THE ABSOLUTE GALOIS GROUP OF A PSEUDO p-ADICALLY CLOSED FIELD

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Introduction

The main problem in Galois theory is to describe the absolute Galois group $G(K)$ of a field K. This problem is solved in the local case, i.e., when K is algebraically, real or p-adically closed. In the first case $G(K)$ is trivial, in the second $G(K) \cong \mathbb{Z}/2\mathbb{Z}$ and in the third case it is given by generators and relations (Jannsen-Wingberg [JW] and Wingberg [W]). The next case to consider is when K is "pseudo closed". A field K is called **pseudo algebraically** (resp., **real, p-adically closed**) (abbreviation : PAC, PRC and PpC, respectively) if every absolutely irreducible variety V defined over K has a K-rational point, provided V has a $\overline{K}$-rational simple point for each algebraic (resp., real, p-adic) closure $\overline{K}$ of K. The absolute Galois group of a pseudo closed field is best described in terms of solvability of Γ-embedding problems, where Γ is 1 (resp., $\mathbb{Z}/2\mathbb{Z}$, $G(\mathbb{Q}_p)$):

Let G be a profinite group. Consider a diagram

$$\begin{array}{ccc} & & G \\ (1) & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow{\alpha} & A \end{array}$$

where α is an epimorphism of finite groups and φ is a homomorphism. We call (1) a finite Γ-**embedding problem** for G if for each closed subgroup H of G which is isomorphic to Γ there exists a homomorphism $\gamma_H\colon H \to B$ such that $\alpha \circ \gamma_H = \mathrm{Res}_H \varphi$. The Γ-embedding problem (1) is **solvable** if there exists a homomorphism $\gamma\colon G \to B$ such that $\alpha \circ \gamma = \varphi$. We call G Γ-**projective** if every finite Γ-embedding problem for G is solvable, and if the collection of all closed subgroups of G which are isomorphic to Γ is topologically closed. For $\Gamma = 1$ (resp., $\Gamma = \mathbb{Z}/2\mathbb{Z}$, $\Gamma = G(\mathbb{Q}_p)$) we obtain **projective** (resp., **real projective, p-adically projective**) groups. Note that the local-global principle included in the definition of pseudo closed fields is also reflected in the definition of Γ-projective groups.

THEOREM: *If K is a* PAC *(resp.,* PRC, PpC*) field, then $G(K)$ is projective (resp., real projective, p-adically projective). Conversely, if G is a projective (resp. real projective, p-adically projective) group, then there exists a* PAC *(resp.,* PRC, PpC*) field K such that $G(K) \cong G$.*

Ax [A1, p. 269] and Lubotzky-v.d. Dries [LD, p. 44] prove the theorem for PAC fields. We prove the theorem for PRC fields in [HJ]. The goal of this work is to prove the theorem for PpC fields.

As in the PRC case, the easier direction is to prove that if K is PpC, then $G(K)$ is p-adically projective. For the converse we must develop a theory of $G(\mathbb{Q}_p)$-structures, which replaces the Artin-Schreier structures of the PRC case.

*There are two intrinsic difficulties in going over from PRC fields to PpC fields. The first one is that the group Γ is no longer the finite group $\mathbb{Z}/2\mathbb{Z}$ but rather the infinite group $G(\mathbb{Q}_p)$. Fortunately $G(\mathbb{Q}_p)$ is finitely generated and with a trivial center. So we consider in Part A of the work a finitely generated profinite group Γ with a trivial center and define a Γ-***structure*** as a structure $\mathbf{G} = \langle G, X, d\rangle$, where G is a profinite group which acts continuously and regularly on a Boolean space X (i.e., for each $x \in X$ and $\sigma \in G$ the equality $x^\sigma = x$ implies $\sigma = 1$), and d is a continuous map from X into $\mathrm{Hom}(\Gamma, G)$ which commutes with the action of G. The assumption that Γ is finitely generated implies that $\mathrm{Hom}(\Gamma, G)$ is a Boolean space. The regularity assumption is essential in constructing cartesian squares of Γ-structures. The latter are essential in reducing arbitrary embedding problems to finite embedding problems. In Section 5 we associate a Γ-structure $\mathbf{G}$ with each Γ-projective group G and prove that $\mathbf{G}$ is projective. The proof depends on an extra assumption which we make on Γ. For each e and m, $0 \le e \le m$, we consider the free product $\Gamma_{e,m}$ of e copies of Γ and the free profinite group $\widehat{F}_{m-e}$. We assume that Γ has a finite quotient $\overline{\Gamma}$ with this property: each closed subgroup H of $\Gamma_{e,m}$ which is a quotient of Γ and has $\overline{\Gamma}$ as a quotient (we call it a* **large quotient** *of Γ) is isomorphic to Γ.*

The second difficulty that arises in dealing with PpC fields is that two p-adic closures E and F of a field K are not necessarily K-isomorphic. Fortunately Macintyre [M] gives a criterion for isomorphism: $E \cong_K F$ if and only if $K \cap E^n = K \cap F^n$ for each $n \in \mathbb{N}$. As $E^\times/(E^\times)^n \cong \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$, E is characterized up to K-isomorphism by a homomorphism $\varphi\colon K^\times \to \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$ with $K^\times \cap E^n$ as the kernel of the induced map $K^\times \to \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$, $n \in \mathbb{N}$. In addition, the unique p-adic valuation defines a place $\pi\colon K \to \mathbb{Q}_p \cup \{\infty\}$ such that $\pi(u) \in \mathbb{Q}_p^\times$ implies $\pi(u) = \varphi(u)$. Here we have identified

$\mathbb{Q}_p^\times$ *as a subgroup of* $\Phi = \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$. *We let* $\Theta = \mathbb{Q}_p \cup \{\infty\} \cup \Phi$ *and call* $\theta = (\pi, \varphi)$ *a* Θ-**site** *of* K. *An extension of* $\mathbb{Q}_p$ *to* $\tilde{\mathbb{Q}}_p$ *replaces* Θ *by* $\tilde{\Theta}$ *and* Θ*-sites by* $\tilde{\Theta}$*-sites. With every Galois extension* L/K *we associate the* **space of sites** $X(L/K)$. *This is the collection of all* $\tilde{\Theta}$*-sites* θ *of* L *such that* $\theta(K) \subseteq \Theta$. *It is a Boolean space and* $\mathcal{G}(L/K)$ *acts continuously and regularly on it. It replaces the space of orderings of Artin-Schreier structures. We also use regularity to define a map* d: $X(L/K) \to \mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L/K))$ *which commutes with the action of* $\mathcal{G}(L/K)$. *A version of Krasner's lemma proves that* d *is continuous. Thus* $\mathbf{G}(L/K) = \langle \mathcal{G}(L/K), X(L/K), d \rangle$ *is a* $G(\mathbb{Q}_p)$*-structure.*

In Section 11 we generalize a theorem of Neukirch and characterize $\tilde{\mathbb{Q}} \cap \mathbb{Q}_p$ *by a large finite quotient of* $G(\mathbb{Q}_p)$. *Then we realize, for* $\Gamma = G(\mathbb{Q}_p)$, *each* $\Gamma_{e,m}$ *as absolute Galois group of a field* K, *algebraic over* K *(Section 12). The combination of these results shows that the above assumptions on* Γ *are satisfied in this case.*

In part C we construct for each $G(\mathbb{Q}_p)$*-structure* $\mathbf{G}$ *a Galois extension* F/E *such that* E *is PpC and* $\mathbf{G}(F/E) \cong \mathbf{G}$. *The restriction map* Res: $\mathbf{G}(\widetilde{E}/E) \to \mathbf{G}(F/E)$ *is a* **cover** *(i.e., if* $x, x' \in X(\widetilde{E}/E)$ *are mapped onto the same element of* $X(F/E)$, *then* $x' = x^\sigma$ *for some* $\sigma \in G(E)$*). Hence, if* $\mathbf{G}$ *is projective, Res has a section and therefore* $\mathbf{G} \cong \mathbf{G}(\widetilde{E}/E_1)$ *for some algebraic extension* E_1 *of* E. *Unfortunately unlike for PAC and PRC fields,* E_1 *need not be PpC. However, an extra transcendental construction finally proves the existence of a PpC field* K *such that* $\mathbf{G}(K) \cong \mathbf{G}$. *In particular* $G(K) \cong G$. *This concludes the proof of the Theorem for PpC fields.*

Notation

$\widetilde{K}$ = *the algebraic closure of a field* K.

If K *is a field of characteristic 0, then* $K_{\rm alg} = K \cap \tilde{\mathbb{Q}}$,

$G(K)$ = *the absolute Galois group of* K.

For a place π *of a field* K, $O_\pi = \{x \in K|\, \pi(x) \neq \infty\}$ *is the valuation ring and* $U_\pi = \{u \in K|\, \pi(u) \neq 0, \infty\}$ *is the group of units and* $\pi(K) = \pi(O_\pi)$ *is the residue field of* π.

If S *is a set of automorphisms of a field* F, *then* $F(S)$ *is the fixed field of* S *in* F. *In particular for* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$, $F(\boldsymbol{\sigma})$ *is the fixed field of* $\sigma_1, \ldots, \sigma_m$ *in* F.

For an abelian group A *and a prime* l, A_l *is the* l*-torsion part of* A.

$\mathbb{Q}_p$ = *the field of* p*-adic numbers.*

$\mathbb{Q}_{p,\rm alg}$ = *the algebraic part of* $\mathbb{Q}_p$.

$\mathbb{Z}_p$ = *the ring of* p*-adic integers.*

$\mathbb{Z}_p^\times$ = *the group of units of* $\mathbb{Z}_p$.

$\mathbb{F}_p$ = *the field with* p*-elements.*

In Part A, Γ *is a fixed finitely generated group; in Parts B and C,* $\Gamma = G(\mathbb{Q}_p)$.

Part A. Γ-Structures.

We fix for all of Part A a finitely generated profinite group Γ. In particular Γ has for each $n \in \mathbb{N}$ only finitely many open subgroups of index n. In Sections 1 and 2 we define and discuss Γ-structures. Later (Section 3) we require that Γ share some properties with $G(\mathbb{Q}_p)$. This is used to prove properties of Γ-projective groups and projective Γ-structures (Section 4 and 5).

1. Definition of Γ-structures.

Recall that a **Boolean space** *X is an inverse limit of finite discrete spaces. Alternatively X is a totally disconnected compact Hausdorff space [HJ, Definition 1.1]. A* **profinite transformation group** *is a pair (X, G), with X a Boolean space and G a profinite group that acts continuously on X: $(x, \sigma) \mapsto x^\sigma$.*

For each profinite group G consider the collection $\mathrm{Hom}(\Gamma, G)$ of continuous homomorphisms from Γ into G. Each homomorphism $h\colon G \to G'$ naturally induces a map $h_*\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G')$ by $h_*(\psi) = h \circ \psi$. Thus $\mathrm{Hom}(\Gamma, G) = \varprojlim \mathrm{Hom}(\Gamma, G/N)$, where N ranges over all open normal subgroups of G. Since each $\psi \in \mathrm{Hom}(\Gamma, G/N)$ is determined by its values on a finite set of generators of Γ, and since G/N is finite, $\mathrm{Hom}(\Gamma, G/N)$ is a finite set. It follows that $\mathrm{Hom}(\Gamma, G)$ is a Boolean space. Obviously, the above map h_* is continuous.

The group G acts continuously on $\mathrm{Hom}(\Gamma, G)$ by

$$\psi^\tau(g) = \tau^{-1}\psi(g)\tau, \qquad \psi \in \mathrm{Hom}(\Gamma, G),\ \ \tau \in G,\ \ g \in \Gamma.$$

Thus $(\mathrm{Hom}(\Gamma, G), G)$ is a profinite transformation group and (h_*, h) is a morphism of profinite transformation groups (i.e., $h_*(\psi^\tau) = h_*(\psi)^{h(\tau)}$ for $\psi \in \mathrm{Hom}(\Gamma, G)$ and $\tau \in G$).

For a profinite group G denote the set of all closed subgroups of G by $\mathrm{Subg(G)}$. Each homomorphism $h\colon G \to G'$ maps closed subgroups of G onto closed subgroups of G' and thus naturally induces a map $h_*\colon \mathrm{Subg(G)} \to \mathrm{Subg(G')}$. Compactness of G implies that $\mathrm{Subg(G)} = \varprojlim \mathrm{Subg(G/N)}$, where N ranges over all open normal subgroups. Thus $\mathrm{Subg(G)}$ is a Boolean space.

Let Im: $\mathrm{Hom}(\Gamma, G) \to \mathrm{Subg(G)}$ *be the map that assigns to each* $\psi \in \mathrm{Hom}(\Gamma, G)$ *its image* $\mathrm{Im}(\psi) = \psi(\Gamma)$ *in* G. *For an open normal subgroup* N *of* G *let* $\psi_N \in \mathrm{Hom}(\Gamma, G/N)$ *be the homomorphism induced by* ψ. *A standard compactness argument shows that* $\mathrm{Im}(\psi) = \varprojlim \mathrm{Im}(\psi_N)$. *Therefore* Im: $\mathrm{Hom}(\Gamma, G) \to \mathrm{Subg(G)}$ *is the inverse limit of the maps* Im: $\mathrm{Hom}(\Gamma, G/N) \to \mathrm{Subg(G/N)}$. *In particular Im is a continuous map.*

DEFINITION 1.1: A **weak Γ-structure** is a system $\mathbf{G} = \langle G, X, d\rangle$, where G is a profinite group, X is a Boolean space on which G continuously acts, and d: $X \to \mathrm{Hom}(\Gamma, G)$ is a continuous map such that

$$d(x^\sigma) = d(x)^\sigma \text{ for all } x \in X \text{ and } \sigma \in G. \tag{1}$$

Call $\mathbf{G}$ a Γ-**structure** if in addition the action of G on X is **regular**, i.e.,

(2) for each $x \in X$, $x^\sigma = x$ implies $\sigma = 1$.

We call X the **space of sites**, d the **forgetful map** and X/G the **space of orbits of G**. The latter quotient space is Boolean [HJ, Claim 1.6]. For $x \in X$ we call $D(x) = \mathrm{Im}(d(x))$ the **decomposition group** of x. By (1), $D(x^\sigma) = D(x)^\sigma$ for all $x \in X$ and $\sigma \in G$. Since Im is continuous so is the map $x \mapsto D(x)$ from X into Subg(G).

Unless explicitly stated otherwise, the underlying group, the space of sites and the forgetful map of a Γ-structure $\mathbf{G}$ will be denoted by G, $X(\mathbf{G})$ and d, respectively. Analogously for $\mathbf{H}$, $\mathbf{A}$, $\mathbf{B}$, etc.

A weak Γ structure $\mathbf{G}$ is said to be **finite** if both G and $X(\mathbf{G})$ are finite.

DEFINITION 1.2: A **morphism** φ: $\mathbf{H} \to \mathbf{G}$ of (weak) Γ-structures is a pair consisting of a continuous homomorphism φ: $H \to G$ and a continuous map φ: $X(\mathbf{H}) \to X(\mathbf{G})$ such that

(3a) $\varphi(x^\sigma) = \varphi(x)^{\varphi(\sigma)}$ for all $x \in X(\mathbf{H})$ and $\sigma \in H$; and

(3b) $d(\varphi(x)) = \varphi \circ d(x)$ for all $x \in X(\mathbf{H})$.

Call a morphism φ: $\mathbf{H} \to \mathbf{G}$ an **epimorphism** if $\varphi(H) = G$ and $\varphi(X(\mathbf{H})) = X(\mathbf{G})$. The epimorphism φ is a **cover** if

(3c) for all $x, x' \in X(\mathbf{H})$ such that $\varphi(x) = \varphi(x')$ there exists $\sigma \in H$ such that $x^\sigma = x'$.

If φ: $\mathbf{H} \to \mathbf{G}$ is a morphism, then the map φ: $X(\mathbf{H}) \to X(\mathbf{G})$ induces a continuous map $\bar{\varphi}$: $X(\mathbf{H})/H \to X(\mathbf{G})/G$ of the respective orbit spaces. Note that φ is a cover if and only if

(3c$'$) $\varphi(H) = G$ and $\bar{\varphi}$ is a bijection (therefore a homeomorphism).

Also

(3d) if $\mathbf{H}$ and $\mathbf{G}$ are Γ structures, then σ in (3c) is unique (by (2)) and $\sigma \in \mathrm{Ker}(\varphi)$ (by (3a)).

Next we consider quotients of weak Γ-structures. Let $\mathbf{G} = \langle G, X, d\rangle$ be a weak Γ-structure and N a closed normal subgroup of G. Let $\varphi_N = (h, \eta)$: $(X, G) \to (X/N, G/N)$ be the canonical quotient map of transformation groups [HJ, Claim 1.6]. Define $\bar{d}$: $X/N \to \mathrm{Hom}(\Gamma, G/N)$ by $\bar{d}(h(x)) = \eta \circ d(x)$, for $x \in X$. Thus the homomorphism η_*: $\mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G/N)$ induced by η (Section 1) satisfies $\eta_* \circ d = \bar{d} \circ h$. Since the maps η_* and d are continuous and h is open [HJ, Claim 1.6], $\bar{d}$ is continuous. It follows that $\mathbf{G}/N = \langle G/N, X/N, \bar{d}\rangle$ is a weak Γ-structure and φ_N: $\mathbf{G} \to \mathbf{G}/N$ is a cover. Moreover, if $\mathbf{G}$ is a Γ-structure, then so is $\mathbf{G}/N$. Conversely, each morphism φ: $\mathbf{G} \to \mathbf{G}'$ of weak Γ-structures with $N \le \mathrm{Ker}(\varphi)$ canonically induces a morphism $\bar{\varphi}$: $\mathbf{G}/N \to \mathbf{G}'$ such that $\bar{\varphi} \circ \varphi_N = \varphi$. If $\mathbf{G}'$ is a Γ-structure, φ is a cover and $\mathrm{Ker}(\varphi) = N$, then $\bar{\varphi}$ is an isomorphism.

An inverse limit of (weak) Γ-structures is a (weak) Γ-structure. Conversely, each weak Γ-structure $\mathbf{G}$ is equal to $\varprojlim \mathbf{G}/N$, where N ranges over all open normal subgroups of G.

Let (X, G) be a profinite transformation group. Recall [HJ, Section 1] that a **partition** of X is a finite collection $Y = \{V_1, \dots, V_n\}$ of disjoint nonempty open-closed subsets of X such that $X = V_1 \cup \cdots \cup V_n$. A partition Y' of X is **finer** than Y if for each $V' \in Y'$ there is $V \in Y$ such that $V' \subseteq V$. Call Y a G-**partition** if in addition for each $\sigma \in G$ and each i, $1 \le i \le n$, there exists j, $1 \le j \le n$, such that $V_i^\sigma = V_j$.

LEMMA 1.3: *Every (weak) Γ-structure $\mathbf{G}$ is an inverse limit of finite (weak) Γ-structures which are epimorphic images of $\mathbf{G}$.*

Proof: By the above remarks we may assume that the group G is finite. Let $\mathcal{P}$ be the family of G-partitions Y of $X(\mathbf{G})$ which

(4a) *are finer than* $\{d^{-1}(\psi)|\ \psi \in \mathrm{Hom}(\Gamma, G)\}$ *(hence* $d_Y(U) = d(x)$ *for* $U \in Y$ *and* $x \in U$ *defines a continuous map* $d_Y\colon Y \to \mathrm{Hom}(\Gamma, G)$*); and*

(4b) *if* $\mathbf{G}$ *is a* Γ*-structure, then* $U^\tau \cap U = \emptyset$ *for all* $U \in Y$ *and* $\tau \in G - \{1\}$.

Each $Y \in \mathcal{P}$ *defines a finite (weak)* Γ*-structure* $\mathbf{G}_Y = \langle G, Y, d_Y\rangle$. *If* Y' *is finer than* Y, *then the map* $U' \mapsto U$ *for* $U \in Y$, $U' \in Y'$ *and* $U' \subseteq U$ *gives a canonical epimorphism* $\mathbf{G}_{Y'} \to \mathbf{G}_Y$. *Moreover, the map* $x \mapsto U$, *for* $U \in Y$ *and* $x \in U$ *defines an epimorphism* $\mathbf{G} \to \varprojlim \mathbf{G}_Y$. *Since both* X *and* $\varprojlim Y$ *are compact and Hausdorff it suffices to prove that this map is injective. In other words, for distinct* $x_1, x_2 \in X$ *show that there exists* $Y \in \mathcal{P}$ *such that* $d_Y(x_1) \neq d_Y(x_2)$.

Indeed, let V *be an open-closed neighborhood of* x_1 *such that* $x_2 \notin V$. *Let* Y' *be a* G*-partition of* X *finer than* $\{V, X - V\}$ *[HJ, Lemma 1.4]. If* $\mathbf{G}$ *is not a* Γ*-structure let* $Y = \{V \cap d^{-1}(\psi), (X - V) \cap d^{-1}(\psi)|\ \psi \in \mathrm{Hom}(\Gamma, G)\}$. *If* $\mathbf{G}$ *is a* Γ*-structure, then each* $x \in X$ *has an open-closed neighborhood* U_x *such that* $x^\tau \notin U_x$ *for each* $\tau \in G - \{1\}$. *Replace* U_x *by* $d^{-1}(d(x)) \cap V \cap U_x - \bigcup_{\tau \in G - \{1\}} U_x^\tau$, *if necessary, to assume that* $U_x \subseteq V \cap d^{-1}(d(x))$ *and* $U_x^\tau \cap U_x = \emptyset$ *for each* $\tau \in G - \{1\}$. *Since* X *is compact, finitely many of these neighborhoods cover* X. *Then there exists a partition* Y_0 *of* X *such that for each* $U \in Y_0$ *and* $x \in X$ *either* $U \subseteq U_x$ *or* $U \cap U_x = \emptyset$. *Finally use [HJ, Lemma 1.4] to choose a* G*-partition* Y *of* X, *finer than* Y_0. *Then* $Y \in \mathcal{P}$. *In each case* $d_Y(x) \neq d_Y(x')$. ∎

LEMMA 1.4: *Each weak* Γ*-structure* $\mathbf{G}$ *with an injective forgetful map is an inverse limit of finite weak* Γ*-structures with injective forgetful maps which are epimorphic images of* $\mathbf{G}$.

Proof: For each open normal subgroup N *of* G *let* $\eta_N\colon G \to G/N$ *be the canonical map. The finite weak* Γ*-structure*

$$\mathbf{G}_N = \langle G/N, \{\eta_N \circ d(x)|\ x \in X(\mathbf{G})\}, \mathrm{inclusion}\rangle$$

is obviously an epimorphic image of G. *If* $x, y \in X(\mathbf{G})$ *and* $x \neq y$, *then* $d(x) \neq d(y)$. *Hence there exists* N *such that* $\eta_N \circ d(x) \neq \eta_N \circ d(y)$. *It follows that* $\mathbf{G} = \varprojlim \mathbf{G}_N$. ∎

2. Basic properties of Γ-structures.

A crucial ingredient in our construction is the existence of fibred products in the category of Γ*-structures. Let* $\alpha_1\colon \mathbf{B}_1 \to \mathbf{A}$ *and* $\alpha_2\colon \mathbf{B}_2 \to \mathbf{A}$ *be morphisms of weak* Γ*-structures. Consider the fibred products* $B_1 \times_A B_2$ *and* $X(\mathbf{B}_1) \times_{X(\mathbf{A})} X(\mathbf{B}_2)$*. For* $i = 1, 2$ *let* $\pi_i\colon B_1 \times_A B_2 \to B_i$ *and* $\pi_i\colon X(\mathbf{B}_1) \times_{X(\mathbf{A})} X(\mathbf{B}_2) \to X(\mathbf{B}_i)$ *be the projection maps. For each* $(x_1, x_2) \in X(\mathbf{B}_1) \times_{X(\mathbf{A})} X(\mathbf{B}_2)$*, we have* $\alpha_1(d(x_1)) = d(\alpha_1(x_1)) = d(\alpha_2(x_2)) = \alpha_2(d(x_2))$*. Hence there exists a unique homomorphism* $\hat{d}(x_1, x_2)\colon \Gamma \to B_1 \times_A B_2$ *such that the following diagram is commutative*

$$\begin{tikzcd}
 & \Gamma \arrow[ddl, "d(x_1)"'] \arrow[d, "\hat{d}(x_1,x_2)"] \arrow[ddr, "d(x_2)"] & \\
 & B_1 \times_A B_2 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
B_1 \arrow[dr, "\alpha_1"'] & & B_2 \arrow[dl, "\alpha_2"] \\
 & A &
\end{tikzcd}$$

Check that the map $\hat{d}\colon X(\mathbf{B}_1) \times_{X(\mathbf{A})} X(\mathbf{B}_2) \to \mathrm{Hom}(\Gamma, B_1 \times_A B_2)$ *defined in this way is continuous. Further let* $B_1 \times_A B_2$ *operate on* $X(\mathbf{B}_1) \times_{X(\mathbf{A})} X(\mathbf{B}_2)$ *componentwise and verify condition (1) of Section 1 for* $\hat{d}$ *to conclude that* $\mathbf{B}_1 \times_{\mathbf{A}} \mathbf{B}_2 = \langle B_1 \times_A B_2, X(\mathbf{B}_1) \times_{X(\mathbf{A})} X(\mathbf{B}_2), \hat{d}\rangle$ *is a weak* Γ*-structure. We call it the* **fibred product** *of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *over* $\mathbf{A}$*. The coordinate projection* $\pi_i\colon \mathbf{B}_1 \times_{\mathbf{A}} \mathbf{B}_2 \to \mathbf{B}_i$*, is a morphism,* $i = 1, 2$*. If both* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are* Γ*-structures, so is* $\mathbf{B}_1 \times_{\mathbf{A}} \mathbf{B}_2$*. If the forgetful maps of both* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are injective so is the forgetful map of* $\mathbf{B}_1 \times_{\mathbf{A}} \mathbf{B}_2$*.*

The proof of the following characterization of fibred products is standard (e.g., [HL, Lemma 1.1]). It is left to the reader.

Lemma 2.1: *Consider a commutative diagram of weak* Γ*-structures.*

(1)
$$\begin{tikzcd}
\mathbf{B} \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & \mathbf{B}_2 \arrow[d, "\alpha_2"] \\
\mathbf{B}_1 \arrow[r, "\alpha_1"] & \mathbf{A}
\end{tikzcd}$$

The following statements are equivalent:

(a) $\mathbf{B}$ *is isomorphic to the fibred product* $\mathbf{B}_1 \times_{\mathbf{A}} \mathbf{B}_2$ *(i.e., there is an isomorphism* $\beta\colon \mathbf{B} \to \mathbf{B}_1 \times_{\mathbf{A}} \mathbf{B}_2$ *such that* $\beta_1 \circ \beta^{-1}$ *and* $\beta_2 \circ \beta^{-1}$ *are the coordinate projections);*

(b) *for each pair of morphisms* $\psi_i\colon \mathbf{C} \to \mathbf{B}_i$, $i = 1, 2$, *of weak* Γ*-structures such that* $\alpha_1 \circ \psi_1 = \alpha_2 \circ \psi_2$ *there is a unique morphism* $\psi\colon \mathbf{C} \to \mathbf{B}$ *such that* $\beta_i \circ \psi = \psi_i$, $i = 1, 2$; *and*

(c) **1.** *for each* $\sigma_i \in B_i$, $i = 1, 2$, *such that* $\alpha_1(\sigma_1) = \alpha_2(\sigma_2)$ *there exists a unique* $\sigma \in B$ *such that* $\beta_i(\sigma) = \sigma_i$, $i = 1, 2$; *and* **2.** *for each* $x_i \in X(\mathbf{B}_i)$, $i = 1, 2$, *such that* $\alpha_1(x_1) = \alpha_2(x_2)$ *there exists a unique* $x \in X(\mathbf{B})$, *such that* $\beta_i(x) = x_i$, $i = 1, 2$.

We call a diagram (1) a **cartesian square** *if it satisfies one of the equivalent conditions of Lemma 2.1.*

LEMMA 2.2: *If in the cartesian square (1)* $\mathbf{A}$ *is a* Γ*-structure and* α_2 *is a cover, then so is* β_1.

Proof: Let $x, x' \in X(\mathbf{B})$ and $\beta_1(x) = \beta_1(x')$. Then $\alpha_2(\beta_2(x)) = \alpha_2(\beta_2(x'))$. Hence there exists $\sigma_2 \in B_2$ such that $\beta_2(x)^{\sigma_2} = \beta_2(x')$. Therefore $\alpha_2(\beta_2(x))^{\alpha_2(\sigma_2)} = \alpha_2(\beta_2(x')) = \alpha_2(\beta_2(x))$. Since $\mathbf{A}$ is a Γ-structure $\alpha_2(\sigma_2) = 1$. Conclude that there exists $\sigma \in B$ such that $\beta_2(\sigma) = \sigma_2$ and $\beta_1(\sigma) = 1$. Thus $\beta_2(x^\sigma) = \beta_2(x')$ and $\beta_1(x^\sigma) = \beta_1(x')$. From Lemma 2.1(c) $x^\sigma = x'$. It follows that β_1 is a cover. ∎

LEMMA 2.3: *Let* $\beta_1\colon \mathbf{B} \to \mathbf{B}_1$ *be an epimorphism of* Γ*-structures and let* K *be a closed normal subgroup of* B *such that* $K \cap \mathrm{Ker}(\beta_1) = 1$. *Let* $\beta_2\colon \mathbf{B} \to \mathbf{B}/K$ *and* $\alpha_1\colon \mathbf{B}_1 \to \mathbf{B}_1/\beta_1(K)$ *be the quotient maps. Denote the unique epimorphism such that* $\alpha_1 \circ \beta_1 = \alpha_2 \circ \beta_2$ *by* $\alpha_2\colon \mathbf{B}/K \to \mathbf{B}_1/\beta_1(K)$. *Then the following diagram is a cartesian square*

$$(2)\qquad \begin{array}{ccc} \mathbf{B} & \xrightarrow{\beta_2} & \mathbf{B}/K \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ \mathbf{B} & \xrightarrow{\alpha_1} & \mathbf{B}_1/\beta_1(K) \end{array}$$

Proof: We leave the proof of 1. of Lemma 2.1(c) to the reader and prove 2. of Lemma 2.1(c). To prove the existence let $x_1 \in X(\mathbf{B}_1)$ and $x_2 \in X(\mathbf{B})/K$ with $\alpha_1(x_1) = \alpha_2(x_2)$.

There exists $x \in X(\mathbf{B})$ such that $\beta_2(x) = x_2$. Since $\alpha_1(x_1) = \alpha_2(\beta_2(x)) = \alpha_1(\beta_1(x))$, there exists $\sigma \in K$ such that $x_1 = \beta_1(x)^{\beta_1(\sigma)} = \beta_1(x^\sigma)$. Finally $x_2 = \beta_2(x) = \beta_2(x^\sigma)$.

For the uniqueness consider $x, x' \in X(\mathbf{B})$ that satisfy $\beta_i(x') = \beta_i(x)$, $i = 1, 2$. There exists $\tau \in K$, such that $x' = x^\tau$. Hence $\beta_1(x) = \beta_1(x') = \beta_1(x)^{\beta_1(\tau)}$. Since $\mathbf{B}_1$ is a Γ-structure, $\beta_1(\tau) = 1$. Conclude from $K \cap \mathrm{Ker}(\beta_1) = 1$ that $\tau = 1$. Hence $x' = x$. ∎

Let (X, G) be a profinite transformation group. A subset X_0 of X is a **system of representatives** *for the G-orbits of X, if for each $x \in X$ there exist $x_0 \in X_0$ and $\sigma \in G$ such that $x = x_0^\sigma$, and if $x_0, x_1 \in X_0$, $\sigma \in G$ and $x_0^\sigma = x_1$ imply $x_0 = x_1$.*

Lemma 2.4: *Let G be a profinite group that acts regularly (Definition 1.1) (and continuously) on a Boolean space X. Then*

(a) *the quotient map $\pi\colon X \to X/G$ has a continuous section; and*

(b) *X has a closed system X_0 of representatives for the G-orbits.*

Proof: Note that assertions (a) and (b) are equivalent. Indeed, if $\lambda\colon X/G \to X$ is a continuous section of π, then $X_0 = \lambda(X/G)$ satisfies (b). If (b) holds, then the restriction of π to X_0 is a homeomorphism onto X/G. Its inverse is a continuous section of π.

We first prove (b) for G finite. Regularity implies that each $x \in X$ has an open-closed neighborhood U_x such that $x^\sigma \notin U_x$ for each $\sigma \in G$, $\sigma \neq 1$. Replace U_x by $U_x - \bigcup_{\sigma\neq 1} U_x^\sigma$, if necessary, to assume that $U_x \cap U_x^\sigma = \emptyset$ for each $\sigma \neq 1$. Since X is compact, a finite collection of such sets, say $U_1, \ldots, U_n$, covers X. Then

$$X_0 = \bigcup_{j=1}^{n} [U_j - (\bigcup_{i=1}^{j-1} \bigcup_{\sigma\in G} U_i^\sigma)]$$

is a closed system of representatives for the G-orbits of X. Indeed, for $x \in X$ let j be the smallest positive integer for which there exists $\sigma \in G$ such that $x^\sigma \in U_j$. Then $x^\sigma \in X_0$ represents x. Also if $x_0, x_1 \in X_0$ and $x_0^\sigma = x_1$ for some $\sigma \in G$, then there exists j, $1 \leq j \leq n$, such that $x_0, x_1 \in U_j$. Hence $\sigma = 1$. From the preceding paragraph (a) is also true.

Now we prove (a) in the general case. Let $\mathcal{L}$ be the collection of all pairs (L, λ), where L is a closed normal subgroup of G and λ is a continuous section of the quotient map $\pi_{L,G}$: $X/L \to X/G$. Partially order $\mathcal{L}$ by defining $(L',\lambda') \geq (L,\lambda)$ if $L' \leq L$ and $\pi_{L',L} \circ \lambda' = \lambda$. By Zorn's Lemma $\mathcal{L}$ has a maximal element (L,λ). If $L \neq 1$, then L has a proper open subgroup L' which is normal in G. Since L/L' is finite $\pi_{L',L}$: $X/L' \to X/L$ has a continuous section , say θ. Then $(L',\theta\circ\lambda) \in \mathcal{L}$ and $(L',\theta\circ\lambda) > (L,\lambda)$, a contradiction. Thus $L = 1$ and (a) holds. ∎

COROLLARY 2.5: *Let α: $\mathbf{G} \to \mathbf{A}$ be a cover of Γ-structures. Then α: $X(\mathbf{G}) \to X(\mathbf{A})$ has a continuous section, and $X(\mathbf{G})$ has a closed system of representatives for its G-orbits.*

Proof: We may assume that α is the quotient map $X(\mathbf{G}) \to X(\mathbf{G})/\mathrm{Ker}(\alpha)$. Now apply Lemma 2.4. ∎

LEMMA 2.6: *Let X be a Boolean space, A a profinite group and d_0: $X \to \mathrm{Hom}(\Gamma, A)$ a continuous map. Then there exists a Γ-structure $\mathbf{A} = \langle A, X\times A, d\rangle$ such that X is a closed system of representatives for the A-orbits of $X(\mathbf{A}) = X \times A$ and $\mathrm{Res}_X d = d_0$.*

Proof: Define the action of A on the Boolean space $X \times A$ by $(x,a)^{a'} = (x, aa')$. Then the map d: $X\times A \to \mathrm{Hom}(\Gamma, A)$ defined by $d(x,a) = d_0(x)^a$ is continuous and $\mathbf{A} = \langle A, X\times A, d\rangle$ is a Γ-structure. Finally identify X with $X \times 1$ to find that X is a closed system of representatives for the A-orbits of $X \times A$ and $\mathrm{Res}_X d = d_0$. ∎

The following lemma asserts that the Γ-structure $\mathbf{A}$ of Lemma 2.6 is unique up to an isomorphism.

LEMMA 2.7: *Let $\mathbf{A}$ be a weak Γ-structure, let $\mathbf{B}$ be a Γ-structure, and let X a closed system of representatives of the B-orbits of $X(\mathbf{B})$. Also, let α_0: $B \to A$ be a continuous homomorphism and α_1': $X \to X(\mathbf{A})$ a continuous map such that $d(\alpha_1'(x)) = \alpha_0 \circ d(x)$ for each $x \in X$. Then α_1' uniquely extends to a map α_1: $X(\mathbf{B}) \to X(\mathbf{A})$ such that $\alpha = (\alpha_0, \alpha_1)$: $\mathbf{B} \to \mathbf{A}$ is a morphism of weak Γ-structures.*

Moreover, α is an epimorphism if and only if α_0 is an epimorphism and $\alpha_1'(X)$ contains a representative of each A-orbit of $X(\mathbf{A})$.

If $\mathbf{A}$ is a Γ-structure, then α is a cover if and only if α_0 is an epimorphism, α_1' is injective and $\alpha_1'(X)$ is a system of representatives of the A-orbits of $X(\mathbf{A})$.

Proof: *The map* $(x,\sigma) \mapsto x^\sigma$, $x \in X$ *and* $\sigma \in B$, *gives an isomorphism of transformation groups* $(X(\mathbf{B}), B) \cong (X \times B, B)$, *where* B *acts on* $X \times B$ *by multiplication from the right on the second factor. Define the map* $\alpha_1 \colon X(\mathbf{B}) \to X(\mathbf{A})$ *by* $\alpha_1(x^\sigma) = \alpha_1'(x)^{\alpha_0(\sigma)}$. *It extends* α_1' *and* $\alpha \colon \mathbf{B} \to \mathbf{A}$ *is a morphism of weak* Γ*-structures. The rest of the lemma follows from Definitions 1.1 and 1.2.* ∎

The following lemma shows that each finite weak Γ*-structure* $\mathbf{A}$ *has a unique minimal cover* $\hat{\mathbf{A}}$ *which is a finite* Γ*-structure.*

LEMMA 2.8: *Let* $\mathbf{A} = \langle A, X, d \rangle$ *be a finite weak* Γ*-structure. Then there exists a finite* Γ*-structure* $\hat{\mathbf{A}} = \langle A, \widehat{X}, \hat{d} \rangle$ *and a cover* $\pi \colon \hat{\mathbf{A}} \to \mathbf{A}$ *such that for every (epi)morphism* $\alpha \colon \mathbf{B} \to \mathbf{A}$ *from a* Γ*-structure* $\mathbf{B}$ *there exists an (epi)morphism* $\hat{\alpha} \colon \mathbf{B} \to \hat{\mathbf{A}}$ *such that* $\pi \circ \hat{\alpha} = \alpha$.

Proof: *Let* X_0 *be a system of representatives for the* A*-orbits of* X. *Since* X_0 *is finite, Lemma 2.6 gives a* Γ*-structure* $\hat{\mathbf{A}} = \langle A, \widehat{X}, \hat{d} \rangle$ *such that* $\widehat{X} \cong X_0 \times A$, X_0 *is a closed system of representatives for the* A*-orbits of* $\widehat{X}$ *and* $\hat{d}(x_0) = d(x_0)$ *for each* $x_0 \in X_0$. *The map* $\mathrm{id} \colon \mathrm{A} \to \mathrm{A}$ *and the map* $X_0 \times A \to X$ *given by* $(x_0, \sigma) \mapsto x_0^\sigma$, *for* $x_0 \in X_0$ *and* $\sigma \in A$ *define a cover* $\pi \colon \hat{\mathbf{A}} \to \mathbf{A}$ *(Lemma 2.7). In particular* $d(\pi(x)) = \hat{d}(x)$ *for each* $x \in \widehat{X}$.

Let now $\mathbf{B}$ *be a* Γ*-structure and* $\alpha \colon \mathbf{B} \to \mathbf{A}$ *a morphism. By Corollary 2.5,* $X(\mathbf{B})$ *has a closed system* Y_0 *of representatives for its* B*-orbits. Choose a map* $\rho \colon \alpha(Y_0) \to \widehat{X}$ *such that* $\pi(\rho(x)) = x$ *for each* $x \in \alpha(Y_0)$. *Since* $\alpha(Y_0)$ *is finite,* ρ *is continuous. Denote the restriction of* $\alpha \colon X(\mathbf{B}) \to X(\mathbf{A})$ *to* Y_0 *by* α_1' *and let* $\hat{\alpha}_1' = \rho \circ \alpha_1'$. *Then* $\hat{d}(\hat{\alpha}_1'(y_0)) = \alpha \circ d(y_0)$ *for each* $y_0 \in Y_0$. *By Lemma 2.7,* $\alpha_1' \colon Y_0 \to X$ *and* $\alpha \colon B \to A$ *extend to a morphism* $\hat{\alpha} \colon \mathbf{B} \to \mathbf{A}$ *such that* $\pi \circ \hat{\alpha} = \alpha$. ∎

3. The Γ-structure $\boldsymbol{\Gamma}_{e,m}$.

For integers $0 \le e \le m$ take e copies $\Gamma_1, \ldots, \Gamma_e$ of Γ. Let

$$\Gamma_{e,m} = \Gamma_1 * \cdots * \Gamma_e * \widehat{F}_{m-e} \tag{1}$$

be the free product (in the category of profinite groups) of $\Gamma_1, \ldots, \Gamma_e$ and the free profinite group $\widehat{F}_{m-e}$ of rank $m-e$. We view $\Gamma_1, \ldots, \Gamma_e$ and $\widehat{F}_{m-e}$ as closed subgroups of $\Gamma_{e,m}$. Each $(e+1)$-tuple $(\gamma_1, \ldots, \gamma_e, \gamma_{e+1})$ of homomorphisms of $\Gamma_1, \ldots, \Gamma_e, \widehat{F}_{m-e}$, respectively, into a profinite group G uniquely extends to a homomorphism $\gamma\colon \Gamma_{e,m} \to G$. The results about projectivity obtained in Sections 4 and 5 depend on the following assumptions on Γ and $\Gamma_{e,m}$.

ASSUMPTION 3.1: *The profinite group Γ satisfies the following conditions:*

- (a) Γ *is finitely generated and nontrivial.*
- (b) *The center of Γ is trivial.*
- (c) *Suppose that closed subgroups H, H' of $\Gamma_{e,m}$ are isomorphic to Γ. Then*
  - (c1) *H is conjugate to one of the groups $\Gamma_1, \ldots, \Gamma_e$;*
  - (c2) *if $\sigma \in \Gamma_{e,m}$ satisfies $H^\sigma = H$, then $\sigma \in H$; and*
  - (c3) *if $H' \neq H$, then $H' \cap H = 1$.*
- (d) *Γ has a finite quotient $\overline{\Gamma}$ with the following property: if a closed subgroup H of $\Gamma_{e,m}$ is a quotient of Γ and has $\overline{\Gamma}$ as a quotient, then $H \cong \Gamma$ (hence, by (c), H is conjugate to one of the subgroups $\Gamma_1, \ldots, \Gamma_e$).*

DEFINITION 3.2: We call a quotient H of Γ **large** if $\overline{\Gamma}$ is a quotient of H. Assumption 3.1(d) says that any closed subgroup of $\Gamma_{e,m}$ which is a large quotient of Γ is isomorphic to Γ.

LEMMA 3.3: *The subgroups $\Gamma_1, \ldots, \Gamma_e$ of $\Gamma_{e,m}$ are mutually nonconjugate. The centralizer of Γ_i in $\Gamma_{e,m}$, is trivial, $i = 1, \ldots, e$.*

Proof: The identity maps $\Gamma_i \to \Gamma_i$, $i = 1, \ldots, e$ and the trivial map $\widehat{F}_{m-e} \to 1$ give a homomorphism φ of $\Gamma_{e,m}$ onto the direct product $\Gamma_1 \times \cdots \times \Gamma_e$. Since $\Gamma \neq 1$ (Assumption 3.1(a)), $\Gamma_1, \ldots, \Gamma_e$ are mutually nonconjugate as subgroups of $\Gamma_1 \times \cdots \times \Gamma_e$. Hence they *are mutually nonconjugate as subgroups of $\Gamma_{e,m}$. The assertion about the centralizer follows from 3.1(b) and 3.1(c2).* ∎

REMARK 3.4: Assumption 3.1(c) does not hold for arbitrary Γ. For example, let $\Gamma = \mathbb{Z}_p$, $e = m = 2$. Then $\Gamma_{2,2}$ is the free product of subgroups $\langle a\rangle$ and $\langle b\rangle$, each isomorphic to $\mathbb{Z}_p$. Consider the map $\alpha\colon \Gamma_{2.2} \to \mathbb{Z}_p \times \mathbb{Z}_p$ defined by $a \mapsto (1,0)$ and $b \mapsto (0,1)$. Then $\alpha(ab) = (1,1)$ generates a group isomorphic to $\mathbb{Z}_p$ but conjugate to none of the components of $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the subgroup $\langle ab\rangle$ of $\Gamma_{2,2}$ contains a subgroup which is isomorhic to $\mathbb{Z}_p$ but conjugate to neither $\langle a\rangle$ nor $\langle b\rangle$. Thus Assumption 3.1(c1) is not fulfilled. Herfort and Ribes [Thm. B' of HR] prove Assumptions 3.1(c2) and 3.1(c3) for arbitrary Γ by group theoretic methods. We however verify Assumption 3.1 for $\Gamma \cong G(\mathbb{Q}_p)$, the only case we need, by field theoretic methods (Proposition 12.10). ∎

LEMMA 3.5: *Let $0 \le e \le m$ be integers. In the above notation choose for each i, $1 \le i \le m$, an isomorphism $\psi_i\colon \Gamma \to \Gamma_i$. Let $X = \{\psi_i^\sigma \mid i = 1,\ldots,e;\ \ \sigma \in \Gamma_{e,m}\}$ and let $d\colon X \to \mathrm{Hom}(\Gamma,\Gamma_{e,m})$ be the inclusion map. Then*

(a) $\mathbf{\Gamma}_{e,m} = \langle \Gamma_{e,m}, X, d\rangle$ *is a Γ-structure (Definition 1.1);*

(b) *the elements of X are embeddings of Γ into $\Gamma_{e,m}$;*

(c)
$$\begin{aligned}\{D(x)\mid x \in X\} &= \{\Gamma_i^\sigma \mid i = 1,\ldots,e;\ \ \sigma \in \Gamma_{e,m}\}\\ &= \{H \le \Gamma_{e,m} \mid H \text{ is a large quotient of } \Gamma\};\end{aligned}$$

and

(d) *for $x,y \in X$, $D(x) = D(y)$ if and only if there exists $\sigma \in D(x)$ such that $y = x^\sigma$; if $D(x) \ne D(y)$ then $D(x) \cap D(y) = 1$.*

Proof: To prove (a) it suffices to check the regularity of the action of $\Gamma_{e,m}$ on X. Indeed, if $\psi_i^\sigma = \psi_i$ for some i, $1 \le i \le n$ and $\sigma \in \Gamma_{e,m}$, then σ belongs to the centralizer of Γ_i in $\Gamma_{e,m}$. Therefore Assumptions 3.1(c2) and 3.1(b) imply that $\sigma = 1$. Assertion (c) follows from Assumption 3.1(d). Finally assertion (d) is a combination of (c), Assumption 3.1(c2) and Assumption 3.1(d). ∎

COROLLARY 3.6: *Let $\mathbf{B}$ be a finite weak Γ-structure. Then, for suitable $0 \le e \le m$,* *there exists a cover* $\beta\colon \mathbf{\Gamma}_{e,m} \to \mathbf{B}$.

Proof: *Let* $x_1, \ldots, x_e$ *represent the* B*-orbits of* $X(\mathbf{B})$ *and let* $m = e + \mathrm{rank}(B)$*. Define an epimorphism* $\beta\colon \Gamma_{e,m} \to B$ *such that its restriction to* Γ_i *is* $d(x_i) \circ \psi_i^{-1}$, $i = 1, \ldots, e$ *(in the notation of Lemma 3.5) and its restriction to* $\widehat{F}_{m-e}$ *maps this group onto* B*. Now define a surjective map* $\beta\colon X \to X(\mathbf{B})$ *by* $\beta(\psi_i^\sigma) = x_i^{\beta(\sigma)}$, *for* $\sigma \in \Gamma_{e,m}$ *and* $i = 1, \ldots, e$*. Then* $\beta\colon \mathbf{\Gamma}_{e,m} \to \mathbf{B}$ *is a cover (Definition 1.2).* ∎

4. Γ-projective groups.

Let G *be a profinite group. A* **conjugacy domain** *of subgroups of* G *is a collection of closed subgroups of* G *which is closed under conjugation by elements of* G*. In particular, the collection of all subgroups of* G *which are isomorphic to* Γ *is a conjugacy domain. We denote it by* $\mathcal{D}(G)$*. Since* Γ *is finitely generated each* $\psi \in \mathrm{Hom}(\Gamma, G)$ *with* $\psi(\Gamma) \in \mathcal{D}(G)$ *is an embedding [R, p. 69]. We say that a conjugacy domain of subgroups of* G *is* **closed** *if it is a closed subset of the Boolean space* Subg(G) *(Section 1).*

Definition 4.1: Let $\mathcal{D}$ be a closed conjugacy domain of subgroups of a profinite group G which are isomorphic to Γ. A $\mathcal{D}$-embedding problem for G is a diagram

$$
\begin{array}{ccc}
 & & G \\
 & & \big\downarrow{\scriptstyle\varphi} \\
B & \xrightarrow{\alpha} & A
\end{array}
\tag{1}
$$

(abbreviated "(φ, α)"), where α is an epimorphism of profinite groups, φ is a homomorphism and for each $H \in \mathcal{D}$ there exists a homomorphism $\gamma_H\colon H \to B$ such that $\alpha \circ \gamma_H = \mathrm{Res}_H \varphi$. The problem is **finite** if B is a finite group. A **solution** to (1) is a homomorphism $\gamma\colon G \to B$ such that $\alpha \circ \gamma = \varphi$. We say that G is $\mathcal{D}$-**projective** if every finite $\mathcal{D}$-embedding problem for G is solvable.

We say that G is Γ-**projective** if $\mathcal{D}(G)$ is topologically closed in Subg(G) and if G is $\mathcal{D}(G)$-projective. In this case we refer to a $\mathcal{D}(G)$-embedding problem also as a Γ-**embedding problem**.

The condition on G to be $\mathcal{D}$-projective may be considered as a local-global principle. Thus (1) is solvable if for each $H \in \mathcal{D}$ the local problem associated to H is solvable.

REMARK 4.2: Note that if X is a subset of $\mathrm{Hom}(\Gamma, G)$ such that $\{\psi(\Gamma) |\, \psi \in X\} = \mathcal{D}$, then each $\psi \in X$ is an embedding. Thus (1) is a $\mathcal{D}$-embedding problem if and only if for each $\psi \in X$ there exists $\rho \in \mathrm{Hom}(\Gamma, B)$ such that $\alpha \circ \rho = \varphi \circ \psi$. ∎

EXAMPLE 4.3: *For each e and m, $1 \le e \le m$, $\Gamma_{e,m}$ is a Γ-projective group.* Indeed let (1) with $G = \Gamma_{e,m}$ be a finite embedding problem for $\Gamma_{e,m}$. Then for each i, $1 \le i \le e$, there exists a homomorphism $\gamma_i\colon \Gamma_i \to B$ such that $\alpha \circ \gamma_i = \mathrm{Res}_{\Gamma_i}\varphi$. Also, as a free profinite group $\widehat{F}_{m-e}$ is projective. Therefore there exists a homomorphism $\gamma_{e+1}\colon \widehat{F}_{m-e} \to B$ such that $\alpha\circ\gamma_{e+1}$ is the restriction of φ to $\widehat{F}_{m-e}$. Combine $\gamma_1, \ldots, \gamma_{e+1}$ to a solution γ of (1). ∎

LEMMA 4.4: *In the notation of Definition 4.1, if G is $\mathcal{D}$-projective, then every $\mathcal{D}$-embedding problem (1) in which A is finite and* $\mathrm{rank}(B) \le \aleph_0$ [J3, Sec.1] *has a solution.*

Proof: There exists a descending sequence $\mathrm{Ker}(\alpha) = N_0 \ge N_1 \ge N_2 \ge \cdots$ *of open normal subgroups of B with a trivial intersection. Identify A with B/N_0 and let $\varphi_0 = \varphi$ and $\alpha_0 = \alpha$. For each i and j, $j \ge i \ge 0$, let $\alpha_i\colon B \to B/N_i$ and $\alpha_{ji}\colon B/N_j \to B/N_i$ be the quotient maps.*

Claim: Let $i \ge 0$ and let $\varphi_i\colon G \to B/N_i$ be a homomorphism such that (φ_i, α_i) is a $\mathcal{D}$-embedding problem for G. Then there exists $\varphi_{i+1} \in \mathrm{Hom}(G, B/N_{i+1})$ *such that $\alpha_{i+1,i} \circ \varphi_{i+1} = \varphi_i$ and $(\varphi_{i+1}, \alpha_{i+1})$ is a $\mathcal{D}$-embedding problem for G.*

Use the claim to inductively construct $\varphi_{i+1} \in \mathrm{Hom}(G, B/N_{i+1})$ *such that* $\alpha_{i+1,i} \circ \varphi_{i+1} = \varphi_i$. *The maps φ_i define* $\gamma \in \mathrm{Hom}(G, B)$ *such that $\alpha \circ \gamma = \varphi$.*

Without loss prove the claim for $i = 0$. For each j the pair (φ, α_{j0}) is a $\mathcal{D}$-embedding problem for G. For each $\beta \in \mathrm{Hom}(G, B/N_j)$ *let* $\beta \circ \mathrm{Hom}(\Gamma, G) = \{\beta \circ \psi |\, \psi \in \mathrm{Hom}(\Gamma, G)\}$. *It is a subset of the finite set* $\mathrm{Hom}(\Gamma, B/N_j)$. *Thus, since G is $\mathcal{D}$-projective, the finite collection of sets*

$$\mathcal{Z}_j = \{\beta \circ \mathrm{Hom}(\Gamma, G) |\, \beta \in \mathrm{Hom}(G, B/N_j),\ \ \alpha_{j0} \circ \beta = \varphi\}$$

is nonempty. The map $\beta \circ \mathrm{Hom}(\Gamma, G) \mapsto \alpha_{j+1,j} \circ \beta \circ \mathrm{Hom}(\Gamma, G)$ *maps $\mathcal{Z}_{j+1}$ into $\mathcal{Z}_j$. It follows that $\varprojlim \mathcal{Z}_j \ne \emptyset$, i.e., there exist homomorphisms $\beta_j\colon G \to B/N_j$ such that $\alpha_{j0} \circ \beta_j = \varphi$ and*

$$\alpha_{j+1,j} \circ \beta_{j+1} \circ \mathrm{Hom}(\Gamma, G) = \beta_j \circ \mathrm{Hom}(\Gamma, G), \qquad j = 0, 1, 2, \ldots \quad . \tag{2}$$

In particular $\alpha_{j0} \circ \beta_1 = \varphi$. Apply Remark 4.2 to show that (β_1, α_1) is a $\mathcal{D}$-embedding problem for G. Indeed, let $\psi_1 \in \mathrm{Hom}(\Gamma, G)$ and use (2) to inductively construct $\psi_j \in \mathrm{Hom}(\Gamma, G)$ such that $\alpha_{j+1,j} \circ \beta_{j+1} \circ \psi_{j+1} = \beta_j \circ \psi_j$, $j = 1, 2, 3, \ldots$. The maps $\beta_j \circ \psi_j \colon \Gamma \to B/N_j$ define $\rho \in \mathrm{Hom}(\Gamma, B)$ such that $\alpha_1 \circ \rho = \beta_1 \circ \psi_1$. This concludes the proof of the claim for $i = 0$. ∎

LEMMA 4.5: *In the notation of Definition 4.1 suppose that G is a $\mathcal{D}$-projective group. Then*

(a) *if $H_1 \le G$ is a large quotient of Γ (Definition 3.2), then $H_1 \in \mathcal{D}$; therefore $\mathcal{D} = \mathcal{D}(G)$, G is Γ-projective and $\mathcal{D}(G)$ is topologically closed in* $\mathrm{Subg(G)}$*);*

(b) *if $H, H' \in \mathcal{D}$ and $H \ne H'$, then $H \cap H' = 1$; and*

(c) *if $H_2 \in \mathcal{D}$ and $\sigma \in G$ satisfies $H_2^\sigma = H_2$, then $\sigma \in H_2$.*

Proof: *Let $H_1 \le G$ be a large quotient of Γ, let $H, H', H_2 \in \mathcal{D}$ and let $\sigma \in G$ such that $H \ne H'$ and $H_2^\sigma = H_2$. Since $\mathcal{D}$ is closed in $\mathrm{Subg(G)} = \varprojlim \mathrm{Subg(G/N)}$, where N ranges over all open normal subgroups of G, there is N such that, with $A = G/N$, the quotient map $\varphi \colon G \to A$ satisfies*

(3a) *$\varphi(H_1), \varphi(H), \varphi(H')$ and $\varphi(H_2)$ are large quotients of Γ;*

(3b) *$\varphi(H_1) \notin \varphi(\mathcal{D})$ if $H_1 \notin \mathcal{D}$;*

(3c) *$\varphi(H) \ne \varphi(H')$; and*

(3d) *$\varphi(\sigma) \notin \varphi(H_2)$ if $\sigma \notin H_2$.*

Let $\alpha_1, \ldots, \alpha_e$ be a listing of all $\alpha \in \mathrm{Hom}(\Gamma, A)$ such that $\alpha(\Gamma) \in \varphi(\mathcal{D})$. With $\Gamma_1 = \cdots = \Gamma_e = \Gamma$ the maps $\alpha_i \colon \Gamma_i \to A$ together with a suitable epimorphism $\alpha_{e+1} \colon \widehat{F}_{m-e} \to A$ (for some $m \ge e$) define an epimorphism α of $\Gamma_{e,m} = \Gamma_1 * \cdots * \Gamma_e * \widehat{F}_{m-e}$ onto A such that (φ, α) is a $\mathcal{D}$-embedding problem. By Lemma 4.4 there exists a homomorphism $\gamma \colon G \to \Gamma_{e,m}$ such that $\alpha \circ \gamma = \varphi$.

From (3a) and Assumption 3.1(d), $\gamma(H_1), \gamma(H), \gamma(H')$ and $\gamma(H_2)$ are isomorphic to Γ. By Assumption 3.1(c1), each of the groups $\gamma(H_1), \gamma(H), \gamma(H')$, and $\gamma(H_2)$ is conjugate to some Γ_i, $i = 1, \ldots, e$. Therefore $\varphi(H_1) \in \varphi(\mathcal{D})$. Conclude from (3b) that $H_1 \in \mathcal{D}$. This proves (a).

From (3c), $\gamma(H) \neq \gamma(H')$. By Assumption 3.1(c3), $\gamma(H) \cap \gamma(H') = 1$. Now note that since $\gamma(H) \cong H \cong \Gamma$, the restriction of γ to H is injective. Therefore $H \cap H' = 1$.

Finally observe for (c) that since $\gamma(H_2)^{\gamma(\sigma)} = \gamma(H_2)$, Assumption 3.1(c2) implies that $\gamma(\sigma) \in \gamma(H_2)$. Hence $\varphi(\sigma) \in \varphi(H_2)$. Conclude from (3d) that $\sigma \in H_2$. ∎

REMARK 4.6: The group $\mathrm{Aut}(\Gamma)$ of all automorphisms of Γ is profinite [Sm, Thm. 1.3]. It acts on $\mathrm{Hom}(\Gamma, G)$ by the following rule:

$$\psi^\omega = \psi \circ \omega, \qquad \psi \in \mathrm{Hom}(\Gamma, G), \text{ and } \omega \in \mathrm{Aut}(\Gamma).$$

Note that the actions of $\mathrm{Aut}(\Gamma)$ and G on $\mathrm{Hom}(\Gamma, G)$ commute. Also, let $\psi, \psi' \in \mathrm{Hom}(\Gamma, G)$.

(a) If $\psi(\Gamma) = \psi'(\Gamma)$ and ψ is an embedding, then there exists $\omega \in \mathrm{Aut}(\Gamma)$ such that $\psi' = \psi^\omega$.

(b) For $g \in \Gamma$ let $[g]$ be the inner automorphism of Γ determined by g. Then $\psi^{[g]} = \psi^{\psi(g)}$. Thus there exists $g \in \Gamma$ such that $\psi' = \psi^{[g]}$ if and only if there exists $\sigma \in \psi(\Gamma)$ such that $\psi' = \psi^\sigma$.

LEMMA 4.7: *Suppose that a profinite group G is Γ-projective. Then there exists a closed subset X of $\mathrm{Hom}(\Gamma, G)$, closed under the action of G, such that $\{\psi(\Gamma) |\, \psi \in X\} = \mathcal{D}(G)$ and for each $\psi, \psi' \in X$,*

(4) $\psi(\Gamma) = \psi'(\Gamma)$ if and only if there exists $\sigma \in \psi(\Gamma)$ such that $\psi' = \psi^\sigma$.

Proof: The set $Y = \{\psi \in \mathrm{Hom}(\Gamma, G) |\, \psi(\Gamma) \in \mathcal{D}(G)\}$ is closed under the actions of G and $\mathrm{Aut}(\Gamma)$ on $\mathrm{Hom}(\Gamma, G)$. By Lemma 4.5(a) the collection $\mathcal{D}(G)$ is topologically closed in $\mathrm{Subg(G)}$. Since $\mathrm{Im}\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Subg(G)}$ is continuous (beginning of Section 1), Y is topologically closed in $\mathrm{Hom}(\Gamma, G)$. The quotient space Y/G is Boolean [HJ, Section 1]. Since the actions of G and $\mathrm{Aut}(\Gamma)$ on Y commute, $\mathrm{Aut}(\Gamma)$ acts on Y/G. By Remark 4.6(b), the group of inner automorphisms $\mathrm{Inn}(\Gamma)$ of Γ, acts trivially on Y/G. Hence $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ acts on Y/G. We claim that this action is regular (Definition 1.1). Indeed, if for $\psi \in Y$, $\omega \in \mathrm{Aut}(\Gamma)$ and $\sigma \in G$ we have $\psi^\omega = \psi^\sigma$, then $\psi(\Gamma) = \psi(\Gamma)^\sigma$. Hence $\sigma \in \psi(\Gamma)$ (Lemma 4.5(c)). Thus $\sigma = \psi(g)$, with $g \in \Gamma$. By Remark 4.6(b), $\psi^\omega = \psi^{[g]}$. But since ψ is an embedding, $\omega = [g]$, which proves our claim.

By Lemma 2.4(b) there exists a closed system of representatives $\overline{X}$ *for the* $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$*-orbits of* Y/G*. Let* X *be the preimage of* $\overline{X}$ *under the map* $Y \to Y/G$*. Then* $\{\psi(\Gamma) |\ \psi \in X\} = \mathcal{D}(G)$*. If* $\psi, \psi' \in X$ *and* $\psi(\Gamma) = \psi'(\Gamma)$*, then there exists* $\omega \in \mathrm{Aut}(\Gamma)$ *such that* $\psi' = \psi^\omega$ *(Remark 4.6(a)). By the definition of* X *there exists* $\sigma \in G$ *such that* $\psi' = \psi^\sigma$*. Lemma 4.5(c) implies that* $\sigma \in \psi(\Gamma)$*. The converse implication of (4) is trivial.* ∎

LEMMA 4.8: *Suppose that* G *is a* Γ*-projective profinite group. Let* X *be as in Lemma 4.7 and let* $\mathbf{G} = \langle G, X, \mathrm{inclusion}\rangle$ *be the corresponding weak* Γ*-structure. Consider an epimorphism* $\alpha\colon \mathbf{B} \to \mathbf{A}$ *of finite weak* Γ*-structures, a morphism* $\varphi\colon \mathbf{G} \to \mathbf{A}$ *and an open normal subgroup* N_0 *of* G*. Then there exists a commutative diagram*

$$\begin{array}{ccccc} & & \mathbf{G} & & \\ & & \Big\downarrow{\scriptstyle\hat\varphi} & & \\ \widehat{\mathbf{B}} & \xrightarrow{\hat\alpha} & \widehat{\mathbf{A}} & & \varphi \\ {\scriptstyle\pi'}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} & & \\ \widehat{\mathbf{B}} & \xrightarrow{\alpha} & \widehat{\mathbf{A}} & & \end{array} \tag{5}$$

in which $\hat\alpha$ *is an epimorphism of weak* Γ*-structures with injective forgetful maps (inclusion, for simplicity), such that* $\mathrm{Ker}(\hat\varphi) \le N_0$*;*

(a) *for each* $\lambda \in X(\widehat{\mathbf{B}})$, $\mathrm{Ker}(\hat\alpha) \cap \lambda(\Gamma) = 1$ *(i.e., the restriction of* $\hat\alpha$ *to* $\lambda(\Gamma)$ *is injective);*

(b) *if* $\rho, \rho' \in X(\widehat{\mathbf{A}})$ *and there exists* $\omega \in \mathrm{Aut}(\Gamma)$ *such that* $\rho' = \rho^\omega$*, then there exists* $g \in \Gamma$ *such that* $\pi \circ \rho' = \pi \circ \rho^{[g]}$*; and*

(c) *for each* $\psi \in X$ *the group* $\hat\varphi(\psi(\Gamma))$ *is a large quotient of* Γ *(Definition 3.2).*

Proof: *By Corollary 3.6 there exists an epimorphism* $\beta\colon \underline{\Gamma}_{e,m} \to \mathbf{B}$ *for suitable* e, m*. Since the forgetful map of* $\underline{\Gamma}_{e,m}$ *is injective,* β *induces an epimorphism* $\bar\beta$ *of a finite weak* Γ*-structure* $\mathbf{B}_1$ *with an injective forgetful map onto* $\mathbf{B}$ *(Lemma 1.4). Replace* $\mathbf{B}$ *by* $\mathbf{B}_1$ *and* α *by* $\alpha \circ \bar\beta$*, if necessary, to assume that the forgetful map of* $\mathbf{B}$ *is injective.*

Let N *be an open normal subgroup of* G *which is contained in* $N_0 \cap \mathrm{Ker}(\varphi)$ *and denote the quotient map* $G \to G/N$ *by* $\hat\varphi$*. Then* $\widehat{\mathbf{A}} = \mathbf{G}_N = \langle G/N, \{\hat\varphi \circ \psi |\ \psi \in X\}, \mathrm{inclusion}\rangle$ *is a finite weak* Γ*-structure. Lemma 1.4 implies that if* N *is sufficiently small, then the map* $G/N \to A$ *defined by* φ *can be completed to a morphism* $\pi\colon \widehat{\mathbf{A}} \to \mathbf{A}$

such that $\varphi = \pi \circ \hat{\varphi}$. With $\widehat{\mathbf{B}} = \mathbf{B} \times_{\mathbf{A}} \widehat{\mathbf{A}}$ (Section 2), we obtain a commutative diagram (5). Since the injective maps of both $\mathbf{B}$ and $\widehat{\mathbf{A}}$ are injective so is the injective map of $\widehat{\mathbf{B}}$ (Section 2). Our aim now is to choose N sufficiently small such that (a), (b) and (c) hold.

To achieve (a) let Δ be the intersection of all $\mathrm{Ker}(\lambda)$ with $\lambda \in X(\widehat{\mathbf{B}})$. Since Γ is finitely generated, Δ is an open normal subgroup of Γ. For each open normal subgroup M of G let $Z(M) = \{\psi \in X \mid \psi^{-1}(M) \leq \Delta\}$. If $\psi \in Z(M)$ and $\psi' \in X$ coincides with ψ modulo M, then $\psi' \in Z(M)$. Thus $Z(M)$ is open in X. For each $\psi \in X$ there exists M such that $\psi(\Gamma) \cap M \leq \psi(\Delta)$. Since ψ is an embedding, $\psi^{-1}(M) \leq \Delta$ and therefore $\psi \in Z(M)$. Thus the collection of all $Z(M)$'s covers X. By compactness there exist open normal subgroups $M_1, \dots, M_m$ of G such that $X = Z(M_1) \cup \dots \cup Z(M_m)$. Choose $N \leq M_1 \cap \dots \cap M_m$. Then $\mathrm{Ker}(\hat{\varphi} \circ \psi) = \psi^{-1}(N) \leq \Delta \leq \mathrm{Ker}(\lambda)$ for every $\psi \in X$ and $\lambda \in X(\widehat{\mathbf{B}})$. Now, for each $\lambda \in X(\widehat{\mathbf{B}})$, $\hat{\alpha} \circ \lambda$ is an element of $X(\widehat{\mathbf{A}})$. Thus there exists $\psi \in X$ such that $\hat{\alpha} \circ \lambda = \hat{\varphi} \circ \psi$. Conclude that $\mathrm{Ker}(\hat{\alpha} \circ \lambda) \leq \mathrm{Ker}(\pi' \circ \lambda)$, i.e., $\pi'(\lambda(\Gamma) \cap \mathrm{Ker}(\hat{\alpha})) = 1$. Hence, by Lemma 2.1(c) 1. we get that $\mathrm{Ker}(\hat{\alpha}) \cap \lambda(\Gamma) = 1$.

To achieve (b) let $Y_1, \dots, Y_n$ be the distinct $\mathrm{Inn}(\Gamma)$-orbits of $X(\mathbf{A})$. Then $X_i = \varphi^{-1}(Y_i) = \{\psi \in X \mid \varphi \circ \psi \in Y_i\}$, $i = 1, \dots, n$, are open-closed subsets of X. If $\psi_i \in X_i$ and $\psi_j \in X_j$, for $i \neq j$, then ψ_i and ψ_j are not in the same $\mathrm{Inn}(\Gamma)$-orbit. From Remark 4.6(b) there exists no $\sigma \in \psi_i(\Gamma)$ such that $\psi_j = \psi_i^\sigma$. Hence by (4), $\psi_i(\Gamma) \neq \psi_j(\Gamma)$ and therefore ψ_i and ψ_j are not in the same $\mathrm{Aut}(\Gamma)$-orbit. That is, the closed subsets $\psi_i^{\mathrm{Aut}(\Gamma)}$ and $\psi_j^{\mathrm{Aut}(\Gamma)}$ of X are disjoint. Hence, if N is sufficiently small, $\hat{\varphi} \circ \psi_i^{\mathrm{Aut}(\Gamma)}$ is disjoint from $\hat{\varphi} \circ \psi_j^{\mathrm{Aut}(\Gamma)}$. Obviously, if ψ_i' and ψ_j' coincide with ψ_i and ψ_j, respectively, modulo N, then $\hat{\varphi} \circ \psi_i'^{\mathrm{Aut}(\Gamma)}$ is disjoint from $\hat{\varphi} \circ \psi_j'^{\mathrm{Aut}(\Gamma)}$. Use the compactness of $X_i \times X_j$ to find an N such that $\hat{\varphi} \circ \psi_i^{\mathrm{Aut}(\Gamma)} \cap \hat{\varphi} \circ \psi_j^{\mathrm{Aut}(\Gamma)} = \emptyset$ for all $i \neq j$ and each $\psi_i \in X_i$ and $\psi_j \in X_j$.

If $\rho, \rho' \in X(\widehat{\mathbf{A}})$ and $\rho' = \rho^\omega$ for some $\omega \in \mathrm{Aut}(\Gamma)$, then there exists $\psi, \psi' \in X$ such that $\rho = \hat{\varphi} \circ \psi$ and $\rho' = \hat{\varphi} \circ \psi'$. By the choice of N, ψ and ψ' lie in the same X_i. Hence $\pi \circ \rho = \varphi \circ \psi$ and $\pi \circ \rho' = \varphi \circ \psi'$ belong to the same Y_i. Conclude that there exists $g \in \Gamma$ such that $\pi \circ \rho' = \pi \circ \rho^{[g]}$. This proves (b).

Finally, to achieve (c), note that for each $\psi \in X$ the group $\psi(\Gamma)$ is isomorphic to

Γ. *Therefore* $\hat\varphi(\psi(\Gamma))$ *is a large quotient of* Γ *if* N *is sufficiently small. The same holds for* $\psi' \in X$ *if* $\hat\varphi \circ \psi' = \hat\varphi \circ \psi$, *i.e., if* ψ' *lies near* ψ. *Use the compactness of* X *to choose* N *such that* $\hat\varphi(\psi(\Gamma))$ *is a large quotient of* Γ *for each* $\psi \in X$. ∎

5. Projective Γ-structures.

We define projective Γ*-structure and prove that the underlying group of each of them is* Γ*-projective. Conversely we show that the* Γ*-structure associated in Lemma 4.8 with a* Γ*-projective group is projective.*

DEFINITION 5.1: *Let* $\mathbf{G}$ *be a* Γ*-structure. A diagram*

(1)
$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\alpha} & A \end{array}$$

(abbreviated by "(φ,α)*") where* φ *is a morphism and* α *is an epimorphism of weak* Γ*-structures is called a* **weak embedding problem** *for* $\mathbf{G}$. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* Γ*-structures and* α *is a cover, we call* (φ,α) *an* **embedding problem** *for* $\mathbf{G}$. *The problem is* **finite** *if* $\mathbf{B}$ *is finite. A* **solution** *to* (φ,α) *is a morphism* $\gamma\colon \mathbf{G} \to \mathbf{B}$ *such that* $\alpha \circ \gamma = \varphi$. *The structure* $\mathbf{G}$ *is* **projective** *if every finite weak embedding problem for* $\mathbf{G}$ *has a solution.*

LEMMA 5.2: *If* $\mathbf{G}$ *is a projective* Γ*-structure, then every embedding problem for* $\mathbf{G}$ *has a solution.*

Proof: Consider embedding problem (1) for $\mathbf{G}$. *Let* $K = \mathrm{Ker}(\alpha)$ *and assume without loss that* $\mathbf{A} = \mathbf{B}/K$ *and* α *is the quotient map (Section 1). Divide the rest of the proof into two parts.*

PART A: K *is finite.* *Then there exists an open normal subgroup* N_0 *of* B *such that* $N_0 \cap K = 1$. *By Lemma 1.3 there exists an epimorphism* β *of* $\mathbf{B}$ *onto a finite* Γ*-structure* $\mathbf{B}_0$ *such that* $\mathrm{Ker}(\beta) \le N_0$. *Now use Lemma 2.3 to construct a cartesian diagram of epimorphisms of* Γ*-structures*

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\alpha} & \mathbf{A} \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ \mathbf{B}_0 & \xrightarrow{\alpha_0} & \mathbf{A}_0 \end{array}$$

in which α_0 is a cover. By assumption there exists a morphism γ_0: $\mathbf{G} \to \mathbf{B}_0$ such that $\alpha_0 \circ \gamma_0 = \alpha_1 \circ \varphi$. Thus, Lemma 2.1(b) gives a morphism γ: $\mathbf{G} \to \mathbf{B}$ such that $\alpha \circ \gamma = \varphi$.

PART B: *The general case.* *Let Λ be the family of pairs (L, λ), where L is a closed normal subgroup of B contained in K and λ: $\mathbf{G} \to \mathbf{B}/L$ is a morphism such that*

$$\begin{tikzcd} & \mathbf{G} \arrow[dl, "\lambda"'] \arrow[d, "\varphi"] \\ \mathbf{B}/L \arrow[r, "\alpha_L"'] & \mathbf{B}/K \end{tikzcd}$$

commutes (α_L is the cover induced by $L \le K$). Partially order Λ by letting $(L', \lambda') \ge (L, \lambda)$ mean that $L' \le L$ and

(2)
$$\begin{tikzcd} & & G \arrow[dll, "\lambda'"'] \arrow[dl, "\lambda"] \\ \mathbf{B}/L' \arrow[r] & \mathbf{B}/L & \end{tikzcd}$$

commutes. Then Λ is inductive and by Zorn's Lemma it has a maximal element (L, λ). If $L \ne 1$, there is an open normal subgroup N in B such that $L \not\le N$; hence $L' = N \cap L$ is a proper open normal subgroup of L. Since L/L' is finite Part A gives a morphism λ': $\mathbf{G} \to \mathbf{B}/L'$ such that (2) commutes. Then $(L', \lambda') \in \Lambda$ and $(L', \lambda') > (L, \lambda)$, a contradiction. Conclude that $L = 1$, as required. ∎

LEMMA 5.3: *Each projective Γ-structure $\mathbf{G}$ has the following properties.*

(a) *the forgetful map d: $X(\mathbf{G}) \to \mathrm{Hom}(\Gamma, G)$ is injective;*

(b) *for each $x \in X(\mathbf{G})$ the map $d(x)$: $\Gamma \to G$ is injective (therefore $D(x) \cong \Gamma$);*

(c) *if $H \le G$ is a large quotient of Γ (Definition 3.2), then $H \cong \Gamma$ and there exists $x \in X(\mathbf{G})$ such that $D(x) = H$ (Definition 1.1);*

(d) *if $x, y \in X(\mathbf{G})$,then $D(x) = D(y)$ if and only if there exists $\sigma \in D(x)$ such that $y = x^\sigma$; if $D(x) \ne D(y)$, then $D(x) \cap D(y) = 1$; and*

(e) *the set $\mathcal{D}(G) = \{H \le G |\ H \cong \Gamma\}$ is closed in $\mathrm{Subg(G)}$ and possesses a closed system of representatives for the conjugacy classes.*

Proof: *Let φ be an epimorphism of $\mathbf{G}$ onto a finite Γ-structure $\mathbf{A}$. By Corollary 3.6 there exists a cover α: $\underline{\Gamma}_{e,m} \to \mathbf{A}$, for some e and m, $0 \le e \le m$. Since $\mathbf{G}$ is projective, there exists a morphism γ: $\mathbf{G} \to \underline{\Gamma}_{e,m}$ such that $\alpha \circ \gamma = \varphi$. Recall that $\mathbf{G}$ is the inverse limit of finite Γ-structures (Lemma 1.3). Since $\mathbf{\Gamma}_{e,m}$ has properties* (a)-*(e) above (Lemma 3.5), a suitable choice of $\mathbf{A}$ will imply these properties for $\mathbf{G}$.*

Proof of (a): *Suppose that $x, x' \in X(\mathbf{G})$ and $x \neq x'$. Choose φ such that $\varphi(x) \neq \varphi(x')$. Then $\gamma(x) \neq \gamma(x')$. Hence $d(\gamma(x)) \neq d(\gamma(x'))$. It follows that $\gamma(d(x)) \neq \gamma(d(x'))$. Therefore $d(x) \neq d(x')$.*

Proof of (b): *The right hand side of $\gamma \circ d(x) = d(\gamma(x))$ is injective. Hence $d(x)$ is injective.*

Proof of (c): *Choose A such that $\varphi(H)$ is a large quotient of Γ. Then $\gamma(H)$ is a large quotient of Γ. It follows that $\gamma(H) \cong \Gamma$. Since H is also a quotient of Γ, $H \cong \Gamma$ [R, p. 69]. Now the map D: $X(\mathbf{G}) \to \mathrm{Subg(G)}$ is continuous (Section 1). Since $X(\mathbf{G})$ is compact, $\{D(x) |\, x \in X(\mathbf{G})\}$ is closed in $\mathrm{Subg(G)}$ and it is the inverse limit of $\{D(a) |\, a \in X(\mathbf{A})\}$, where $\mathbf{A}$ ranges over all finite quotients of $\mathbf{G}$. If $H \neq D(x)$ for all $x \in X(\mathbf{G})$, then we may choose $\mathbf{A}$ such that $\varphi(H) \neq D(a)$ for all $a \in X(\mathbf{A})$. Therefore $\gamma(H) \neq D(y)$ for all $y \in X(\mathbf{\Gamma}_{e,m})$. Since $\gamma(H) \cong \Gamma$, this is impossible.*

Proof of (d): *Obviously, if $y = x^\sigma$ with $\sigma \in D(x)$, then $D(x) = D(y)$. Conversely if $D(x) = D(y)$, then $D(\gamma(x)) = D(\gamma(y))$. Hence $\gamma(x)$ and $\gamma(y)$ lie in the same $\Gamma_{e,m}$-orbit. Therefore $\varphi(x)$ and $\varphi(y)$ lie in the same A-orbit. Since this holds for each $\mathbf{A}$ and since the G-orbit of x is closed, x and y lie in the same G-orbit. Finally suppose that $D(x) \neq D(y)$. Choose $\mathbf{A}$ such that $D(\varphi(x)) \neq D(\varphi(y))$. Then $D(\gamma(x)) \neq D(\gamma(y))$. Hence $\gamma(D(x)) \cap \gamma(D(y)) = 1$. Since the restriction of γ to $D(x)$ is injective (by (b)), $D(x) \cap D(y) = 1$.*

Proof of (e): *By (b) and (c), $\mathcal{D}(G) = \{D(x) |\, x \in X(\mathbf{G})\}$. Hence $\mathcal{D}(G)$ is closed in* $\mathrm{Subg(G)}$. *Now let X_0 be a closed system of representatives for the G-orbits of $X(\mathbf{G})$ (Corollary 2.5). Then (d) implies that $\{D(x) |\, x \in X_0\}$ is a closed system of representatives for the conjugacy classes of $\mathcal{D}(G)$.* ∎

PROPOSITION 5.4: (a) *If* $\mathbf{G}$ *is a projective* Γ*-structure, then* G *is a* Γ*-projective group.*

(b) *Conversely, let* G *be a* Γ*-projective group. Then there exists a closed subset* X *of* $\mathrm{Hom}(\Gamma, G)$*, closed under the action of* G *such that* $\mathcal{D}(G) = \{\psi(\Gamma) | \ \psi \in X\}$ *and for all* $\psi, \psi' \in X$*,* $\psi(\Gamma) = \psi'(\Gamma)$ *if and only if there exists* $\sigma \in \psi(\Gamma)$ *such that* $\psi^\sigma = \psi'$*. For each such* X*,* $\mathbf{G} = \langle G, X, \mathrm{inclusion}\rangle$ *is a projective* Γ*-structure.*

Proof of (a): *From Lemma 5.3,* $\mathcal{D}(G)$ *is topologically closed in* Subg(G) *and we may assume that the forgetful map of* $\mathbf{G}$ *is an inclusion. Choose a closed system* X_0 *of representatives for the* G*-orbits of* $X(\mathbf{G})$ *(Corollary 2.5). As in (1) of Definition 4.1 let* (φ, α) *be a finite* $\mathcal{D}(G)$*-embedding problem for* G*. Then* $\overline{Y}_0 = \{\varphi \circ \psi | \ \psi \in X_0\}$*, as a subset of* $\mathrm{Hom}(\Gamma, A)$*, is finite, and for each* $\bar{\rho} \in \overline{Y}_0$ *we may choose* $\rho \in \mathrm{Hom}(\Gamma, B)$ *such that* $\alpha \circ \rho = \bar{\rho}$*. Let* $Y_0 = \{\rho | \ \bar{\rho} \in \overline{Y}_0\}$*. Define regular actions of* A *and* B *on* $\overline{Y}_0 \times A$ *and* $Y_0 \times B$ *by* $(\bar{\rho}, a)^{a'} = (\bar{\rho}, aa')$ *and* $(\rho, b)^{b'} = (\rho, bb')$*, respectively. Define maps* $d_A\colon \overline{Y}_0 \times A \to \mathrm{Hom}(\Gamma, A)$ *and* $d_B\colon Y_0 \times B \to \mathrm{Hom}(\Gamma, B)$ *by* $d_A(\bar{\rho}, a) = \bar{\rho}^a$ *and* $d_B(\rho, b) = \rho^b$*, respectively. Then* $\mathbf{A} = \langle A, \overline{Y}_0 \times A, d_A\rangle$ *and* $\mathbf{B} = \langle B, Y_0 \times B, d_B\rangle$ *are finite* Γ*-structures. Since* $(X(\mathbf{G}), G)$ *and* $(X_0 \times G, G)$ *are isomorphic as transformation groups, the map* $\psi^\sigma \mapsto (\varphi \circ \psi, \varphi(\sigma))$ *for* $\psi \in X_0$ *and* $\sigma \in G$ *together with the homomorphism* $\varphi\colon G \to A$ *is a morphism* $\varphi\colon \mathbf{G} \to \mathbf{A}$*. Similarly the map* $(\rho, b) \mapsto (\bar{\rho}, \alpha(b))$ *gives together with the homomorphism* $\alpha\colon B \to A$ *an epimorphism* $\alpha\colon \mathbf{B} \to \mathbf{A}$*. Since* $\mathbf{G}$ *is projective, there exists a morphism* $\gamma\colon \mathbf{G} \to \mathbf{B}$ *such that* $\alpha \circ \gamma = \varphi$*. The underlying homomorphism* $\gamma\colon G \to B$ *solves the* Γ*-embedding problem for* G*.*

Proof of (b): *The existence of* X *is the content of Lemma 4.7. So we only have to prove that* $\mathbf{G}$ *is a projective* Γ*-structure. Note first that the action of* G *on* X *is regular. Indeed, suppose that* $\psi = \psi^\sigma$ *for some* $\psi \in X$ *and* $\sigma \in G$*. Since* $\psi(\Gamma) \in \mathcal{D}(G)$*, Lemma 4.5(c) implies that* $\sigma \in \psi(\Gamma)$*. But then* σ *belongs to the center of* $\psi(\Gamma) \cong \Gamma$*. Conclude from Assumption 3.1(b) that* $\sigma = 1$*. Thus* $\mathbf{G}$ *is a* Γ*-structure.*

To prove that $\mathbf{G}$ *is projective we solve each finite weak embedding problem* (φ, α) *as in Definition 5.1. Replace* $\alpha\colon \mathbf{B} \to \mathbf{A}$ *by* $\hat{\alpha}\colon \mathbf{B} \to \widehat{A}$ *and* φ *by* $\hat{\varphi}$ *of (5) of Lemma 4.8 to assume that the forgetful maps of* $\mathbf{A}$ *and* $\mathbf{B}$ *are embeddings and*

(3) for each $\lambda \in X(\mathbf{B})$*,* α *is injective on* $\lambda(\Gamma)$*.*

Apply Lemma 4.8 again to obtain a commutative diagram (5) of finite weak Γ-structures with injective forgetful maps such that

(4) *for $\rho, \rho' \in X(\widehat{A})$, if there exists $\omega \in \mathrm{Aut}(\Gamma)$ such that $\rho' = \rho^{\omega}$, then there exists $g \in \Gamma$ such that $\pi \circ \rho' = \pi \circ \rho^{[g]}$; and*

(5) *$\hat{\varphi}(\psi(\Gamma))$ is a large quotient of Γ for each $\psi \in X$.*

Choose for suitable e, m an epimorphism $\hat{\beta}\colon \mathbf{\Gamma}_{e,m} \to \widehat{\mathbf{B}}$ (Corollary 3.6). Then $(\hat{\varphi}, \hat{\alpha} \circ \hat{\beta})$ is a Γ-embedding problem for G. Indeed if $H \in \mathcal{D}(G)$, then there exists $\psi \in X$ such that $\psi(\Gamma) = H$. Since $\hat{\alpha} \circ \hat{\beta}\colon X(\mathbf{\Gamma}_{e,m}) \to X(\widehat{A})$ is surjective and $\hat{\varphi} \circ \psi \in X(\widehat{A})$ there exists $\delta \in X(\mathbf{\Gamma}_{e,m})$ such that $\hat{\alpha} \circ \hat{\beta} \circ \delta = \hat{\varphi} \circ \psi$. As ψ is injective, there is an isomorphism $\theta\colon H \to \Gamma$ such that $\psi \circ \theta = \mathrm{id}$. Thus $\hat{\alpha} \circ \hat{\beta} \circ \delta \circ \theta = \mathrm{Res}_H \hat{\varphi}$. Now, since G is Γ-projective there exists $\gamma' \in \mathrm{Hom}(G, \Gamma_{e,m})$ such that $\hat{\alpha} \circ \hat{\beta} \circ \gamma' = \hat{\varphi}$ (Lemma 4.4). Let $\hat{\gamma} = \hat{\beta} \circ \gamma'$ and $\gamma = \pi' \circ \hat{\gamma}$. To show that γ defines a solution to the embedding problem (φ, α) of Γ-structures it suffices now to prove for each $\psi \in X$ that $\gamma \circ \psi \in X(\mathbf{B})$.

Indeed, by (5), $(\gamma' \circ \psi)(\Gamma)$ is a large quotient of Γ. Hence by Lemma 3.5(c) there exists $\lambda' \in X(\mathbf{\Gamma}_{e,m})$ such that $\lambda'(\Gamma) = (\gamma' \circ \psi)(\Gamma)$. Moreover λ' and $\gamma' \circ \psi$ are embeddings. Hence there exists $\omega \in \mathrm{Aut}(\Gamma)$ such that $\gamma' \circ \psi = \lambda' \circ \omega$. Both $\hat{\alpha} \circ \hat{\beta} \circ \gamma' \circ \psi = \hat{\varphi} \circ \psi$ and $\hat{\alpha} \circ \hat{\beta} \circ \lambda'$ belong to $X(\widehat{A})$ and $\hat{\alpha} \circ \hat{\beta} \circ \gamma' \circ \psi = \hat{\alpha} \circ \hat{\beta} \circ \lambda' \circ \omega$. Thus (4) gives $g \in \Gamma$ such that $\pi \circ \hat{\alpha} \circ \hat{\beta} \circ \gamma' \circ \psi = \pi \circ \hat{\alpha} \circ \hat{\beta} \circ (\lambda')^{[g]}$. Rewrite this as $\alpha \circ \pi' \circ \hat{\beta} \circ \gamma' \circ \psi = \alpha \circ \pi' \circ \hat{\beta} \circ (\lambda')^{[g]}$. Since $\pi' \circ \hat{\beta} \circ (\lambda')^{[g]} \in X(\mathbf{B})$, (3) implies that α is injective on $(\pi' \circ \hat{\beta} \circ \gamma' \circ \psi)(\Gamma) = (\pi' \circ \hat{\beta} \circ (\lambda')^{[g]})(\Gamma)$. Hence $\gamma \circ \psi = \pi' \circ \hat{\beta} \circ \gamma' \circ \psi = \pi' \circ \hat{\beta} \circ (\lambda')^{[g]} \in X(\mathbf{B})$, as required.
∎

DEFINITION 5.5: We call a morphism $\varphi\colon \mathbf{G} \to \mathbf{H}$ of Γ-structures **rigid** if for each $x \in X(\mathbf{G})$ we have $\mathrm{Ker}(d(x)) = \mathrm{Ker}(d(\varphi(x)))$. This condition is equivalent to $\mathrm{Ker}(\varphi) \cap D(x) = 1$ and also to "φ induces an isomorphism of $D(x)$ onto $D(\varphi(x))$". It is satisfied if $D(y) \cong \Gamma$ for each $y \in X(\mathbf{H})$. ∎

LEMMA 5.6: *Let $\varphi\colon \mathbf{G} \to \mathbf{H}$ be a rigid morphism of Γ-structures. Then each open normal subgroup M of G contains an open normal subgroup K of G such that the induced morphism $\hat{\varphi}\colon \mathbf{G}/K \to \mathbf{H}/\varphi(K)$ is rigid.*

Proof: Let $x \in X(\mathbf{G})$ and $y = \varphi(x) \in X(\mathbf{H})$. Denote the collection of all open normal subgroups N of G contained in M by $\mathcal{N}$. For each $N \in \mathcal{N}$ let $x_N \in X(\mathbf{G})/N$ and $y_N \in X(\mathbf{H})/\varphi(N)$ be the respective images of x and y. Note that $d(x_N)$ is the composed map $\Gamma \xrightarrow{d(x)} G \to G/N$ and $d(y_N)$ is the composed map $\Gamma \xrightarrow{d(y)} H \to H/\varphi(N)$. Therefore

$$\bigcap_{N \in \mathcal{N}} \mathrm{Ker}(d(y_N)) = \mathrm{Ker}(d(y)) = \mathrm{Ker}(d(x)) \le \mathrm{Ker}(d(x_M)).$$

Since the latter group is open in Γ *there exists* $N \in \mathcal{N}$ *such that*

$$\mathrm{Ker}(d(y_N)) \le \mathrm{Ker}(d(x_M)). \tag{6}$$

As $\mathrm{Hom}(\Gamma, G/M)$ *and* $\mathrm{Hom}(\Gamma, H/\varphi(N))$ *are finite, there exists an open neighborhood* U *of* x *in* $X(\mathbf{G})$ *such that for each* $x' \in U$ *and* $y' = \varphi(x')$, $d(x'_M) = d(x_M)$ *and* $d(y'_N) = d(y_N)$. *Thus (6) holds also for* x' *and* y'. *Use the compactness of* $X(\mathbf{G})$ *to assume that (6) holds for all* $x \in X(\mathbf{G})$.

Let $K = M \cap \varphi^{-1}(\varphi(N))$. *Then* $K \in \mathcal{N}$, $N \subseteq K \subseteq \varphi^{-1}(\varphi(N))$ *and* $\varphi(K) = \varphi(N)$. *Thus for each* $x \in X(\mathbf{G})$ *and* $y = \varphi(x)$ *we have* $y_K = y_N$. *By (6)*

$$\begin{aligned}\mathrm{Ker}(d(x_K)) &= d(x)^{-1}(K) = d(x)^{-1}(M) \cap d(x)^{-1}(\varphi^{-1}(\varphi(N))) \\ &= d(x)^{-1}(M) \cap d(y)^{-1}(\varphi(N)) = \mathrm{Ker}(d(x_M)) \cap \mathrm{Ker}(d(y_N)) = \mathrm{Ker}(d(y_K)).\end{aligned}$$

This means that $\overline{\varphi}\colon \mathbf{G}/K \to \mathbf{H}/\varphi(K)$ *is rigid.* ∎

Part B. The $G(\mathbb{Q}_p)$-structure associated with Galois extension.

For the rest of this work we fix a prime p. In Section 11 we characterize the p-adic closures of $\mathbb{Q}$ as algebraic extensions of $\mathbb{Q}$ whose absolute Galois groups are large quotients of $G(\mathbb{Q}_p)$. Since $G(\mathbb{Q}_p)$ is finitely generated we may speak about $G(\mathbb{Q}_p)$-structures. To each field K of characteristic 0 we associate its absolute $G(\mathbb{Q}_p)$-structure $\mathbf{G}(K)$. The elements of the space of sites of $\mathbf{G}(K)$ are essentially the p-adic closures of K. If L is a Galois extension of K, then the relative $G(\mathbb{Q}_p)$-structure $\mathbf{G}(L/K)$ is the quotient structure $\mathbf{G}(K)/G(L)$. Most of Part B (Sections 7, 8, 9 and 10) is dedicated to describe the elements of the space of sites, $X(L/K)$, of $\mathbf{G}(L/K)$ in terms of L/K. The orbit of each site in $X(L/K)$ is uniquely determined by the following data: a field L_0 between K and L (the decomposition field), a place $\pi_0\colon L_0 \to \mathbb{Q}_p \cup \{\infty\}$ and a homomorphism $\varphi_0\colon L_0^\times \to \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^m$. It satisfies the following conditions: the place π_0 is trivial on $\mathbb{Q}$, it does not extend to a $\mathbb{Q}_p$-valued place of a proper extension of L_0 in L, and $\pi_0(u) \neq 0, \infty$ implies $\pi_0(u) = \varphi_0(u)$. In Section 12 we define pseudo p-adically closed fields and realize each $\Gamma_{e,m}$ as the absolute Galois group of a pseudo p-adically closed field, algebraic over $\mathbb{Q}$. We combine this with the results of Section 11 to conclude that $\Gamma = G(\mathbb{Q}_p)$ satisfies Assumption 3.1.

6. p-adically closed fields.

A **valued field** *is a pair (K, v), where K is a field and v is a valuation of K. The valuation v is called p-***adic** *if the residue field is $\mathbb{F}_p$ and $v(p)$ is the smallest positive element of the value group $v(K^\times)$. A field K which admits a p-adic valuation is* **formally** *p-adic; it must be of characteristic 0. As with formally real fields, the existence of a p-adic valuation can be expressed in terms of the field. The p-adic substitution for the square operator X^2 is the* **Kochen operator**

$$\gamma(X) = \frac{1}{p}\frac{X^p - X}{(X^p - X)^2 - 1} \tag{1}$$

LEMMA 6.1: *If (K, v) is a p-adically valued field, then $\gamma(x)$ is defined for each $x \in K$ and $v(\gamma(x)) \geq 0$. Conversely, let K be a field of characteristic 0. If $ap^{-1} \neq$*

$f(\gamma(x_1), \ldots, \gamma(x_n))$ for each $a \in \mathbb{Z}$, $a \neq 0$ relatively prime to p, for each polynomial $f \in \mathbb{Z}[X_1, \ldots, X_n]$ and each $x_1, \ldots, x_n \in K$, then K is formally p-adic.

Proof: [PR, pp. 95 and 99]. ∎

*A p-adically valued field (K, v) which has no proper p-adically valued algebraic extension is p-**adically closed**. Zorn's Lemma implies that each p-adically valued field (K, v) has an algebraic extension $(\overline{K}, \bar{v})$ which is p-adically closed. This is a p-**adic closure** of (K, v). Its isomorphism type over K is determined by the following theorem of Macintyre [M].*

PROPOSITION 6.2: *Let (K, v) be a p-adically valued field. Two p-adic closures (L_1, v_1) and (L_2, v_2) of (K, v) are isomorphic over K if and only if for each $n \in \mathbb{N}$, $L_1^n \cap K = L_2^n \cap K$.*

Proof: [PR, p. 57]. ∎

*The p-adically closed fields are characterized among all p-adically valued fields by the following result. Recall that a $\mathbb{Z}$-**group** is an ordered abelian group A with a smallest positive integer 1 such that $(A : nA) = n$ for each $n \in \mathbb{N}$.*

PROPOSITION 6.3: *Let (K, v) be a p-adically valued field. Then (K, v) is p-adically closed if and only if (K, v) is Henselian and $v(K^\times)$ is a $\mathbb{Z}$-group. In particular, if (K, v) is p-adically closed, then v is the unique p-adic valuation of K.*

Proof: [PR, pp. 34 and 37]. ∎

*Let (K, v) be a p-adically closed field. Using the uniqueness of v we also refer to K as p-**adically closed**.*

PROPOSITION 6.4: *Let (K, v) be a p-adically closed field that extends a p-adically valued field (K_0, v_0).*

(a) *If K_0 is algebraically closed in K, then (K_0, v_0) is p-adically closed.*

(b) *If (K_0, v_0) is p-adically closed, then K is an elementary extension of K_0.*

(c) *Let V be an absolutely irreducible variety defined over K. A necessary and sufficient condition for v to extend to the function field of V is that $V_{\mathrm{sim}}(K) \neq \emptyset$ ($V_{\mathrm{sim}}(K)$ is the set of K-rational simple points of V).*

Proof: [PR, pp. 38, 86 and 145]. ∎

The field $\mathbb{Q}$ admits a unique p-adic valuation v_p. The p-adic closure of $\mathbb{Q}$ coincides with its Henselization with respect to v_p. Hence it is unique up to isomorphism. We denote it by $\mathbb{Q}_{p,\mathrm{alg}}$ and consider $\mathbb{Q}_{p,\mathrm{alg}}$ as the algebraic part of the field $\mathbb{Q}_p$ of p-adic numbers.

PROPOSITION 6.5: *$G(\mathbb{Q}_p)$ is finitely generated and has a trivial center.*

Proof: Jannsen and Wingberg [JW] and [W] give for $p \neq 2$ a presentation of $G(\mathbb{Q}_p)$ by 4 generators and relations. For $p = 2$, Diekert [Di] presents an open subgroup of $G(\mathbb{Q}_2)$ of index 2 by 5 generators and relations. Thus $G(\mathbb{Q}_2)$ is generated by at most 6 elements.

That $G(\mathbb{Q}_p)$ has trivial center follows from the basic results of local class field theory (e.g., [I, p. 7]). ∎

For each field K of characteristic 0 let $K_{\mathrm{alg}} = \tilde{\mathbb{Q}} \cap K$.

COROLLARY 6.6: *Let K be a p-adically closed field. Then $K_{\mathrm{alg}} \cong \mathbb{Q}_{p,\mathrm{alg}}$, $\tilde{\mathbb{Q}}K = \widetilde{K}$ and $G(K) \cong G(\mathbb{Q}_p)$.*

Proof: By Proposition 6.4, K_{alg} is p-adically closed. Since its unique p-adic valuation extends v_p, $K_{\mathrm{alg}} \cong \mathbb{Q}_{p,\mathrm{alg}}$. Without loss identify K_{alg} with $\mathbb{Q}_{p,\mathrm{alg}}$. By Proposition 6.5, $\mathbb{Q}_p$ has for each $n \in \mathbb{N}$ only finitely many extensions of degree $\leq n$ (see also [L2, p. 64]). Since $\mathbb{Q}_p$ and K are elementary extension of $\mathbb{Q}_{p,\mathrm{alg}}$ (Proposition 6.4(b)), $\mathbb{Q}_{p,\mathrm{alg}}$ and K have for each n only finitely many extensions of degree $\leq n$. Moreover, each extension of K of degree $\leq n$ is the compositum of K with an extension of $\mathbb{Q}_{p,\mathrm{alg}}$ of degree $\leq n$. Thus $\tilde{\mathbb{Q}}K = \widetilde{K}$. It follows that $G(K) \cong G(\mathbb{Q}_{p,\mathrm{alg}}) \cong G(\mathbb{Q}_p)$. ∎

It is convenient to shift our point of view from p-adic valuations to the corresponding coarse valuations [PR, p. 25] or rather to their associated $\mathbb{Q}_p$-valued places. We do not distinguish between equivalent p-adic valuations (i.e., p-adic valuations with the same valuation ring).

LEMMA 6.7: *Let K be a field. There is a canonical bijection $v \mapsto \pi_v$ between p-adic valuations v of K and places $\pi\colon K \to \mathbb{Q}_p \cup \{\infty\}$. A p-adically valued field (L, w) extends*

(K, v) *if and only if* (L, π_w) *extends* (K, π_v).

Proof: Let v *be a* p*-adic valuation of* K*, with a valuation ring* O_v*. Each element* $a \in O_v$ *can be uniquely written as* $a = a_0 + b_1 p$*, with* $0 \le a_0 < p$ *and* $b_1 \in O_v$*. Thus,* a *defines by induction a sequence* $a_0, a_1, a_2, \ldots$ *of integers between 0 and* $p-1$ *such that* $a \equiv a_0 + a_1 p + \cdots + a_n p^n \bmod p^{n+1} O_v$*,* $m \in \mathbb{N}$*. This gives a homomorphism* $\pi_v\colon O_v \to \mathbb{Z}_p$*,* $\pi_v(a) = \sum_{n=0}^{\infty} a_n p^n$*, with* $\mathrm{Ker}(\pi_v) = \bigcap_{n=1}^{\infty} p^n O_v$*. The local ring* $O_{\dot v}$ *of* O_v *at* $\mathrm{Ker}(\pi_v)$*, as an overring of a valuation ring, is a valuation ring. Hence* π_v *uniquely extends to a place* $\pi_v\colon K \to \mathbb{Q}_p \cup \{\infty\}$ *with* $O_{\dot v}$ *as the valuation ring. Obviously the restriction of* π_v *to* $\mathbb{Q}$ *and hence to* K_{alg} *is an embedding into* $\mathbb{Q}_p$*. Observe that if* v *and* v' *are equivalent* p*-adic valuations, then* $\pi_v = \pi_{v'}$*.*

Note that $O_v = \{x \in O_{\dot v} |\, \pi_v(x) \in \mathbb{Z}_p\}$*. Indeed, if* x *belongs to the right hand side but* $x \notin O_v$*, then* $x^{-1} \in pO_v$*. Hence* $1 = \pi_v(x^{-1})\pi_v(x) \in p\mathbb{Z}_p$*, a contradiction. It follows that the map* $v \mapsto \pi_v$ *is injective. We show that it is also surjective.*

Let $\pi\colon K \to \mathbb{Q}_p \cup \{\infty\}$ *be a place with a valuation ring* $\dot O$*. Then* $O = \{x \in \dot O |\, \pi(x) \in \mathbb{Z}_p\}$ *is a valuation ring with* pO *as the maximal ideal. Since* π *is the identity map on* $\mathbb{Q}$*, we have* $\mathrm{Ker}(\mathrm{Res}_O \pi) = \bigcap_{n=1}^{\infty} p^n O$*. Denote the corresponding valuation by* v*. Then* $O/pO \cong \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ *and* $v(p)$ *is the smallest positive integer of* $v(K^\times)$*. Thus* v *is a* p*-adic valuation. Moreover, each* $x \in O$ *has for each* $n \in \mathbb{N}$ *a unique representation* $x \equiv x_0 + x_1 p + \cdots + x_n p^n \bmod p^{n+1} O$*, with* $0 \le x_i < p$*,* $i = 0, \ldots, n$*. Hence* $\pi(x) \equiv \pi_v(x) \bmod p^{n+1}\mathbb{Z}_p$*,* $n = 1, 2, 3, \ldots$ *. Conclude that* π *coincides with* π_v *on* O *and therefore on the valuation ring* $O_{\dot v}$*. It follows that* $O = O_{\dot v}$ *and* $\pi = \pi_v$*.*

To prove the second assertion of the lemma check that $K \cap O_w = O_v$ *if and only if* $K \cap O_{\dot w} = O_{\dot v}$*.* ∎

The following lemma gives information about the multiplicative group $\mathbb{Q}_p^\times$ *and its profinite completion* $\Phi = \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$*.*

Lemma 6.8: (a) *The canonical map* $\mathbb{Q}_p^\times \to \Phi$ *is injective; we consider* $\mathbb{Q}_p^\times$ *as a subgroup of* Φ*.*

(b) *For each* $n \in \mathbb{N}$

(b1) $\mathbb{Q}^\times(\mathbb{Q}_p^\times)^n = \mathbb{Q}_p^\times$*;*

(b2) $\mathbb{Q}_{p,\mathrm{alg}}^\times \cap \Phi^n = (\mathbb{Q}_{p,\mathrm{alg}}^\times)^n$ *and* $\mathbb{Q}_{p,\mathrm{alg}}^\times \Phi^n = \Phi$; *and*

(b3) $\zeta \in \Phi$ *and* $\zeta^n = 1$ *implies* $\zeta \in \mathbb{Q}_{p,\mathrm{alg}}^\times$.

Proof of (a): *The multiplicative group* $\mathbb{Q}_p^\times$ *of* $\mathbb{Q}_p$ *has a canonical decomposition* $\mathbb{Q}_p^\times = \langle p\rangle \times \mathbb{Z}_p^\times$. *The discrete group* $\langle p\rangle$ *generated by* p *is isomorphic to* $\mathbb{Z}$. *The group of units* $\mathbb{Z}_p^\times$ *of* $\mathbb{Q}_p$, *is compact and isomorphic to* $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ *if* $p \neq 2$ *and to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2$ *if* $p = 2$. *It follows that* $\bigcap_{n\in\mathbb{N}} (\mathbb{Q}_p^\times)^n = 1$. *Hence the canonical map* $x \mapsto (x(\mathbb{Q}_p^\times)^n)_{n\in\mathbb{N}}$ *of* $\mathbb{Q}_p^\times$ *into* Φ *is injective. We identify* x *with its image in* Φ.

Proof of (b1): *Let* $n \in \mathbb{N}$. *From the proof of (a) it suffices to show that each* $x \in \mathbb{Z}_p^\times$ *belongs to* $\mathbb{Q}^\times(\mathbb{Q}_p^\times)^n$. *Indeed* $x = a + p^{2v_p(n)+1}b$ *with* $a \in \mathbb{Z}$, $a \neq 0$ *and* $b \in \mathbb{Z}_p$. *By the Hensel-Rychlik-Newton Lemma* $c = 1 + p^{2v_p(n)+1}a^{-1}b \in (\mathbb{Q}_p^\times)^n$. *Hence* $x = ac \in \mathbb{Q}^\times(\mathbb{Q}_p^\times)^n$.

Proof of (b2): *The group* Φ^n *is the closure of* $(\mathbb{Q}_p^\times)^n$ *in* Φ. *From [L2, p. 47]* $(\mathbb{Q}_p^\times)^n$ *is a closed subgroup of* $\mathbb{Q}_p^\times$ *of finite index. Therefore* $(\mathbb{Q}_p^\times)^n$ *is open in* $\mathbb{Q}_p^\times$. *It follows that* $\mathbb{Q}_p^\times \cap \Phi^n = (\mathbb{Q}_p^\times)^n$. *Obviously* $\mathbb{Q}_{p,\mathrm{alg}}^\times \cap (\mathbb{Q}_p^\times)^n = (\mathbb{Q}_{p,\mathrm{alg}}^\times)^n$. *Hence* $\mathbb{Q}_{p,\mathrm{alg}}^\times \cap \Phi^n = (\mathbb{Q}_{p,\mathrm{alg}}^\times)^n$. *Also,* $\mathbb{Q}_p^\times \Phi^n = \Phi$. *Therefore* $\mathbb{Q}_{p,\mathrm{alg}}^\times \Phi^n = \Phi$ *follows from (b1).*

Proof of (b3): *From (a),* $\Phi \cong \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times$. *Since* $\hat{\mathbb{Z}}$ *is torsion free, each* $\zeta \in \Phi$ *with* $\zeta^n = 1$ *belongs to* $\mathbb{Z}_p^\times$, *hence to* $\mathbb{Q}_{p,\mathrm{alg}}^\times$. ∎

7. F-closed fields.

In Section 6 we have associated a place π: $K \to \mathbb{Q}_p \cup \{\infty\}$ to each p-adically closed field (K, v). The results we achieve depend only on $\mathrm{char}(\mathbb{Q}_p) = 0$. *Also, in Section 9 we consider places into the algebraic closure of $\mathbb{Q}_p$. Thus, to gain more clarity and generality, we replace $\mathbb{Q}_p$ by some fixed field F of characteristic 0 and consider pairs (K, π) where π: $K \to F \cup \{\infty\}$ is a place. Call each such pair an* **F-valued placed field**. *Let $O_\pi = \{x \in K|\, \pi(x) \in F\}$ be the valuation ring of π. Denote the group $\{u \in K|\, \pi(u) \in F^\times\}$ of π-units of K by U_π and denote the residue field of π by $\pi(K)$. Let (K', π') be an F-valued placed field that extends (K, π). Take valuations v and v' of K and K', corresponding to π and π' respectively such that v' extends v. We say that (K', π') is an* **unramified extension** *of (K, π) if $v(K^\times) = v({K'}^\times)$. Lemmas 7.1-7.3 give information on the existence and uniqueness of extensions of F-valued placed fields.*

LEMMA 7.1: *Let (K, π) be an F-valued placed field. Denote the valuation of K that corresponds to π by v. Let α be an element of the divisible closure $\mathbb{Q} \otimes v(K^\times)$ of $v(K^\times)$ and let n be the smallest positive integer such that $n\alpha \in v(K^\times)$. Choose an element $a \in K^\times$ such that $v(a) = n\alpha$, let $x = a^{1/n}$ and $L = K(x)$. Then π uniquely extends to an F-valued place π' of L with v' the corresponding valuation such that $[L : K] = (v(L^\times) : v'(K^\times)) = n$.*

Proof: Extend π to an $\widetilde{F}$-valued place π' of L and let v' be the corresponding valuation that extends v. Then $v'(x) = \alpha$ and

$$n \leq (\langle v(K^\times), \alpha\rangle : v(K^\times)) \leq (v'(L^\times) : v(K^\times)) \leq [L : K] \leq n.$$

Hence

$$(v'(L^\times) : v(K^\times)) = [L : K] = n. \tag{1}$$

Now let ${v'}_1, \ldots, {v'}_g$ be all extensions of v to L, and let ${L'}_1, \ldots, {L'}_g$ be their residue fields. Then, for the residue field K' of v we have [Ri, p. 228]

$$\sum_{i=1}^{g} ({v'}_i(L^\times) : v(K^\times))[{L'}_i : K'] \leq [L : K]. \tag{2}$$

Conclude from (1) and (2) that $g = 1$, v' is the unique extension of v to L and the residue field of v' is K'. Thus π' is F-valued. If π'' is another extension of π to L, then π'' is equivalent to π'. That is, there exists an automorphism σ of K' (which is the residue field of both π' and π'') such that $\pi'' = \sigma \circ \pi'$. For each $x' \in K'$ take $x \in K$ such that $\pi(x) = x'$. Then $\sigma(x') = \sigma(\pi'(x)) = \pi''(x) = x'$. Conclude that $\pi'' = \pi'$. ∎

LEMMA 7.2: *Let $\pi\colon K \to F \cup \{\infty\}$ be a place and let $K' = \pi(K)$ be its residue field. If K' is algebraically closed in F, then π maps K_{alg} isomorphically onto F_{alg}. Now suppose that L' is an algebraic extension of K' contained in F. Then π extends to a place $\rho\colon L \to F \cup \{\infty\}$ such that $L' = \rho(L)$ and (L, ρ) is an unramified extension of (K, π).*

Proof: Note that the restriction of π to K_{alg} is an embedding into F_{alg}. If $\widetilde{K}' \cap F = K'$, then $\tilde{\mathbb{Q}} \cap F = \tilde{\mathbb{Q}} \cap \pi(K) = \pi(\tilde{\mathbb{Q}} \cap K)$. Thus π maps K_{alg} isomorphically onto F_{alg}.

Next suppose that L'/K' is algebraic and $L' \subseteq F$. Use Zorn's lemma to reduce the existence of ρ to the case where L'/K' is finite, say of degree n. Choose a primitive element z' for L'/K' and let $f' = \mathrm{irr}(z', K')$. Take a monic polynomial $f \in O_\pi[X]$ with $\deg(f) = \deg(f')$ such that $\pi(f) = f'$. Let z be a root of f and let $L = K(z)$. Extend π to a place $\rho_1\colon L \to \widetilde{K}'$. Then $z'_1 = \rho_1(z)$ is a root of f'. Hence

$$n = [K'(z'_1) : K'] \le [\rho_1(L) : K'] \le [L : K] \le n.$$

It follows that $[L : K] = n$ and f is irreducible. Thus π extends to a place π' of L such that $\pi'(z) = z'$. Since $L' \subseteq \pi'(L)$ we have $n = [L' : K'] \le [\pi'(L) : K'] \le [L : K] = n$. Thus $L' = \pi'(L)$ and π' is an F-valued place. Also, (2) implies that $(L, \pi')/(K, \pi)$ is unramified. ∎

LEMMA 7.3: *Let $\pi\colon K \to F \cup \{\infty\}$ be a place and let $\pi_0 = \mathrm{Res}_{K_{\mathrm{alg}}}\pi$. Consider an algebraic extension L_0 of K_{alg} and an extension $\pi'_0\colon L_0 \to F$. Then for $L = L_0K$ there exists a unique place $\pi'\colon L \to F \cup \{\infty\}$ which extends both π and π'_0. Moreover, $(L, \pi')/(K, \pi)$ is an unramified extension. In particular, if (K, π) has no unramified extension to a proper algebraic extension of K, then π maps K_{alg} isomorphically onto F_{alg}.*

Proof: *Let O be the valuation ring of π. Since* $\mathrm{char}(F) = 0$, *π_0' is an embedding of fields. Without loss assume that L_0/K_{alg} is a finite extension with a primitive element z. Since L_0 is linearly disjoint from K over K_{alg} there exists a homomorphism π': $O[z] \to F$ which extends both π_0' and π. The discriminant of z over K is a nonzero element of K_{alg}, hence a unit of O. Therefore, since O is integrally closed, $O[z]$ is the integral closure of O in L [ZS, p. 264]. It follows that the local ring of $O[z]$ with respect to* $\mathrm{Ker}(\pi')$ *is a valuation ring [L3, p. 18]. Conclude that π uniquely extends to a place π': $L \to F \cup \{\infty\}$ such that* $\mathrm{Res}_{L_0}\pi' = \pi_0'$.

To prove the second assertion of the lemma consider $f = \mathrm{irr}(z, K_{\mathrm{alg}})$. *Let $\pi(f) = f_1 \cdots f_r$ be a factorization into irreducible factors over $K' = \pi(K)$. For each i, $1 \le i \le r$, take a root ${z'}_i$ of f_i and extend π to a place ρ_i of L such that $\rho_i(z) = {z'}_i$. Then* $\deg(f_i) \le [\rho_i(L) : K']$. *Since the restriction of π to K_{alg} is injective $f_1, \ldots, f_r$ are distinct. Therefore $\rho_1(L), \ldots, \rho_r(L)$ are mutually nonisomorphic over K' and $\rho_1, \ldots, \rho_r$ are nonequivalent places. Let v be a valuation of K that corresponds to v. Let w_i be a valuation of L that corresponds to ρ_i, $i = 1, \ldots, r$. From (2)*

$$[L : K] = [L_0 : K_{\mathrm{alg}}] = \deg(f) = \sum_{i=1}^{r} \deg(f_i)$$

$$\le \sum_{i=1}^{r} (w_i(L^\times) : v(K^\times))[\rho_i(L) : K'] \le [L : K].$$

Hence $\rho_1, \ldots, \rho_r$ represent all equivalent classes of places of L that extend π. Also $w_i(L^\times) = v(K^\times)$, that is, ρ_i is unramified over K, $i = 1, \ldots, r$. In particular, π', which is equivalent to one of the ρ_i's, is unramified over K.

To prove the last assertion note that if $\pi_0(K_{\mathrm{alg}})$ is properly contained in F_{alg}, then π_0 extends to an embedding π_0' of a proper algebraic extension L_0 into F. Then use the two first parts of the lemma. ∎

*Call an F-valued placed field (K, π), F-***closed** *if π does not extend to a place π': $K' \to F \cup \{\infty\}$ of a proper algebraic extension K' of K. If in addition (K, π) is an extension of an F-valued field (K_0, π_0) and K is algebraic over K_0, then (K, π) is an F-***closure** *of (K_0, π_0). The existence of an F-closure of a given F-valued field (K_0, π_0) is a straightforward application of Zorn's lemma.*

Remark 7.4: From Lemma 6.7, a p-adically valued field (K,v) is p-adically closed if and only if the corresponding $\mathbb{Q}_p$-placed field (K,π_v) is $\mathbb{Q}_p$-closed. ∎

The following characterization of F-closed placed fields overlaps with [PR, Thm. 3.1].

Lemma 7.5: *Let (K,π) be an F-valued placed field and let v be the valuation of K that corresponds to π. The following three conditions are equivalent:*

(3) *(K,π) is F-closed;*

(4a) *every proper algebraic extension (K',π') of (K,π) to an F-valued placed field is ramified (i.e., $v(K^\times)$ is a proper subgroup of $v(K'^\times)$); and*

(4b) *$v(K^\times)$ is a divisible group;*

and

(5a) *the residue field $K_0 = \pi(K)$ is algebraically closed in F;*

(5b) *(K,v) is Henselian; and*

(5c) *$v(K^\times)$ is a divisible group.*

Proof that (3) implies (4): Condition (3) implies that (K,π) has no proper algebraic extensions to F-valued placed fields. Thus (4a) is trivially fulfilled and (4b) follows from Lemma 7.1.

Proof that (4) implies (5): Condition (5a) follows from (4a) by Lemma 7.2. Since in the transfer from (K,v) to its Henselian closure neither the residue field nor the value group are changed (4a) implies that (K,v) is Henselian.

Proof that (5) implies (3): Let (L,ρ) be an F-valued finite extension of (K,π) and let w be the unique (by (5b)) extension of v to L. By (5a), $\pi(K) = \rho(L)$. Since L/K is algebraic $w(L^\times)$ is contained in the divisible hull of $v(K^\times)$. Hence, by (5c), $v(K^\times) = w(L^\times)$. As $\mathrm{char}(F) = 0$ and K is Henselian, $[L:K] = [\rho(L):\pi(K)][w(L^\times):v(K^\times)] = 1$ [A2, Prop. 15]. Conclude that (K,π) is F-closed. ∎

Lemma 7.6: *Let (K,π) be an F-closed placed field.*

(a) *The place π maps K_{alg} isomorphically onto F_{alg}.*

(b) *Suppose that for a positive integer m, $F_{\mathrm{alg}}^\times (F^\times)^m = F^\times$. Then $K_{\mathrm{alg}}^\times (K^\times)^m = K^\times$.*

Proof: Lemma 7.5(5a) and Lemma 7.2 imply (a). To prove (b) let $x \in K^\times$. Denote the valuation of K that corresponds to π by v. By Lemma 7.5(5c) there exists $y \in K^\times$ such that $mv(y) = v(x)$. Then, for $z = xy^{-m}$ we have $v(z) = 0$ and therefore $\pi(z) \in F^\times$. By assumption there exist $b \in F_{\mathrm{alg}}^\times$ and $c \in F^\times$ such that $\pi(z) = bc^m$. Choose $u \in K_{\mathrm{alg}}$ such that $\pi(u) = b$. Observe that c solves the equation $\pi(u)T^m = \pi(z)$. Apply Hensel's lemma (Lemma 7.5(5b)) to the polynomial $uT^m - z$ to conclude the existence of $t \in K^\times$ such that $ut^m = z$. Thus $x = u(ty)^m \in K_{\mathrm{alg}}^\times (K^\times)^m$. ∎

LEMMA 7.7: *Let $\pi\colon K \to F \cup \{\infty\}$ be a place, with v the corresponding valuation such that the value group $v(K^\times)$ is divisible. Let (K_1, π_1) and (K_2, π_2) be F-closures of (K, π). Then there exists a unique K-isomorphism $\sigma\colon K_1 \to K_2$ such that $\pi_1 = \pi_2 \circ \sigma$.*

Proof: For $i = 1, 2$ let v_i be the valuation of K_i corresponding to π_i. Since (K_i, v_i) is Henselian (by (5b)), it contains a Henselization (K_i^h, v_i^h) of (K, v). The residue field K' of K with respect to v is the residue field of K_i^h with respect to v_i^h. Extend π_i to a place $\widetilde{\pi}_i$ of $\widetilde{K}$ with residue field $\widetilde{K}'$ and let $\widetilde{v}_i$ be the corresponding valuation. Since $\widetilde{v}_i(\widetilde{K}^\times)$ is the divisible hull of $v(K^\times)$ [Ri, p.256] it coincides with $v(K^\times)$, i.e., $\widetilde{v}_i$ is unramified over K. In addition, since $\mathrm{char}(K') = 0$, the extension $(\widetilde{K}, \widetilde{v}_i)/(K, v)$ is defectless. Therefore the inertia subgroup $I(\widetilde{v}_i) = \{\kappa \in G(K_i) | \widetilde{\pi}_i \circ \kappa = \widetilde{\pi}_i\}$ of $\widetilde{v}_i/v$ is trivial [E, p.184] and the map $L \mapsto \widetilde{\pi}_i(L)$ is a bijective correspondence between the set of algebraic extensions of K_i^h and the algebraic extensions of K' [E, p.162].

Suppose now that $\sigma, \tau\colon K_1 \to K_2$ are K-isomorphisms such that $\pi_2 \circ \sigma = \pi_1 = \pi_2 \circ \tau$. Extend σ, τ to $\widetilde{\sigma}, \widetilde{\tau} \in G(K)$. Then there exists $\widetilde{\rho} \in G(K_1)$ such that $\widetilde{\pi}_2 \circ \widetilde{\sigma} \circ \widetilde{\rho} = \widetilde{\pi}_2 \circ \widetilde{\tau}$ [L1, p. 247]. Therefore $\widetilde{\sigma}\widetilde{\rho} \cdot (\widetilde{\tau})^{-1}$ belongs to $I(\widetilde{v}_2)$. Thus $\widetilde{\sigma}\widetilde{\rho}(\widetilde{\tau})^{-1} = 1$. Restrict this equality to K_2 to conclude that $\sigma = \tau$. This proves the uniqueness of σ.

To prove the existence of σ note first that there exists a K-isomorphism $\sigma^h\colon K_1^h \to K_2^h$ such that $v_1^h = v_2^h \circ \sigma^h$ [Ri, p. 176]. Hence there exists an automorphism ρ of K' such that $\rho \circ \pi_1^h = \pi_2^h \circ \sigma^h$. Apply both sides on the elements of O_v to conclude that $\rho = 1$. Extend σ^h further to $\widetilde{\sigma} \in G(K)$ such that $\widetilde{\pi}_1 = \widetilde{\pi}_2 \circ \widetilde{\sigma}$ [L1, p. 247]. By (5a)

$$\widetilde{\pi}_2(\widetilde{\sigma}K_1) = \pi_1(K_1) = \widetilde{K}' \cap F = \pi_2(K_2) = \widetilde{\pi}_2(K_2).$$

Since both $\widetilde{\sigma}K_1$ and K_2 are algebraic extensions of K_2^h the first paragraph of the proof *implies that* $\widetilde{\sigma}K_1 = K_2$. *Let* σ *be the restriction of* $\widetilde{\sigma}$ *to* K_1 *and obtain* $\pi_1 = \pi_2 \circ \sigma$. ∎

8. Sites.

Let $\overline{K}$ *be a* p*-adic closure of a formally* p*-adic field. Proposition 6.2 characterizes* $\overline{K}$ *up to* K*-isomorphism by the sequence* $K \cap \overline{K}^n$, $n = 1, 2, 3, \ldots$. *For each* $n \in \mathbb{N}$, $K^\times \cap \overline{K}^n$ *is the kernel of the canonical homomorphism* $K^\times \to \overline{K}^\times/(\overline{K}^\times)^n$. *Observe that* $K_{\rm alg}^\times \cap (K^\times)^n = (K_{\rm alg}^\times)^n$. *Since* $\mathbb{Q}_{p,{\rm alg}}^\times(\mathbb{Q}_p^\times)^m = \mathbb{Q}_p^\times$ *(Lemma 6.8(b1)) Lemma 7.6(b) with* $F = \mathbb{Q}_p$ *implies that* $K_{\rm alg}^\times(K^\times)^n = K^\times$. *Thus, by Lemma 7.6(a)*

$$\overline{K}^\times/(\overline{K}^\times)^n \cong \overline{K}_{\rm alg}^\times/(\overline{K}_{\rm alg}^\times)^n \cong \mathbb{Q}_{p,{\rm alg}}^\times/(\mathbb{Q}_{p,{\rm alg}}^\times)^n \cong \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n.$$

Therefore $\overline{K}$ *induces a compatible sequence of homomorphims* $\varphi_n\colon K^\times \to \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$, *such that* $K \cap \overline{K}^n = {\rm Ker}(\varphi_n)$, $n = 1, 2, 3, \ldots$. *It defines a homomorphism* $\varphi\colon K^\times \to \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$.

As in Section 7 we replace $\mathbb{Q}_p$ *by a field* F *of characteristic 0 and* $\varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$ *by a group* Φ. *The properties (a)-(d) of Lemma 6.8 that* $\mathbb{Q}_p$ *and* $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$ *have are made here as assumptions on* F *and* Φ.

ASSUMPTION 8.1: (a) $F^\times$ *is a subgroup of* Φ.

(b) *For each* $n \in \mathbb{N}$

- (b1) $F_{\rm alg}^\times(F^\times)^n = F^\times$;
- (b2) $F_{\rm alg}^\times \cap \Phi^n = (F_{\rm alg}^\times)^n$ *and* $F_{\rm alg}^\times\Phi^n = \Phi$; *and*
- (b3) $\zeta \in \Phi$ *and* $\zeta^n = 1$ *implies* $\zeta \in F_{\rm alg}^\times$.

Note that Lemma 6.8(b1) is somewhat stronger for $F = \mathbb{Q}_p$ *than Assumption 8.1(b1). We denote the set theoretic union* $F \cup \{\infty\} \cup \Phi$ *by* Θ.

DEFINITION 8.2: *Let* K *be a field of characteristic 0,* $\pi\colon K \to F \cup \{\infty\}$ *a place and* $\varphi\colon K^\times \to \Phi$ *a homomorphism. We say that the pair* (π, φ) *is a* **Θ-site** *of* K *if* $\varphi(u) = \pi(u)$ *for every* $u \in U_\pi$ *(see Notation).*

Let $\theta = (\pi, \varphi)$ *and* $\theta' = (\pi', \varphi')$ *be* Θ*-sites of fields* K *and* K'*, respectively. We say that* (K', θ') **extends** (K, θ) *if* $K \subseteq K', \pi'$ *extends* π *and* φ' *extends* φ. *If* σ *is an isomorphism of a field* K_0 *onto* K*, then* $\theta \circ \sigma = (\pi \circ \sigma, \varphi \circ \sigma)$ *is a* Θ*-site of* K_0.

REMARK 8.3: Most of this section and Sections 9 and 10 holds if we replace Assumption 8.1(a) by a weaker assumption saying that there exists a homomorphism $\eta\colon F^\times \to \Phi$. The connection between π and φ in a Θ-site has to be modified to $\varphi(u) = \eta(\pi(u))$ for each $u \in U_\pi$.

In this version ordered fields may also be viewed as Θ-sites. Here $F = \mathbb{R}$, $\Phi = \{\pm 1\}$ and $\eta\colon \mathbb{R}^\times \to \Phi$ is the sign function. Then the obvious modification of Assumption 8.1(b) is true. If $(K, \leq)$ is an ordered field, then the ring of "finite elements" $O = \{x \in K \mid \exists r \in \mathbb{Q}\colon |x| \leq r\}$ is a valuation ring of K. The corresponding place π defined for $x \in O$ by $\pi(x) = \sup\{r \in \mathbb{Q} \mid r < x\}$ maps K into $\mathbb{R} \cup \{\infty\}$. The homomorphism $\varphi\colon K^\times \to \Phi$ is defined by $\varphi(x) = 1$ if and only if $x > 0$. If $\pi(u) \in \mathbb{R}^\times$, then $0 < r < |u| < s$ for some $r, s \in \mathbb{Q}$, hence $r \leq |\pi(u)| < s$ and therefore $\varphi(u) = \eta(\pi(u))$. Thus (π, φ) is a Θ-site. Conversely, if (π, φ) is a Θ-site, then "$x > 0$ if and only if $\varphi(x) = 1$" defines an ordering of K.

LEMMA 8.4: *Let $\theta = (\pi, \varphi)$ be a Θ-site of K and v the valuation corresponding to π. Let α be an element of the divisible hull of $v(K^\times)$ and let n be the smallest positive integer such that $n\alpha \in v(K^\times)$. Choose $a \in K^\times$ such that $v(a) = n\alpha$, let $x = a^{1/n}$ and let $L = K(x)$. Suppose that there exists $\omega \in \Phi$ such that $\omega^n = \varphi(a)$. Then θ extends to a unique Θ-site $\theta' = (\pi', \varphi')$ of L such that $\varphi'(x) = \omega$.*

Proof: By Lemma 7.1 it suffices to prove only the existence and uniqueness of φ'. Write each $y \in L^\times$ in the form $y = \sum_{i=0}^{n-1} b_i x^i$ with $b_i \in K$. Let v' be the valuation of L that corresponds to π'. If $0 \leq i < j < n$, then, since $v'(x) = \alpha$, $v'(b_i x^i) \neq v'(b_j x^j)$. Hence there exists $i \in \mathbb{Z}$ and $b \in K^\times$ such that $v'(y) = v'(bx^i)$. In particular $\pi'(yb^{-1}x^{-i}) \in F^\times$. Define $\varphi'(y) = \varphi(b)\omega^i \pi'(yb^{-1}x^{-i})$. If also $j \in \mathbb{Z}$ and $c \in K^\times$ are elements such that $v'(y) = v'(cx^j)$, then n divides $j - i$ and hence $u = bc^{-1}x^{i-j} \in U_\pi$. It follows that

$$
\begin{aligned}
\varphi(c)\omega^j \pi'(yc^{-1}x^{-j}) &= \varphi(c)\omega^i \omega^{j-i} \pi(u) \pi'(yb^{-1}x^{-i}) \\
&= \varphi(c)\omega^i \varphi(a^{(j-i)/n}) \varphi(u) \pi'(yb^{-1}x^{-i}) \\
&= \varphi(c)\omega^i \varphi(bc^{-1}) \pi'(yb^{-1}x^{-i}) = \varphi(b)\omega^i \pi'(yb^{-1}x^{-i}).
\end{aligned}
$$

Thus, φ' is well defined. Moreover, one easily checks that φ' is a homomorphism of $L^\times$ into Φ and that it extends φ.

If (π', φ'') is another Θ-site which extends θ such that $\varphi''(x) = \omega$, and y is as above, then apply φ'' on the identity $y = bx^i \cdot yb^{-1}x^{-i}$ to obtain

$$\varphi''(y) = \varphi(b)\omega^i\varphi''(yb^{-1}x^{-i}) = \varphi(b)\omega^i\pi'(yb^{-1}x^{-i}) = \varphi'(y).$$

This proves the uniqueness of θ'. ∎

LEMMA 8.5: *Let $\theta = (\pi, \varphi)$ be a Θ-site of K and let L an algebraic extension of K. Let π' be an F-place of L, unramified over K and which extends π. Then φ uniquely extends to a homomorphism $\varphi'\colon L^\times \to \Phi$ such that (π', φ') is a Θ-site. Moreover*

$$\varphi'(L^\times) \subseteq \varphi'(K^\times) \cdot \pi'(L)^\times. \tag{1}$$

Proof: Let v' be a valuation of L that corresponds to π'. By assumption $v'(L^\times) = v'(K^\times)$. Hence, for each $y \in L^\times$ there exists $b \in K^\times$ such that $v'(y) = v'(b)$ and therefore $\pi'(yb^{-1}) \in F^\times$. Define $\varphi'(y) = \varphi(b)\pi'(yb^{-1})$. As in the proof of Lemma 8.4 this definition is independent of b, it is unique and gives the desired extension (π', φ') of θ such that (1) holds. ∎

*Let $\theta = (\pi, \varphi)$ be a Θ-site of a field K. We say that (K, θ) is Θ-**closed** if θ does not extend to a Θ-site of a proper algebraic extension of K. If in addition (K, θ) is an extension of a Θ-site (K_0, θ_0) and K/K_0 is algebraic, then (K, θ) is a Θ-**closure** of (K_0, θ_0). Note that if (K, π) is F-closed, then (K, θ) is Θ-closed. The converse of this is less obvious but equally true.*

LEMMA 8.6: *Let $\theta = (\pi, \varphi)$ be a Θ-site of K such that (K, θ) is Θ-closed. Then (K, π) is F-closed.*

Proof: By Lemma 8.5, π has no unramified extension to an F-place of a proper algebraic extension of K. Hence, by Lemma 7.3, π maps K_{alg} isomorphically onto F_{alg}. Let v be the valuation of K corresponding to π. By Lemma 7.5 it suffices to show that $v(K^\times)$ is divisible.

Let α be an element of the divisible hull of $v(K^\times)$ and let n be the smallest positive integer such that $n\alpha = v(a)$ with $a \in K^\times$. By Assumption 8.1(b2), there exists $u_0 \in F_{\mathrm{alg}}^\times$ and $\omega \in \Phi$ such that $u_0 = \varphi(a)\omega^{-n}$. Let a_0 be the element of $K_{\mathrm{alg}}^\times$ such that $\pi(a_0) = u_0$.

Then $\varphi(a_0) = \pi(a_0) = \varphi(a)\omega^{-n}$. *Thus* $\varphi(aa_0^{-1}) = \omega^n$ *and, since* $v(a_0) = 0$, n *is the smallest positive integer such that* $v(aa_0^{-1}) = n\alpha$. *By Lemma 8.4,* θ *extends to a* Θ*-site of* $L = K((aa_0^{-1})^{1/n})$. *But then* $L = K$ *and therefore* $\alpha = v((aa_0^{-1})^{1/n}) \in v(K^\times)$. *Thus* $v(K^\times)$ *is divisible.* ∎

PROPOSITION 8.7: *Let* $\theta = (\pi, \varphi)$ *be a* Θ*-site of a field* K. *Then* (K, θ) *has a* Θ*-closure* $(\overline{K}, \bar{\theta})$. *If* (K', θ') *is another* Θ*-closure of* (K, θ), *then there exists a unique* K*-isomorphism* $\sigma\colon \overline{K} \to K'$ *such that* $\bar{\theta} = \theta' \circ \sigma$.

Proof: The existence of $(\overline{K}, \bar{\theta})$ *follows from Zorn's lemma. To prove the existence and uniqueness of* σ *apply Zorn's lemma again to construct a maximal extension* (K_1, θ_1) *of* (K, θ) *such that* $(\overline{K}, \bar{\theta})$ *extends* (K_1, θ_1) *and for which there exists a unique* K*-embedding* $\sigma\colon K_1 \to K'$ *such that* $\theta' = \theta \circ \sigma$ *on* K_1. *If we show that* (K_1, θ_1) *is* Θ*-closed, then so will be* $(\sigma(K_1), \theta_1 \circ \sigma)$ *and therefore* $\sigma(K_1) = K'$.

Without loss assume that σ *is the identity. Otherwise extend* σ *to an automorphism of* $\widetilde{K}$, *replace* (K_1, θ_1) *by* $(\sigma(K_1), \theta_1 \circ \sigma)$ *and* $(\overline{K}, \bar{\theta})$ *by* $(\sigma(\overline{K}), \bar{\theta} \circ \sigma)$. *Further, replace* (K, θ) *by* (K_1, θ_1) *to assume that* (K, θ) *has no proper extension* (K_2, θ_2) *for which there exists a unique* K*-embedding* $\sigma\colon K_2 \to K'$ *such that* $\theta' = \theta \circ \sigma$ *on* K_2. *We have to show that* $K = \overline{K}$.

Let $\theta = (\pi, \varphi)$, $\bar{\theta} = (\bar{\pi}, \bar{\varphi})$ *and* $\theta' = (\pi', \varphi')$. *Denote the valuation of* K *(resp.,* $\overline{K}$, K'*) that corresponds to* π *(resp,* $\bar{\pi}$, π'*) by* v *(resp.,* $\bar{v}$, v'*). We divide the rest of the proof into three parts.*

PART A: $K_{\mathrm{alg}} = \overline{K}_{\mathrm{alg}}$ *and* π *maps* K_{alg} *isomorphically onto* F_{alg}. *By Lemma 8.6* $(\overline{K}, \bar{\pi})$ *and* (K', π') *are* F*-closed. Therefore, by Lemmas 7.5(5a) and 7.2,* $\bar{\pi}$ *(resp.,* π'*) maps* $\overline{K}_{\mathrm{alg}}$ *(resp.,* ${K'}_{\mathrm{alg}}$*) isomorphically onto* F_{alg}. *Thus there exists a unique* K_{alg}*-isomorphism* $\sigma_0\colon \overline{K}_{\mathrm{alg}} \to {K'}_{\mathrm{alg}}$ *such that* $\bar{\pi} = \pi' \circ \sigma_0$ *on* $\overline{K}_{\mathrm{alg}}$. *Since* $\overline{K}_{\mathrm{alg}}$ *and* K *are linearly disjoint over* K_{alg} , σ_0 *uniquely extends to a* K*-isomorphism* $\sigma\colon \overline{K}_{\mathrm{alg}}K \to {K'}_{\mathrm{alg}}K$. *By Lemma 7.3,* $\bar{\pi} = \bar{\pi}' \circ \sigma$ *on* $\overline{K}_{\mathrm{alg}}K$. *Moreover, the restriction of* $\bar{\pi}$ *to* $\overline{K}_{\mathrm{alg}}K$ *is an unramified extension of* π. *Hence, by Lemma 8.5,* $\bar{\varphi} = \varphi' \circ \sigma$ *on* $\overline{K}_{\mathrm{alg}}K$. *Thus* $\bar{\theta} = \theta' \circ \sigma$ *on* $\overline{K}_{\mathrm{alg}}K$. *Conclude that* $\overline{K}_{\mathrm{alg}}K = K$, $\overline{K}_{\mathrm{alg}} = K_{\mathrm{alg}}$ *and* π *maps* K_{alg} *isomorphically onto* F_{alg}.

PART B: *$v(K^\times)$ is divisible.* *Let α be an element of the divisible hull of $v(K^\times)$. Let n be the smallest positive integer such that $n\alpha \in v(K^\times)$. As in the proof of Lemma 8.6 (use Part A instead of Lemma 7.3) find $a \in K^\times$ and $\omega \in \Phi$ such that $v(a) = n\alpha$ and $\varphi(a) = \omega^n$. By Assumption 8.1(b1) and Lemma 7.6 $\overline{K}^\times_{\mathrm{alg}}(\overline{K}^\times)^n = \overline{K}^\times$. Hence there exists $b \in \overline{K}^\times_{\mathrm{alg}}$ such that $ab \in (\overline{K}^\times)^n$. Thus $\varphi(b) \in \Phi^n$. By Assumption 8.1(b2), $\varphi(b) \in (F^\times_{\mathrm{alg}})^n$. Hence, by Part A, $b \in (\overline{K}^\times_{\mathrm{alg}})^n$. Conclude that there exists $y \in \overline{K}^\times$ such that $y^n = a$. Apply $\bar{\varphi}$ to obtain $\bar{\varphi}(y)^n = \varphi(a) = \omega^n$. From Assumption 8.1(b3), $\bar{\varphi}(y)\omega^{-1} \in F^\times_{\mathrm{alg}}$. Hence Part A gives an nth root of unity $z \in K_{\mathrm{alg}}$ such that $\varphi(z) = \bar{\varphi}(y)\omega^{-1}$. Thus $x = yz^{-1}$ satisfies $x^n = a$ and $\bar{\varphi}(x) = \omega$. If $x_1 \in \overline{K}^\times$ also satisfies $x_1^n = a$ and $\bar{\varphi}(x_1) = \omega$, then $(xx_1^{-1})^n = 1$. In particular $xx_1^{-1} \in \overline{K}^\times_{\mathrm{alg}}$ and $\bar{\varphi}(xx_1^{-1}) = 1$. From Part A $x = x_1$.*

Similarly there exists a unique $x' \in K'$ such that $(x')^n = a$ and $\varphi'(x') = \omega$. By Lemma 7.1, the polynomial $X^n - a$ is irreducible over K. Hence there exists a unique K-embedding $\sigma\colon K(x) \to K'$ such that $\sigma(x) = x'$. By Lemma 7.1, $\bar{\pi} = \pi' \circ \sigma$ on $K(x)$. Since $\varphi'(\sigma(x)) = \varphi'(x') = \omega = \bar{\varphi}(x)$, Lemma 8.4 implies that $\bar{\varphi} = \varphi' \circ \sigma$ on $K(x)$. Finally observe that if $\sigma'\colon K(x) \to K'$ is a K-embedding such that $\bar{\theta} = \theta' \circ \sigma'$, then $\varphi'(\sigma'(x)) = \bar{\varphi}(x) = \omega$ and $\sigma'(x)^n = \sigma'(a) = a$. Thus the uniqueness of x' implies that $\sigma'(x) = x'$ and $\sigma' = \sigma$. Conclude that $K(x) = K$ and therefore $n = 1$.

PART C: *Conclusion.* *By Part B and Lemma 7.7 there exists a unique K-embedding $\sigma\colon \overline{K} \to K'$ such that $\bar{\pi} = \pi' \circ \sigma$. From Lemma 8.5, $\bar{\varphi}' \circ \sigma$. Conclude that $\overline{K} = K$. That is, (K, π) is F-closed.* ∎

LEMMA 8.8: *Let $\theta = (\pi, \varphi)$ and $\theta' = (\pi', \varphi')$ be Θ-sites of a field K. Then*

(a) *$\pi(x) = 0$ if and only if $\varphi(1 + x) = \varphi(1 - x) = 1$; and*

(b) *$\varphi = \varphi'$ implies $\pi = \pi'$.*

Proof of (a): *If $\pi(x) = 0$, then $\varphi(1 \pm x) = \pi(1 \pm x) = 1$. If $\pi(x) = \infty$, then $\pi(x^{-1}) = 0$, hence $\varphi(1 \pm x^{-1}) = 1$. Therefore $\varphi(1+x) = \varphi(x)$ and $\varphi(1-x) = \varphi(-x) = \pi(-1)\varphi(x) = -\varphi(x) \neq \varphi(1 + x)$. If $\pi(x) = -1$, then $\varphi(1 - x) = \pi(1 - x) = 2 \neq 1$. Finally if $\pi(x) \neq -1, 0, \infty$, then $\pi(1 + x) \neq 0, 1, \infty$, hence $\varphi(1 + x) = \pi(1 + x) \neq 1$.*

Proof of (b): *Apply (a) to $x \in K^\times$:*

$$\pi(x) = 0 \iff \varphi'(1 \pm x) = \varphi(1 \pm x) = 1 \iff \pi'(x) = 0$$
$$\pi(x) = \infty \iff \pi(x^{-1}) = 0 \iff \pi'(x^{-1}) = 0 \iff \pi'(x) = \infty.$$

It follows that $U_\pi = U_{\pi'}$. For $x \in U_\pi$ we have $\pi(x) = \varphi(x) = \varphi'(x) = \pi'(x)$. Conclude that $\pi = \pi'$. ∎

The following result is restricted to the case $F = \mathbb{Q}_p$ and $\Phi = \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$.

PROPOSITION 8.9: *Let (K, π) be a $\mathbb{Q}_p$-closed placed field.*

(a) *There exists a unique homomorphism φ: $K^\times \to \Phi$ which is the identity on $\mathbb{Q}^\times$. Moreover, (π, φ) is a Θ-site.*

(b) *π is the only $\mathbb{Q}_p$-place of K.*

(c) *If K/K_0 is an algebraic extension, then* $\mathrm{Aut}(K/K_0) = 1$.

Proof of (a): *Let $x \in K^\times$ and $n \in \mathbb{N}$. By Lemma 6.8(b1) there exists $a_n \in \mathbb{Q}^\times$ such that $x \in a_n(K^\times)^n$. If b is another element of $\mathbb{Q}^\times$ such that $x \in b(K^\times)^n$, then $a_n b^{-1} \in \mathbb{Q} \cap (K_{\mathrm{alg}}^\times)^n$. Hence $a_n b^{-1} = \pi(a_n b^{-1}) \in (\mathbb{Q}_p^\times)^n$. Therefore a_n is unique modulo $(\mathbb{Q}_p^\times)^n$. This implies for $m|n$ that $a_n \in a_m(\mathbb{Q}_p^\times)^m$. Thus there is a unique $\varphi(x) \in \Phi$ such that $\varphi(x) \in a_n\Phi^n$ for each $n \in \mathbb{N}$. Obviously φ: $K^\times \to \Phi$ is a homomorhism with $\varphi(x) = x$ for each $x \in \mathbb{Q}^\times$.*

If ψ: $K^\times \to \Phi$ is another homomorhism which is the identity on $\mathbb{Q}^\times$, then $\varphi(x)\psi(x)^{-1} = (\varphi(x)a_n^{-1})(a_n\psi(x)^{-1}) \in \Phi^n$ for each $n \in \mathbb{N}$. Therefore $\varphi(x) = \psi(x)$.

If $x \in U_\pi$, then, since also $a_n \in U_\pi$, so is xa_n^{-1}. Hence $\pi(x)a_n^{-1} = \pi(xa_n^{-1}) \in (\mathbb{Q}_p^\times)^n$. Conclude that $\pi(x) = \varphi(x)$ and that (π, φ) is a Θ-site.

Proof of (b): *If π' is a $\mathbb{Q}_p$-place of K, then, by (a), (π', φ) is a Θ-site of K. Conclude from Lemma 8.8 that $\pi' = \pi$.*

Proof of (c): *Let $\theta = (\pi, \varphi)$ and $\theta_0 = \mathrm{Res}_{K_0}\theta$. Then (K, θ) is a Θ-closure of (K_0, θ_0). By (a) and (b) each $\sigma \in \mathrm{Aut}(K/K_0)$ satisfies $\theta = \theta \circ \sigma$. Conclude from Proposition 8.7 that $\sigma = 1$.* ∎

COROLLARY 8.10: *For each $\mathbb{Q}_p$-place π of a field K there exists a homomorphism $\varphi\colon K^\times \to \Phi$ such that (π, φ) is a Θ-site.*

Proof: *Let $(\overline{K}, \bar\pi)$ be a $\mathbb{Q}_p$-closure of (K, π). By Proposition 8.9(a) $\overline{K}$ has a Θ-site $(\bar\pi, \bar\varphi)$. Then $(\pi, \mathrm{Res}_{\overline{K}/K}\bar\varphi)$ is a Θ-site of K.* ∎

REMARK: Note that Proposition 8.7 implies Macintyre's result (Proposition 6.2). Indeed in the notation of Proposition 6.2 let π_i be the $\mathbb{Q}_p$-place that corresponds to v_i. Let φ_i be the unique homomorphism $\varphi_i\colon L_i^\times \to \Phi$ such that (π_i, φ_i) is a Θ-site, $i = 1, 2$ (Proposition 8.9). Suppose that $K \cap L_1^n = K \cap L_2^n$ for $n = 1, 2, 3, \ldots$. For each $n \in N$ and each $x \in L_i^\times$ there exists $a \in \mathbb{Q}^\times$ such that $\varphi_1(x) \equiv a \bmod \Phi^n$ (Lemma 6.8). By Lemma 6.8 there exists $b \in \mathbb{Q}^\times$ such that $xa^{-1}b^{-1} \in \Phi^n$. Then $b \in \mathbb{Q} \cap \Phi^n \subseteq \mathbb{Q}_p^n$. It follows that $\varphi_2(x) \equiv a \bmod \Phi^n$. Hence $\varphi_1(x) \equiv \varphi_2(x) \bmod \Phi^n$. Since this is true for each n we have $\varphi_1(x) = \varphi_2(x)$. Conclude from Proposition 8.7 that $L_1 \cong_K L_2$. ∎

9. $\widetilde{\Theta}$-sites.

Each F-place π of a field K extends to an $\widetilde{F}$-place of $\widetilde{K}$. An analogue of this holds for sites. For each algebraic extension E of F define a group

$$\Phi_E = E^\times \times \Phi / \{(a^{-1}, a) |\, a \in F^\times\}.$$

For $x \in E^\times$ and $\omega \in \Phi$ define the class of (x, ω) modulo the subgroup $\{(a^{-1}, a) | a \in F^\times\}$ to be $[x, w]$. In particular $[x, \omega] = [xa^{-1}, \omega a]$ for every $a \in F^\times$. Both $E^\times$ and Φ can be embedded in Φ_E by $x \mapsto [x, 1]$ and $\omega \mapsto [1, \omega]$, respectively. These embeddings coincide on $F^\times$.

The case $E = \widetilde{F}$ deserves special attention. We write $\widetilde{\Phi}$ for $\Phi_{\widetilde{F}}$. Note that $\widetilde{\Phi}$ is the union $\bigcup \Phi_E$ where E ranges over all finite extensions of F.

LEMMA 9.1: *The group $\widetilde{\Phi}$ is divisible.*

Proof: *Let n be a positive integer. For $x \in \widetilde{F}^\times$ and $\omega \in \Phi$ choose $b \in F_{\mathrm{alg}}^\times$ and $\omega_1 \in \Phi$ such that $\omega b = \omega_1^n$ (Assumption 8.1(b2)). Let $y \in \widetilde{F}^\times$ satisfies $x = by^n$. Then $[x, \omega] = [xb^{-1}, \omega b] = [y, \omega_1]^n$.* ∎

LEMMA 9.2: *$\widetilde{F}$ and $\widetilde{\Phi}$ satisfy Assumption 8.1.*

Proof: *Since* $\widetilde{F} \cap \tilde{\mathbb{Q}} = \tilde{\mathbb{Q}}$ *and* $(\widetilde{F}^\times)^n = \widetilde{F}^\times$ *Assumption 8.1(b1) is trivial and (b2) follows from Lemma 9.1. Thus we have only to prove Assumption 8.1(b3).*

Let $[x, \omega] \in \widetilde{\Phi}$ *with* $[x, \omega]^n = 1$. *Then there exists* $t \in F^\times$ *such that* $x^n = t^{-1}$ *and* $\omega^n = t$. *By Assumption 8.1(b1), there exist* $a \in F_{\rm alg}^\times$ *and* $s \in F^\times$ *such that* $ats^n = 1$. *By Assumption 8.1(b2),* $a \in F_{\rm alg}^\times \cap \Phi^n = (F_{\rm alg}^\times)^n$. *Let* $b \in F_{\rm alg}^\times$ *such that* $b^n = a$. *Then* $(\omega sb)^n = 1$ *and by Assumption 8.1(b3)* $c = \omega sb \in F_{\rm alg}^\times$. *Conclude that* $(xs^{-1}b^{-1}c)^n = 1$ *and* $[x, \omega] = [xs^{-1}b^{-1}c, \omega sbc^{-1}] = [xs^{-1}b^{-1}c, 1] \in \widetilde{F}^\times$. *Since* $[x, \omega]^n = 1$ *we have* $[x, \omega] = \widetilde{F}_{\rm alg}^\times$. ∎

We abbreviate $E \cup \{\infty\} \cup \Phi_E$ *by* Θ_E *and write* $\widetilde{\Theta}$ *for* $\Theta_{\widetilde{F}}$. *A* $\widetilde{\Theta}$-**site** *of a field* L *is a pair* $\theta = (\pi, \varphi)$, *where* $\pi\colon L \to \widetilde{F} \cup \{\infty\}$ *is a place and* $\varphi\colon L^\times \to \widetilde{\Phi}$ *is a homomorphism such that* $\varphi(u) = \pi(u)$ *for each* $u \in U_\pi$. *For a subfield* K *of* L, $\mathrm{Res}_K\theta = (\mathrm{Res}_K(\pi), \mathrm{Res}_{K^\times}(\varphi))$ *is a* $\widetilde{\Theta}$*-site of* K. *Write* $\theta(L) \subseteq \Theta$ *if* $\pi(L) \subseteq F \cup \{\infty\}$ *and* $\varphi(L^\times) \subseteq \Phi$. *In this case* $\mathrm{Res}_K\theta$ *is a* Θ*-site of* K. *Lemma 9.2 implies that the results of Section 8 except Proposition 8.9 may be applied to* $\widetilde{\Theta}$*-sites.*

PROPOSITION 9.3: *Let* θ_0 *be a* Θ*-site of a field* K *and let* L *be a Galois extension of* K. *Then*

(a) θ_0 *extends to a* $\widetilde{\Theta}$*-site* θ *of* L;

(b) *if another* $\widetilde{\Theta}$*-site* θ' *of* L *extends* θ_0, *then there exists a unique* $\sigma \in \mathcal{G}(L/K)$ *such that* $\theta = \theta' \circ \sigma$.

Proof: *Consider* θ_0 *as a* $\widetilde{\Theta}$*-site. Use Lemma 9.2 and apply Proposition 8.7 on* (K, θ_0) *to obtain a* $\widetilde{\Theta}$*-closure* $(\overline{K}, \bar{\theta})$, *with* $\bar{\theta} = (\bar{\pi}, \bar{\varphi})$. *In particular* $(\overline{K}, \bar{\pi})$ *is* $\widetilde{F}$*-closed (Lemma 8.6). Hence* $\overline{K} = \widetilde{K}$. *Then* $\theta = \mathrm{Res}_L\bar{\theta}$ *is an extension of* θ_0 *to* L. *This proves (a).*

To prove (b) extend θ' *as above to a* $\widetilde{\Theta}$*-site* $\bar{\theta}'$ *of* $\widetilde{K}$. *By Proposition 8.7 there exists a unique* $\tau \in G(K)$ *such that* $\bar{\theta} = \bar{\theta}' \circ \tau$. *Hence* $\theta = \theta' \circ \mathrm{Res}_L\tau$. ∎

Define an action of $G(F)$ *on* $\widetilde{\Phi}$:

$$g[x, \omega] = [g(x), \omega], \qquad g \in G(F),\ x \in \widetilde{F}^\times \text{ and } \omega \in \Phi.$$

If E *is an algebraic extension of* F *and* $[x, \omega] \in \widetilde{\Phi}$ *is fixed under the action of* $G(E)$, *then for each* $g \in G(E)$ *there exists* $a \in F^\times$ *such that* $(g(x), \omega) = (x, \omega)(a^{-1}, a)$. *Hence*

$a = 1$ and $g(x) = x$. *Thus Φ_E is the fixed subgroup of $\widetilde{\Phi}$ under $G(E)$. Since $G(F)$ acts on $\widetilde{F}$, this defines an action of $G(F)$ on $\widetilde{\Theta}$. The fixed subset of $\widetilde{\Theta}$ under $G(E)$ is Θ_E. For a $\widetilde{\Theta}$-site $\theta = (\pi, \varphi)$ of a field L and $g \in G(F)$ we define $g \circ \theta$ to be $(g \circ \pi, g \circ \varphi)$. Then $g \circ \theta$ is also a $\widetilde{\Theta}$-site of L. Also, for $x \in L^\times$, we write $\theta(x)$ for $(\pi(x), \varphi(x))$.*

Definition 9.4: *Let L/K be a Galois extension and θ a $\widetilde{\Theta}$-site of L such that $\theta(K) \subseteq \Theta$. For each $g \in G(F)$ we have $\mathrm{Res}_K(g \circ \theta) = \mathrm{Res}_K(\theta)$. Thus Proposition 9.3(b) gives a unique element $d_\theta(g) \in \mathcal{G}(L/K)$ such that $g \circ \theta = \theta \circ d_\theta(g)$. We call $D(\theta) = \{d_\theta(g) |\, g \in G(F)\}$ the* **decomposition group** *of θ. The fixed field in L of $D(\theta)$ is the* **decomposition field** *of θ.*

Lemma 9.5: *Let L/K be a Galois extension and let θ be a $\widetilde{\Theta}$-site of L such that $\theta(K) \subseteq \Theta$.*

- (a) *If L'/K' is a Galois extension such that $K \subseteq K'$ and $L \subseteq L'$, and θ' is a $\widetilde{\Theta}$-site of L' that extends θ such that $\theta'(K') \subseteq \Theta$, then $d_\theta(g) = \mathrm{res}_L(d_{\theta'}(g))$ for each $g \in G(F)$ and therefore $D(\theta) = \mathrm{res}_L D(\theta')$.*
- (b) *The decomposition field L_0 of θ (Definition 9.4) is the unique maximal field such that $K \subseteq L_0 \subseteq L$ and $\theta(L_0) \subseteq \Theta$. If $L = \widetilde{K}$, then $(L_0, \mathrm{Res}_{L_0}\theta)$ is Θ-closed.*
- (c) *For each finite extension K' of K which is contained in L there exists a finite extension F' of F such that $[F' : F] \leq [K' : K]$ and $\theta(K') \subseteq \Theta_{F'}$.*
- (d) *The map $d_\theta\colon G(F) \to \mathcal{G}(L/K)$ is a continuous homomorphism.*
- (e) *For each $\sigma \in \mathcal{G}(L/K)$ and each $g \in G(F)$ we have $d_{\theta\circ\sigma}(g) = \sigma^{-1} d_\theta(g)\sigma$.*

Proof of (a): *Restrict $g \circ \theta' = \theta' \circ d_{\theta'}(g)$ to L to obtain $g \circ \theta = \theta \circ \mathrm{res}_L(d_{\theta'}(g))$. Conclude that $\mathrm{res}_L(d_{\theta'}(g)) = d_\theta(g)$.*

Proof of (b): *If $x \in L_0^\times$, then $g(\theta(x)) = \theta(d_\theta(g)(x)) = \theta(x)$, for all $g \in G(F)$. Hence $\theta(x) \in \Theta$. Conversely, let M be a field between K and L such that $\theta(M) \subseteq \Theta$. For each $g \in G(F)$ we have $\mathrm{Res}_M(\theta \circ d_\theta(g)) = \mathrm{Res}_M(g \circ \theta) = \mathrm{Res}_M(\theta)$. The existence part of Proposition 9.3(b) for L/M gives $\tau \in \mathcal{G}(L/M)$ such that $\theta \circ d_\theta(g) = \theta \circ \tau$. The uniqueness part of Proposition 9.3(b) implies that $d_\theta(g) = \tau$. It follows that $\mathcal{G}(L/L_0) \leq \mathcal{G}(L/M)$ and therefore $M \subseteq L_0$.*

Proof of (c): *Extend* θ *to a* $\widetilde{\Theta}$*-site, also denoted* $\theta = (\pi, \varphi)$*, of* $\widetilde{K}$*. Let* $\overline{K}$ *be the decomposition field of* θ*. By (b),* $(\overline{K}, \mathrm{Res}_{\overline{K}}\theta)$ *is* Θ*-closed, and by Lemma 8.6,* $(\overline{K}, \mathrm{Res}_{\overline{K}}\pi)$ *is* F*-closed. In particular* $\mathrm{Res}_{\overline{K}}\pi$ *is unramified in* $K'\overline{K}$ *(Lemma 7.5(5c)). Let* $F' = \pi(K'\overline{K})$*. Then* $[F' : F] \leq [K'\overline{K} : \overline{K}] \leq [K' : K]$*. By Lemma 8.5,* $\varphi((K')^\times) \subseteq \varphi((K'\overline{K})^\times) \subseteq \varphi(\overline{K}^\times)(F')^\times \subseteq \Phi_{F'}$*. Therefore* $\theta(K') \subseteq \Theta_{F'}$ *.*

Proof of (d): *The multiplicativity of* d_θ *is an immediate consequence of the definition of* $d_\theta(g)$*. To prove its continuity let* K' *be a finite Galois extension of* K *contained in* L*. By (c) there exists a finite extension* F' *of* F *such that* $\theta(K') \subseteq \Theta_{F'}$*. Then for each* $g \in G(F')$ *we have* $\theta \circ d_\theta(g) = g \circ \theta = \theta$ *on* K'*. Apply the uniqueness part of Lemma 9.3 to the extension* K'/K *to conclude that* $\mathrm{Res}_{K'} d_\theta(g) = 1$*. Thus* $d_\theta(G(F')) \leq \mathcal{G}(L/K')$*. Conclude that* d_θ *is continuous.*

Proof of (e): *By definition* $\theta \circ \sigma \circ d_{\theta\circ\sigma}(g) = g \circ \theta \circ \sigma = \theta \circ d_\theta(g) \circ \sigma = \theta \circ \sigma \circ \sigma^{-1} d_\theta(g)\sigma$*. The uniqueness of* $d_{\theta\circ\sigma}$ *implies* $d_{\theta\circ\sigma} = \sigma^{-1} d_\theta(g)\sigma$*.* ∎

10. The space of sites of a Galois extension.

From now on we consider only the case $F = \mathbb{Q}_p$ and $\Phi = \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$. Thus $\Theta = \mathbb{Q}_p \cup \{\infty\} \cup \Phi$ and $\widetilde{\Theta} = \tilde{\mathbb{Q}}_p \cup \{\infty\} \cup \widetilde{\Phi}$. The goal of this section is to associate a $G(\mathbb{Q}_p)$-structure $\mathbf{G}(L/K)$ with each Galois extension L/K. The space of sites of $\mathbf{G}(L/K)$ is the collection of all $\widetilde{\Theta}$-site θ of L such that $\theta(K) \subseteq \Theta$. The forgetful map maps θ onto d_θ (Definition 9.4).

Endow $\tilde{\mathbb{Q}}_p^\times$ with the p-adic (locally compact) topology. Observe that Φ, as a profinite group, is compact. Equip $\widetilde{\Phi} = (\tilde{\mathbb{Q}}_p^\times \times \Phi)/\{(a, a^{-1})|\, a \in \mathbb{Q}_p^\times\}$ with the quotient topology of the product topology of

Then the canonical embeddings of $\tilde{\mathbb{Q}}_p^\times$ and Φ into $\widetilde{\Phi}$ (Section 9) are continuous. Since the action of $G(\mathbb{Q}_p)$ on $\tilde{\mathbb{Q}}_p^\times$ is continuous so is the action of $G(\mathbb{Q}_p)$ on $\widetilde{\Phi}$.

LEMMA 10.1: *If a topological group G has an open subgroup H of finite index and H is profinite, then so is G.*

Proof: G is a union of finitely many disjoint cosets modulo H. Each coset gH is a Boolean space (Section 1). Therefore so is G. It follows that G is a profinite group [R, p. 16]. ∎

LEMMA 10.2: *Let E be a finite extension of $\mathbb{Q}_p$. Then the subgroup Φ_E of $\widetilde{\Phi}$ is profinite.*

Proof: Let t be a prime element of E, U the group of units of E and e the ramification index of E over $\mathbb{Q}_p$. Then $V = \langle t^e \rangle \times U = \langle p \rangle \times U$ is an open subgroup of $E^\times$ of finite index which contains $\mathbb{Q}_p^\times$. By Lemma 10.1 it suffices to prove that $W = (V \times \Phi)/\{(a, a^{-1})|\, a \in \mathbb{Q}_p^\times\}$ is profinite. Indeed, use $\mathbb{Q}_p^\times = \{p^n u|\, n \in \mathbb{Z},\ u \in \mathbb{Z}_p^\times\}$ and $\Phi = \{p^m v | m \in \hat{\mathbb{Z}}^\times,\ v \in \mathbb{Z}_p^\times\}$ to define a continuous open homomorphism $V \times \Phi \to \hat{\mathbb{Z}} \times U$ by $(p^n u, p^m v) \mapsto (p^{n+m}, uv)$, for $n \in \mathbb{Z}$, $u \in U$, $m \in \hat{\mathbb{Z}}$ and $v \in \mathbb{Z}_p^\times$. The kernel is $\{(a, a^{-1})|\, a \in \mathbb{Q}_p^\times\}$. Thus $W \cong \hat{\mathbb{Z}} \times U$. Since U is compact [L2, p.46] so is W. ∎

For a Galois extension L/K we denote the set of all $\widetilde{\Theta}$-sites $\theta = (\pi, \varphi)$ of L such that $\theta(K) \subseteq \Theta$ by $X(L/K)$. Since $\pi\colon L \to \tilde{\mathbb{Q}}_p \cup \{\infty\}$ and $\varphi\colon L^\times \to \widetilde{\Phi}$ are maps, consider $X(L/K)$ as a subset of $Y = (\tilde{\mathbb{Q}}_p \cup \{\infty\})^L \times \widetilde{\Phi}^{L^\times}$. Equip $\tilde{\mathbb{Q}}_p \cup \{\infty\}$ with the topology of one point compactification. Then Y and $X(L/K)$ are topological spaces.

If L'/K' is another Galois extension such that $K \subseteq K'$ and $L \subseteq L'$, then the *obvious restriction map* $\mathrm{Res}_{L'/L}\colon X(L'/K') \to X(L/K)$ *is continuous. Moreover*

$$X(L/K) \cong \varprojlim X(L_0/K), \tag{1}$$

where L_0/K *ranges over all finite Galois subextensions of* L/K.

Let $K \subseteq K' \subseteq L$ *and* $\theta \in X(L/K)$, *and suppose that* $D(\theta) \leq \mathcal{G}(L/K')$ *(Definition 9.4). Then* $\theta(K') \subseteq \Theta$ *(Lemma 9.5(b)). Conclude that*

$$X(L/K') = \{\theta \in X(L/K) |\, D(\theta) \leq \mathcal{G}(L/K')\}. \tag{2}$$

LEMMA 10.3: (a) *For each Galois extension* L/K, $X(L/K)$ *is a Boolean space.*

(b) *The collection of sets*

$$\{(\pi, \varphi) \in X(L/K) |\, \varphi(y) \in V\}, \tag{3}$$

where y *ranges over* $L^\times$ *and* V *ranges through a basis of* $\widetilde{\Phi}$, *is a subbasis for the topology of* $X(L/K)$.

Proof of (a): *By (1), it suffices to consider the case where* L/K *is finite. Denote the compositum of all extensions of* $\mathbb{Q}_p$ *of degree* $\leq [L : K]$ *by* E. *It is a finite extension of* $\mathbb{Q}_p$ *(Proposition 6.5). By Lemma 9.5(c),* $\theta(L) \subseteq \Theta_E$ *for each* $\theta \in X(L/K)$. *Thus* $X(L/K)$ *is a subspace of* $Y_E = (E \cup \{\infty\})^L \times \Phi_E^{L^\times}$. *Since generalized addition and multiplication in* $E \cup \{\infty\}$ *are continuous,* $X(L/K)$ *is closed in* Y_E. *As* E *is a locally compact totally disconnected Hausdorff space,* $E \cup \{\infty\}$ *is Boolean. By Lemma 10.2, so is* Φ_E. *Hence the product space* Y_E *is Boolean and therefore so is* $X(L/K)$.

Proof of (b): *The map* $(\pi, \varphi) \mapsto \varphi$ *of* $X(L/K)$ *into* $\widetilde{\Phi}^{L^\times}$ *is injective, by Lemma 8.8(b). By (a) it is a homomorphism of* $X(L/K)$ *onto its image in* $\widetilde{\Phi}^{L^\times}$. ∎

If L/K *is a finite extension,* E *is the compositum of all finite extensions of* $\mathbb{Q}_p$ *of degree at most* $[L : K]$, *then* $\theta(L^\times) \subseteq \Theta_E$ *for each* $\theta \in X(L/K)$ *(Lemma 9.5(c)). Hence, in order to get a subbasis of* $X(L/K)$, *it suffices to allow* V *in (3) to run through a basis of* Φ_E.

Remark 10.4: *The action of $\mathcal{G}(L/K)$ on $X(L/K)$.* Define the action of $\mathcal{G}(L/K)$ on $X(L/K)$ by $\theta^\sigma = \theta \circ \sigma$. Each $\sigma \in \mathcal{G}(L/K)$ maps the set (3) onto the set

$$\{(\pi, \varphi) \in X(L/K) |\, \varphi(\sigma^{-1}V) \in V\}.$$

Moreover $\theta \circ \sigma = \theta$ implies $\sigma = 1$ (Proposition 9.3(b)). Hence this action is continuous and regular and $(X(L/K), \mathcal{G}(L/K))$ is a profinite transformation group (Lemma 10.3). If L'/K' is another Galois extension such that $K \subseteq K'$ and $L \subseteq L'$, then $\mathrm{Res}_{L'/L}$: $(X(L'/K'), \mathcal{G}(L'/K')) \to (X(L/K), \mathcal{G}(L/K))$ is a morphism of transformation groups. ∎

Remark 10.5: *The space $X(K)$.* We write $X(K)$ for $X(K/K)$, the set of all Θ-sites of K. The subbasis for its topology given by

$$\{(\pi, \varphi) \in X(K) |\, \varphi(a) \equiv \omega \bmod \Phi^m\}, \qquad a \in K^\times,\ \ \omega \in \Phi \ \text{ and } \ m \in \mathbb{N},$$

(Lemma 10.3(b)) consists of open-closed sets.

By Lemma 10.3(a) each open-closed subset H of $X(K)$ is compact. Hence it is a finite union of finite intersections of subbasis sets

$$H = \bigcup_{i=1}^{k} \bigcap_{j=1}^{l(i)} \{(\pi, \varphi) \in X(K) |\, \varphi(a_{ij}) \equiv \omega_{ij} \bmod \Phi^{m_{ij}}\} \tag{4}$$

with $a_{ij} \in K^\times$, $\omega_{ij} \in \Phi$ and $m_{ij} \in \mathbb{N}$. Let m be a common multiple of all m_{ij}'s. Since $\Phi^{m_{ij}}/\Phi^m \cong (\mathbb{Q}_p^\times)^{m_{ij}}/(\mathbb{Q}_p^\times)^m$ is finite, we may enlarge each $l(i)$, if necessary, to assume that $m_{ij} = m$ for each i and j. Lemma 6.8 gives $b_{ij} \in \mathbb{Q}^\times$ such that $b_{ij} \equiv \omega_{ij} \bmod \Phi^m, i = 1, \ldots, k$ and $j = 1, \ldots, l(i)$. Then $\varphi(b_{ij}) = \pi(b_{ij}) = b_{ij}$. Replace a_{ij} by $b_{ij}^{-1} a_{ij}$ if necessary and use De-Morgan laws to change the order of the union and intersection in (4) and add trivial conditions if necessary like $\varphi(1) \in \Phi^m$ to represent H as

$$H = \bigcap_{i=1}^{r} \bigcup_{j=1}^{n} \{(\pi, \varphi) \in X(K) |\, \varphi(a_{ij}) \in \Phi^m\}. \qquad ∎$$

Again, let L/K be a Galois extension. Define a map

$$d: X(L/K) \to \mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L/K))$$

by $d(\theta) = d_\theta$, where $d_\theta\colon \Gamma \to \mathcal{G}(L/K)$ is the unique homomorphism for which $\theta \circ d_\theta(g) = g \circ \theta$ for every $g \in G(\mathbb{Q}_p)$ (Definition 9.4). For each $\sigma \in \mathcal{G}(L/K)$

$$\theta^\sigma \circ d_{\theta^\sigma}(g) = g \circ \theta^\sigma = g \circ \theta \circ \sigma = \theta \circ \sigma \circ \sigma^{-1} \circ d_\theta(g) \circ \sigma = \theta^\sigma \circ d_\theta(g)^\sigma.$$

Hence $d(\theta)^\sigma = d(\theta^\sigma)$. Continuity of d is a consequence of the next result which is a version of Krasner's lemma.

LEMMA 10.6: *Let L/K be a finite Galois extension and let $\theta \in X(L/K)$. Then θ has an open neighborhood V_θ such that $d_\theta = d_{\theta'}$ for each $\theta' \in V_\theta$.*

Proof: We first fix an element $g \in G(\mathbb{Q}_p)$ and construct an open neighborhood $V_{\theta,g}$ of $\theta = (\pi, \varphi)$ such that $d_\theta(g) = d_{\theta'}(g)$ for each $\theta' \in V_{\theta,g}$.

Indeed let $\sigma = d_\theta(g)$. Let $\tau \in \mathcal{G}(L/K)$, $\tau \neq \sigma$. Proposition 9.3(b) implies that $\theta \circ \tau \neq \theta \circ \sigma$. Hence $\varphi \circ \tau \neq \varphi \circ \sigma$ (Lemma 8.8(b)). Choose $a_\tau \in L^\times$ such that $(\varphi \circ \tau)(a_\tau) \neq (\varphi \circ \sigma)(a_\tau)$.

Let E be the compositum of all finite extensions of $\mathbb{Q}_p$ of degree at most $[L : K]$. E is a finite extension of $\mathbb{Q}_p$ (Proposition 6.5) and $\varphi'(L^\times) \subseteq \Phi_E$ for each $(\pi', \varphi') \in X(L/K)$ (Lemma 9.5(c)).

Since L/K is finite and Φ_E is profinite (Lemma 10.2) Φ_E has an open subgroup U such that

$$(\varphi \circ \tau)(a_\tau) \not\equiv (\varphi \circ \sigma)(a_\tau) \bmod U \qquad \textit{for each } \tau \in \mathcal{G}(L/K),\ \tau \neq \sigma. \tag{9}$$

Replace U, if necessary, by $\bigcap_{h \in \mathcal{G}(E/\mathbb{Q}_p)} h(U)$ to assume that $g(U) = U$.

Now define $V_{\theta,g}$ to be the set of all $\theta' = (\pi', \varphi') \in X(L/K)$ such that

$$(\varphi' \circ \kappa)(a_\tau) \equiv (\varphi \circ \kappa)(a_\tau) \bmod U \qquad \textit{for all } \kappa, \tau \in \mathcal{G}(L/K). \tag{10}$$

It is an open neighborhood of θ. In particular, for $\kappa = 1$, $\varphi'(a_\tau) \equiv \varphi(a_\tau) \bmod U$, and therefore $(g \circ \varphi')(a_\tau) \equiv (g \circ \varphi)(a_\tau) \bmod U$ for all $\tau \in \mathcal{G}(L/K)$. Thus

$$(\varphi' \circ d_{\theta'}(g))(a_\tau) \equiv (\varphi \circ \sigma)(a_\tau) \bmod U \qquad \textit{for every } \tau \in \mathcal{G}(L/K). \tag{11}$$

Substitute $\kappa = d_{\theta'}(g)$ in (10) to obtain

$$(\varphi' \circ d_{\theta'}(g))(a_\tau) \equiv (\varphi \circ d_{\theta'}(g))(a_\tau) \bmod U \qquad \textit{for every } \tau \in \mathcal{G}(L/K). \tag{12}$$

It follows from (11) and (12) that

$$(13) \qquad (\varphi \circ d_{\theta'}(g))(a_\tau) \equiv (\varphi \circ \sigma)(a_\tau) \bmod U \qquad \text{for every } \tau \in \mathcal{G}(L/K).$$

Thus (9) and (13) imply that $d_{\theta'}(g) = \sigma = d_\theta(g)$.

Finally let $V_\theta = \bigcap_g V_{\theta,g}$, *where* g *ranges over a finite set* G_0 *of generators of* $G(\mathbb{Q}_p)$ *(Proposition 6.5). Then* V_θ *is an open neighborhood of* θ *such that for each* $\theta' \in V_\theta$ *and each* $g \in G_0$ *we have* $d_\theta(g) = d_{\theta'}(g)$. *Since* d_θ *and* $d_{\theta'}$ *are continuous homomorphisms (Lemma 9.5(d))* $d_\theta = d_{\theta'}$. ∎

PROPOSITION 10.7: *Let* L/K *be a Galois extension. Then*

(a) $\mathbf{G}(L/K) = \langle \mathcal{G}(L/K), X(L/K), d \rangle$ *is a* $G(\mathbb{Q}_p)$*-structure;*

(b) *if* L_0/K_0 *is a Galois extension such that* $K_0 \subseteq K$ *and* $L_0 \subseteq L$, *then*

$$\mathrm{Res}_{L/L_0}\colon \mathbf{G}(L/K) \to \mathbf{G}(L_0/K_0)$$

is a morphism of $G(\mathbb{Q}_p)$*-structures;*

(c) *in particular if* $K_0 = K$, *then* Res_{L/L_0} *is a cover of* $G(\mathbb{Q}_p)$*-structures.*

Proof: *By Lemma 9.5(a) and in the notation of (b), the following diagram commutes:*

$$(10) \qquad \begin{array}{ccc} X(L/K) & \xrightarrow{d} & \mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L/K)) \\ {\scriptstyle \mathrm{Res}_{L/L_0}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Res}_{L/L_0}} \\ X(L_0/K_0) & \xrightarrow{d} & \mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L_0/K_0)) \end{array}$$

Since $X(L/K) = \varprojlim X(L_0/K)$ *(by (1)) and*

$$\mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L/K)) = \varprojlim \mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L_0/K))$$

where L_0/K *ranges over all finite Galois subextensions of* L/K *(Section (1)), the map* $d\colon X(L/K) \to \mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L/K))$ *is the inverse limit of the maps* $d\colon X(L_0/K) \to \mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L_0/K))$. *By Lemma 10.6,* d *is continuous. Combine this with Remark 10.4 to conclude that* $\mathbf{G}(L/K)$ *is a* $G(\mathbb{Q}_p)$*-structure. Similarly conclude (b). Assertion (c) follows from Proposition 9.3.* ∎

If $L = \widetilde{K}$ *we write* $\mathbf{G}(K)$ *for* $\mathbf{G}(\widetilde{K}/K)$ *and call* $\mathbf{G}(K)$ *the* **absolute** $G(\mathbb{Q}_p)$**-structure** *of* K.

LEMMA 10.8: *Let K be a field and let $\theta, \theta' \in X(\widetilde{K}/K)$. Denote the decomposition field of θ (resp., θ') by M (resp., M'). Then*

(a) *M is p-adically closed;*

(b) *the map $d_\theta\colon G(\mathbb{Q}_p) \to G(K)$ is injective, $M\tilde{\mathbb{Q}} = \widetilde{M}$ and $G(M) \cong G(\mathbb{Q}_p)$;*

(c) *$M = M'$ if and only if there exists $\sigma \in G(M)$ such that $\theta' = \theta \circ \sigma$;*

(d) *the forgetful map $d\colon X(\widetilde{K}/K) \to \mathrm{Hom}(G(\mathbb{Q}_p), G(K))$ of $\mathbf{G}(K)$ is injective; and*

(e) *every p-adically closed field L, with $K \subseteq L \subseteq \widetilde{K}$ is the decomposition field of some $\theta \in X(\widetilde{K}/K)$.*

Proof of (a): *Combine Lemma 8.6 with Lemma 9.5(c). By Lemma 9.5(b), $(M, \mathrm{Res}_M \theta)$ is Θ-closed. Hence, with $\theta = (\pi, \varphi)$, $(M, \mathrm{Res}_M \pi)$ is $\mathbb{Q}_p$-closed. That is, M is p-adically closed (Remark 7.4).*

Proof of (b): *By (a) and Corollary 6.6, $G(M) \cong G(\mathbb{Q}_p)$ and $M\tilde{\mathbb{Q}} = \widetilde{M}$. Since $d_\theta\colon G(\mathbb{Q}_p) \to G(M)$ is surjective and $G(\mathbb{Q}_p)$ is finitely generated (Proposition 6.5), d_θ is injective.*

Proof of (c): *If $M = M'$, then, by (3), $\theta, \theta' \in X(\widetilde{K}/M)$. By (a) and Proposition 8.9, $\mathrm{res}_M \theta = \mathrm{res}_M \theta'$. Hence, by Proposition 9.3(b), there exists $\sigma \in G(M)$ such that $\theta' = \theta \circ \sigma$. Conversely, if the latter condition holds, then $d_{\theta'}(g) = \sigma^{-1} d_\theta(g) \sigma$ for each $g \in G(\mathbb{Q}_p)$ (Lemma 9.5(e)). Hence $M = M'$.*

Proof of (d): *If $d_{\theta'} = d_\theta$, then, from the proof of (c), σ belongs to the center of $G(M)$. Since the latter is trivial ((b) and Proposition 6.5) $\sigma = 1$. Thus $\theta' = \theta$.*

Proof of (e): *By Proposition 8.9, L has a (unique) Θ-site θ_0. Let $\theta \in X(\widetilde{K}/L) \subseteq X(\widetilde{K}/K)$ be an extension of θ_0 (Proposition 9.3(a)). Since (L, θ_0) is Θ-closed, L is the decomposition field of θ (Lemma 9.5(b)).* ∎

11. Characterization of $\mathbb{Q}_{p,\mathrm{alg}}$ by a large quotient of $G(\mathbb{Q}_p)$.

J. Neukirch proves in [N2] that if K is an algebraic extension of $\mathbb{Q}$ and $G(K) \cong G(\mathbb{Q}_p)$, then $K \cong \mathbb{Q}_{p,\mathrm{alg}}$. The main result of this section generalizes this to the case where $G(K)$ is a priori only a quotient of $G(\mathbb{Q}_p)$ which maps surjectively onto a "large" finite quotient of $G(\mathbb{Q}_p)$. Throughout this section we use l (resp., p) to denote a prime number and ζ_l (resp., ζ_p) to denote primitive lth (resp., pth) root of unity.

For a prime l and a profinite group G denote the maximal pro-l quotient of G by $G(l)$ and let $\mathrm{rank}_l G = \mathrm{rank}(G(l)) = \dim_{\mathbb{F}_l} \mathrm{Hom}(G, \mathbb{Z}/l\mathbb{Z})$.

Lemma 11.1: *Every finitely generated profinite group G has an open normal subgroup G_0 such that G/G_0 is an l-group and for each open normal subgroup N of G contained in G_0, $\mathrm{rank}_l G/N = \mathrm{rank}_l G$.*

Proof: There are only finitely many homomorphisms of G into $\mathbb{Z}/l\mathbb{Z}$. Take G_0 to be the intersection of the kernels of these homomorphisms. ∎

Lemma 11.2: *Let G be a finitely generated profinite group. Suppose that $G(l)$ is not a free pro-l-group. Then G has an open normal subgroup G_0 with G/G_0 an l-group such that if G_1 is a closed normal subgroup of G and $G_1 \le G_0$, then G/G_1 is not a free pro-l-group.*

Proof: Choose G_0 such that G/G_0 is an l-group and $\mathrm{rank}(G/G_0) = \mathrm{rank}(G(l))$ (Lemma 11.1). Let $G_1 \le G_0$ be a closed normal subgroup of G. If G/G_1 is a free pro-l-group, then it is a quotient of $G(l)$, on one hand, and has G/G_0 as a quotient on the other hand. Hence $\mathrm{rank}(G/G_1) = \mathrm{rank}(G(l))$. Conclude that $G(l)$ is also a free pro-l-group [R, p. 69], a contradiction. ∎

The l-ranks are well known for $G = G(E)$, where E is an algebraic extension of $\mathbb{Q}_p$ such that $l^\infty \nmid [E : \mathbb{Q}_p]$ [N2, Satz 4]:

$$(1) \qquad \mathrm{rank}_l G(E) = \begin{cases} 1 & \text{if } l \ne p \text{ and } \zeta_l \notin E \\ 2 & \text{if } l \ne p \text{ and } \zeta_l \in E \\ 1 + [E : \mathbb{Q}_p] & \text{if } l = p \text{ and } \zeta_p \notin E \\ 2 + [E : \mathbb{Q}_p] & \text{if } l = p \text{ and } \zeta_p \in E. \end{cases}$$

In what follows we denote the Brauer group of a field L by $\mathrm{Br}(L) = H^2(G(L), L_s^\times)$.

Let $\mathrm{Br}(L)_l = \{a \in \mathrm{Br}(L) | \; la = 0\}$ *be its* l*th torsion part. All groups are assumed to operate trivially on* $\mathbb{Z}/l\mathbb{Z}$.

LEMMA 11.3: *Let* L *be an algebraic extension of* $\mathbb{Q}$ *which contains* ζ_l *(resp.,* $\sqrt{-1} \in L$ *if* $l = 2$*). If* $\mathrm{Br}(L)_l \neq 0$*, then* $\mathrm{Br}(L')_l \neq 0$ *for each finite extension* L' *of* L.

Proof: Consider the following short exact sequence

$$1 \longrightarrow U_l \longrightarrow \tilde{\mathbb{Q}}^\times \xrightarrow{l} \tilde{\mathbb{Q}}^\times \longrightarrow 1,$$

where l *means raising to the* l*th power. It induces a four term exact sequence,*

$$H^1(G(L), \tilde{\mathbb{Q}}^\times) \longrightarrow H^2(G(L), U_l) \longrightarrow H^2(G(L), \tilde{\mathbb{Q}}^\times) \xrightarrow{l} H^2(G(L), \tilde{\mathbb{Q}}^\times). \tag{2}$$

By Hilbert's Theorem 90 the first term of (2) is trivial. Since $U_l \subseteq L$*, the second term is isomorphic to* $H^2(G(L), \mathbb{Z}/l\mathbb{Z})$*. Thus, (2) turns to be*

$$0 \longrightarrow H^2(G(L), \mathbb{Z}/l\mathbb{Z}) \longrightarrow \mathrm{Br}(L) \xrightarrow{l} \mathrm{Br}(L).$$

It follows that

$$\mathrm{Br}(L)_l \cong H^2(G(L), \mathbb{Z}/l\mathbb{Z}). \tag{3}$$

Consider now the induced module $A = \mathrm{Ind}_{G(L')}^{G(L)} \mathbb{Z}/l\mathbb{Z}$ *and an appropriate short exact sequence*

$$1 \longrightarrow A_1 \longrightarrow A \xrightarrow{\pi} \mathbb{Z}/l\mathbb{Z} \longrightarrow 0$$

of trivial $G(L)$*-modules. It induces an exact sequence of cohomology groups*

$$H^2(G(L), A) \xrightarrow{\bar{\pi}} H^2(G(L), \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^3(G(L), A_1).$$

Since $\mathrm{cd}_l L \leq 2$ *[R, p. 303], the right term in this sequence is 0. Therefore* $\bar{\pi}$ *is surjective. Hence, by (3),* $H^2(G(L), A) \neq 0$*. By Shapiro's lemma [R, p. 146],* $H^2(G(L), A) \cong H^2(G(L'), \mathbb{Z}/l\mathbb{Z})$*. Conclude from (3), with* L' *replacing* L*, that* $\mathrm{Br}(L')_l \neq 0$. ∎

LEMMA 11.4 (F.K. Schmidt): (a) *A field* K *which is not separably closed can be Henselian with respect to at most one 1-rank valuation.*

(b) *Let L/K be a Galois extension of fields. If L is Henselian with respect to a rank-1 valuation v and L is not separably closed, then K is Henselian with respect to $\mathrm{Res}_K v$.*

Proof: See Engler [En, pp. 5 and 7] for a generalization to higher rank valuations. ∎

PROPOSITION 11.5: *For each prime p there exists a finite Galois extension E of $\mathbb{Q}_p$ with this property: if K is an algebraic extension of $\mathbb{Q}$ and there exist epimorphisms $\varphi: G(\mathbb{Q}_p) \to G(K)$ and $\psi: G(K) \to \mathcal{G}(E/\mathbb{Q}_p)$, then $K \cong \mathbb{Q}_{p,\mathrm{alg}}$.*

Proof: Choose a prime $p' \notin \{2,3,p\}$ and let $S = \{p,p'\}$. Denote the compositum of all extensions of $\mathbb{Q}_p$ with degree at most $\max\{p-1, p'-1\}$ by E_0. This is a finite Galois extension of $\mathbb{Q}_p$. Since $[\mathbb{Q}_p(\zeta_l) : \mathbb{Q}_p] \le l-1$, it contains ζ_l for each $l \in S$.

By Proposition 6.5 and Lemma 11.1, E_0 has a finite extension E_1 such that for each $l \in S$ and for each Galois extension ${E'}_1$ of E_0 which contains E_1

$$\mathrm{rank}_l \mathcal{G}({E'}_1/E_0) = \mathrm{rank}_l G(E_0). \tag{4}$$

Since for each $l \in S$, $\zeta_l \in E_0$, the maximal l-quotient of $G(E_0)$ is not l-free [Se, p. II-30]. Therefore, by Lemma 11.2, E_0 has a proper finite Galois l-extension E_l such that for each Galois extension ${E'}_l$ of E_0 which contains E_l the group $\mathcal{G}({E'}_l/E_0)$ is not a free pro-l-group.

Let E be the compositum of all finite extensions of $\mathbb{Q}_p$ of degree at most $m = \max\{[E_1 : \mathbb{Q}_p], [E_p : \mathbb{Q}_p], [E_{p'} : \mathbb{Q}_p]\}$. It is a finite Galois extension of $\mathbb{Q}_p$. Let K be as in the theorem and denote the fixed field in $\tilde{\mathbb{Q}}_p$ of $\mathrm{Ker}(\varphi)$ by N. Then $\mathcal{G}(N/\mathbb{Q}_p) \cong G(K)$. Also, for the fixed field E' of $\mathrm{Ker}(\psi \circ \varphi)$, we have $\mathcal{G}(E'/\mathbb{Q}_p) \cong \mathcal{G}(E/\mathbb{Q}_p)$. Therefore E' is a compositum of extensions of $\mathbb{Q}_p$ of degree at most m. Hence $E' \subseteq E$. Since both fields have the same degree over $\mathbb{Q}_p$, $E' = E$. Thus, since $\mathrm{Ker}(\varphi) \le \mathrm{Ker}(\psi \circ \varphi)$, we have $E \subseteq N$. We prove in two parts that $K \cong \mathbb{Q}_{p,\mathrm{alg}}$.

PART A: *K is a Henselian field.* *By construction, each $l \in S$ divides $[N : E_0]$. Let $E_0^{(l)}$ be the maximal l-extension of E_0. Then $E_l \subseteq N \cap E_0^{(l)}$. Hence, the maximal pro-l quotient $\mathcal{G}(N \cap E_0^{(l)}/E_0)$ of $\mathcal{G}(N/E_0)$, is not l-free. It follows [R, p. 255] that*

$$\mathrm{cd}_l \mathcal{G}(N/E_0) > 1. \tag{5}$$

Let L_0 be the fixed field in $\tilde{\mathbb{Q}}$ of $\varphi(G(E_0))$. It is a finite Galois extension of K, $G(L_0) \cong \mathcal{G}(N/E_0)$ and $\mathcal{G}(E_0/\mathbb{Q}_p) \cong \mathcal{G}(L_0/K)$. In particular L_0 contains every finite extension of K of degree $\le l-1$. Since $[K(\zeta_l):K] \le l-1$, we have $\zeta_l \in L_0$. Since $p'-1 \ge 2$, we have $\sqrt{-1} \in L_0$. By (5), $\mathrm{cd}_l G(L_0) > 1$. Hence [R, p. 261] L_0 has a finite extension L_l such that

$$\mathrm{Br}(L_l)_l \neq 0. \tag{6}$$

Let L_1 be a finite Galois extension of K that contains both L_p and $L_{p'}$. By Lemma 11.3, $\mathrm{Br}(L_1)_l \neq 0$ for $l = p, p'$. Also, since $G(L_1)$ is isomorphic to a subgroup of $\mathcal{G}(N/\mathbb{Q}_p)$, $G(L_1)$ is prosolvable. Thus, Neukirch's Satz 1 of [N1] asserts that L_1 is Henselian. Now apply Lemma 11.4 to the Galois extension L_1/K and conclude that K is Henselian.

PART B: $K \cong \mathbb{Q}_{p,\mathrm{alg}}$. *Denote the characteristic of the residue field of K with respect to its Henselian valuation by q. Then K contains an isomorphic copy of $Q_{q,\mathrm{alg}}$. Assume without loss that $\mathbb{Q}_{q,\mathrm{alg}} \subseteq K$. By (6), $\mathrm{Br}(L_p)_p \neq 0$. Hence $p^\infty \not| [L_p : \mathbb{Q}_{q,\mathrm{alg}}]$ [R, p. 291] and therefore*

$$p^\infty \not| [L_0 : \mathbb{Q}_{q,\mathrm{alg}}]. \tag{7}$$

On one hand (4) and (1) give

$$\mathrm{rank}_p G(L_0) = \mathrm{rank}_p \mathcal{G}(N/E_0) = \mathrm{rank}_p G(E_0) = 2 + [E_0 : \mathbb{Q}_p]. \tag{8}$$

On the other hand (1) implies

$$\mathrm{rank}_p G(L_0) = \begin{cases} 2 & \text{if } p \neq q \\ 2 + [L_0 : \mathbb{Q}_{q,\mathrm{alg}}] & \text{if } p = q. \end{cases} \tag{9}$$

Clearly, (8) and (9) can be reconciled only if $p = q$ and $[E_0 : \mathbb{Q}_p] = [L_0 : \mathbb{Q}_{q,\mathrm{alg}}]$. But $[E_0 : \mathbb{Q}_p] = [L_0 : K]$, so necessarily $K = \mathbb{Q}_{p,\mathrm{alg}}$. ∎

12. Pseudo p-adically closed fields.

We call a field extension E/K **totally p-adic** *if the map $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ (Section 10) is surjective.*

Lemma 12.1:

(a) *A regular extension E/K is totally p-adic if and only if the $\mathrm{Res}_{\widetilde{E}/\widetilde{K}}\colon X(\widetilde{E}/E) \to X(\widetilde{K}/K)$ is a surjective map.*

(b) *A regular extension E/K is totally p-adic if and only if for each p-adic closure $\overline{K}$ of K, $\overline{K}E/\overline{K}$ is totally p-adic.*

(c) *Let V be an absolutely irreducible variety defined over K and let E be its function field. Then E/K is totally p-adic if and only if $V_{\mathrm{sim}}(\overline{K}) \neq \emptyset$ for each p-adic closure $\overline{K}$ of K.*

Proof of (a): *Suppose that $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ is surjective. Let $\widetilde{\theta} \in X(\widetilde{K}/K)$. Take $\theta_1 \in X(E)$ that extends $\theta = \mathrm{Res}_{\widetilde{K}/K}\widetilde{\theta}$ and extend it to $\theta_1' \in X(\widetilde{E}/E)$ (Proposition 9.3(a)). Let $\theta' = \mathrm{Res}_{\widetilde{E}/\widetilde{K}}\theta_1'$. Since $\mathrm{Res}_{\widetilde{K}/K}\widetilde{\theta} = \mathrm{Res}_{\widetilde{K}/K}\theta'$ Proposition 9.3(b) gives $\sigma \in G(K)$ such that $\theta' = \widetilde{\theta}^{\sigma}$. Since E/K is regular σ extends to $\tau \in G(E)$. Then $\widetilde{\theta}_1 = (\theta_1')^{\tau^{-1}} \in X(\widetilde{E}/E)$ and extends $\widetilde{\theta}$. Thus $\mathrm{Res}_{\widetilde{E}/\widetilde{K}}\colon X(\widetilde{E}/E) \to X(\widetilde{K}/K)$ is surjective. The converse is trivial.*

Proof of (b): *We use (a). Suppose first that E/K is totally p-adic, let $\overline{K}$ be a p-adic closure of K and let $\theta \in X(\widetilde{K}/\overline{K})$. Extend θ to $\theta' \in X(\widetilde{E}/E)$. Then $\mathrm{Res}_{\widetilde{K}}D(\theta') = D(\theta) \le G(\overline{K})$ (Lemma 9.5(a)). Hence $D(\theta') \le G(\overline{K}E)$. By (3) of section 10, $\theta' \in X(\overline{K}E/\overline{K})$.*

The converse holds, since each $\theta \in X(\widetilde{K}/K)$ belongs to $X(\widetilde{K}/\overline{K})$, where $\overline{K}$ is the decomposition field of θ.

Proof of (c): *By (b) we may assume that K is p-adically closed. Suppose first that $V_{\mathrm{sim}}(K) \neq \emptyset$. Let (π, φ) be the unique Θ-site of K (Proposition 8.9). Then π extends to a $\mathbb{Q}_p$-place π' of E (Proposition 6.4(c) and Lemma 6.7). By Corollary 8.10, E has a Θ-site (π', φ'). Since $(\pi, \mathrm{Res}_{E/K}(\varphi'))$ is a Θ-site of K, the uniqueness of $\bar{\varphi}$ implies that $\varphi = \mathrm{Res}_{E/K}(\varphi')$. Conversely, if (π, φ) extends to a Θ-site of E, then Proposition 6.4(c) implies that $V_{\mathrm{sim}}(K) \neq \emptyset$.* ∎

DEFINITION 12.2: We call a field K of characteristic 0 **pseudo p-adically closed** (PpC) if every absolutely irreducible variety V defined over K has a K-rational point, provided that the function field of V is totally p-adic over K (i.e., $V_{\mathrm{sim}}(\overline{K}) \neq \emptyset$ for every p-adic closure $\overline{K}$ of K).

Note that we do not assume that K has a p-adic valuation; a PpC with no p-adic valuation is pseudo algebraically closed.

We shall construct a class of PpC fields contained in $\tilde{\mathbb{Q}}$ with finitely many p-adic valuations. Since the p-adic closure of a formally p-adic number field is its Henselization, i.e., an isomorphic copy of $\mathbb{Q}_{p,\mathrm{alg}}$, we may use the results of Heinemann and Prestel [HP] to simplify the definition of algebraic PpC fields.

LEMMA 12.3: *let K be a subfield of $\tilde{\mathbb{Q}}$ and let $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}, \ldots, \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e}$ be p-adic closures of K. Suppose that every absolutely irreducible polynomial $f \in K[X,Y]$ has a K-rational zero, provided that for each i, $1 \le i \le e$, there exist $a, b \in \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}$ such that $f(a,b) = 0$ and $\frac{\partial f}{\partial Y}(a,b) \neq 0$. Then K is* PpC *and its only p-adic valuations are those induced from $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}, \ldots, \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e}$.*

Proof: Let $f \in K[X,Y]$ be an absolutely irreducible polynomial that admits a $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}$-rational simple point for $i = 1, \ldots, e$. After a linear transformation of the coordinates, we may assume that for each i, $1 \le i \le e$, there exist $a, b \in \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}$ such that $f(a,b) = 0$ and $\frac{\partial f}{\partial Y}(a,b) \neq 0$. By assumption, f has a K-rational zero. It follows from [HP, Thm. 1.8] that K is PpC, and from [HP, Lemma 1.6] that the only p-adic valuations on K are those induced from $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}, \ldots, \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e}$. ∎

Fix integers $0 \le e \le m$. For each $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in G(\mathbb{Q})^m$ let

$$\mathbb{Q}_{\boldsymbol{\sigma}} = \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1} \cap \cdots \cap \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e} \cap \tilde{\mathbb{Q}}(\sigma_{e+1}) \cap \cdots \cap \tilde{\mathbb{Q}}(\sigma_m).$$

Also, for $\boldsymbol{\sigma}, \boldsymbol{\lambda} \in G(\mathbb{Q})^m$ write $\boldsymbol{\sigma\lambda}$ for $(\sigma_1\lambda_1, \ldots, \sigma_m\lambda_m)$. In the following result we use the term "almost all" in the sense of the Haar measure of $G(\mathbb{Q})^m$.

LEMMA 12.4: *Let $\boldsymbol{\tau} \in G(\mathbb{Q})^m$ and let $L \subseteq \mathbb{Q}_{\boldsymbol{\tau}}$ be a finite extension of $\mathbb{Q}$. Then almost all $\boldsymbol{\lambda} \in G(L)^m$ have this property: if $f \in L[X,Y]$ is an absolutely irreducible polynomial and for each $1 \le i \le e$ there exist $a_{0i}, b_{0i} \in \mathbb{Q}_{p,\mathrm{alg}}^{\tau_i}$ such that $f(a_{0i}, b_{0i}) = 0$ and $\frac{\partial f}{\partial Y}(a_{0i}, b_{0i}) \neq 0$, then f has a $\mathbb{Q}_{\boldsymbol{\tau\lambda}}$-rational zero.*

Proof: Let $n = \deg_Y f$. Without loss assume that for some d between 1 and e, $\mathbb{Q}_{p,\mathrm{alg}}^{\tau_1}, \ldots, \mathbb{Q}_{p,\mathrm{alg}}^{\tau_d}$ represent the L-isomorphism classes of the set $\{\mathbb{Q}_{p,\mathrm{alg}}^{\tau_1}, \ldots, \mathbb{Q}_{p,\mathrm{alg}}^{\tau_e}\}$. In particular $\mathbb{Q}_{p,\mathrm{alg}}^{\tau_1}, \ldots, \mathbb{Q}_{p,\mathrm{alg}}^{\tau_d}$ induce distinct p-adic valuations on L.

Since L is Hilbertian we may use [G, Lemma 3.4] to inductively find $a_1, a_2, a_3, \ldots \in L$ *and* $b_1, b_2, b_3, \ldots \in \tilde{\mathbb{Q}}$ *such that for each* $j \geq 1$

(1a) a_j *lies near* a_{0i} *in* $\mathbb{Q}_{p,\mathrm{alg}}^{\tau_i}, i = 1, \ldots, d$;

(1b) $f(a_j, Y)$ *is irreducible over L of degree n, and* $f(a_j, b_j) = 0$; *and*

(1c) *for* $L_j = L(b_j)$, *the sequence* $L_1, L_2, L_3, \ldots$ *is linearly disjoint over L.*

Condition (1a) and $f(a_{0i}, b_{0i}) = 0$ *imply by Krasner's lemma [Ri, p. 190] that* $f(a_j, Y)$ *has a root in* $\mathbb{Q}_{p,\mathrm{alg}}^{\tau_i}$, $i = 1, \ldots, d$. *By the choice of d this also holds for* $i = d+1, \ldots, e$. *Thus, by (1b), there exist* $\lambda_{j1}, \ldots, \lambda_{je} \in G(L)$ *such that* $L_j^{\lambda_{ji}} \subseteq \mathbb{Q}_{p,\mathrm{alg}}^{\tau_i}$, $i = 1, \ldots, e$. *Let* $\lambda_{ji} = \tau_i$ *for* $i = e+1, \ldots, m$ *and* $\boldsymbol{\lambda}_j = (\lambda_{j1}, \ldots, \lambda_{jm})$.

Condition (1c) implies by [J1, Lemma 6.3] that for almost all $\boldsymbol{\lambda} \in G(L)^m$ *there exists* $j \geq 1$ *such that* $\mathrm{res}_{L_j} \boldsymbol{\lambda}^{-1} = \mathrm{res}_{L_j} \boldsymbol{\lambda}_j$. *But then* $L_j^{\lambda_i^{-1}} = L_j^{\lambda_{ji}} \subseteq \mathbb{Q}_{p,\mathrm{alg}}^{\tau_i}$, *hence* $L_j \subseteq \mathbb{Q}_{p,\mathrm{alg}}^{\tau_i \lambda_i}$, $i = 1, \ldots, e$. *Also,* $\mathrm{res}_{L_j} \lambda_i^{-1} = \mathrm{res}_{L_j} \tau_i$, *hence* $L_j \subseteq \tilde{\mathbb{Q}}(\tau_i \lambda_i)$, $i = e+1, \ldots, m$. *Conclude that* $L_j \subseteq \mathbb{Q}_{\boldsymbol{\tau\lambda}}$. *Thus* (a_j, b_j) *is a* $\mathbb{Q}_{\boldsymbol{\tau\lambda}}$*-rational zero of f.* ∎

LEMMA 12.5: *For almost all* $\boldsymbol{\sigma} \in G(\mathbb{Q})^m$ *the field* $\mathbb{Q}_{\boldsymbol{\sigma}}$ *is* PpC *and has at most e distinct p-adic valuations.*

Proof: Fix a countable dense subset T of $G(\mathbb{Q})^m$. Let $\boldsymbol{\sigma} \in G(\mathbb{Q})^m$ and consider an absolutely irreducible polynomial $f \in \mathbb{Q}_{\boldsymbol{\sigma}}[X, Y]$ which has a $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}$-rational zero (a_i, b_i) such that $\frac{\partial f}{\partial Y}(a_i, b_i) \neq 0$ for $i = 1, \ldots, e$. Let $L \subseteq \mathbb{Q}_\sigma$ be a finite extension of $\mathbb{Q}$ that contains the coefficients of f. Consider $\boldsymbol{\tau} \in T \cap \boldsymbol{\sigma} G(L)^m$. Since $\mathbb{Q}_{p,\mathrm{alg}}^{\tau_i}$ is isomorphic to $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}$ over L, f has a $\mathbb{Q}_{p,\mathrm{alg}}^{\tau_i}$-rational zero (a'_i, b'_i) such that $\frac{\partial f}{\partial Y}(a'_i, b'_i) \neq 0$, $i = 1, \ldots, e$. Hence, by Lemma 12.4, f has a $\mathbb{Q}_{\boldsymbol{\sigma}}$-rational point, unless σ belongs to a zero subset of $\boldsymbol{\tau} G(L)^m$. Use that a countable union of zero sets is again a zero set to exclude such a case. Conclude from Lemma 12.3 that $\mathbb{Q}_{\boldsymbol{\sigma}}$ is PpC and has at most e distinct p-adic valuations. ∎

REMARK 12.6: *Regular action.* A regular action of a finite group on a finite set X is unique up to a permutation of X. More precisely, if two groups G and G' act regularly on X (Definition 1.1) and there exists an isomorphism $\varphi\colon G \to G'$, then there exists a permutation s of X such that

$$x^{\varphi(g)s} = x^{sg}, \qquad \text{for } g \in G \text{ and } x \in X. \tag{2}$$

Indeed, let X_0 be a system of representatives for the G-orbits of X. Then each $x \in X$ can be uniquely written as $x = x_0^g$ with $x_0 \in X_0$ and $g \in G$. It follows that $|X_0| = |X|/|G|$. Similarly, a system X'_0 of representatives for the G'-orbits of X has $|X|/|G'|$ elements. Thus there exists a bijective map $s\colon X'_0 \to X_0$. Extend s to a permutation of X by the rule $x_0^{\varphi(g)s} = x_0^{sg}$, for $x_0 \in X'_0$ and $g \in G$. Obviously, it satisfies (2). ∎

NOTATION 12.7: Let $\Gamma_{e,m} = \Gamma_1 * \cdots * \Gamma_e * \widehat{F}_{m-e}$ be the free product in the category of profinite groups of e copies $\Gamma_1, \ldots, \Gamma_e$ of $G(\mathbb{Q}_p)$, and $\widehat{F}_{m-e}$, the free profinite group on $m - e$ generators (c.f., (1) of Section 3).

LEMMA 12.8: *For almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^m$*

$$G(\mathbb{Q}_\sigma) \cong \Gamma_{e,m}. \tag{3}$$

Proof: We follow Geyer's proof [G] for the case $e = m$. The case $e = 0$ is treated in [J2, Thm. 5.1]. So assume $e \geq 1$. Since both sides of (3) are finitely generated, it suffices to prove that they have the same finite quotients. But $G(\mathbb{Q}_\sigma)$, being generated by $G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}) \cong G(\mathbb{Q}_p)$, $i = 1, \ldots, e$, and $\langle \sigma_{e+1}, \ldots, \sigma_m \rangle$ is a quotient of $\Gamma_{e,m}$. Thus it suffices to consider finite groups of the form $G = \langle G_1, \ldots, G_{e+1} \rangle$ where $G_i \cong \mathcal{G}(E/\mathbb{Q}_p)$, $i = 1, \ldots, e$, the field E is a finite Galois extension of $\mathbb{Q}_p$, and G_{e+1} is generated by $m - e$ elements, and to prove that G is a quotient of $G(\mathbb{Q}_\sigma)$ for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^m$.

Let x_1 be a primitive element for the extension $E/\mathbb{Q}_p$ and let $x_1, \ldots, x_s$ be the conjugates of x_1 over $\mathbb{Q}_p$. Note that $n = |G|$ is a multiple of $s = |G_i|$, $i = 1, \ldots, e$. Take integers $k_1, \ldots, k_{n/s}$ such that $x_i + k_j \neq x_r + k_t$ if $(i,j) \neq (r,t)$. Then $f(X) = \prod_{i=1}^{s} \prod_{j=1}^{n/s} (X - x_i - k_j)$ is a monic polynomial with coefficients in $\mathbb{Q}_p$ with $n = \deg(f)$ distinct roots. Each of the roots is a primitive element for $E/\mathbb{Q}_p$. Hence $\mathcal{G}(E/\mathbb{Q}_p)$ acts regularly on them.

Use Hilbert irreducibility theorem and [G, Lemma 3.4] to inductively construct a sequence $f_1, f_2, f_3, \dots$ of monic polynomials in $\mathbb{Q}[X]$ of degree n and a sequence $L_1, L_2, L_3, \dots$ of Galois extensions of $\mathbb{Q}$ such that for each $j \geq 1$

(4a) *L_j is the splitting field of f_j over $\mathbb{Q}$, and $\mathcal{G}(L_j/\mathbb{Q}) \cong S_n$;*

(4b) *f_j is p-adically close to f; and*

(4c) *$L_1, L_2, L_3, \dots$ are linearly disjoint over $\mathbb{Q}$.*

(cf. the proof of [J1, Lemma 2.2]).

Condition (4b) implies by Krasner's lemma [Ri, pp. 190-197] that the splitting field of f_j over $\mathbb{Q}_p$ coincides with that of f, namely with E. Moreover each of the roots of f_j is p-adically close to a root of f and therefore generates E over $\mathbb{Q}_p$. Thus $\mathcal{G}(L_j/L_j \cap \mathbb{Q}_p) \cong \mathcal{G}(E/\mathbb{Q}_p)$ regularly acts on the set of roots R_j of f_j, and $|R_j| = n$. On the other hand G acts regularly on itself by multiplication from the right. So identify G as a subgroup of $\mathcal{G}(L_j/\mathbb{Q})$, which is by (4a) the full permutation group of R_j. Denote the image of G_i under this identification by $G_{ji}, i = 1, \dots, e+1$. Choose an isomorphism φ_{ji}: $G_{ji} \to \mathcal{G}(L_j/L_j \cap \mathbb{Q}_p)$. By Remark 12.6 there exists $\sigma_{ji} \in \mathcal{G}(L_j/\mathbb{Q})$ such that $x^{\varphi_{ji}(g)\sigma_{ji}} = x^{\sigma_{ji} g}$ for each $x \in R_j$ and $g \in G_{ji}$. Thus $\sigma_{ji}^{-1}\varphi_{ji}(g)\sigma_{ji} = g$ for each $g \in G_{ji}$. It follows that $G_{ji} = \mathcal{G}(L_j/L_j \cap \mathbb{Q}_p)^{\sigma_{ji}}$. Also, let $\sigma_{j,e+1}, \dots, \sigma_{jm}$ be generators of $G_{j,e+1}$.

By [J2, Lemma 4.1], for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^m$ there exists $j \geq 1$ such that the restriction of $\boldsymbol{\sigma}$ to L_j is $(\sigma_{j1}, \dots, \sigma_{jm})$. Therefore

$$\begin{aligned}\mathcal{G}(L_j\mathbb{Q}_\sigma/\mathbb{Q}_\sigma) &\cong \mathcal{G}(L_j/L_j \cap \mathbb{Q}_\sigma)\\ &= \langle \mathcal{G}(L_j/L_j \cap \mathbb{Q}_p)^{\sigma_{j1}}, \dots, \mathcal{G}(L_j/L_j \cap \mathbb{Q}_p)^{\sigma_{je}}, \sigma_{j,e+1}, \dots, \sigma_{j,m}\rangle\\ &= \langle G_{j1}, \dots, G_{je}, G_{j,e+1}\rangle = G.\end{aligned}$$

Thus G is a quotient of $G(\mathbb{Q}_\sigma)$. ∎

PROPOSITION 12.9: *The following statements hold for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^m$:*

(a) *$\mathbb{Q}_\sigma$ is a PpC field;*

(b) *$G(\mathbb{Q}_\sigma) = G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}) * \cdots * G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e}) * \langle \sigma_{e+1}, \dots, \sigma_m\rangle \cong \Gamma_{e,m}$;*

(c) *$\mathbb{Q}_\sigma$ has exactly e p-adic valuations; they are induced by the p-adic Henselizations $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}, \dots, \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e}$ of $\mathbb{Q}_\sigma$; and*

(d) *if M and M' are two distinct p-adic Henselizations of $\mathbb{Q}_\sigma$, then $MM' = \tilde{\mathbb{Q}}$.*

Proof of (a): *See Lemma 12.5.*

Proof of (b): *The isomorphisms $G(\mathbb{Q}_p) \to G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i})$, $i = 1, \ldots, e$, and $\widehat{F}_{m-e} \to \langle \sigma_{e+1}, \ldots, \sigma_m \rangle$ combine to an epimorphism $\varphi\colon \Gamma_{e,m} \to G(\mathbb{Q}_\sigma)$. Since, by Lemma 12.8, both groups are finitely generated and isomorphic, $\bar{\varphi}$ is an isomorphism [R, p. 69].*

Proof of (c): *Map $G(\mathbb{Q}_\sigma)$ homomorphically onto the direct product $G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}) \times \cdots \times G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e})$ to conclude that $G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}), \ldots, G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e})$ are pairwise nonconjugate in $G(\mathbb{Q}_\sigma)$. Thus $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_1}, \ldots, \mathbb{Q}_{p,\mathrm{alg}}^{\sigma_e}$ induce e distinct p-adic valuations $v_1, \ldots, v_e$ on $\mathbb{Q}_\sigma$. Since $\mathbb{Q}_\sigma$ has at most e p-adic valuations (Lemma 12.5), $v_1, \ldots, v_e$ are all of them.*

Proof of (d): *Extend the p-adic valuations v of M and v' of M' to $\tilde{\mathbb{Q}}$. Since M and M' are the respective decomposition fields of v and v', these valuations are distinct on $\tilde{\mathbb{Q}}$ and therefore on MM'. Thus MM' is Henselian with respect to two distinct 1-rank valuations. Use Lemma 11.4 to conclude that $MM' = \tilde{\mathbb{Q}}$.* ∎

We conclude this section by a proposition that allows us to apply the results of Sections 3, 4 and 5 to $\Gamma = G(\mathbb{Q}_p)$.

PROPOSITION 12.10: *The group $\Gamma = G(\mathbb{Q}_p)$ satisfies Assumption 3.1.*

Proof: Proposition 6.5 says that $G(\mathbb{Q}_p)$ satisfies conditions (a) and (b) of Assumption 3.1. As to the other conditions let $\boldsymbol{\sigma}$ be an element of $G(\mathbb{Q})^m$ that satisfies the conclusions of Proposition 12.9. In particular $G(\mathbb{Q}_\sigma) \cong \Gamma_{e,m}$. Let E be the finite Galois extension of $\mathbb{Q}_p$ mentioned in Proposition 11.5. Consider a closed subgroup H of $G(\mathbb{Q}_\sigma)$. Suppose that H is a quotient of $G(\mathbb{Q}_p)$ and has $\mathcal{G}(E/\mathbb{Q}_p)$ as its quotient (i.e., H is a large quotient of $G(\mathbb{Q}_p)$). Then $\tilde{\mathbb{Q}}(H) \cong \mathbb{Q}_{p,\mathrm{alg}}$. Hence $H \cong G(\mathbb{Q}_p)$ (this gives Assumption 3.1(d)). Also, $\tilde{\mathbb{Q}}(H)$ induces p-adic valuation on $\mathbb{Q}_\sigma$. By Proposition 12.9, it coincides with the valuation induced by some $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}$. Therefore $\tilde{\mathbb{Q}}(H)$ is $\mathbb{Q}_\sigma$-isomorphic to $\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i}$, and H is conjugate to $G(\mathbb{Q}_{p,\mathrm{alg}}^{\sigma_i})$. Thus Assumption 3.1(c1) holds. Assumption 3.1(c2) follows from Proposition 8.9(c). Finally Proposition 12.9(d) implies Assumption 3.1(c3). ∎

Part C. Projective $G(\mathbb{Q}_p)$-structures as absolute $G(\mathbb{Q}_p)$-Galois structures.

From now on we replace the term "$G(\mathbb{Q}_p)$-projective group" by "p-adically projective group". The absolute $G(\mathbb{Q}_p)$-structure $\mathbf{G}(K)$ of a PpC *field K is projective and the absolute Galois group of K is p-adically projective (Theorem 15.1). Most of Part C proves the converse. For each projective $G(\mathbb{Q}_p)$-structure $\mathbf{G}$ there exists a* PpC *field K such that $\mathbf{G} \cong \mathbf{G}(K)$ (Theorem 15.3) and for each p-adically projective group G there exists a* PpC *field K such that $G \cong G(K)$ (Theorem 15.4). Section 13 prepares the proof by showing the existence of continuous sections to the maps $\mathrm{Res}_{F/L}\colon X(F/E) \to X(L/K)$ in various cases. In particular Proposition 13.11 asserts that for each Boolean space X there exists a* PpC *field K such that $X \cong X(K)$. In Section 14 we prove that for each p-adic structure $\mathbf{G}$ (not necessarily projective) there exists a Galois extension F/E, with E* PpC, *such that $\mathbf{G} \cong \mathbf{G}(F/E)$.*

13. Restriction maps of spaces of sites.

The restriction map $\mathrm{Res}_{L'/L}\colon X(L'/K') \to X(L/K)$ for two Galois extensions L'/K' and L/K with $K \subseteq K'$ and $L \subseteq L'$ is continuous (Section 10). Since spaces of sites are compact and Hausdorff, $\mathrm{Res}_{L'/L}$ is a closed map. In this section we prove openness results and investigate the existence of continuous sections for these maps.

LEMMA 13.1: *Let E/K be a finite extension. Then $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ is an open map. Moreover, $X(E)$ has a partition $\{V_i\}_{i=1}^n$ such that for each i, $1 \le i \le n$, $\mathrm{Res}_{E/K}\colon V_i \to \mathrm{Res}_{E/K}(V_i)$ is a homeomorphism.*

Proof: By compactness, it suffices to find for each $\theta \in X(E)$ an open-closed neighborhood V on which $\mathrm{Res}_{E/K}$ is injective and such that $\mathrm{Res}_{E/K}(V)$ is open-closed.

Indeed let L be a finite Galois extension of K that contains E. Consider the following commutative diagram

$$\begin{array}{ccc} X(L/E) & \xrightarrow{i} & X(L/K) \\ {\scriptstyle \mathrm{Res}_{L/E}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{L/K}} \\ X(E) & \xrightarrow[\mathrm{Res}_{E/K}]{} & X(K) \end{array}$$

Here i is the inclusion map. By (2) of Section 10, $X(L/E)$ consists of all $\theta \in X(L/K)$ such that $D(\theta) \le \mathcal{G}(L/E)$. Since D: $X(L/K) \to \mathrm{Subg}(\mathcal{G}(\mathrm{L/K}))$ is continuous and

$\mathcal{G}(L/K)$ is finite, $X(L/E)$ is open in $X(L/K)$. Hence i is open. The vertical maps are quotient maps by $\mathcal{G}(L/E)$ and $\mathcal{G}(L/K)$, respectively, and therefore open [HJ, Claim 1.6]. Conclude for each open subset V of $X(E)$ that $\mathrm{Res}_{E/K}(V) = \mathrm{Res}_{L/K}(\mathrm{Res}_{L/E}^{-1}(V))$ is open in $X(K)$.

Now extend θ to $\theta' \in X(L/E)$. Since $(\theta')^\sigma \neq \theta'$ for each σ, $1 \neq \sigma \in \mathcal{G}(L/K)$ (Proposition 9.3(b)), θ' has an open-closed neighborhood $V' \subseteq X(L/E)$ such that $\theta' \notin (V')^\sigma$ for each σ, $1 \neq \sigma \in \mathcal{G}(L/K)$. Replace V' by $V' - \bigcap_{\sigma \neq 1} (V')^\sigma$ to assume that $V' \cap (V')^\sigma = \emptyset$ for each $\sigma \neq 1$. It follows that $\mathrm{Res}_{L/K}$ is injective on V' (Proposition 9.3(b)). Hence $\mathrm{Res}_{E/K}$ is injective on the open-closed neighborhood $V = \mathrm{Res}_{L/E}(V')$ of θ. ∎

LEMMA 13.2: *Let L/K be a Galois extension, T an ordered set of algebraically independent elements over L and ε a function from T into $\{\pm 1\}$. Consider $E = K(T)$, $F = L(T)$ and for each $t \in T$ let $L_t = L(t_0 \in T |\, t_0 < t)$. Then each $\theta \in X(L/K)$ uniquely extends to $\theta_T = (\pi_T, \varphi_T) \in X(F/E)$ such that*

$$\pi_T(at) = 0 \text{ for all } a \in L_t \text{ and } \varphi_T(t) = \varepsilon(t). \tag{1}$$

Moreover, for each $t \in T$ and each $f \in L_t[X]$ with $f(0) \neq 0$

$$\varphi_T(f(t)) = \varphi_T(f(0)). \tag{2}$$

Finally, the map $\theta \mapsto \theta_T$ is a continuous section of $\mathrm{Res}_{F/L}\colon X(F/E) \to X(L/K)$.

Proof: *Replace T if necessary by $\{\varepsilon(t)t |\; t \in T\}$ to assume that $\varepsilon(t) = 1$ for all $t \in T$. The uniqueness part of the Lemma reduces the infinite case to the finite case. The latter follows by induction on $|T|$ from the case $|T| = 1$. So assume that $T = \{t\}$.*

Each element $a \in F^\times$ has a unique presentation,

$$a = a_0 t^m \frac{1 + b_1 t + \cdots + b_k t^k}{1 + c_1 t + \cdots + c_l t^l},$$

where $a_0 \in L^\times$, $m \in \mathbb{Z}$ and $1 + b_1 t + \cdots + b_k t^k$ and $1 + c_1 t + \cdots + c_l t^l$ are relatively prime polynomials in $L[t]$.

Let $\theta = (\pi, \varphi) \in X(L/K)$ and let $\theta' = (\pi', \varphi') \in X(F/E)$ be an extension of θ which satisfies (1). Then

$$\pi'(1 + b_1 t + \cdots + b_k t^k) = \pi'(1 + c_1 t + \cdots + c_l t^l) = 1. \tag{3}$$

Hence

$$\pi'(a) = \pi'(a_0 t^m) = \begin{cases} 0 & \text{if } m > 0 \\ \pi(a_0) & \text{if } m = 0 \\ \infty & \text{if } m < 0. \end{cases} \tag{4}$$

By (3), $\varphi'(1 + b_1 t + \cdots + b_k t^k) = \varphi'(1 + c_1 t + \cdots + c_l t^l) = 1$. Hence

$$\varphi'(a) = \varphi(a_0). \tag{5}$$

This proves the uniqueness of θ' and (2).

To prove the existence, use (4) and (5) as definitions for π' and φ' and check that indeed $\theta' = (\pi', \varphi') \in X(F/E)$.

The continuity of the map $\theta \mapsto \theta'$ follows from (4) and (5) by (1) of Section 10.

∎

DEFINITION 13.3: Let $\varepsilon(t) = 1$ for all $t \in T$. We call $\theta_T \in X(F/E)$ of Lemma 13.2 the **infinitesimal** extension of θ to $X(F/E)$ with respect to T.

PROPOSITION 13.4: *Let E/K be a finitely generated extension and let H_E be an open-closed subset of $X(E)$. Then $H_K = \mathrm{Res}_{E/K}(H_E)$ is open-closed in $X(K)$ and the restriction map $\mathrm{Res}_{E/K}\colon H_E \to H_K$ has a continuous section.*

Proof: First note that if $K \subseteq K' \subseteq E$, $H_{K'} = \mathrm{Res}_{E/K'}(H_E)$ and the proposition holds for the maps $\mathrm{Res}_{E/K'}\colon H_E \to H_{K'}$ and $\mathrm{Res}_{K'/K}\colon H_{K'} \to H_K$, then it also holds for their composition $\mathrm{Res}_{E/K}\colon H_E \to H_K$. This reduces the proposition to the case where E/K is a simple extension. Also, by compactness, it suffices to find for each $\theta \in H_K$ an open-closed neighborhood V in H_K and a continuous map $s\colon V \to H_E$ such that for each $\theta \in V$, $s(\theta)$ extends θ. If E/K is finite this follows from Lemma 13.1. So, assume that $E = K(t)$ and t is transcendental over K.

Let $\theta_0 = (\pi_0, \varphi_0) \in H_K$ and let $\theta_0' = (\pi_0', \varphi_0') \in H_E$ be an extension of θ_0 to E. By Remark 10.5, θ_0' has an open-closed neighborhood ${H'}_E \subseteq H_E$ of the form

$${H'}_E = \{(\pi', \varphi') \in X(E) |\, \varphi'(f_i(t)) \in \Phi^m, \qquad i = 1, \ldots, r\}$$

where $f_1(t), \ldots, f_r(t) \in E^\times$, and $m \in \mathbb{N}$. Let $f_i(t) = g_i(t)/g_0(t)$, with $g_0(t), \ldots, g_r(t) \in K[t]$. Replace $f_i(t)$ if necessary by $f_i(t)g_0(t)^{m-1}$ to assume that $f_i(t) \in K[T]$, $i = 1, \ldots, r$. The rest of the proof splits into two parts.

PART A: *A special case.* *Suppose first that there exists $a \in K$ such that*

$$f_i(a) \neq 0 \text{ and } \varphi_0'(f_i(a)) \in \Phi^m, \qquad i = 1, \ldots, r.$$

Replace t if necessary by $t - a$ to assume that $a = 0$. By Lemma 13.2 $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ has a continuous section s such that each $(\pi', \varphi') \in s(X(K))$ satisfies $\varphi'(f_i(t)) = \varphi(f_i(0))$, $i = 1, \ldots, r$. In particular s maps the open-closed neighborhood of θ_0,

$$V = \{(\pi, \varphi) \in X(K) |\, \varphi(f_i(0)) \in \Phi^m, \quad i = 1, \ldots, r\}$$

into H_E.

PART B: *Reduction of the general case to the case of Part A.* *Let $(\overline{E}, \bar{\theta}_0)$, with $\bar{\theta}_0 = (\bar{\pi}_0, \bar{\varphi}_0)$ be a Θ-closure of (E, θ_0') (Proposition 8.7). Then $\overline{E}$ is $\mathbb{Q}_p$-closed (Lemma 8.6), so $\overline{E}$ is p-adically closed (Remark 7.4). By Lemma 7.6 and Lemma 6.8(b), $\bar{\varphi}_0$ induces an isomorphism of $\overline{E}^\times/(\overline{E}^\times)^m$ onto Φ/Φ^m. Since $\bar{\varphi}_0(f_i(t)) \in \Phi^m$ there exists $z_i \in \overline{E}^\times$ such that $f_i(t) = z_i^m, i = 1, \ldots, r$.*

The field $\overline{K} = \widetilde{K} \cap \overline{E}$ is p-adically closed (Proposition 6.4(a)). Hence, $\overline{E}$ is an elementary extension of $\overline{K}$ (Proposition 6.4(b)). In particular there exist $a \in \overline{K}$ and $c_1, \ldots, c_r \in \overline{K}^\times$ such that $f_i(a) = c_i^m$ for $i = 1, \ldots, r$. Let $L = K(a, c_1, \ldots, c_r)$, $F = L(t)$, $\theta_1' = (\pi_1', \varphi_1') = \mathrm{Res}_F \bar{\theta}_0$, $\theta_1 = \mathrm{Res}_L \theta_1'$,

$$H'_F = \mathrm{Res}_{F/E}^{-1}(H'_E) = \{(\pi', \varphi') \in X(F) |\, \varphi'(f_i(t)) \in \Phi^m, \quad i = 1, \ldots, r\}$$

and $H'_L = \mathrm{Res}_{F/L}(H'_F)$. Then $\mathrm{Res}_{L/K}(H'_L) \subseteq H_K$. By Part A, θ_1 has an open-closed neighborhood V_1 and there exists a continuous map $s_1\colon V_1 \to H'_F$ such that $s_1(\theta)$ extends θ for each $\theta \in V_1$. Since L/K is finite, the beginning of the proof implies that $V = \mathrm{Res}_{L/K}(V_1)$ is an open-closed neighborhood of θ_0 and $\mathrm{Res}_{L/K}\colon V_1 \to V$ has a continuous section s_0. Clearly $s = \mathrm{Res}_{F/E} \circ s_1 \circ s_0\colon V \to H_E$ is a continuous map and $s(\theta)$ extends θ for each $\theta \in V$. ∎

LEMMA 13.5: *Let K be a field and let H be an open-closed subset of $X(K)$. Then there exists a finitely generated regular extension E of K such that $\mathrm{Res}_{E/K}X(E) = H$.*

Proof: We divide the proof into three parts.

PART A: *Construction of E.* *Write H in the form*

$$H = \bigcap_{i=1}^{r} \bigcup_{j=1}^{n} \{(\pi, \varphi) \in X(K) | \ \varphi(a_{ij}) \in \Phi^m\},$$

with $a_{ij} \in K^\times$ and $m \in \mathbb{N}$ (Remark 10.5). Let $S \subseteq \mathbb{Z}$ be a finite set of representatives for $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^m$ (Lemma 6.8(b1)). By Lemma 7.6(b), S represents $M^\times/(M^\times)^m$ for every p-adically closed field M. Choose $k \in \mathbb{N}$ such that

$$k > 2v_p(m) + 2v_p(s) \qquad \text{for all } s \in S. \tag{6}$$

Consider the algebraic subset V of the affine space $\mathbb{A}^{(n+2)r}$ defined by the system of equations

$$(Y_{i1}^m - a_{i1}) \cdots (Y_{in}^m - a_{in}) = a_{i1} \cdots a_{in} p^{kn}(\gamma(X_{i1}) + \gamma(X_{i2})), \qquad i = 1, \ldots, r, \tag{7}$$

where $\gamma(X)$ is the Kochen operator ((1) of Section 6). By a theorem of Schinzel [Sc], each of the equations in (7) is absolutely irreducible. Since the equations are algebraically independent, V is an absolutely irreducible variety defined over K. Its function field E is a finitely generated regular extension of K.

PART B: $\mathrm{Res}_{E/K}X(E) \subseteq H$. *Let $\theta = (\pi, \varphi) \in X(E)$. As in Part B of the proof of Proposition 13.4, let $(\overline{E}, \bar{\theta})$, with $\bar{\theta} = (\bar{\pi}, \bar{\varphi})$, be a Θ-closure of (E, θ). By construction there exist $y_{i1}, \ldots, y_{in} \in E^\times$ and $x_{i1}, x_{i2} \in E$ such that*

$$(y_{i1}^m - a_{i1}) \cdots (y_{in}^m - a_{in}) = a_{i1} \cdots a_{in} p^{kn}(\gamma(x_{i1}) + \gamma(x_{i2})), \qquad i = 1, \ldots, r. \tag{8}$$

For each i and j take $b_{ij} \in \overline{E}^\times$ and $s_{ij} \in S$ such that $a_{ij} = b_{ij}^m s_{ij}$. Let $z_{ij} = y_{ij}/b_{ij}$. Divide (8) by $b_{i1}^m \cdots b_{in}^m$ to obtain

$$(z_{i1}^m - s_{i1}) \cdots (z_{in}^m - s_{in}) = s_{i1} \cdots s_{in} p^{kn}(\gamma(x_{i1}) + \gamma(x_{i2})), \qquad i = 1, \ldots, r. \tag{9}$$

Apply the p-adic valuation $\bar v$ of $\overline{E}$ associated with $\bar\pi$ on (9) and use Lemma 6.1:

$$\sum_{j=1}^{n}\bar v(z_{ij}^m - s_{ij}) \ge \sum_{j=1}^{n}(v_p(s_{ij})+k), \qquad i=1,\dots,r. \tag{10}$$

For each i, $1\le i\le r$, (10) gives $j=j(i)$ such that

$$\bar v(z_{ij}^m - s_{ij}) \ge \bar v(s_{ij})+k > \bar v(s_{ij}). \tag{11}$$

Therefore $\bar v(z_{ij}^m)=\bar v(s_{ij})$. Hence, by (6) and (11),

$$\bar v(mz_{ij}^{m-1}) = \bar v(m)+\frac{m-1}{m}\bar v(z_{ij}^m) \le \bar v(m)+\bar v(s_{ij}) < \frac{1}{2}k \le \frac{1}{2}\bar v(z_{ij}^m - s_{ij}).$$

Since $\overline{E}$ is Henselian with respect to $\bar v$ (Lemmas 8.6 and 7.5) we may apply the Hensel-Rychlik lemma to $Z^m - s_{ij}$ and obtain $c_{ij}\in\overline{E}^\times$ such that $c_{ij}^m = s_{ij}$, $i=1,\dots,r$. It follows that $\varphi(a_{ij})=\bar\varphi(b_{ij}c_{ij})^m\in\Phi^m$. This means that $\mathrm{Res}_{E/K}\theta\in H$.

PART C: $H\subseteq \mathrm{Res}_{E/K}X(E)$. *Let $\theta=(\pi,\varphi)\in H$. Extend (K,θ) to a Θ-closure $(\overline{K},\bar\theta)$, with $\bar\theta=(\bar\pi,\bar\varphi)$ (Proposition 8.7). By Lemma 8.6, $\overline{K}$ is p-adically closed. Hence $\bar\varphi$ induces an isomorphism of $\overline{K}^\times/(\overline{K}^\times)^m$ onto Φ/Φ^m (Lemma 7.6 and Lemma 6.8(c)). In particular, for each $i, 1\le i\le r$, there exist $j(i)$, $1\le j(i)\le n$, and $y_{i,j(i)}\in\overline{K}^\times$ such that $y_{i,j(i)}^m = a_{i,j(i)}$. Let $y_{ij}=0$ for each $j\ne j(i)$ and $x_{i1}=x_{i2}=0$. Then $\{(y_{i1},\dots,y_{in},x_{i1},x_{i2})|\ i=1,\dots,r)\}$ is a $\overline{K}$-rational simple point of V. Extend $\bar\pi$ to a $\mathbb{Q}_p$-place π_1 of $\overline{K}E$ (Proposition 6.4(c)). The p-adic closure $(\overline{E},\bar\pi)$ of $(\overline{K}E,\pi_1)$ has a unique Θ-site θ' whose restriction to $\overline{K}$ is the unique Θ-site $\bar\theta$ of $\overline{K}$ (Proposition 8.9). The p-adic closure $(\overline{E},\bar\pi)$ of $(\overline{K}E,\bar\pi_1)$ has a unique Θ-site θ' whose restriction to $\overline{K}$ is the unique Θ-site $\bar\theta$ of $\overline{K}$ (Proposition 8.9). Conclude that $\theta=\mathrm{Res}_{E/K}(\mathrm{Res}_{\overline{E}/E}\theta')\in \mathrm{Res}_{E/K}X(E)$.* ∎

LEMMA 13.6: *Let K be a field and let C be a closed subset of $X(K)$. Then there exists a regular extension E of K such that $\mathrm{Res}_{E/K}X(E)=C$, and $\mathrm{Res}_{E/K}\colon X(E)\to C$ has a continuous section.*

Proof: The set C is the intersection of open-closed sets, $C=\bigcap_{\lambda<m}H_\lambda$, where λ ranges over all ordinals smaller than some cardinal number m. For each $\mu\le m$ let $C_\mu=$

$\bigcap_{\lambda<\mu} H_\lambda$. Thus $C_0 = X(K)$ and $C_m = C$. If $\lambda < \lambda' \le m$, then $C_{\lambda'} \subseteq C_\lambda$. Denote the inclusion map $C_{\lambda'} \to C_\lambda$ by $i_{\lambda',\lambda}$. Finally let $E_0 = K$ and let s_0 be the identity map of $X(K)$.

Let $\mu \le m$. Suppose, by transfinite induction, that for each $\lambda < \mu$ we have constructed

(12a) a regular extension E_λ of K such that $\mathrm{Res}_{E_\lambda/K} X(E_\lambda) = C_\lambda$; and

(12b) a continuous section $s_\lambda\colon C_\lambda \to X(E_\lambda)$ of $\mathrm{Res}_{E_\lambda/K}$;

such that for every $\lambda \le \lambda' < \mu$

$$E_\lambda \subseteq E_{\lambda'} \text{ and } \mathrm{Res}_{E_{\lambda'}/E_\lambda} \circ s_{\lambda'} = s_\lambda \circ i_{\lambda',\lambda}. \tag{13}$$

If μ is a limit ordinal, let $E_\mu = \bigcup_{\lambda<\mu} E_\lambda$. Then E_μ/K is regular and $X(E_\mu) = \varprojlim_{\lambda<\mu} X(E_\lambda)$. Hence $\mathrm{Res}_{E_\mu/K} X(E_\mu) = \bigcap_{\lambda<\mu} C_\lambda = C_\mu$. Also, the maps $s_\lambda \circ i_{\mu,\lambda}$, with $\lambda < \mu$, define a section $s_\mu\colon C_\mu \to X(E_\mu)$ of $\mathrm{Res}_{E_\mu/K}$ such that $\mathrm{Res}_{E_\mu/E_\lambda} \circ s_\mu = s_\lambda \circ i_{\mu,\lambda}$, for every $\lambda < \mu$.

If $\mu = \lambda + 1$, then $C_\mu = C_\lambda \cap H_\lambda$. Hence $C'_\mu = \mathrm{Res}^{-1}_{E_\lambda/K}(C_\mu) = \mathrm{Res}^{-1}_{E_\lambda/K}(H_\lambda)$ is an open-closed subset of $X(E_\lambda)$ and $s_\lambda(C_\mu) \subseteq C'_\mu$. By Lemma 13.5, E_λ has a finitely generated regular extension E_μ such that $\mathrm{Res}_{E_\mu/E_\lambda} X(E_\mu) = C'_\mu$. By Proposition 13.4, $\mathrm{Res}_{E_\mu/E_\lambda}\colon X(E_\mu) \to C'_\mu$ has a continuous section $s'_\mu\colon C'_\mu \to X(E_\mu)$. Obviously $\mathrm{Res}_{E_\mu/K} X(E_\mu) = C_\mu$ and the map $s_\mu = s'_\mu \circ s_\lambda \circ i_{\mu,\lambda}$ is a continuous section of $\mathrm{Res}_{E_\mu/K}\colon X(E_\mu) \to C_\mu$ such that $\mathrm{Res}_{E_\mu/E_\lambda} \circ s_\mu = s_\lambda \circ i_{\mu,\lambda}$. Thus E_μ and s_μ satisfy the induction hypothesis.

Let $E = E_m$ and $s = s_m$. Then $\mathrm{Res}_{E/K} X(K) = C_m = C$ and $s\colon C \to X(E)$ is a continuous section of $\mathrm{Res}_{E/K}$. ∎

LEMMA 13.7: *Let K be a field and let C be a closed subset of $X(K)$. Then K has a regular extension E such that $\mathrm{Res}_{E/K}$ maps $X(E)$ homeomorphically onto C.*

Proof: Let $E_0 = K$ and $C_0 = C$. Suppose by induction that for $n \in \mathbb{N}$ there exists a tower $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n$ of regular extensions and for each i, $1 \le i \le n$, $\mathrm{Res}_{E_i/E_{i-1}}(X(E_i)) = C_{i-1}$ and $X(E_i)$ has a closed subset C_i which $\mathrm{Res}_{E_i/E_{i-1}}$ maps homeomorphically onto C_{i-1}. By Lemma 13.6, E_n has a regular extension E_{n+1} such *that* $\mathrm{Res}_{E_{n+1}/E_n}(X(E_{n+1})) = C_n$ *and* $\mathrm{Res}_{E_{n+1}/E_n}\colon X(E_{n+1}) \to C_n$ *has a continuous section* s_n. *Then* $C_{n+1} = s_n(C_n)$ *is a closed subset of* $X(E_{n+1})$, *and* $\mathrm{Res}_{E_{n+1}/E_n}$ *maps* C_{n+1} *homeomorphically onto* C_n.

Now let $E = \bigcup_{n=1}^{\infty} E_n$. *Then* $X(E) = \varprojlim C_n$. *Conclude that for each* n, Res_{E/E_n} *maps* $X(E)$ *homeomorphically onto* C_n. ∎

Definition 13.8: Recall that an extension of fields E/K is totally p-adic if

$$\mathrm{Res}_{E/K}\colon X(E) \to X(K)$$

is surjective (Section 12). We say that E/K is **exactly p-adic** if $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ is a homeomorphism.

The field K is **existentially closed** in E if each formula without quantifiers in the language of fields with coefficients in K which is satisfiable in E is satisfiable in K.

Lemma 13.9: *Let K be a field.*

(a) *If K is* PpC *(Definition 12.2), then K is existentially closed in every regular totally p-adic extension.*

(b) *If K is existentially closed in every regular exactly p-adic extension, then K is* PpC.

Proof of (a): *Let E be a regular totally p-adic extension of K. We have to show that if* $f_1, \ldots, f_r, g_1, \ldots, g_s \in K[X_1, \ldots, X_n]$ *and the system*

$$f_i(\mathbf{X}) = 0, \quad i = 1, \ldots, r; \qquad g_j(\mathbf{X}) \neq 0, \quad j = 1, \ldots, s \tag{14}$$

has a solution $\mathbf{x} \in E^n$, *then it also has a solution in* K^n. *Replace* $g_j(\mathbf{X}) \neq 0$, $j = 1, \ldots, s$, *if necessary, by the equation* $g_1(\mathbf{X}) \cdots g_s(\mathbf{X}) X_{n+1} - 1 = 0$ *to assume that* $s = 0$. *Since* $K(\mathbf{x})/K$ *is a regular extension,* $\mathbf{x}$ *generates over* K *an absolutely irreducible variety* V. *Since* $K(\mathbf{x})/K$ *is totally p-adic, Lemma 12.1(c) implies that* $V_{\mathrm{sim}}(\overline{K}) \neq \emptyset$ *for each p-adic closure* $\overline{K}$ *of* K. *Conclude from K being* PpC *that V has a K-rational point* $\mathbf{x}'$. *This point solves (14).*

Proof of (b): *Let V be an absolutely irreducible variety defined over K. Denote the function field of V by E and assume that E/K is totally p-adic. By Proposition 13.4,* *the surjective map* $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ *has a continuous section* s. *Its image* $s(X(K))$ *is closed in* $X(E)$. *By Lemma 13.7,* E *has a regular extension* F *such that* $\mathrm{Res}_{F/E}$ *maps* $X(F)$ *homeomorphically onto* $s(X(K))$. *Hence* $\mathrm{Res}_{F/K}\colon X(F) \to X(K)$ *is a homeomorphism. Thus* F/K *is a regular exactly* p*-adic extension. It follows that* K *is existentially closed in* F. *Since* V *has an* F*-rational point it also has a* K*-rational point.* ∎

PROPOSITION 13.10: *Let* L/K *be a Galois extension and let* C *be a closed subset of* $X(L/K)$ *which is closed under the action of* $\mathcal{G}(L/K)$. *Then there exists a Galois extension* F/E *such that* E *is a regular* PpC *extension of* K, $LE = F$, *the map* $\mathrm{Res}_{F/L}\colon \mathcal{G}(F/E) \to \mathcal{G}(L/K)$ *is an isomorphism, and* $\mathrm{Res}_{F/L}$ *maps* $X(F/E)$ *homeomorphically onto* C.

Proof: Let $C_0 = \mathrm{Res}_{L/K}(C)$. *Lemma 13.6 gives a regular extension* K' *of* K *such that* $\mathrm{Res}_{K'/K}$ *maps* $X(K')$ *homeomorphically onto* C_0. *Denote the class of regular exactly* p*-adic extensions of* K' *by* $\mathcal{E}$. *Clearly,* $\mathcal{E}$ *is closed under union of chains. Hence* $\mathcal{E}$ *has a member* E *which is existentially closed in each* $E' \in \mathcal{E}$ *that contains* E *[D, p. 28]. If* E'' *is an exactly* p*-adic extension of* E, *then* E'' *is an exactly* p*-adic extension of* K'. *Hence* $E'' \in \mathcal{E}$ *and therefore* E *is existentially closed in* E''. *Conclude from Lemma 13.9(b) that* E *is* PpC. *By construction* E *is a regular extension of* K *and* $\mathrm{Res}_{E/K}$ *maps* $X(E)$ *homeomorphically onto* C_0. *In particular, for* $F = LE$, $\mathrm{Res}_{F/L}\colon \mathcal{G}(F/E) \to \mathcal{G}(L/K)$ *is an isomorphism.*

If $\theta' \in X(F/E)$, *then* $\mathrm{Res}_{L/K}(\mathrm{Res}_{F/L}\theta') = \mathrm{Res}_{E/K}(\mathrm{Res}_{F/E}\theta') \in C_0$. *Since* $\mathrm{Res}_{L/K}\colon \mathbf{G}(L/K) \to \mathbf{G}(K/K)$ *is a cover and* C *is closed under the action of* $\mathcal{G}(L/K)$, *we have* $\mathrm{Res}_{F/L}\theta' \in C$. *Conversely, if* $\theta \in C$, *then there exists* $\theta_0' \in X(E)$ *such that* $\mathrm{Res}_{L/K}\theta = \mathrm{Res}_{E/K}\theta_0'$. *Extend* θ_0' *to* $\theta'' \in X(F/E)$. *Then* $\mathrm{Res}_{L/K}(\mathrm{Res}_{F/L}\theta'') = \mathrm{Res}_{L/K}(\theta)$. *Hence there exists* $\sigma \in \mathcal{G}(L/K)$ *such that* $(\mathrm{Res}_{F/L}\theta'')^\sigma = \theta$. *Extend* σ *to* $\sigma' \in \mathcal{G}(F/E)$. *Then* $\mathrm{Res}_{F/L}$ *maps* $(\theta'')^{\sigma'}$ *onto* θ. *If* $\mathrm{Res}_{F/L}$ *maps* $\theta_1', \theta_2' \in X(F/E)$ *onto the same element* $\theta \in X(L/K)$, *then, since* $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ *is injective, there exists* $\sigma' \in \mathcal{G}(F/E)$ *such that* $\theta_2' = (\theta_1')^{\sigma'}$. *Hence,* $\theta = \theta^\sigma$, *where* $\sigma = \mathrm{Res}_{F/L}\sigma'$. *Since the action of* $\mathcal{G}(L/K)$ *on* $X(L/K)$ *is regular,* $\sigma = 1$. *Hence* $\sigma' = 1$. *Thus* $\mathrm{Res}_{F/L}\colon X(F/K) \to C$ *is a bijective continuous map. Conclude that it is a homeomor-*

phism. ∎

The following Proposition is the p-adic analogue of a result of Craven [C, Thm 5] for spaces of orderings.

PROPOSITION 13.11: *For every Boolean space X there exists a* PpC *field E such that $X(E)$ is homeomorphic to X.*

Proof: By Proposition 13.10 it suffices to construct a field K and an embedding of X into $X(K)$. Since every Boolean space is homeomorphic to a closed subset of the space $\{\pm 1\}^T$, for a suitable set T [HJ, Definition 1.1], we may assume that $X = \{\pm 1\}^T$. Assume without loss that T is an ordered set of algebraically independent elements over $\mathbb{Q}$ and let $E = \mathbb{Q}(T)$. Denote the unique Θ-site of $\mathbb{Q}$ by θ. For each $\varepsilon \in X$ (i.e., ε: $T \to \{\pm 1\}$) let $\theta_\varepsilon = (\pi_\varepsilon, \varphi_\varepsilon) \in X(E)$ be the unique extension of θ to E such that $\pi_\varepsilon(at) = 0$ for each $t \in T$ and each $a \in \mathbb{Q}(t_0 |\, t_0 < t)$, and $\varphi_\varepsilon(t) = \varepsilon(t)t$ (Lemma 13.2).

The map $\varepsilon \mapsto \theta_\varepsilon$ from X into $X(E)$ is obviously injective. To show that it is continuous consider $a_1, \ldots, a_n \in E$. Let T_0 be a finite subset of T such that $a_1, \ldots, a_n \in \mathbb{Q}(T_0)$. If two elements $\varepsilon, \varepsilon' \in X$ coincide on T_0, then $\varphi_\varepsilon(t) = \varphi_{\varepsilon'}(t)$ for each $t \in T_0$. By the uniqueness part of Lemma 13.2, $\theta_\varepsilon = \theta_{\varepsilon'}$. Conclude from Lemma 10.5 that the map $\varepsilon \mapsto \theta_\varepsilon$ is continuous. ∎

14. Realization of $G(\mathbb{Q}_p)$-structures as $\mathbf{G}(F/E)$.

PROPOSITION 14.1: *Let L/K be a Galois extension, $\mathbf{G}$ a $G(\mathbb{Q}_p)$-structure and $\alpha\colon \mathbf{G} \to \mathbf{G}(L/K)$ an epimorphism. Then there exists a Galois extension F/E such that E is a regular* PpC *totally p-adic extension of K, $L \subseteq F$ and there exists a commutative diagram*

(1)
$$\begin{tikzcd} \mathbf{G} \arrow[rr, "\rho"] \arrow[dr, "\alpha"'] & & \mathbf{G}(F/E) \arrow[dl, "\mathrm{Res}_{F/L}"] \\ & \mathbf{G}(L/K) & \end{tikzcd}$$

in which ρ is an epimorphism of $G(\mathbb{Q}_p)$-structures and the underlying map of groups $\rho\colon G \to \mathcal{G}(F/E)$ is an isomorphism. Moreover, if the forgetful map of $\mathbf{G}$ is injective, then $\rho\colon \mathbf{G} \to \mathbf{G}(F/E)$ is an isomorphism.

Proof: It suffices to prove the existence of a commutative diagram (1) such that E is a regular extension of K, F/E is Galois, $L \subseteq F$, $\rho\colon \mathbf{G} \to \mathbf{G}(F/E)$ is a morphism and $\rho\colon G \to \mathcal{G}(F/E)$ is an isomorphism. Indeed, use Proposition 13.10 to construct a Galois extension F'/E' such that E' is a regular PpC extension of E, $FE' = F'$ and

$$\mathrm{Res}_{F'/F}\colon \langle \mathcal{G}(F'/E'), X(F'/E'), d\rangle \longrightarrow \langle \mathcal{G}(F/E), \rho(X(\mathbf{G})), d\rangle$$

is an isomorphism of $G(\mathbb{Q}_p)$-structures. Then replace ρ, E and F in (1), respectively, by $\rho' = \mathrm{Res}_{F'/F}^{-1} \circ \rho$, E' and F' to obtain a commutative diagram with the required conditions. Note that since α is an epimorphism, so is $\mathrm{res}_{F/L}\colon X(E) \to X(K)$. Hence E'/K is a totally p-adic extension. Also, if the forgetful map of $\mathbf{G}$ is injective and for $x, x' \in X(\mathbf{G})$, $\rho'(x) = \rho'(x')$, then $\rho' \circ d(x) = \rho' \circ d(x')$. Hence $d(x) = d(x')$ and $x = x'$. Thus in this case ρ' is an isomorphism of $G(\mathbb{Q}_p)$-structures.

The rest of the proof splits into five parts.

PART A: *Reduction to the case where α is rigid (Definition 5.5).* Let L' be a Galois extension of K that contains L such that $D(\theta) \cong G(\mathbb{Q}_p)$ for each $\theta \in X(L'/K)$. For example, by Lemma 10.8(b), this is the case for $L' = \widetilde{K}$. Construct a cartesian square

(Lemma 2.1):

$$\begin{tikzcd} \mathbf{G}' \arrow[r, "\alpha'"] \arrow[d, "\pi"'] & \mathbf{G}(L'/K) \arrow[d, "\mathrm{Res}_{L'/L}"] \\ \mathbf{G} \arrow[r, "\alpha"] & \mathbf{G}(L/K) \end{tikzcd}$$

Then α' is a rigid morphism (i.e., $\alpha'\colon D(y') \to D(\alpha'(y'))$ is an isomorphism for each $y' \in X(\mathbf{G}')$). Suppose that there is a Galois extension F'/E such that E is a regular extension of K, $L' \subseteq F'$, and there is a morphism $\rho'\colon \mathbf{G}' \to \mathbf{G}(F'/E)$ such that $\rho'\colon G' \to \mathcal{G}(F'/E)$ is an isomorphism of groups, and the upper face of the following diagram commutes:

$$\begin{tikzcd} \mathbf{G}' \arrow[rr, "\rho'"] \arrow[dr, "\alpha'"] \arrow[dd, "\pi"'] & & \mathbf{G}(F'/E) \arrow[dl, "\mathrm{Res}_{F'/L'}"'] \arrow[dd, "\mathrm{Res}_{F'/F}"] \\ & \mathbf{G}(L'/K) \arrow[dd, "\mathrm{Res}_{L'/L}"] & \\ \mathbf{G} \arrow[rr, "\rho"] \arrow[dr, "\alpha"'] & & \mathbf{G}(F/E) \arrow[dl, "\mathrm{Res}_{F/L}"] \\ & \mathbf{G}(L/K) & \end{tikzcd}$$

Since $\mathrm{Res}_{L'/L}\colon \mathbf{G}(L'/K) \to \mathbf{G}(L/K)$ is a cover, so is π (Lemma 2.2). Let F be the fixed field of $\rho'(\mathrm{Ker}(\pi))$. Then ρ' induces a morphism ρ such that the back face of (2) commutes. Also

$$\begin{aligned} \mathrm{Res}_{F'/L}(\mathcal{G}(F'/F)) &= \mathrm{Res}_{L'/L} \circ \mathrm{Res}_{F'/L'} \circ \rho'(\mathrm{Ker}(\pi)) \\ &= \mathrm{Res}_{L'/L} \circ \alpha'(\mathrm{Ker}(\pi)) = \alpha \circ \pi(\mathrm{Ker}(\pi)) = 1. \end{aligned}$$

Hence $L \subseteq F$ and the right face of (2) commutes. Conclude from the surjectivity of π that the lower face of (2) commutes.

So we may assume that α is a rigid morphism.

PART B: *Definition of E and F.* *Let $\mathcal{N}$ be the family of open normal subgroups N of G for which the induced morphism $\alpha_N\colon \mathbf{G}/N \to \mathbf{G}(L/K)/\alpha(N)$ is rigid. By Lemma 5.6, $\mathcal{N}$ is a basis for the open neighborhoods of 1 in G. For each $N \in \mathcal{N}$ choose $a_N \in L$*

such that $K(a_N)$ is the fixed field of $\alpha(N)$ in L. Thus $\alpha_N\colon \mathbf{G}/N \to \mathbf{G}(K(a_N)/K)$ is a rigid morphism.

Let $\mathcal{C} = \{N\sigma |\ N \in \mathcal{N},\ \ \sigma \in G\}$ and let $T = \{t_C |\ C \in \mathcal{C}\}$ be a set of algebraically independent elements over L. Define an action of G on $F = L(T)$ by the following rules:

$$\begin{cases} z^\sigma = z^{\alpha(\sigma)}, & z \in L \text{ and } \sigma \in G; \text{ and} \\ (t_C)^\sigma = t_{C\sigma} & C \in \mathcal{C} \text{ and } \sigma \in G. \end{cases}$$

Then G acts faithfully on T and therefore also on F. The stabilizer of $z \in L$ is $\alpha^{-1}(\mathcal{G}(L/K(z)))$ and the stabilizer of $t_{N\sigma}$ is N. Both are open subgroups of G. Hence the stabilizer of each element of F is open in G.

Let E be the fixed field of G in F. By [W, Thm. 1] there exists an isomorphism $\rho\colon G \to \mathcal{G}(F/E)$ compatible with the action on F. In particular the following diagram of groups commutes: Since G acts on L as $\mathcal{G}(L/K)$, we have $L \cap E = K$. Since F/L is a purely transcendental extension EL/L is regular. Hence E/K is also regular.

PART C: *Purely transcendental extensions.* *Let $x \in X(\mathbf{G})$. Denote the fixed field of $\rho(D(x))$ (resp., $\alpha(D(x))$) in F (resp., L) by M' (resp., M). We prove that M' is a purely transcendental extension of M.*

Indeed, the commutativity of (3) implies that $\mathrm{Res}_{F/L}(\rho(D(x))) = \alpha(D(x))$, therefore $M \subseteq M'$. Since α is injective on $D(x)$ and ρ is an isomorphism, $\mathrm{Res}_{F/L}\colon \mathcal{G}(F/M') \to \mathcal{G}(F/M)$ is an isomorphism. Thus $LM' = F$ and $L \cap M' = M$.

The group $D(x)$ acts on T (as a subgroup of G). Let $\mathcal{T}_x$ be the collection of $D(x)$-orbits of T. Each $S \in \mathcal{T}_x$ has the form $S = \{t_{N\sigma\delta} |\ \delta \in D(x)\}$, with $N \in \mathcal{N}$ and $\sigma \in G$. Since N is the stabilizer of each element of S, $|S| = (D(x) : D(x) \cap N) = (D(x)N : N)$. So, if $\delta_1, \ldots, \delta_n \in D(x)$ represent $D(x)N/N$, then $S = \{t_{N\sigma\delta_i} | i = 1, \ldots, n\}$ with $n = |S|$. Let

$$u_{S,j} = \sum_{i=1}^{n} a_N^{(j-1)\delta_i} t_{N\sigma\delta_i}, \qquad j = 1, \ldots, n. \tag{4}$$

Since N acts trivially on a_N, the right hand side of (4) is independent of the choice of $\delta_1, \ldots, \delta_n$. In particular $D(x)$ acts trivially on $u_{S,j}$. So $u_{S,j} \in M'$, $j = 1, \ldots, n$.

Since $\alpha_N\colon \mathbf{G}/N \to \mathbf{G}(K(a_N)/K)$ is rigid (Part B), α_N maps $D(x)N/N$ injectively into $\mathcal{G}(K(a_N)/K)$. In particular $a_N^{\delta_1}, \ldots, a_N^{\delta_n}$ are distinct. Hence the coefficients matrix

$(a_N^{(j-1)\delta_i})_{i,j=1}^n$ *of the linear system (4), which is a Vandermonde matrix, is invertible. It follows that*

$$L(u_{S,j}\,|\; j=1,\ldots,n) = L(S). \tag{5}$$

Since S is a set of n algebraically independent elements over L, (5) implies that $u_{S,j}$, $j = 1,\ldots,n$, are also algebraically independent over L.

Let $U_x = \{u_{S,j}\,|\; S \in \mathcal{T}_x$ and $j = 1,\ldots,|S|\}$. Since $L(S)$, $S \in \mathcal{T}_x$, are free over L, the elements of U_x are algebraically independent over L. Moreover, by (5), $L(U_x) = L(T) = F$. Hence, the linear disjointness of L and M' over M gives $[M' : M(U_x)] = [F : L(U_x)] = 1$. Conclude that $M' = M(U_x)$ is purely transcendental over M.

PART D: *Definition of ρ: $X(\mathbf{G}) \to X(F/E)$.* *Fix an ordering of F (as a set). For each $x \in X(\mathbf{G})$, it induces an ordering of U_x (We use the notation of Part C). By (2) of Section 10, $\alpha(x) \in X(L/M)$. Define $\rho(x)$ to be the infinitesimal extension of $\alpha(x)$ with respect to U_x (Definition 13.3). Then $\rho(x) \in X(L(U_x)/M(U_x)) = X(F/M') \subseteq X(F/E)$, and $\mathrm{Res}_{F/L} \circ \rho(x) = \alpha(x)$. The images of both homomorphisms $\rho \circ d(x)$ and $d(\rho(x))$ from $G(\mathbb{Q}_p)$ into $\mathcal{G}(F/E)$ are contained in $\mathcal{G}(F/M')$. Moreover,*

$$\mathrm{Res}_{F/L} \circ \rho \circ d(x) = \alpha \circ d(x) = d(\alpha(x)) = d(\mathrm{Res}_{F/L}(\rho(x))) = \mathrm{Res}_{F/L} \circ d(\rho(x)).$$

Since $\mathrm{Res}_{F/L}$ is injective on $\mathcal{G}(F/M')$, we have $\rho \circ d(x) = d(\rho(x))$.

PART E: *Continuity of ρ: $X(\mathbf{G}) \to X(F/E)$.* *Let $x \in X(\mathbf{G})$. Each open neighborhood of $\rho(x) = (\pi_x, \varphi_x)$ in $X(F/E)$ contains a basic open neighborhood of the form*

$$V = \{(\pi,\varphi) \in X(F/E)|\; \varphi(a_i) \in V_i,\;\; i = 1,\ldots,k\}$$

for some $a_1,\ldots,a_k \in F^\times$ and open subsets $V_1,\ldots,V_k$ of $\widetilde{\Phi}$ (Lemma 10.3(b)). Since $F = L(U_x)$ there are $u_i = u_{S(i),j(i)} \in U_x$, $i = 1,\ldots,r$, such that $a_1,\ldots,a_k \in L(u_1,\ldots,u_r)$. Let N_i be the stabilizer of the elements of $S(i)$, $i = 1,\ldots,r$. There exists an open neighborhood W_1 of x in $X(\mathbf{G})$ such that for each $z \in W_1$, $D(x)N_i = D(z)N_i$, $i = 1,\ldots,r$. Hence $S(1),\ldots,S(r) \in \mathcal{T}_z$ and $u_1,\ldots,u_r \in U_z$. Let $F_0 = L(u_1,\ldots,u_r)$ and $E_0 = K(u_1,\ldots,u_r)$. The definition of $\rho(x)$ and $\rho(z) = (\pi_z,\varphi_z)$ imply that

$\mathrm{Res}_{F/F_0}(\rho(x))$ and $\mathrm{Res}_{F/F_0}(\rho(z))$ are respectively the infinitesimal extensions of $\alpha(x)$ and $\alpha(z)$ to $X(F_0/E_0)$ with respect to $\{u_1,\dots,u_r\}$. Since α and the infinitesimal extension map from $X(L/K)$ into $X(F_0/E_0)$ are continuous (Lemma 13.2) W_1 contains an open neighborhood W_2 of x such that if $z\in W_2$, then $\varphi_z(a_i)\in V_i$, $1,\dots,k$. Therefore $\rho(z)\in V$. Conclude that ρ: $X(\mathbf{G})\to X(F/E)$ is continuous.

PART F: *Conclusion of the proof.* *We still have to ensure that $\rho(x^\sigma)=\rho(x)^{\rho(\sigma)}$ for all $x\in X(\mathbf{G})$ and $\sigma\in G$. Unfortunately this need not be the case. So we have to modify the definition of ρ: $X(\mathbf{G})\to X(F/E)$. By Lemma 2.5, $X(\mathbf{G})$ has a closed system X of representatives for the G-orbits. Denote the restriction of ρ: $X(\mathbf{G})\to X(F/E)$ and α: $X(\mathbf{G})\to X(L/K)$ to X by ρ_0 and α_0, respectively. By Part D, $\mathrm{Res}_{F/L}\circ\rho_0(x)=\alpha_0(x)$ and $d(\rho_0(x))=\rho(d(x))$ for each $x\in X$. Hence, by Lemma 2.7, ρ_0 extends to a map of $X(\mathbf{G})$ into $X(F/E)$ which, together with the group isomorphism ρ: $G\to\mathcal{G}(F/E)$, is a morphism ρ: $\mathbf{G}\to\mathbf{G}(F/E)$ (this is the modified ρ). Moreover, both $\mathrm{Res}_{F/L}\circ\rho$ and α coincide on X with ρ_0 and on G with α_0. Hence, by Lemma 2.7, $\mathrm{Res}_{F/L}\circ\rho=\alpha$.*

The modified morphism ρ satisfies the requirements of the proposition. ∎

COROLLARY 14.2: *Let $\mathbf{G}$ be a $G(\mathbb{Q}_p)$-structure. Then there exists a PpC field E and a Galois extension F of E such that $\mathbf{G}\cong\mathbf{G}(F/E)$.*

Proof: The quotient space $X(\mathbf{G})/G$ is Boolean. Hence, by Proposition 13.11, there exists a PpC field K such that $X(K)\cong X(\mathbf{G})/G$. This isomorphism defines a cover α: $\mathbf{G}\to\mathbf{G}(K/K)$. By Proposition 14.1, K has a PpC extension E which has a Galois extension F, and there exists an epimorphism ρ: $\mathbf{G}\to\mathbf{G}(F/E)$ with a trivial kernel such that the diagram (1), with $L=K$, commutes. Since α is a cover, so is ρ. Hence ρ is indeed an isomorphism. ∎

15. The main results.

We are finally able to characterize the p-adically projective groups as absolute Galois groups of PpC fields. An analogous characterization holds for projective $G(\mathbb{Q}_p)$-structures.

THEOREM 15.1: *Let K be a PpC field. Then*

(a) *$G(K)$ is a p-adically projective group; and*

(b) *$\mathbf{G}(K)$ is a projective $G(\mathbb{Q}_p)$-structure.*

Proof: Let $X = X(\widetilde{K}/K)$ and let $\mathcal{D}$ be the collection of all subgroups $G(M)$ of $G(K)$ where M is a p-adically closed field and $K \subseteq M \subseteq \widetilde{K}$. By Lemma 10.8(d) and (e) the forgetful map d: $X \to \mathrm{Hom}(G(\mathbb{Q}_p), G(K))$ is injective and $\mathcal{D} = \{D(\theta) |\ \theta \in X\}$ is the collection of all decomposition groups of the elements of X. In particular $\mathcal{D}$ is a closed conjugacy domain of subgroups of $G(K)$. Also, for each $\theta, \theta' \in X$, $D(\theta) = D(\theta')$ if and only if there exists $\sigma \in D(\theta)$ such that $\theta^\sigma = \theta'$ (Lemma 10.8(c)). We show that $G(K)$ is $\mathcal{D}$-projective (Definition 4.1).

Consider a finite embedding problem for $G(K)$

$$\begin{array}{ccc} & & G(K) \\ & & \big\downarrow \mathrm{res} \\ B & \xrightarrow{\alpha} & \mathcal{G}(L/K) \end{array}$$

with L/K a finite Galois extension. Let X_0 be a closed system of representatives for the $\mathcal{G}(L/K)$-orbits of $X(L/K)$ (Corollary 2.5). Since $\mathcal{G}(L/K)$ is finite the subset $\{d(\theta) |\ \theta \in X_0\}$ of $\mathrm{Hom}(G(\mathbb{Q}_p), \mathcal{G}(L/K))$ is finite. Let $\overline{\psi}_1, \dots, \overline{\psi}_n$ be a listing of its elements. Choose $\bar{\theta}_i \in X_0$ such that $d(\bar{\theta}_i) = \psi_i$ and let $\theta_i \in X$ be an extension of $\bar{\theta}_i$, $i = 1, \dots, n$. By Remark 4.2 there exists $\psi_i \in \mathrm{Hom}(G(\mathbb{Q}_p), B)$ such that $\alpha \circ \psi_i = \mathrm{res} \circ d(\theta_i) = \overline{\psi}_i$, $i = 1, \dots, n$ (Remark 4.2). Define a map d_0: $X_0 \to \mathrm{Hom}(G(\mathbb{Q}_p), B)$ by the rule, $d_0(\theta) = \psi_i$ if and only if $d(\theta) = \bar{\psi}_i$. Since d is continuous, so is d_0. By Lemma 2.6, there is a $G(\mathbb{Q}_p)$-structure $\mathbf{B}$ with B the underlying group, X_0 a closed system of representatives for the B-orbits of $X(\mathbf{B})$, and such that the forgetful map extends d_0. The epimorphism α together with the identity map define a cover α: $\mathbf{B} \to \mathbf{G}(L/K)$ (Lemma 2.7).

Apply Proposition 14.1 to construct a Galois extension F/E such that E is a regular totally p-adic extension of K and $L \subseteq F$, and a commutative diagram of groups

$$\begin{tikzcd} B \arrow[rr, "\rho"] \arrow[dr, "\alpha"'] & & \mathcal{G}(F/E) \arrow[dl, "\mathrm{res}"] \\ & \mathcal{G}(L/K) & \end{tikzcd}$$

such that ρ is an isomorphism. There will be no loss to assume that $B = \mathcal{G}(F/E)$ and $\alpha = \mathrm{res}_{F/L}$. Also, replace E and F, if necessary, by a sufficiently large finitely generated subextensions of K and L, to assume that E is finitely generated over K.

Let z be a primitive element for F/E, let $f = \mathrm{irr}(z, E)$ and let $c \in E$ be the discriminant of f. Take an integrally closed domain R, finitely generated over K, which contains c^{-1} and the coefficients of f, and such that E is the quotient field of R. By definition of PpC *field (Definition 12.2) there exists a K-homomorphism $\psi\colon R \to K$. Let S be the integral closure of R in F (and note that $L \subseteq S$). Extend ψ to an L-homomorphism $\psi\colon S \to \widetilde{K}$. Let $D(\psi)$ be the decomposition group of ψ in $\mathcal{G}(F/E)$ and let M be the splitting field of the polynomial $\psi(f)$ over K. Then $L \subseteq M$, and $\psi(f)$ has no multiple roots, since $\psi(c) \neq 0$. Then M/K is a Galois extension and ψ induces an isomorphism $\psi_*\colon D(\psi) \to \mathcal{G}(M/K)$ such that $\psi(y)^{\psi_*(\sigma)} = \psi(y^\sigma)$ for each $\sigma \in D(\psi)$ and $y \in S$ [L1, p. 248]. The homomorphism $\psi_*^{-1} \circ \mathrm{res}_{\widetilde{K}/M}\colon G(K) \to \mathcal{G}(F/E)$ solves the embedding problem. Thus $G(K)$ is $\mathcal{D}$-projective.*

By Lemma 4.5(a), $\mathcal{D}$ is the collection $\mathcal{D}(G(K))$ of all closed subgroups of $G(K)$ isomorphic to $G(\mathbb{Q}_p)$. In particular $G(K)$ is p-adically projective. For each $\theta, \theta' \in X$, $D(\theta) = D(\theta')$ if and only if there exists $\sigma \in G(K)$ such that $\theta^\sigma = \theta'$ (Lemma 10.8). Since the forgetful map of $\mathbf{G}(K)$ is injective, the last statement of Proposition 5.4 implies that $\mathbf{G}(K)$ is a projective $G(\mathbb{Q}_p)$-structure. ∎

The proof of Theorem 15.1 gives an additional information on PpC *fields.*

Corollary 15.2: *Let K be a* PpC *field. Then a closed subgroup H of $G(K)$ is isomorphic to $G(\mathbb{Q}_p)$ if and only if its fixed field M is p-adically closed.*

Now we prove the converse of Theorem 15.1.

THEOREM 15.3: *Let $\mathbf{G}$ be a projective $G(\mathbb{Q}_p)$-structure. Let L/K be a Galois extension and $\alpha\colon \mathbf{G} \to \mathbf{G}(L/K)$ an epimorphism. Then there exists a totally p-adic* PpC *extension E of K and a commutative diagram*

$$\begin{tikzcd} \mathbf{G} \arrow[rr, "\rho"] \arrow[dr, "\alpha"'] & & \mathbf{G}(E) \arrow[dl, "\mathrm{res}"] \\ & \mathcal{G}(L/K) & \end{tikzcd}$$

in which ρ is an isomorphism.

Proof: The forgetful map of $\mathbf{G}$ is injective (Lemma 5.3(a)). Hence, Proposition 14.1 gives a totally p-adic PpC extension E_1 of K, a Galois extension F_1 of E_1 that contains L, and an isomorphism ρ_1 such that the following diagram commutes

$$\begin{tikzcd} \mathbf{G} \arrow[rr, "\rho_1"] \arrow[dr, "\alpha"'] & & \mathbf{G}(E) \arrow[dl, "\mathrm{res}"] \\ & \mathbf{G}(L/K) & \end{tikzcd}$$

Since $\mathrm{res}_{\widetilde{E}_1/F_1}\colon \mathbf{G}(E_1) \to \mathbf{G}(F_1/E_1)$ is a cover and $\mathbf{G}$ is projective, there exists a morphism $\alpha_1\colon \mathbf{G} \to \mathbf{G}(E_1)$ such that $\mathrm{res}_{\widetilde{E}_1/F_1} \circ \alpha_1 = \rho_1$ (Lemma 5.2). Since ρ_1 is an isomorphism, $\alpha_1\colon G \to G(E_1)$ and $\alpha_1\colon X(\mathbf{G}) \to X(\widetilde{E}_1/E_1)$ are injective.

Let K_1 be the fixed field of $\alpha_1(G)$ in $\widetilde{E}_1$. Then $\alpha_1(G) = G(K_1)$. We prove also that $\alpha_1(X(\mathbf{G})) = X(\widetilde{K}_1/K_1)$. Indeed, for each $x \in X(\mathbf{G})$, $D(\alpha_1(x)) = \alpha_1(D(x)) \le \alpha_1(G) = G(K_1)$. Hence, by (2) of Section 10, $\alpha_1(x) \in X(\widetilde{K}_1/K_1)$. Conversely, let $\theta \in X(\widetilde{K}_1/K_1)$. Take the unique closed subgroup H of G such that $\alpha_1(H) = D(\theta)$. Since $H \cong D(\theta) \cong G(\mathbb{Q}_p)$ (Lemma 10.8(b)) and since $\mathbf{G}$ is $G(\mathbb{Q}_p)$-projective there exists $x \in X(\mathbf{G})$ such that $D(x) = H$ (Lemma 5.3(c)). Thus $D(\theta) = \alpha_1(H) = D(\alpha_1(x))$. Hence, by Lemma 5.3(d), there exists $\sigma \in G$ such that $\theta = \alpha_1(x)^{\alpha_1(\sigma)} = \alpha_1(x^\sigma) \in \alpha_1(X(\mathbf{G}))$.

It follows that $\alpha_1\colon \mathbf{G} \to \mathbf{G}(K_1)$ is an isomorphism and the following diagram commutes

$$\begin{tikzcd} \mathbf{G} \arrow[r, "\alpha_1"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\rho_1"] & \mathbf{G}(K_1) \arrow[d, "\mathrm{res}"] \\ \mathbf{G}(L/K) & \mathbf{G}(F_1/E_1) \arrow[l, "\mathrm{res}"] \end{tikzcd}$$

We do not know if K_1 is PpC*. So we proceed as follows.*

First observe that since α_1 and ρ_1 are isomorphisms, so is $\mathrm{res}_{\widetilde{K}_1/F_1}\colon \mathbf{G}(K_1) \to \mathbf{G}(F_1/E_1)$*. In particular K_1 is totally p-adic over E_1. Repeat the above construction (with α_1 instead of α) and use induction to obtain an ascending chain of fields $K = K_0 \subseteq E_1 \subseteq K_1 \subseteq E_2 \subseteq \cdots$, and isomorphisms $\alpha_i\colon \mathbf{G} \to \mathbf{G}(K_i)$, $i = 1, 2, \ldots$ such that*

(2a) $\mathrm{res}_{\widetilde{K}_{i+1}/K_i} \circ \alpha_{i+1} = \alpha_i$, $i = 1, 2, \ldots$;

(2b) $E_1, E_2, \ldots$ are PpC fields; and

(2c) E_i/K_{i-1} and K_i/E_i are totally p-adic extensions (therefore so is E_{i+1}/E_i), $i = 1, 2, \ldots$.

Let $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} K_i$. The maps α_i define an isomorphism $\rho\colon \mathbf{G} \to \mathbf{G}(E)$ such that (1) commutes. In particular E/K is totally p-adic. Furthermore, E is PpC*. Indeed, let V be an absolutely irreducible variety defined over E with a function field F, totally p-adic over E. Then there exists $i \geq 1$ such that V is defined over E_i. By (2c), F/E_i is totally p-adic. Hence, the function field of V over E_i (which is a subfield of F) is totally p-adic over E_i. Since, by (2b), E_i is* PpC*, V has an E_i-rational point. This point is also E-rational. Conclude that E is* PpC. ∎

THEOREM 15.4: *For each $G(\mathbb{Q}_p)$-projective group G there exists a* PpC *field E such that $G(E) \cong G$.*

Proof: By Proposition 5.4(b) there exists a $G(\mathbb{Q}_p)$-projective structure $\mathbf{G}$ with G as the underlying group. Proposition 13.11 gives a field K such that $X(\mathbf{G})/G \cong X(K)$. This isomorphism defines a cover $\alpha\colon \mathbf{G} \to \mathbf{G}(K/K)$. Theorem 15.3 gives a P$p$C *field E and an isomorphism $\rho\colon \mathbf{G} \to \mathbf{G}(E)$. In particular $G(E) \cong G$.* ∎

A well known theorem of Artin-Schreier says that each field K with $G(K) \cong \mathbb{Z}/2\mathbb{Z}$ is real closed. The p-adic analogue is unknown.

PROBLEM 15.5: *Is each field K with $G(K) \cong G(\mathbb{Q}_p)$ p-adically closed?*

This question has an affirmative answer if K is algebraic over $\mathbb{Q}$ (Neukirch's theorem [N2]) or K is algebraic over a PpC *field (Corollary 15.2). L. Pop [P] generalizes Neukirch's theorem to the case where $\tilde{\mathbb{Q}}K = \widetilde{K}$.*

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