{K}'_\sigma/\mathcal{D}$, since $\mathcal{D}$ is regular, hence it is also true in $\mathcal{E}$. Q.ED.

The special case where $K=\mathbf{Q}$ and $P_1=\ldots=P_e$=the unique ordering of $\mathbf{Q}$ provides our final characterisation of the theory of Van den Dries-field of corank e.

COROLLARY 6.3. *A sentence Θ of $\mathcal{L}_e$ is true in all Van den Dries-fields of corank e if and only if it is true in $\mathcal{Q}'_\sigma$* for almost all $\sigma \in G(\mathbf{Q})^e$.

Remark. Van den Dries proves in [11, p. 74] that the theory $\overline{OF}_e$ is decidable. In [5] we show that the theory of Geyer-fields of corank e is decidable. It is not difficult to modify the proof of [5] and to get a second proof for the decidability of $\overline{OF}_e$ which is based on Corollary 6.3 and Theorem 5.1.

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